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THE ANALYSIS OF FLOOD LEVEE RELIABILITY

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Introduction

In the design of a flood levee system, the height of the levee is usually used as the principal measure of flood protection and the principal design parameter. Practice has shown, though, that most levee systems do not fail by overtopping but by structural weaknesses, either in the levee or in the soil near it. Bogardi and Zoltán [1] have identified four common modes of failure. These are:

1) Overtopping: the elevation of the flood wave exceeds that of the levee;

2) Structural failure of the levee by water saturation and loss of soil stability: the flood wave causes increased saturation of the levee and an increased pressure gradient through the levee. Decrease in soil strength is associated with increased saturation which, with the increase in the pressure gradient from the height of the flood wave, leads to levee failure through slumping;

3) Boils and hydraulic soil failures: the height of the flood wave and its resulting pressure is transmitted through the foundation soil under the levee and can cause soil failure through rupturing. The ensuing failure usually leads to large inflows of water into the protected areas and to the undermining of the levee's foundation;

4) Wave action: high flood levels give rise to wave action which scours the top of the levee. Such scouring reduces levee strength and causes premature failure.

These four modes of failure are illustrated in Figure 1.

Traditionally, levee design procedures use the height of the levee as the principal measure of flood protection; and the dimensioning of the levee, to protect against failure other than by overtopping, is regarded as being of secondary importance. Yet, most levee failures are not caused by overtopping. The failure of a levee under the load of a particular flood wave depends not only upon the height and shape of the levee, two possible decision variables, but also upon the hydraulic, geologic, and soil properties that vary within and along the levee. The latter variables are random variables; thus the resistance of the levee to floods is a random variable.

This paper looks at the effect upon decisions when resistance of the levee system is considered a random variable.

The analysis considers the load upon the levee due to floods that have been generated by some stochastic process. The levee is defined by two decision variables, the height H and the base width W. The flood discharge at which failure occurs, q_o, is considered as a fixed but unknown quantity and is represented by a probability distribution function $f(q_o)$. The resistance of the levee, therefore, depends upon the occurrence of floods of a particular magnitude and upon the "strength" of the levee. It is conceptually convenient to consider such uncertainty within the framework of Bayesian risk analysis (Cornell, [2]). Higher flood resistance levels lead to higher and stronger levees, but such levee systems are extremely expensive and, if extended far enough, lead to lower net benefits. Certain tradeoffs exist between the objectives of levee reliability and economic benefits. These tradeoffs are particularly significant when the resistance of the levee is considered as a random variable. Bayesian decision analysis with multi-dimensional utility theory provides an adept tool for considering decision making when these tradeoff conditions exist.

A Bayesian analysis of flood levee reliability, flood damages and net benefits is considered and results are given for a typical example.

General Theory of Reliability Analysis

If the resistance of a levee system is deterministic at a given flood discharge $q_o = q_d$, then the reliability of the system against failure is easily found from the probability of failure:

$$p_f = \int_{q_d}^{\infty} f_Q(q)\,dq = 1 - F_Q(q_d) \tag{1}$$

where

p_f = probability of failure

$f_Q(q)$ = the probability density function of flood events,

$F_Q(q_d)$ = the cumulative density function of the resistance q_d.

The reliability of the system is just $1 - p_f$. When uncertainty exists in the parameters of the density function of flood events, the Bayesian procedures set forth by Wood et al. [6] apply. In this case

$$\tilde{p}_f = \int_{q_d}^{\infty} \int_{\underline{\theta}} f_Q(q|\underline{\theta}) \cdot f(\underline{\theta})\, d\underline{\theta}\, dq$$

$$= \int_{q_d}^{\infty} \tilde{f}_Q(q)\, dq = 1 - \tilde{F}_Q(q_d) \qquad (2)$$

where

$f_Q(q|\underline{\theta})$ = the probability density function of flood discharges, conditional upon the uncertain parameter set $\underline{\theta}$,

$f(\underline{\theta})$ = the joint distribution on the uncertain parameter set $\underline{\theta}$,

$\tilde{f}_Q(q)$ = the Bayesian distribution of flood discharges,

$\tilde{p}_f$ = the Bayesian probability of failure.

If the resistance of the levee system is uncertain and if the level of resistance, q_o (maximum discharge before levee failure), is described by the density function, $f_{Q_o}(q_o)$, then the probability of failure $\tilde{p}_f$ is found from

$$\tilde{p}_f = \int_{q_o=0}^{\infty} f_{Q_o}(q_o) \int_{q=q_o}^{\infty} \tilde{f}_Q(q)\, dq\, dq_o$$

$$= \int_{q_o=0}^{\infty} f_{Q_o}(q_o)\, [1 - \tilde{F}_Q(q_o)]\, dq_o \quad . \qquad (3)$$

If there exists uncertainty in the parameters of the resistance, $f_{Q_o}(q_o)$, then $f_{Q_o}(q_o)$ may be replaced by its Bayesian distribution, $\tilde{f}_{Q_o}(q_o)$.

The probability of failure of Equation (3) is the expected probability of failure, $E[\tilde{p}_f]$, of the density function for failure $f(\tilde{p}_f)$. This is shown by applying the principles of derived distribution theory in the following manner.

If two random variables, x and y, are functionally related, y = g(x), and if the function is monotonic and continuous, then the following relationships hold:

$$E[y^n] = \int_x g^n(x) \cdot f(x)\, dx \tag{4}$$

$$f(y) = f(x) \cdot \left|\frac{dx}{dy}\right| \quad . \tag{5}$$

Equations (4) and (5) provide a procedure to obtain the probability density function, $f(\tilde{p}_f)$, and its moments when the levee resistance, q_o, is uncertain and is treated as a random variable. The functional relationship between p_f and q_o is

$$p_f = 1 - F_Q(q_o) \quad . \tag{6}$$

The first moment, from Equation (4), is

$$E[p_f] = \int_{q_o=0}^{\infty} [1 - F_Q(q_o)] \cdot f_{Q_o}(q_o)\, dq_o \tag{7}$$

which is exactly Equation (3). The second moment is

$$E^2[p_f] = \int_{q_o=0}^{\infty} [1 - F_Q(q_o)]^2 \cdot f_{Q_o}(q_o)\, dq_o \quad , \tag{8}$$

and the variance of the failure probability, $V[p_f]$, is calculated from

$$V[p_f] = E[p_f{}^2] - E^2[p_f] \quad . \tag{9}$$

Bayesian Distribution of the Probability of Failure: Model of Flood Events

Consider the hypothetical streamflow trace presented in Figure 2. The discharges of interest are those flows greater than Q_b. It is assumed that the occurrence of independent events larger than Q_b can be described by a Poisson process (the time between events being exponentially distributed), with an average annual arrival rate ν. It is also assumed that the probability density function for the flows larger than Q_b can be represented by a shifted exponential distribution of the form

$$f(q|q \geq Q_b) = \alpha \exp(-\alpha z) \tag{10}$$

where

$$z = q - Q_b$$

$$z \geq 0 \quad .$$

This distribution is a fairly general form, since the upper tails of many distributions may be represented as being exponential. This proposed model has been used for extreme flood discharges by Shane and Lynn [3], Todorovic and Zelenharic [4], and Wood [5].

It can easily be shown (Wood, [5]) that the cumulative distribution of z is

$$F_Z(z) = 1 - \nu t \cdot \exp(-\alpha z) \tag{11}$$

if the following assumptions are valid: that the probability of exceeding z is small, and that the arrival rate of such events is small.

The Bayesian analysis of the flood frequency curve considers the uncertainty in the independent parameters ν and α. If the uncertainty in each of the parameters can be represented by a gamma-1 probability density function, that is,

$$f(\nu|u,s) \propto \exp(-s \cdot \nu) \cdot \nu^u \tag{12}$$

$$f(\alpha|v,\ell) \propto \exp(-\alpha \cdot \ell) \cdot \alpha^v \quad , \tag{13}$$

then the Bayesian distribution of flood discharge can be shown to be (Wood et al., [6])

$$\tilde{f}(z) = \bar{\alpha}\ \bar{\nu}t \left[1 + \frac{\bar{\alpha}z}{v+1}\right]^{-(v+2)} \tag{14}$$

where

$$\bar{\alpha} = E[\alpha] \quad ,$$

$$\bar{\nu} = E[\nu] \quad .$$

The Bayesian exceedance probability, $\tilde{G}_Z(z) = 1 - \tilde{F}_Z(z)$ is just

$$\tilde{G}_Z(z) = \bar{\nu} t \left[1 + \frac{\bar{\alpha} z}{v + 1}\right]^{-(v+1)} . \qquad (15)$$

The Bayesian probability density function and the Bayesian exceedance probability fully account for the parameter uncertainty in the model of flood discharges. In the remaining part of this paper, the exponential exceedance model developed here will be assumed to be the appropriate model for the underlying stochastic process for flood generation.

Model of Levee Resistance

The modes of failure, presented earlier, of a levee system can be divided into two groups. One group consists of failure due to the structural failure of the levee or the soil around it. The other group consists of failure by overtopping. If the levee is built such that the probability of failure of the first type is zero, then the probability distribution of the resistance can be modelled as a delta function of unit area at q_d, the design capacity of the levee system.

As the probability of failure when the discharge is less than q_d increases, the area under the delta function decreases and the cumulative density function of the resistance, evaluated at q_d, $F_R(q_o = q_d)$, increases. Thus, the probability density function for q_o will consist of two parts. One part is a density function for the probability of failure at failure discharge less than q_d, the levee design capacity; the second part consists of a delta function, of area $1 - F_R(q_o = q_d)$ at $q_o = q_d$, that accounts for levee failure by overtopping.

Probability of Structural Failure Uniformly Distributed

Assume that $f_R(z_o)$[1] for $z_o \leq z_d$ is uniformly distributed between z_m and z_d and that the area under the density function

---

[1] $f_R(z_o) = f_{Q_o}(q_o)$ if the condition $f_{Q_o}(q_o \leq Q_b) = 0$ holds, since $z = q - Q_b$, Q_b being a constant. For the remaining part of the paper z_o will replace q_o, z_d for q_d.

is a. Then

$$f_R(z_o) = \frac{a}{z_d - z_m} , \quad \text{for } z_m \leq z_o \leq z_d , \tag{16}$$

$$= \delta_{z_d}(1 - a) , \quad \text{for } z_o = z_d ,$$

$$= 0 \quad \text{otherwise;}$$

$\delta_{z_d}(1 - a)$ is interpreted as a delta function at z_d of area 1 - a.

From Equation (6) the Jacobian transform, $|dz_o/dp_f|$, is

$$\left|\frac{dz_o}{dp_f}\right| = (\bar{\alpha}\ \bar{\nu}t)^{-1} \left[1 + \frac{\bar{\alpha}z_o}{v + 1}\right]^{(v+2)} \tag{17}$$

$$= \bar{\alpha}^{-1}\ p_f^{-(\frac{v+2}{v+1})}\ (\bar{\nu}t)^{(\frac{1}{v+1})} .$$

The probability density function $f(p_f)$ is, from derived distribution theory,

$$f(p_f) = \frac{a}{z_d - z_m} \cdot \bar{\alpha}^{-1}\ p_f^{-(\frac{v+2}{v+1})}\ \bar{\nu}t^{(\frac{1}{v+1})} \tag{18}$$

$$\text{for } \bar{\nu}t\left[1 + \frac{\bar{\alpha}z_d}{v + 1}\right]^{-(v+1)} \leq p_f \leq \bar{\nu}t\left[1 + \frac{\bar{\alpha}z_m}{v + 1}\right]^{-(v+1)} , \text{ and}$$

$$f(p_f) = \delta_{p_f}(1 - a)$$

$$\text{for } p_f = \bar{\nu}t\left[1 + \frac{\bar{\alpha}z_d}{v + 1}\right]^{-(v+1)} .$$

$E[p_f]$ from the application of Equation (5) can be calculated to be

$$E[p_f] = \tilde{p}_f = \frac{-a\ \bar{\nu}t}{\bar{\alpha}(z_d - z_m)} \left(\frac{v + 1}{v}\right) \left\{ \left[1 + \frac{\bar{\alpha}z_m}{v + 1}\right]^{-v} \right. \tag{19}$$

$$\left. - \left[1 + \frac{\bar{\alpha}z_d}{v + 1}\right]^{-v} \right\} + (1 - a)\ \bar{\nu}t \left[1 + \frac{\bar{\alpha}z_d}{v + 1}\right]^{-(v+1)}$$

and the second moment $E[p_f^2]$ is just

$$E[p_f^2] = \frac{a(\bar{\nu}t)^2}{\bar{\alpha}(z_d - z_m)} \left(\frac{v+1}{2v+1}\right) \left\{ \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-2(v+1)} - \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-(2v+1)} \right\} + (1-a)(\bar{\nu}t)^2 \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-2(v+1)} . \quad (20)$$

Probability of Structural Failure Quadratically Distributed

Assume that $f_R(z_o)$ is distributed as follows:

$$\begin{aligned} f_R(z_o) &= b(z_o - z_\ell)^2 , && \text{for } z_m \le z_o \le z_d \\ &= \delta_{z_d}(1 - a) , && \text{for } z_o = z_d \\ &= 0 && \text{otherwise,} \end{aligned} \quad (21)$$

where

$a = F_R(z_o = z_d)$, the probability that failure will occur at $z_o \le z_d$,

$b = 3 \cdot a/[(z_d - z_\ell)^3 - (z_m - z_\ell)^3]$.

Using the Jacobian transform presented in the previous analysis, the probability density function $f(p_f)$ can be found. It is

$$f(p_f) = \left\{ \left[\left(\frac{\bar{\nu}t}{p_f}\right)^{\left(\frac{1}{v+1}\right)} - 1\right] \frac{v+1}{\bar{\alpha}} - z_\ell \right\}^2 \cdot \frac{b}{\bar{\alpha}}\, p_f^{-\left(\frac{v+2}{v+1}\right)} (\bar{\nu}t)^{\left(\frac{1}{v+1}\right)}$$

for $\bar{\nu}t\left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-(v+1)} \le p_f \le \bar{\nu}t\left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-(v+1)}$, and

$$f(p_f) = \delta_{p_f}(1 - a)$$

$$\text{for } p_f = \bar{\nu}t\left[1 + \frac{\bar{\alpha}z_d}{v + 1}\right]^{-(v+1)} . \tag{22}$$

The first moment $E[p_f]$ is, from derived distribution theory,

$$\begin{aligned} E[p_f] = \tilde{p}_f = \frac{\bar{\nu}t\ b}{\bar{\alpha}} \left(\frac{v+1}{v}\right) \Bigg\{ & (z_m - z_\ell)^2 \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-v} \\ & + 2 \left(\frac{z_m - z_\ell}{\bar{\alpha}}\right)\left(\frac{v+1}{v-1}\right) \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-(v-1)} \\ & + \frac{2}{\bar{\alpha}} 2 \frac{(v+1)}{(v-1)(v-2)} \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-(v-2)} \\ & - (z_d - z_\ell)^2 \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-v} \\ & - \frac{2}{\bar{\alpha}} (z_d - z_\ell)\left(\frac{v+1}{v-1}\right) \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-(v-1)} \\ & - \frac{2}{\bar{\alpha}} 2 \frac{(v+1)}{(v-1)(v-2)} \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-(v-2)} \Bigg\} \\ & + (1 - a)\ \bar{\nu}t \left[1 + \frac{\bar{\alpha}z_d}{v+1}\right]^{-(v+1)} . \end{aligned} \tag{23}$$

Similarly, the second moment, $E[p_f^2]$, can be calculated, and is:

$$\begin{aligned} E[p_f^2] = \frac{b(\bar{\nu}t)^2}{\bar{\alpha}} \frac{(v+1)}{(2v+1)} \Bigg\{ & (z_m - z_\ell)^2 \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-(2v+1)} \\ & + \left(\frac{z_m - z_\ell}{\bar{\alpha}}\right)\left(\frac{v+1}{v}\right) \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-2v} \\ & + \frac{(v+1)^2}{\bar{\alpha}^2 v(2v-1)} \left[1 + \frac{\bar{\alpha}z_m}{v+1}\right]^{-(2v-1)} \end{aligned}$$

$$- (z_d - z_\ell)^2 \left[1 + \frac{\bar{\alpha} z_d}{v + 1}\right]^{-(2v+1)}$$

$$- \frac{(z_d - z_\ell)}{\bar{\alpha}} \left(\frac{v + 1}{v}\right) \left[1 + \frac{\bar{\alpha} z_d}{v + 1}\right]^{-2v}$$

$$- \frac{(v + 1)^2}{\bar{\alpha}^2 v(2v - 1)} \left[1 + \frac{\bar{\alpha} z_d}{v + 1}\right]^{-(2v-1)} \Bigg\}$$

$$+ (1 - a)(\bar{\nu} t)^2 \left[1 + \frac{\bar{\alpha} z_d}{v + 1}\right]^{-2(v+1)} \quad . \qquad (24)$$

Bayesian Distribution of Damage with Uncertain Levee Resistance

In a manner similar to the analysis of the failure probability, the damage due to levee failure can also be considered. Assume that the damage function is of the form:

$$D(z, z_o | z_d) = c z_d^{.5} (z - z_o)^2 \quad ; \qquad (25)$$

then the expected damage for a known failure discharge, z_o, is just

$$E[D | z_o, z_d] = \int_z c z_d^{.5} (z - z_o)^2 f(z)\, dz$$

$$= \frac{A}{v - 1} (1 + \beta z_o)^{-(v-1)} \quad . \qquad (26)$$

Now let $E[D | z_o, z_d]$ be designated as D, where

$$A = \frac{2 z_d^{.5} \gamma}{\beta^3 v(v + 1)}$$

$$\gamma = (\bar{\nu} t)\, c\bar{\alpha}$$

$$\beta = \bar{\alpha}/(v + 1)$$

$$f(z) = \bar{\alpha}\, \bar{\nu} t [1 + \beta z]^{-(v+2)} \quad .$$

By using the Jacobian transform

$$\left|\frac{dz_o}{dD}\right| = \beta^{-1} A^{(\frac{1}{v-2})} D^{-(\frac{v}{v-1})} (v-1)^{-(\frac{v}{v-1})} , \quad (27)$$

the distribution of damage, due to the uncertain levee resistance, can be calculated from derived distribution theory. As in the analysis of failure probabilities, the distribution of damage will be calculated for two failure discharge distributions $f(z_o)$--one uniformly distributed and the other quadratically distributed.

$f(z_o)$ Uniformly Distributed

Let $f(z_o)$ be of the form

$$f(z_o) = \frac{a}{z_d - z_m} . \quad (28)$$

Then it can be shown that

$$f(D) = \frac{a}{z_d - z_m} \beta^{-1} (v-1)^{-(\frac{v}{v-1})} D^{-(\frac{v}{v-1})} A^{(\frac{1}{v-1})}$$

for $\frac{A}{v-1} (1 + \beta z_d)^{-(v-1)} \leq D \leq \frac{A}{v-1} (1 + \beta z_m)^{-(v-1)}$, and

$$f(D) = \delta_D (1 - a)$$

for $D = \frac{A}{v-1} (1 + \beta z_d)^{-(v-1)}$. (29)

The first moment $E[D|z_d]$ is, from derived distribution theory,

$$E[D|z_d] = \int_{z_o} D(z_o) \cdot f(z_o)\, dz_o \quad (30)$$

$$= (\frac{a A}{(z_d - z_m)}) \frac{1}{\beta(v-1)(v-2)} \left[(1 + \beta z_m)^{-(v-2)}\right.$$

$$- (1 + \beta z_d)^{-(v-2)} \Big]$$

$$+ (1 - a) \frac{A}{v - 1} (1 + \beta z_d)^{-(v-1)} \quad .$$

In a similar manner the second moment $E[D^2|z_d]$ can be calculated, and is:

$$E[D^2|z_d] = \frac{a}{(z_d - z_m)} \frac{A^2}{(v-1)^2 \beta(2v-3)} \cdot \left[(1 + \beta z_m)^{-(2v-3)} - (1 + \beta z_d)^{-(2v-3)} \right] + (1 - a) \frac{A^2}{(v-1)^2} (1 + \beta z_d)^{-2(v-1)} \quad . \tag{31}$$

<u>$f(z_o)$ Quadratically Distributed</u>

Let $f(z_o)$ be of the form

$$f(z_o) = b(z_o - z_\ell)^2 \; ; \tag{32}$$

then it can be shown that the distribution of damage $f(D)$ is

$$f(D) = \left\{ \left[\left(\frac{A}{(v-1)D}\right)^{\left(\frac{1}{v-1}\right)} - 1 \right] \cdot \frac{1}{\beta} - z_\ell \right\}^2 \cdot \frac{b}{\beta}(v-1)^{-\left(\frac{v}{v-1}\right)} D^{-\left(\frac{v}{v-1}\right)} A^{\left(\frac{1}{v-1}\right)}$$

for $\frac{A}{v-1} (1 + \beta z_d)^{-(v-1)} \leq D \leq \frac{A}{v-1} (1 + \beta z_m)^{-(v-1)}$, and

$$f(D) = \delta_D (1 - a)$$

for $D = \frac{A}{v-1} (1 + \beta z_d)^{-(v-1)}$. (33)

The first moment $E[D|z_d]$ can be calculated as:

$$E[D|z_d] = \frac{bA}{(v-1)}\frac{1}{(v-2)\beta}\left[(z_m - z_\ell)^2 (1+\beta z_m)^{-(v-2)} + \frac{2(z_m - z_\ell)}{\beta(v-3)}(1+\beta z_m)^{-(v-3)} + \frac{2}{\beta^2(v-3)(v-4)}(1+\beta z_m)^{-(v-4)} - (z_d - z_\ell)^2 (1+\beta z_d)^{-(v-2)} - \frac{2(z_d - z_\ell)}{\beta(v-3)}(1+\beta z_d)^{-(v-3)} - \frac{2}{\beta^2(v-3)(v-4)}(1+\beta z_d)^{-(v-4)}\right] + (1-a)\frac{A}{(v-1)}(1+\beta z_d)^{-(v-1)} \; ; \qquad (34)$$

and the second moment $E[D^2|z_d]$ as

$$E^2[D|z_d] = \frac{bA^2}{(2v-3)(v-1)^2\beta}\left[(z_m - z_\ell)^2 (1+\beta z_m)^{-(2v-3)} + \frac{(z_m - z_\ell)}{\beta(v-2)}(1+\beta z_m)^{-2(v-2)} + \frac{1}{\beta^2(v-2)(2v-5)}(1+\beta z_m)^{-(2v-5)} - (z_d - z_\ell)^2 (1+\beta z_d)^{-(2v-3)} - \frac{(z_d - z_\ell)}{\beta(v-2)}(1+\beta z_d)^{-2(v-2)} - \frac{1}{\beta^2(v-2)(2v-5)}(1+\beta z_d)^{-(2v-5)}\right] + (1-a)\frac{A^2}{(v-1)^2}(1+\beta z_d)^{-2(v-1)} \; . \qquad (35)$$

Illustrating Example

The analytical results of the previous sections can be easily applied to analyze the tradeoffs that exist among flood levee strength, levee reliability and flood benefits from levee construction. The decision-making aspects of these tradeoffs will be dealt with in a future paper.

A hypothetical area will be used for an illustrating example, but the numerical values for the functions are similar to those found in Wood et al., [6] for Woonsocket, Rhode Island, which is on the Blackstone River.

A model representing the probability density function for peak flood discharges was developed earlier in the paper and had the form

$$f(z) = \overline{\alpha\nu t}\left[1 + \frac{\alpha z}{v + 1}\right]^{-(v-2)} , \tag{36}$$

where all terms have been defined in Equations (13) and (14). For our example, the parameters have the following values.

$$\bar{\alpha} = .0001415 , \quad ft^{-3} \tag{37}$$

$$\bar{\nu} = .115 , \quad \text{flood events per year}$$

$$v = 7 , \quad \text{flood events} .$$

The peak flood discharges can be "converted" into a peak flood stage with a stage-discharge curve. Figure 2 shows the stage-discharge curve for the upstream end of the area to be protected. Since that area is assumed to be quite small (for example protection works for a city) and the length of the protecting levees is short, it is assumed that after construction of the levees, the upstream stage-discharge curve is still appropriate and that the flood wave attenuation in the levee system is insignificant.

The cost of flood protection is assumed to have the form

$$C(W,H) = K_1 W^{\beta_1} H^{\beta_2} + K_2 \tag{38}$$

where

W = base width of the levee,

H = height of the levee,

and K_1, K_2, β_1 and β_2 are constants. If $\beta_1 > 1.0$ and $\beta_2 > 1.0$, then the marginal cost for width increases with height and vice versa.

K_2 implies that the levees have a fixed cost representing planning, surveying, etc. An illustrative diagram of the cost surface is presented in Figure 3 for the following parameter values:

$$K_1 = \$40,000$$
$$K_2 = \$2 \times 10^6$$
$$\beta_1 = 1.25 \quad (39)$$
$$\beta_2 = 1.25 \ .$$

Those regions where the cost surface is zero represents infeasible sets of widths and heights.

The damage function used in the analysis had the form

$$D(z,z_o|z_d) = c\ z_d^{.5}\ (z - z_o)^2$$

where c is a constant. This form has two interesting properties. One, for the same level of exceedance $(z - z_o)$ higher protection leads to higher damage. This can arise from the feeling of security behind a comprehensive levee system and consequently developments of higher density and quality. Higher damages can also arise in part from higher reconstruction costs of the levee system if it is damaged. The second property of the damage curve that should be noticed is that damage increases quadratically with exceedance discharge. This is often observed in real situations (see Wood et al., [6]) and will be assumed to be appropriate for this discussion. An illustrative description of the two-dimensional damage function is presented in Figure 4.

Effect on the Return Period

The effect of levee strength on the flood frequency curve (return period, or 1/Probability of failure, versus design discharge) due to varying levee strengths is illustrated in Figures 5 and 6, where the expected return periods for various levee strengths are plotted.

It will be remembered that the following definitions hold:

1. a = the cumulative distribution $F_{Z_o}(z_o = z_d)$, for z_o, the exceedance discharge where failure occurs, evaluated at z_d, the design discharge.

2. The uniform density function for z_o had the form

$$
\begin{aligned}
f(z_o) &= \frac{a}{z_d - z_m} , && \text{for } z_m \leq z_o \leq z_d \\
&= \delta_{z_d} (1 - a) , && \text{for } z_m - z_d \\
&= 0 && \text{otherwise.}
\end{aligned}
\qquad (40)
$$

3. The quadratic density function for z_o had the form

$$
\begin{aligned}
f(z_o) &= b(z_o - z_\ell)^2 , && \text{for } z_m \leq z_o \leq z_d \\
&= \delta_{z_d} (1 - a) , && \text{for } z_m - z_d \\
&= 0 && \text{otherwise;}
\end{aligned}
\qquad (41)
$$

where

$$b = 3 \cdot a/\left[(z_d - z_\ell)^3 - (z_m - z_\ell)^3\right] .$$

It is interesting to note that for a design discharge of 35,000 cu ft/sec and a uniform failure probability density function, a levee that will fail only by overtopping (a = 0) has an expected return period of almost 200 years, while a levee that has a 90% chance of failing before overtopping has an expected return period of 70 years--only 1/3 the value of the former. With a quadratic failure probability distribution, a levee of the same "strength" has an expected return period of 90 years or about half that of the deterministic levee.

Equations (18) and (22) present the probability density functions for P_f, the probability of flood failure. Figures 7 and 8 illustrate the first part of these density functions at a design discharge of 27,500 and 35,000 cu ft/sec respectively for the uniform failure probability.

The second part of the density functions consists of a delta function which varies in area between 0 and 1 as $F_{Z_o}(z_o = z_d)$ varies between 1 and 0. This delta function is attached to the first part of the density functions at the upper end of the return period (i.e. 88 and 188.3 years respectively).

Effect on Costs and Benefits

For a given design discharge, the stochastic nature of the levee strength affects the cost of levee construction and the resulting flood benefits.

Figures 9 and 10 present the annual benefit and cost curves for the condition of a uniform and a quadratic failure probability density function, respectively. The design parameters for the levees are given in Table 1, and cost and damage coefficients in Table 2. Figures 11 and 12 show the net benefit curve for the uniform and the quadratic failure probability density function, respectively.

In terms of an efficiency criterion, the optimal decision is to build the levee system quite high but quite weak.

For the cost and damage functions used, the decision to build high, strong levees cannot be justified on economic grounds. From Figures 5 and 6, it can be seen that economically preferable, weak levees have a higher probability of failure. The decision maker is faced with the dilemma of trading off economic efficiency with failure probabilities. This decision problem will be addressed in a forthcoming paper.

The probability density function for the damage was calculated in Equations (29) and (33). Like the distributions for the probability of failure, the distributions of flood damage consist of two parts--a delta function at the damage level corresponding to failure at the design discharge, and a continuous function between the damage at the minimum exceedance discharge, where failure will occur, and the level corresponding to failure at the design discharge.

The probability density function for flood benefits can be easily found. Annual flood benefits due to a particular decision, d, are taken to be the expected annual flood damage averted; that is

$$B_d = D_o - D_d \quad , \tag{42}$$

where D_o is the expected annual damage without protection.

In an analysis, it is assumed that D_o is known and is not a random variable. Therefore,

$$f(B_d) = f(D_d)\big|_{D_d = D_o - B_d} \quad , \tag{43}$$

and $f(B_d)$ can be easily calculated from Equations (28) and (33).

Conclusions

This paper analyzes the uncertainty in the probability of failure and the expected flood benefits due to the uncertainty in the strength of a flood levee.

Experience has shown that during a flood, most levees fail structurally, rather than by the flood waters overtopping the levee. Present day analyses rarely include the probability of structural failure in an explicit manner. This can result in significantly overestimating the protection offered by a levee system, and underestimating the expected damage that may occur. For the example presented here, a deterministic analysis could overestimate the expected return period by up to 300%, and underestimate expected annual damage by 50%.

The procedures developed here can also be applied to the analysis of other systems--for example, the distribution of isotherms from thermal power plant outfalls, the reliability of large water resource systems for flood control or water supply, etc. Many of the extensions may require numerical analysis as opposed to the analytical derivations presented here, but that should not limit application.

Table 1. Levee design parameters.

Levee Strength	DESIGN DISCHARGE (cfs)											
	25000		30000		35000		40000		45000		50000	
$F_{Z_o}(z_o=z_d)$	Height	Base Width	Height	Base Width	Height	Base Width	Height	Base Width	Height	Base Width	Height	Base Width
.1	8.5	29.75	11.15	39.025	12.5	43.7	13.5	47.11	15	52.5	16.5	57.9
.25	8.5	23.8	11.15	31.22	12.5	35.	13.5	37.7	15	42.	16.5	46.3
.50	8.5	17.	11.15	22.3	12.5	25.	13.5	26.9	15	30.	16.5	33.1
.75	8.5	11.05	11.15	14.49	12.5	16.25	13.5	17.5	15	19.5	16.5	21.5
.90	8.5	8.5	11.15	11.15	12.5	12.5	13.5	13.5	15	15	16.5	16.5

NOTE: Levee height in feet, levee base width in feet

Table 2

Annual Cost Function Parameters

β_1	=	1.0	
β_2	=	2.0	
K_1	=	20.17	$\$/ft^2$
K_2	=	40340.	\$

Damage Function Parameter

c	=	2.155×10^{-4}	\$/cfs

NOTE:

1. Annual costs in dollars, levee base width in feet, levee height in feet, discharge in cu ft/sec.

2. Costs for levee construction have been appropriately discounted into equivalent annual costs for comparison with expected annual flood benefits. The issues of fixing the appropriate interest rate or project life have not been explicitly addressed.

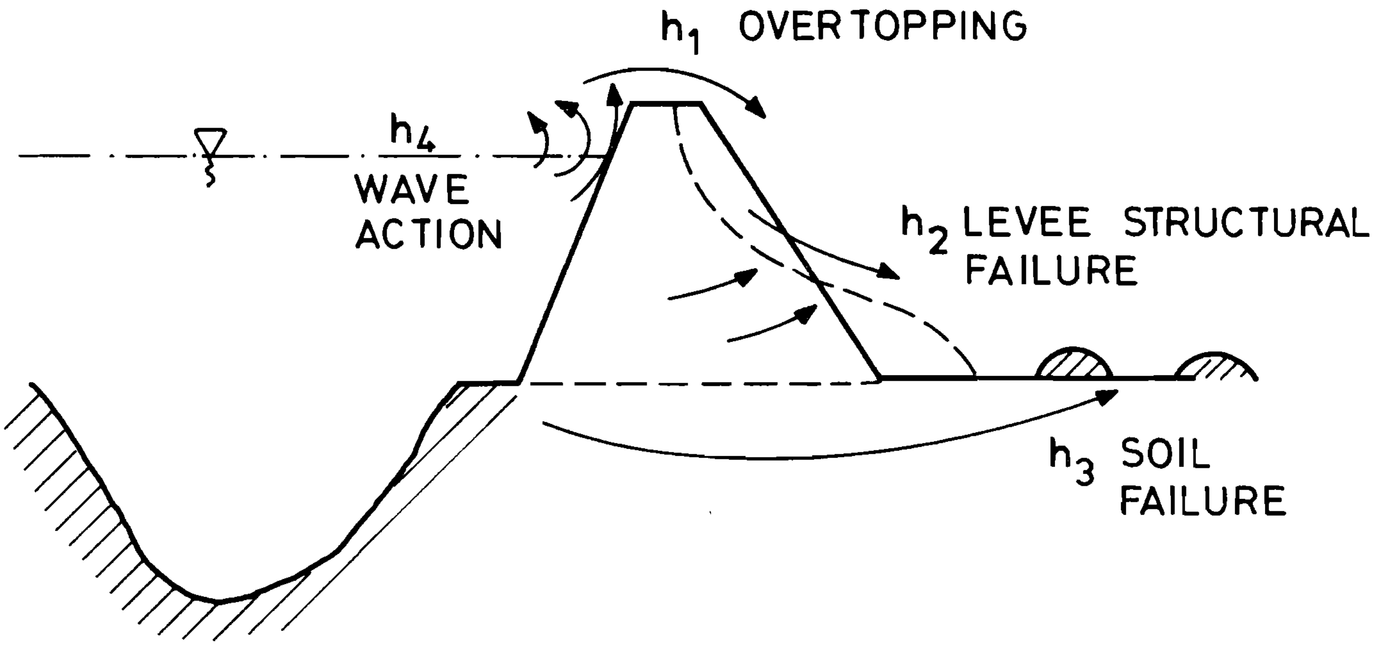

FIGURE 1: FOUR MODES OF LEVEE FAILURE

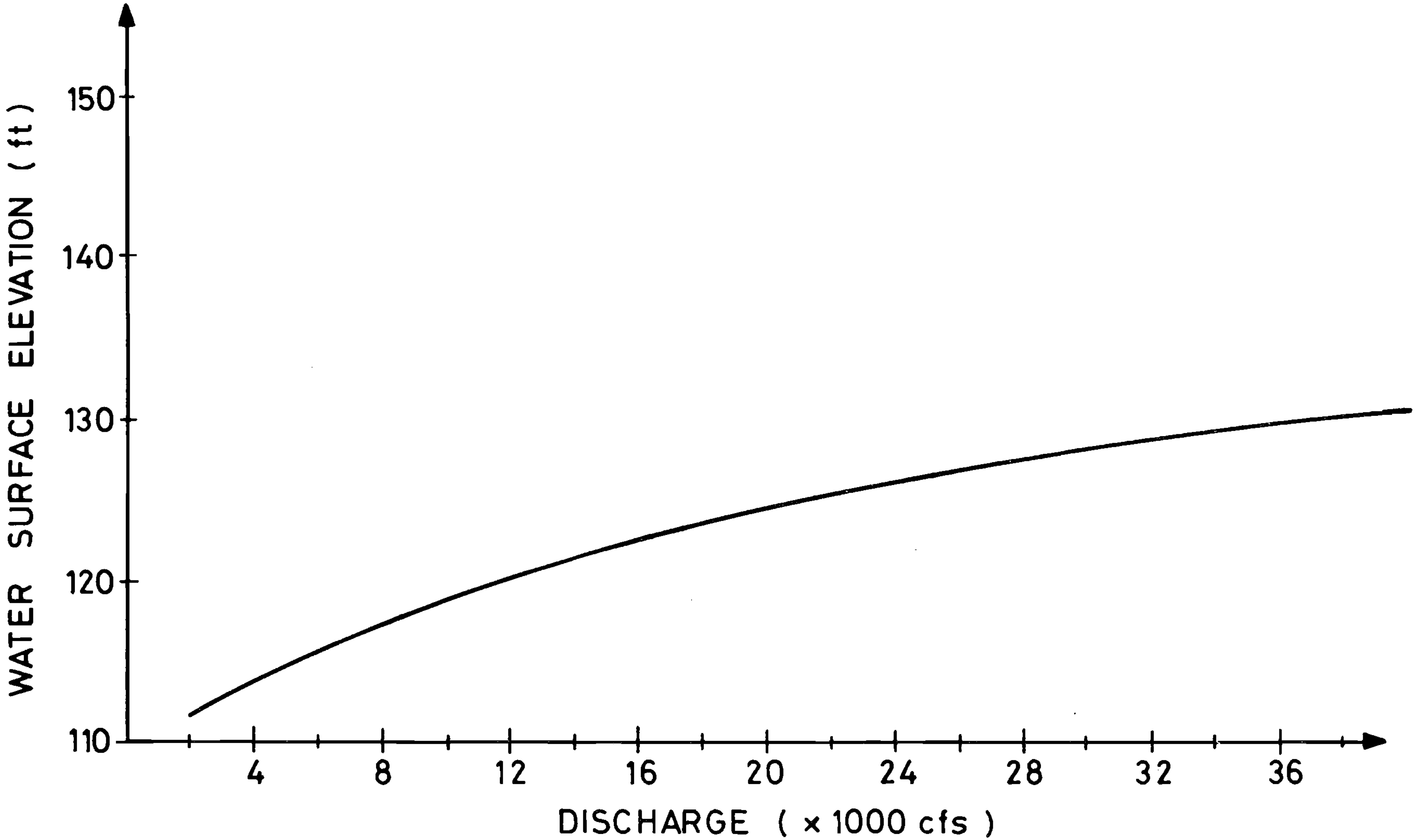

FIGURE 2. STAGE - DISCHARGE CURVE FOR BLACKSTONE RIVER AT WOONSOCKET, R.I.

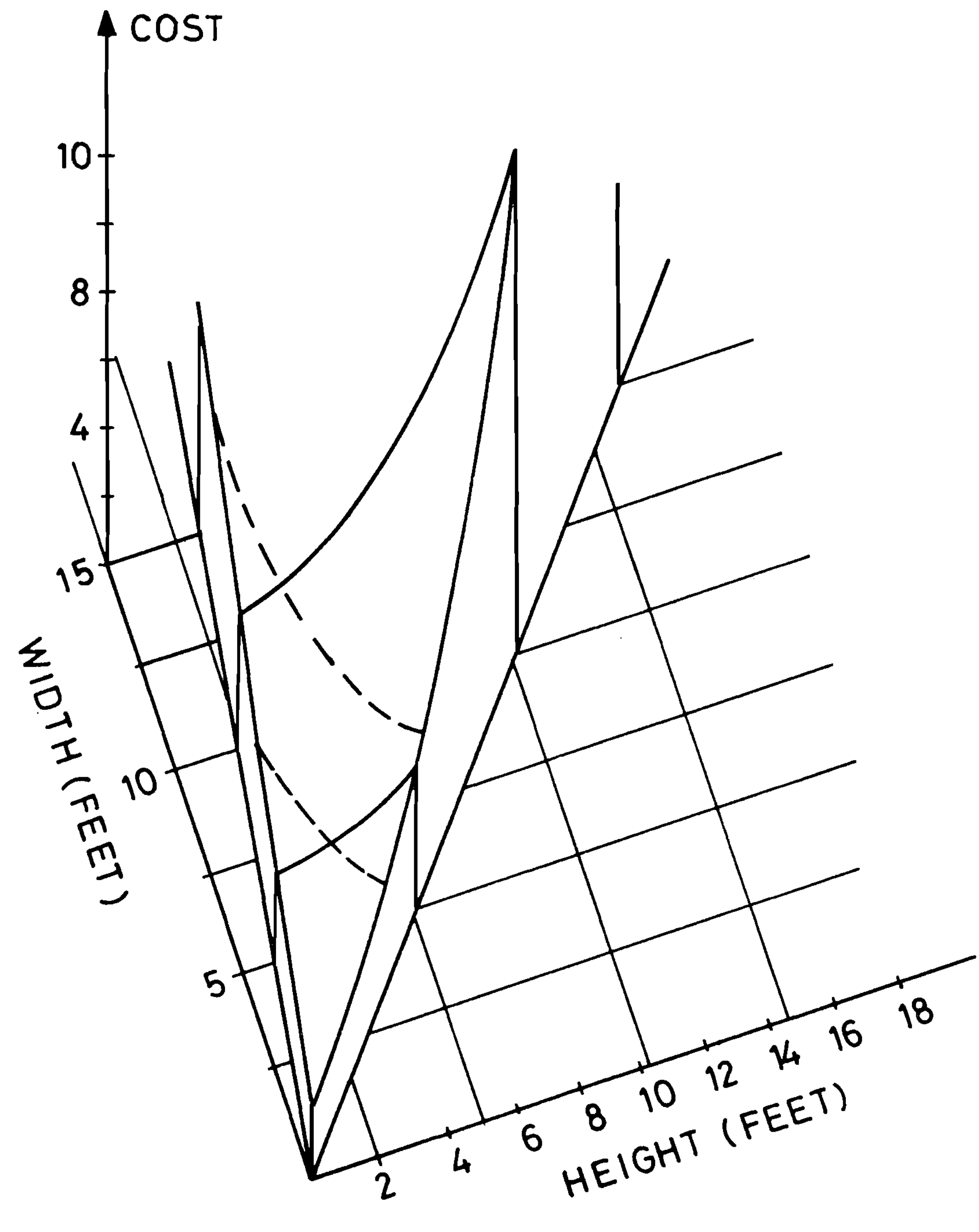

FIGURE 3. ILLUSTRATIVE COST FUNCTION

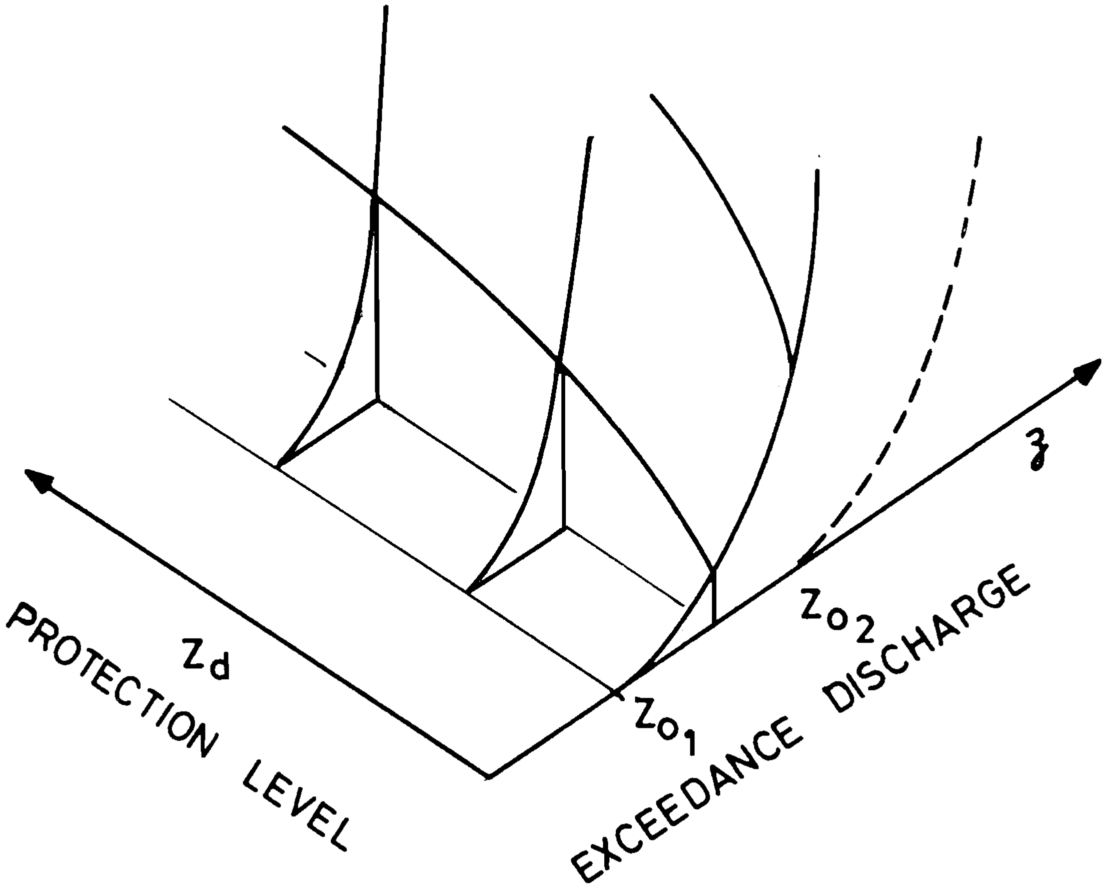

FIGURE 4 : ILLUSTRATIVE DAMAGE FUNCTION

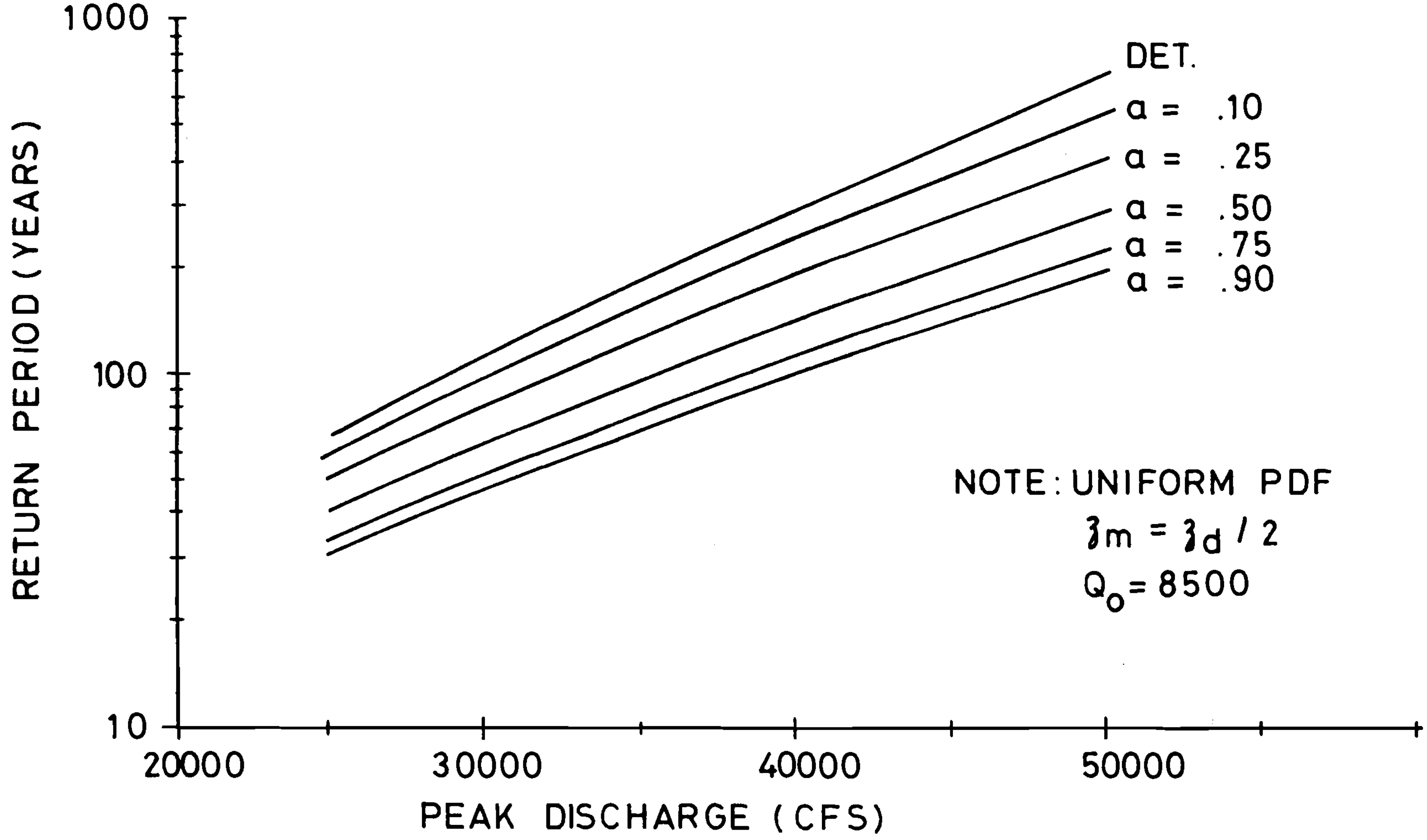

FIGURE 5 : A FLOOD FREQUENCY CURVE FOR UNIFORM pdf.

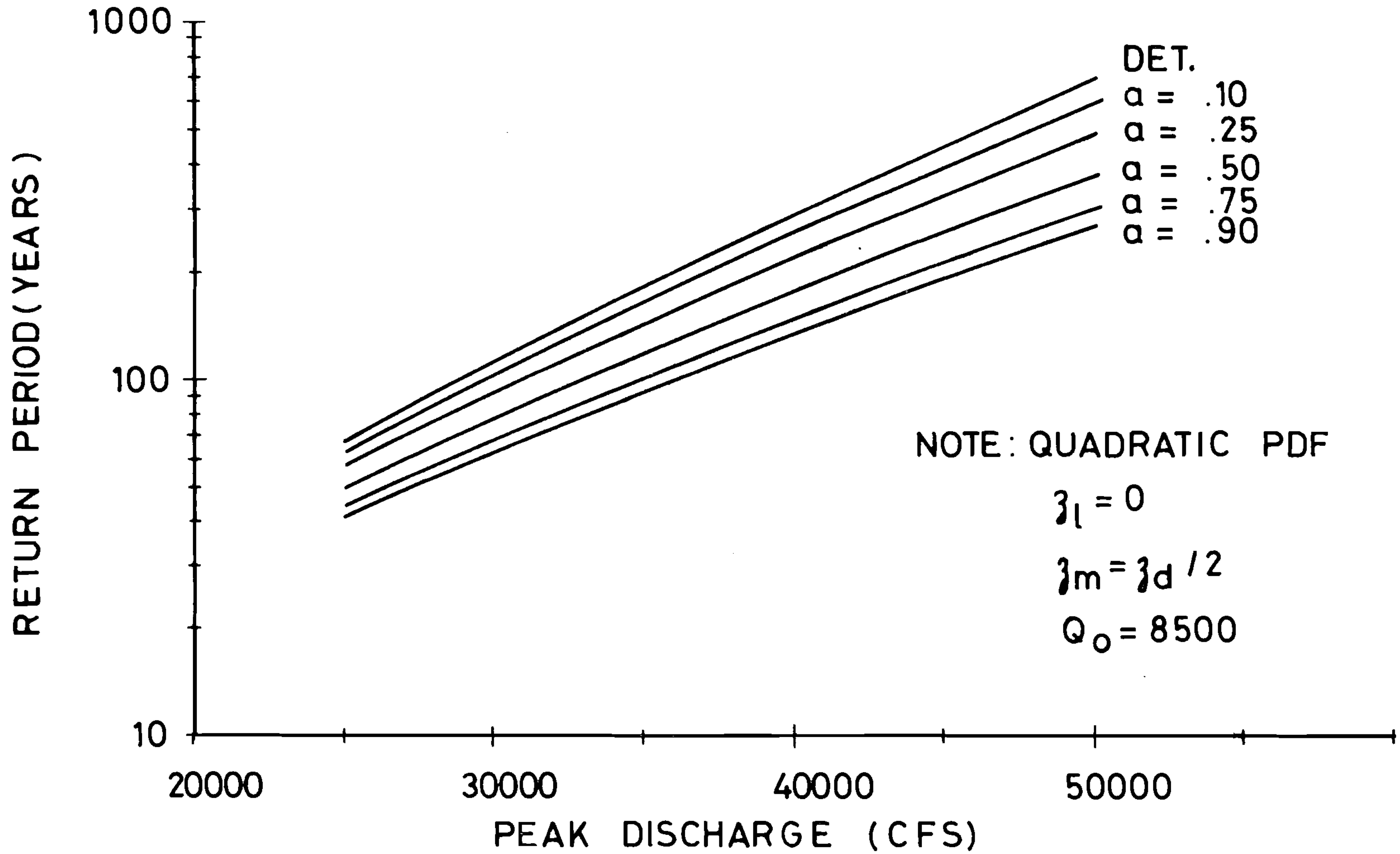

FIGURE 6 : A FLOOD FREQUENCY CURVE FOR QUADRATIC pdf.

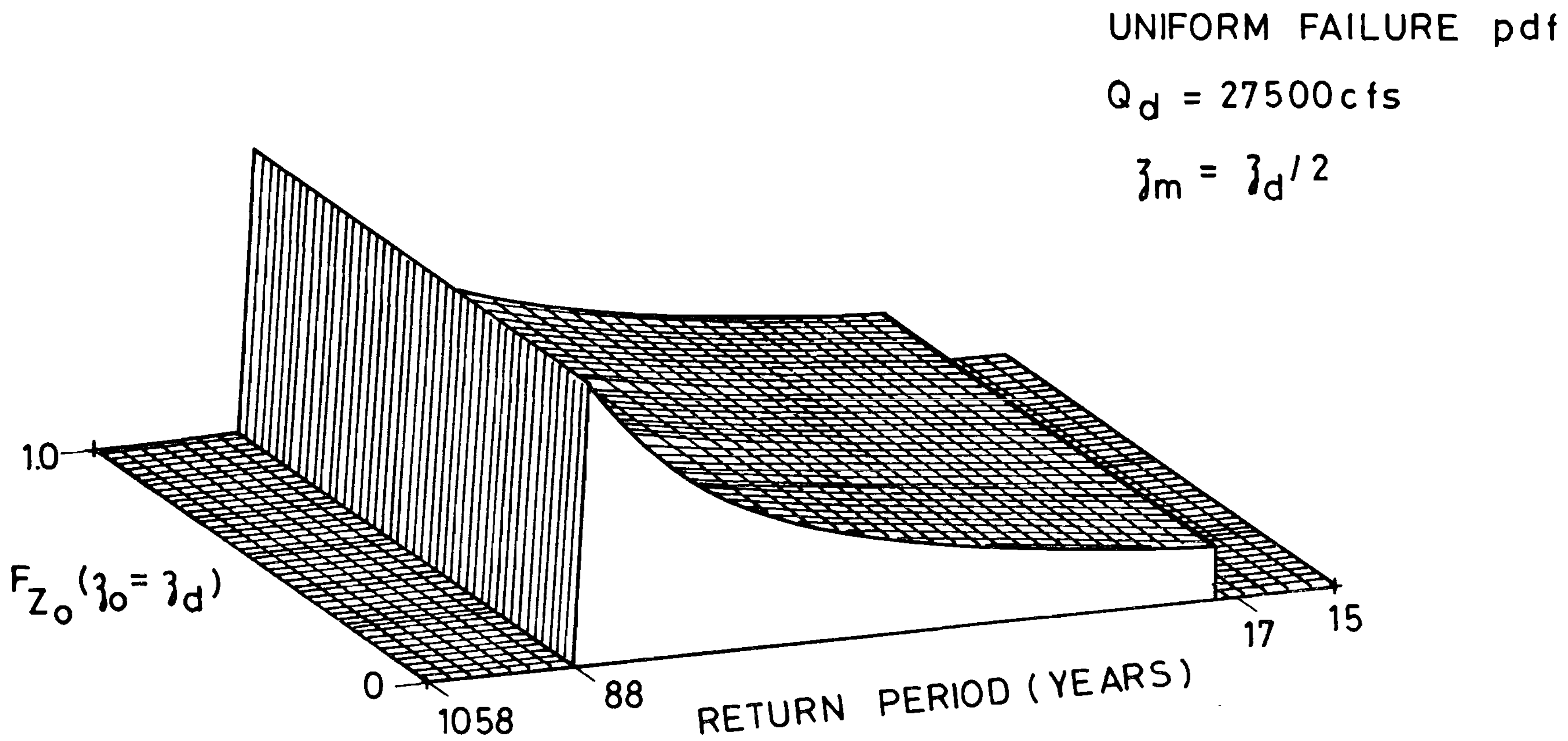

FIGURE 7: A DENSITY FUNCTION FOR THE UNIFORM FLOOD FREQUENCY CURVE.

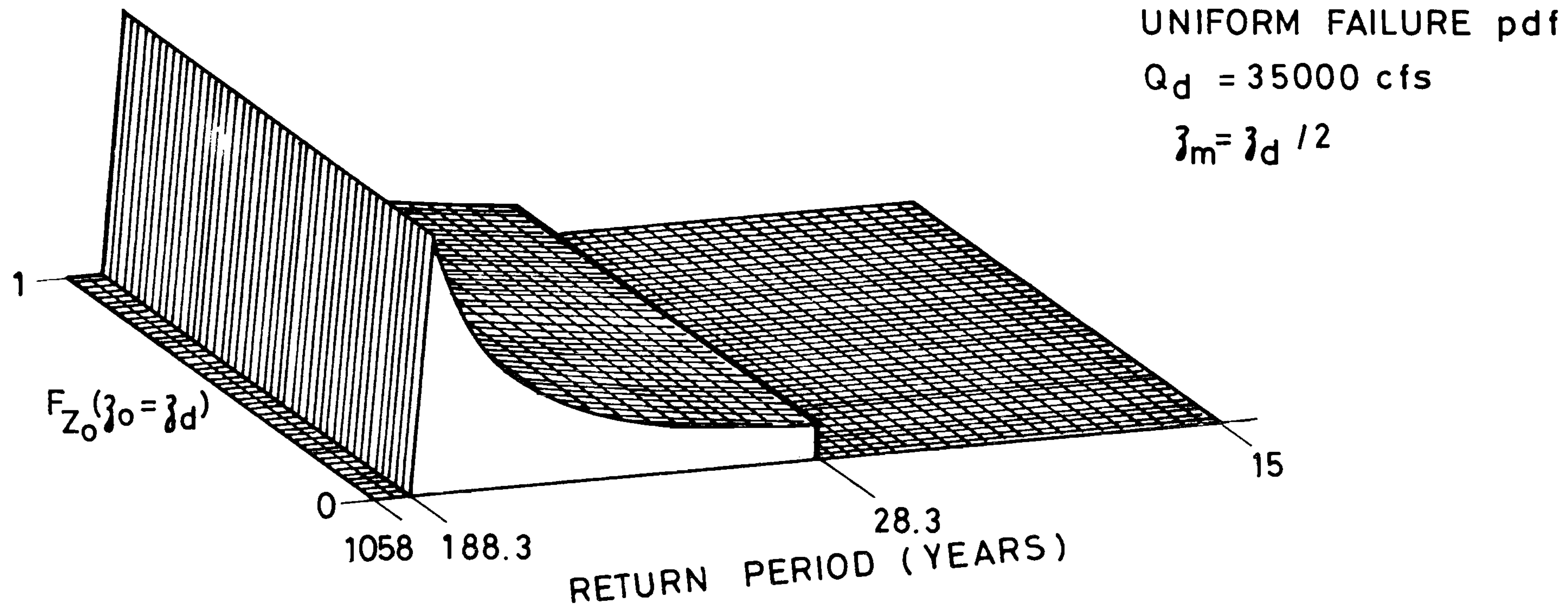

FIGURE 8 : A DENSITY FUNCTION FOR THE QUADRATIC FLOOD FREQUENCY CURVE.

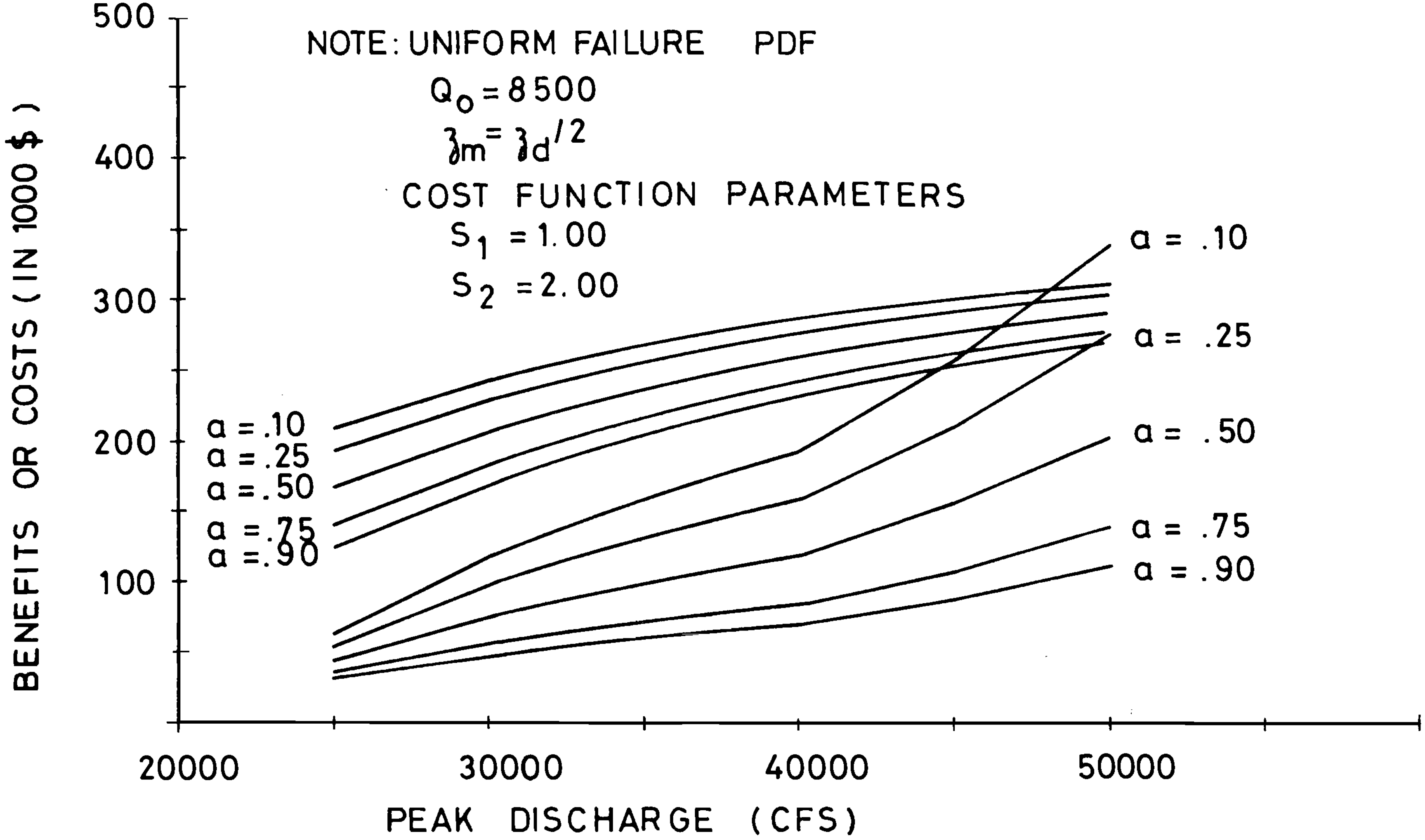

FIGURE 9 : BENEFIT / COST CURVE VERSUS DESIGN DISCHARGE

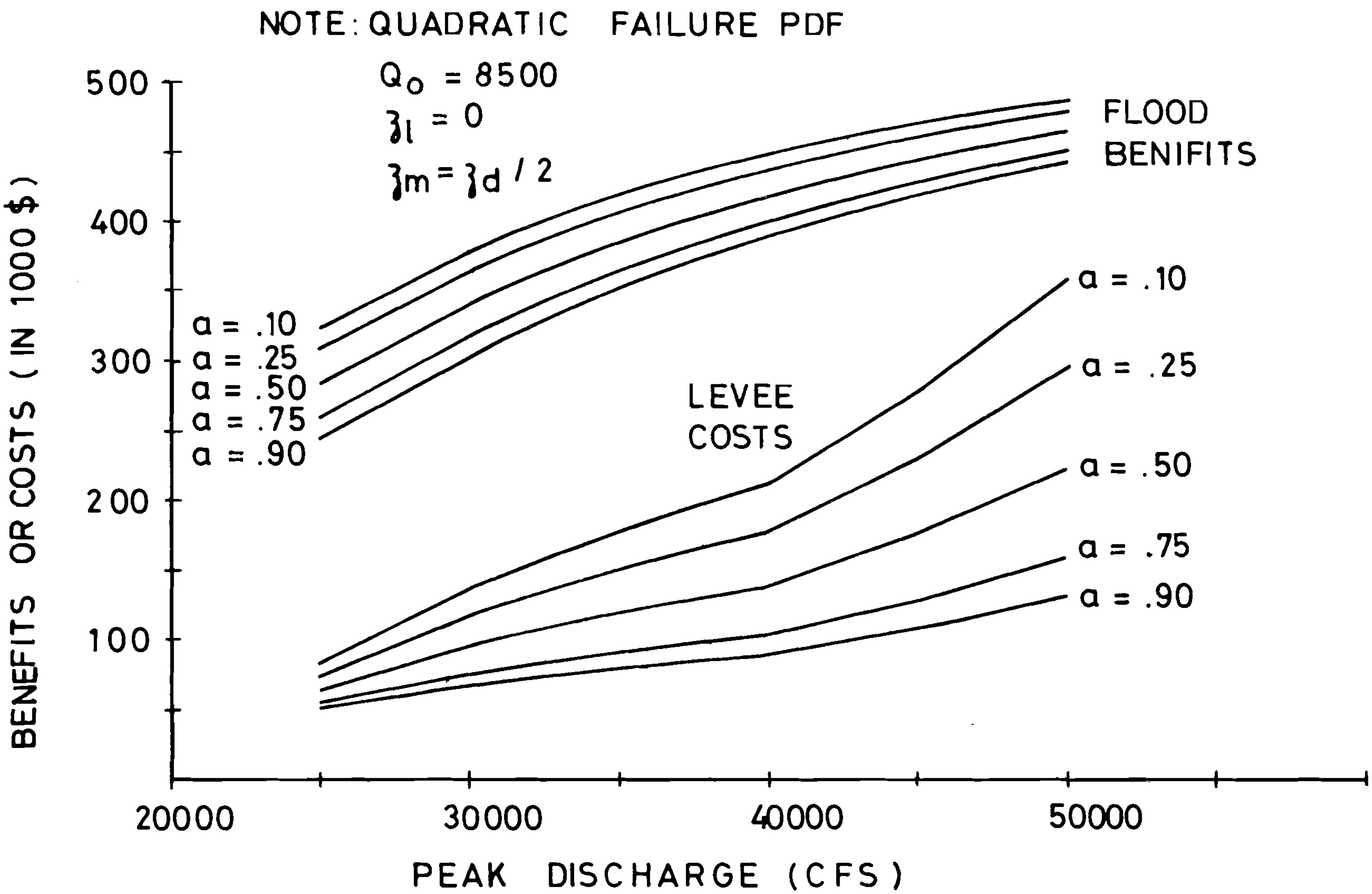

FIGURE 10 : BENEFIT / COST CURVE VERSUS DESIGN DISCHARGE

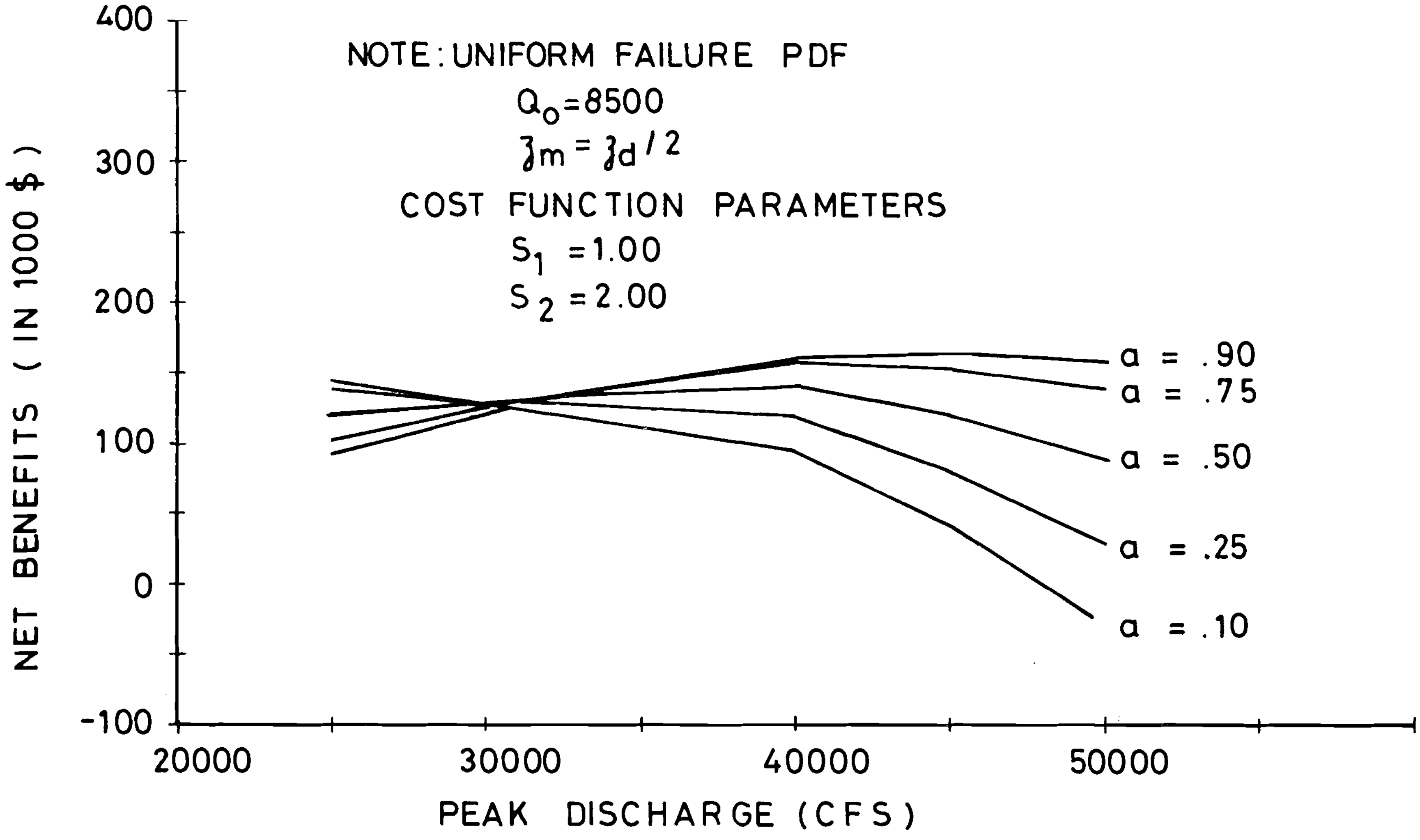

FIGURE 11: NET BENEFITS VERSUS DESIGN DISCHARGE

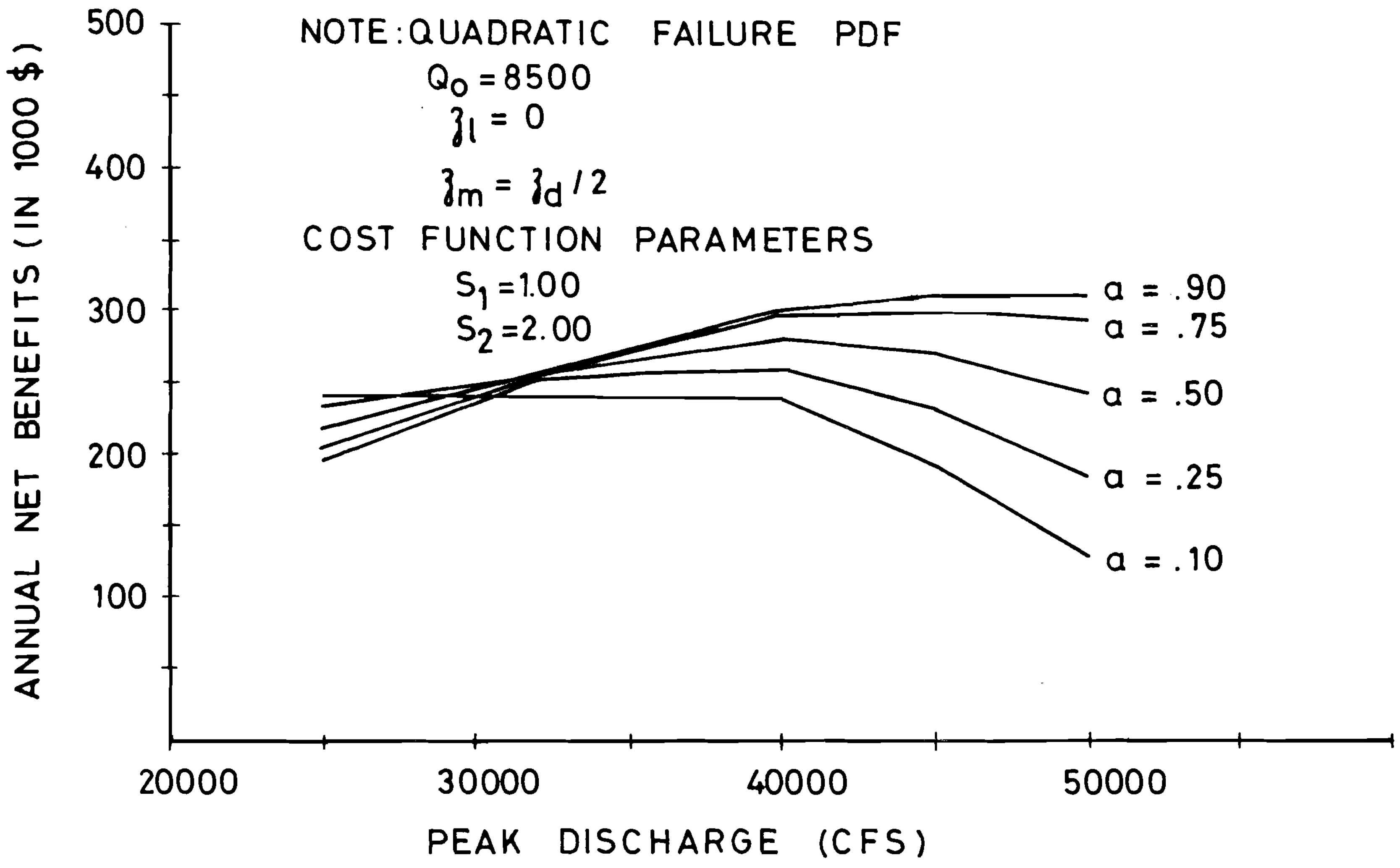

FIGURE 12 : NET BENEFIT VERSUS DESIGN DISCHARGE

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