# **REPORT 1111**

# AN ANALYSIS OF LAMINAR FREE-CONVECTION FLOW AND HEAT TRANSFER ABOUT A FLAT PLATE PARALLEL TO THE DIRECTION OF THE GENERATING BODY FORCE <sup>1</sup>

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## SUMMARY

The free-convection flow and heat transfer (generated by a body force) about a flat plate parallel to the direction of the body force are formally analyzed and the type of flow is found to be dependent on the Grashof number alone. For large Grashof numbers (which are of interest in aeronautics), the flow is of the boundary-layer type and the problem is reduced in a formal manner, which is analogous to Prandtl's forced-flow boundarylayer theory, to the simultaneous solution of two ordinary differential equations subject to the proper boundary conditions.

Velocity and temperature distributions for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000 are computed, and it is shown that velocities and Nusselt numbers of the order of magnitude of those encountered in forced-convection flows may be obtained in free-convection flows. The theoretical and experimental velocity and temperature distributions are in good agreement.

A flow and a heat-transfer parameter, from which the important physical quantities such as shear stress and heat-transfer rate can be computed, are derived as functions of Prandtl number alone. Comparison of theoretically computed values of the heat-transfer parameter with values obtained from an approximate calculation and experiments yielded good agreement over a large range of Prandtl number. Agreement between the theoretical values and those obtained from a frequently used semiempirical heat-transfer law was good only in restricted Prandtl number ranges (depending on an arbitrary constant).

### INTRODUCTION

Two important types of fluid flow problems involving heat transfer are those of forced and those of free convection. By forced-convection flow is meant flows maintained mechanically as, for example, by a pressure drop or an agitator. Free-convection flow, on the other hand, results from the action of body forces on the fluid, that is, forces which are proportional to the mass or the density of the fluid. The flow is generally produced in the following manner: Consider, for example, a fixed object (such as a plate) in a quiescent fluid subject to a body force. When the plate is at the same temperature as the surrounding fluid, the body forces acting on the fluid are in equilibrium with the hydrostatic pressure and no flow ensues in the steady state. If a temperature gradient normal to the body force is imposed by heating (or cooling) the plate, there will exist a defect (or excess) of body force because of the decreased (or increased) density, with the fluid closer to the plate having the greater defect (or excess) than that away from the plate. This unbalance of the forces causes the fluid to be accelerated with the particles nearer the plate moving more rapidly than those farther from the plate. Free-convection flow has usually been considered to be generated in a gravitational field where the previously mentioned defect or excess of body force was the Archemedian (buoyancy) force. However, since centrifugal forces are also proportional to the fluid density, freeconvection flows can also be set up by the action of such forces. (See ref. 1 for a more explicit discussion of the development of free-convection flows by centrifugal forces.)

Free-convection flows produced by centrifugal forces are now of practical importance in aeronautics because many aircraft propulsion systems contain components (such as gas turbines and helicopter ram jets) which rotate at high speeds and in which heat is being transferred. The method of free-convection cooling of gas-turbine rotor blades where the centrifugal forces create a free-convection flow of the coolant in the blade passages is an example of a practical application of the free-convection phenomenon in aeronautics. Also, free-convection flow due to centrifugal force is superimposed on the flow through helicopter ram jets and on the flow of cooling air in hollow rotor blades of air-cooled turbines and, under proper conditions, can appreciably influence the resultant flow and heat transfer.

As a simplification of the many free-convection problems which are now of some consequence in aeronautics, consideration is here given to the special case of free-convection flow about a flat plate parallel to the direction of the generating body force. The experimental and theoretical considerations of Schmidt and Beckmann (ref. 2) concerning the free-convection flow of air subject to the gravitational force about a vertical flat plate constitute the most complete treatment of this subject up to the present time. Eckert (ref. 1) as well as others has further verified and extended the experimental results of Schmidt and Beckmann, and Schuh (ref. 3) has extended the numerical calculations by computing the velocity and temperature distributions for several Prandtl numbers different from that for air. However, all the theoretical work in these references is based on the incompressible equations in which the density (or temperature) variation is introduced in the buoyancy term alone.

: Supersedes NACA TN 2635, "An Analysis of Laminar Free-Convection Flow and Heat Transfer about a Flat Plate Parallel to the Direction of the Generating Body Force" by Simon Ostrach, 1952.

Various terms are omitted from the equations at the start on the basis of either intuitive arguments or no arguments at all. Although a theoretical development made in such a manner led to good final results, the significance of all the important factors associated with the free-convection flow phenomenon is not obtained from such an analysis.

The problem of free-convection flow as produced by a body force about a flat plate in the direction of the body force was studied at the NACA Lewis laboratory during 1951 and is treated in a formal and more general manner herein. The method used is somewhat similar to that used in reference 4 wherein consideration was given to the free-convection flow at high Grashof numbers in a horizontal cylinder which had a variable surface-temperature distribution. The application of this method to the present problem leads to a development which is analogous to Prandtl's treatment of high Reynolds number forced-convection flows. Although the final equations obtained by this method are the same as those of Schmidt and Beckmann, this more general approach not only clearly demonstrates the significance of all the important parameters and assumptions and hence leads to a better understanding of this type of flow but also indicates the quantitative limitations of the theory. In addition, the numerical solutions of references 2 and 3 are herein extended to cover a more complete range of parameters. The new calculations yield information on the free-convection flow for Prandtl numbers corresponding to those of liquid metals, gases, liquids, and very viscous fluids.

#### ANALYSIS

#### STATEMENT OF PROBLEM AND BASIC EQUATIONS

The steady-state equations expressing the conservation of mass, momentum, and energy for a compressible, viscous, and heat-conducting fluid subject to a body force together with an equation of state govern the flow and associated temperature distribution about the plate. These equations in Cartesian tensor notation are (see ref. 5), respectively,

$$\frac{\partial}{\partial X_j} \left( \rho U_j \right) = 0 \tag{1}$$

$$\rho U_{j} \frac{\partial U_{i}}{\partial X_{j}} = \rho f_{i} + \frac{\partial}{\partial X_{j}} \left[ \mu \left( \frac{\partial U_{i}}{\partial X_{j}} + \frac{\partial U_{j}}{\partial X_{i}} \right) \right] - \frac{2}{3} \frac{\partial}{\partial X_{i}} \left( \mu \frac{\partial U_{j}}{\partial X_{j}} \right) - \frac{\partial P}{\partial X_{i}}$$
(2)

$$\rho c_{p} U_{j} \frac{\partial T}{\partial X_{j}} = U_{j} \frac{\partial P}{\partial X_{j}} + \frac{\partial}{\partial X_{j}} \left( k \frac{\partial T}{\partial X_{j}} \right) + \mu \left[ \frac{\partial U_{i}}{\partial X_{j}} \left( \frac{\partial U_{i}}{\partial X_{j}} + \frac{\partial U_{j}}{\partial X_{i}} \right) - \frac{2}{3} \left( \frac{\partial U_{j}}{\partial X_{j}} \right)^{2} \right]$$
(3)

$$\rho = \rho(P,T) \tag{4}$$

(A complete list of the symbols used herein is given in appendix A.) For the two-dimensional case, equations (1) to (4) represent a system of five equations in the five dependent variables  $U_1, U_2, \rho, P$ , and T. For later use, equation (4) can be written

$$d\rho = \rho(K \, dP - \dot{\beta} \, dT) \tag{4a}$$

where K and  $\beta$  are the coefficients of isothermal compressibility and volumetric expansion, respectively (see ref. 6). In addition to a general state equation, such as is given in equations (4) or (4a), it will be convenient at times in the discussion to refer to some specific state equation. To this end, the equation of state for an ideal gas

$$P = \rho R T \tag{4b}$$

will be used.

Particular consideration is here given to the two-dimensional free-convection flow about a semi-infinite vertical flat plate. The  $X_1$ -axis of the coordinate system is taken along the plate and the  $X_2$ -axis, normal to it. No distinction is made as to the specific type of body force acting, for example, gravitational or centrifugal, but the force is assumed to be acting in the vertical direction only (that is, parallel to the plate). Centrifugal and Coriolis forces which are connected with flows on curved paths and with rotating systems generally vary with position and velocity. However, in order not to make the analysis unduly complicated, the body force is taken to be constant.

In order to define the problem clearly, a choice must still be made of the position of the origin of the coordinate sys-Before making a definite decision on this point, note tem. that for constant plate temperatures there are four permutations of the body-force direction (either upward or downward) and the plate thermal condition (either heated or cooled) which will lead to free-convection flows. Once the position of the edge of the plate, which is also to be the origin of the coordinate system, is decided, there are two combinations of the body-force direction and plate thermal condition that will yield flows which proceed away from the edge. It is this type of flow that is amenable to the type of analysis to be made here. This point will be more fully discussed subsequently. If the edge of the plate (recall that a semi-infinite plate has but one edge) is taken at the bottom of the plate (that is, the plate extends to  $+\infty$  in the  $X_1$ -direction), the two combinations leading to flows in the proper direction (upward in this case) are, respectively, the body force acting downward with a heated plate and the body force acting upward with a cooled plate. The equations developed for one of the cases reduce directly to those for the other. The remaining two permutations, namely, the body force acting downward with a cooled plate and the body force acting upward with a heated plate, would yield flows which proceed downward or toward the edge of the plate if this edge were taken at the bottom of the plate. This type of flow would violate a physical condition of the problem which states that the flow starts at the plate edge. The latter combinations hence will not be considered further.

Because the two acceptable configurations can be reduced essentially to one, for the development to be given here, the origin of the coordinate system will be taken at the bottom of a heated plate, with the body force acting downward. The assumption is now made that the viscosity and thermalconductivity coefficients are functions of the temperature alone and obey the following laws:

$$\mu = \mu_{\infty} \left( \frac{T}{T_{\infty}} \right)^{*} \left( k = k_{\infty} \left( \frac{T}{T_{\infty}} \right)^{*} \right)$$
(5)

The choice of the body-force direction together with equations (5) alters equations (2) and (3) so that they become

$$\rho U_{j} \frac{\partial U_{i}}{\partial X_{j}} = -\rho(-f_{i}) + \frac{\mu_{\omega}}{T_{\omega}^{m}} \frac{\partial}{\partial X_{j}} \left[ T^{m} \left( \frac{\partial U_{i}}{\partial X_{j}} + \frac{\partial U_{j}}{\partial X_{i}} \right) \right] - \frac{2}{3} \frac{\mu_{\omega}}{T_{\omega}^{m}} \frac{\partial}{\partial X_{i}} \left( T^{m} \frac{\partial U_{j}}{\partial X_{j}} \right) - \frac{\partial P}{\partial X_{i}}$$
(6)

$$\rho c_{p} U_{j} \frac{\partial T}{\partial X_{j}} = U_{j} \frac{\partial P}{\partial X_{j}} + \frac{k_{\infty}}{T_{\infty}^{n}} \frac{\partial}{\partial X_{j}} \left( T^{n} \frac{\partial T}{\partial X_{j}} \right) + \frac{\mu_{\infty}}{T_{\infty}^{m}} T^{m} \left[ \frac{\partial U_{t}}{\partial X_{j}} \left( \frac{\partial U_{t}}{\partial X_{j}} + \frac{\partial U_{j}}{\partial X_{i}} \right) - \frac{2}{3} \left( \frac{\partial U_{j}}{\partial X_{j}} \right)^{2} \right] \quad (7)$$

Note that the only nonzero component of the body force is the  $X_1$ -component.

#### BOUNDARY CONDITIONS

The boundary conditions associated with the given problem are that:

(a) The fluid must adhere to the plate (the no-slip condition of viscous flows) and the plate must be a streamline, or mathematically,

$$U_1(X_{1,0}) = U_2(X_{1,0}) = 0 \tag{8}$$

(b) The temperature of the fluid at the plate must be equal, to the plate temperature, that is,

$$T(X_{1},0) = T_{0} \tag{9}$$

(c) The velocity  $U_1$  at large distances from the plate must be undisturbed, or

$$U_1(X_1,\infty) = 0 \tag{10}$$

(d) The temperature at large distances from the plate must be equal to the undisturbed fluid temperature, or

$$T(X_1,\infty) = T_\infty \tag{11}$$

#### SIMPLIFICATION OF EQUATIONS

Let a small quantity  $\epsilon$  now be defined as

$$\epsilon = \beta (T_0 - T_{\infty}) \tag{12}$$

which is a measure of the magnitude of temperature variation in the flow field. The coefficient of volumetric expansion  $\beta$ is generally of the order of magnitude between  $10^{-2}$  and  $10^{-4}$ (see table 15 of ref. 7, for example) and for gases,  $\beta = 1/T$ . (Thus, for gases, if  $\beta$  is taken to be constant,  $\epsilon = (T_0 - T_{\infty})/T_{\infty}$ ; that is,  $\epsilon$  is the relative temperature difference.) The coefficient  $\beta$  will be assumed constant. Because in the steady state flow ensues only when there is a temperature variation in the fluid, the free-convection velocity should then depend directly on  $\epsilon$ , and the variations in pressure and density (from the static,  $\epsilon \equiv 0$ , case) due to the temperature differences should also depend on  $\epsilon$ . Thus

$$U_{i} = \epsilon \left( \frac{\rho_{\infty} f_{x} l^{z}}{\mu_{\infty}} \right) u_{i}$$
(13)

$$P = P_{s} + P_{\infty} \epsilon \sigma \tag{14}$$

$$\rho = \rho_s + \rho_{\infty} \epsilon \varphi \tag{15}$$

$$T = T_{\infty} (1 + \epsilon \theta) \tag{16}$$

where  $-f_x$  denotes the  $X_1$ -component of the body force per unit mass,  $u_i$ ,  $\sigma$ ,  $\varphi$ , and  $\theta$  denote dimensionless functions (which, in general, can be functions of  $\epsilon$ ), l is some characteristic length (for example, the distance from the edge of the plate to the point of interest),  $P_i$  and  $\rho_i$  are the pressure and the density, respectively, for the static case ( $U_i \equiv 0$  or  $\epsilon \equiv 0$ ), and  $P_{\infty}$  and  $\rho_{\infty}$  denote constant values of the pressure and the density (that is, the values if no force field were present) defined by the state equation (in the case of a gas, in particular,  $P_{\infty} = \rho_{\infty} R T_{\infty}$ ): Because there is no characteristic velocity associated with the type of flow under consideration, the velocity is dimensionalized by the factor given in parentheses on the right side of equation (13).

In order to determine the static quantities, it will at first be convenient to consider the particular case of a gas. The problem is then considered with the temperature uniform throughout the flow field at the value  $T_{\infty}$  (therefore there will be no flow and  $U_i \equiv 0$ ). For this situation, equations (4b) and (6) become

$$P_{s} = \rho_{s} R T_{\omega} \tag{17}$$

and

$$\frac{\partial P_s}{\partial X_1} + \rho_s f_X = 0 \\
\frac{\partial P_s}{\partial X_2} = 0$$
(18)

(It should be noted that eq. (18) expresses the physical fact previously stated that the body force and hydrostatic pressure are in equilibrium for the static case.) Substitution of equation (17) into equation (18) leads to

$$P_{s} = P_{\infty} \exp\left(-\frac{f_{X}}{RT_{\infty}} X_{1}\right) \tag{19}$$

and equation (19) together with equation (17) and the equation defining  $P_{\infty}$  and  $\rho_{\infty}$  yields

$$\rho_s = \rho_{\infty} \exp\left(-\frac{f_X}{RT_{\infty}} X_1\right) = \rho_{\infty} \exp\left(-\frac{f_X \rho_{\infty}}{P_{\infty}} X_1\right) \quad (20)$$

If the exponential in equation (20) is expressed in terms of its series expansion, that equation becomes

$$\rho_{s} = \rho_{\infty} \left( 1 - \frac{\rho_{\infty} f_{X}}{P_{\infty}} X_{1} + \dots \right)$$
(21)

A computation of the second term in the parentheses of equation (21) for the case of air under normal conditions with  $f_X = g$  and the fact that  $X_1$  is of the order of magnitude l show that  $\rho_{\infty}gl/P_{\infty} \sim 10^{-5}l/\text{foot}$ . For the type of problem under consideration, l will always be of unit order of magnitude so that even if the body force  $f_X$  represents a centrifugal force many times that of gravity, the inequality  $\rho_{\infty}f_XX_1/P_{\infty}\ll 1$  may still be satisfied. Thus, in the subsequent development it will be assumed that  $\rho_{*}\cong \rho_{\infty}$ . This assumption, which was justified by the computation for the case of a gas, is expected to be reasonable for other fluids as well. The physical interpretation of this assumption is that under static conditions ( $\epsilon \equiv 0$ ), the density (or pressure) is not affected by the force field.

In order that all quantities in the following equations be dimensionless, it is further necessary to define  $x_t = X_t/l$ , where the  $x_t$  are now dimensionless space coordinates. Substituting these new coordinates along with equations (13) to . (16) into equations (1), (6), (7), and (4a) and noting equations (18) and that  $\rho_t \cong \rho_{\infty}$  yield, on neglection of terms of higher order in  $\epsilon$  compared with those of order  $\epsilon$ ,

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \tag{22}$$

$$Gr\left(u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}\right)=\Delta u_1-NGr\frac{\partial\sigma}{\partial x_1}-\varphi \qquad (23)$$

$$Gr\left(u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}\right) = \Delta u_2 - NGr\frac{\partial\sigma}{\partial x_2}$$
(24)

$$Gr Pr\left(u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2}\right) = \frac{\gamma}{\gamma - 1} Gr Pr\left(u_1 \frac{\partial \sigma}{\partial x_1} + u_2 \frac{\partial \sigma}{\partial x_2}\right) + \Delta\theta$$
(25)

$$d\varphi = KP_{\omega}d\sigma - \beta T_{\omega}d\theta \qquad (26)$$

where  $P_{\infty}/\rho_{\infty}f_{x}l=NGr$  and the Grashof number Gr and the Prandtl number Pr are defined as

 $Gr = \frac{\rho_{\infty}^{2} f_{X} l^{3} \epsilon}{n^{2}}$ 

and

$$Pr = \frac{c_p \mu_{\infty}}{k_{\infty}}$$

Physically speaking, the Grashof number represents the ratio of the body forces to the viscous forces. The freeconvection flows of interest here are those associated with large Grashof numbers. The factors K and  $\beta$  in equation (26) may well be taken to be constants (see ref. 6).

The boundary conditions (eqs. (8) to (11)) in terms of the new dimensionless variables are

$$u_1(x_1,0) = u_2(x_1,0) = 0 \tag{27}$$

$$\theta(x_1,0) = \frac{1}{\beta T_{\infty}} \tag{28}$$

$$u_1(x_1,\infty) = 0$$
 (29)

$$\theta(x_1,\infty) = 0 \tag{30}$$

Thus, to a first approximation, equations (22) to (26) together with the boundary conditions replace the original equations and boundary conditions. (Note that for gases,  $\beta T_{\infty} = 1.$ )

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The prime assumption made in this analysis is that the higher-order terms in  $\epsilon$  are negligible, which implies that  $\epsilon$ is small, and consequently, that the temperature difference or  $\beta$  is moderately small. It is a consequence of this assumption alone that the basic equations were simplified to equations (22) to (26) wherein the viscosity term in the energy equation is neglected and the only coupling of the momentum and energy equations occurs by means of the body-force term in equation (23). As a result of this assumption, the variations of the viscosity and heat-conductivity coefficients with temperature are also negligible. Without any discussion, the authors of reference 2 start directly from simplified equations of the same form wherein the pressure terms in the energy equation were also neglected. In reference 3 some intuitive arguments are given to justify the simplified equations.

It is now convenient to revert to the more familiar notation where  $x=x_1$ ,  $y=x_2$ ,  $u=u_1$ , and  $v=u_2$ . Equation (22) implies the existence of a stream function  $\psi$  such that

 $u = \frac{\partial \psi}{\partial y} = \psi_{y}$   $v = -\frac{\partial \psi}{\partial x} = -\psi_{x}$ 

(31)

and

where subscripts denote differentiation. Applying equation (31) to equations (23) to (25) yields, respectively,

$$\frac{1}{Gr}(\Delta\psi_{\mathbf{y}}-\varphi)=\psi_{\mathbf{y}}\psi_{\mathbf{z}\mathbf{y}}-\psi_{\mathbf{z}}\psi_{\mathbf{y}\mathbf{y}}+N\sigma_{\mathbf{z}}$$
(32)

$$-\frac{1}{Gr}(\Delta\psi_x) = -\psi_y\psi_{xx} + \psi_x\psi_{xy} + N\sigma_y \qquad (33)$$

$$\frac{1}{GrPr}(\Delta\theta) = \psi_y \theta_x - \psi_x \theta_y - \frac{\gamma}{\gamma - 1} \left(\psi_y \sigma_x - \psi_x \sigma_y\right) \tag{34}$$

The boundary conditions (eqs. (27) to (30)) become

$$\psi_{y}(x,0) = \psi_{x}(x,0) = 0 \tag{35}$$

$$\theta(x,0) = \frac{1}{\beta T_{\infty}} \tag{36}$$

$$\psi_{y}(x,\infty) = \theta(x,\infty) = 0 \tag{37}$$

Equations (32) to (34) and equation (26) form the system of equations for the four unknown functions  $\psi$ ,  $\theta$ ,  $\varphi$ , and  $\sigma$ of the problem. The system is nonlinear, and therefore further simplification of the equations would be desirable. Just as in the case of forced-convection flows where the Reynolds number determines the type of flow or, in mathematical terms, the type of solution, the Grashof number is the prime factor for free-convection flows. For the case of small Grashof number, it can be seen from equations (32) to (34) that a perturbation in the small parameter Gr will yield a system of linear equations. For Grashof numbers of unit order of magnitude, no further important simplification can be made and the solutions would have to be obtained numerically. For the other limiting case, that of large Grashof numbers (which is the case under consideration herein), it would, at first thought, appear that some simplification could be obtained by performing a perturbation in the small parameter 1/Gr. However, this would then imply that the term containing the highest-order derivatives (the left term in equations (32) to (34)) could, among others, be neglected. (This argument would also imply that the body-force term  $\varphi$  in equation (32), which is essentially. causing the flow, could also be neglected.) The omission of the highest-order derivatives from consideration, however. would lead to solutions which would not satisfy all the boundary conditions. Problems of this type are referred to as singular perturbation problems. For further discussions of singular perturbation problems, see references 8 and 9.

Equations in which a small parameter multiplies the highest-order terms are said to be of the boundary-layer type, because in order for solutions which satisfy all the boundary conditions to be obtained, the highest-order terms must be considered near the boundary. This fact implies the existence of a thin region, called the boundary layer, wherein the functions vary rapidly from the value at the boundary to that in the flow outside this layer. The conclusion to be drawn from the preceding discussion is that for large Grashof numbers the flow is of the boundarylayer type. Schmidt and Beckmann (ref. 2) also made the boundary-layer assumptions in their theoretical development, and these assumptions. The Grashof numbers for their experimental observations. The Grashof numbers for

In view of the fact, previously discussed, that highestorder derivatives of each dependent variable as well as of those terms of physical importance (as, for example, the body-force term) must be retained in the boundary layer, it is convenient to make both sides of each of the equations of the same order in Gr. In this way, as will be shown, the equations will be further simplified. It is thus convenient to make the following transformations in the system of equations (32) to (34) and (26) and then to retain only the dominant parts (that is, those multiplied by Gr to the highest power) of each individual term.

Let  $\bar{y} = Gr^r y$ ,  $\psi = Gr^s \bar{\psi}$ ,  $\sigma = Gr^s \bar{\sigma}$ ,  $\varphi = \bar{\varphi}$ , and  $\theta = \theta$ . Then equations (32) to (34) and (26) become

$$Gr^{i+3r-1}\overline{\psi}_{\overline{y}\overline{y}\overline{y}} - Gr^{-1}\overline{\varphi} = Gr^{2s+2r}(\overline{\psi}_{\overline{y}}\overline{\psi}_{\overline{z}\overline{y}} - \overline{\psi}_{\overline{s}}\overline{\psi}_{\overline{y}\overline{y}}) + NGr^{i}\overline{\sigma}_{s} \quad (38)$$

$$-Gr^{i+2r-1}\overline{\psi}_{\overline{y}\overline{y}\overline{y}} = Gr^{2i+r}(-\overline{\psi}_{\overline{y}}\overline{\psi}_{xx} + \overline{\psi}_{x}\overline{\psi}_{x\overline{y}}) + NGr^{i+r}\overline{\sigma}_{\overline{y}} \quad (39)$$

$$\frac{Gr^{2r-1}}{Pr} \hat{\theta}_{\overline{v}\overline{v}} = Gr^{i+r} (\overline{\psi}_{\overline{v}} \overline{\theta}_{x} - \overline{\psi}_{x} \overline{\theta}_{\overline{v}}) - \frac{\gamma}{\gamma - 1} Gr^{i+i+r} (\overline{\psi}_{\overline{v}} \overline{\sigma}_{x} - \overline{\psi}_{x} \overline{\sigma}_{\overline{v}})$$
(40)  
$$d\overline{v} = KP \quad Gr^{i} d\overline{\sigma} - \beta T \quad d\overline{\theta}$$
(41)

It now can be seen that by proper choice of r, s, and t a transformation of the type given provides a means for making the important terms in the differential equations of the same order in Gr. Thus if r=4, s=-4, and t=-1, equations (38) to (41) become

$$\overline{\psi_{yyy}} - \overline{\varphi} = \overline{\psi_y} \overline{\psi_{xy}} - \overline{\psi_x} \overline{\psi_{yy}} + N\overline{\sigma}_x \tag{42}$$

$$N\bar{\sigma}_{r} = O\left(Gr^{-\frac{1}{2}}\right) \simeq 0 \tag{43}$$

$$\vec{\theta}_{\vec{s}\vec{s}} = Pr(\vec{\psi}_{\vec{s}}\vec{\theta}_{\vec{s}} - \vec{\psi}_{\vec{s}}\vec{\theta}_{\vec{s}}) \tag{44}$$

$$d\vec{\varphi} + \beta T_{\omega} d\vec{\theta} = 0 \tag{45}$$

More generally, if N is very much different from unit order of magnitude, a value of t can always be chosen (depending on N) such that equations (42) to (45) are obtained. (For any negative t less than -1, the last term of eq. (42) will also disappear.)

There are now several important points to be discussed concerning the transformation just made and the resulting simplified equations. First, it should be noted that the transformation is merely a formal expression of the boundary-layer assumptions first made by Prandtl and hence the solutions will be asymptotic for large Gr. Second, the second equation of motion here also reduces to state that the pressure across the boundary layer is constant. Third, the pressure terms in the energy and state equations are here found to be negligible. This fact verifies a priori assumptions made by others from the physics of the problem. Finally, note that integration of the general state equation (independent of pressure) as now given by equation (45) leads to

$$\overline{\varphi} + \beta T_{\infty} \overline{\theta} = 0 \tag{46}$$

where the constant of integration has been taken as zero without any loss of generality. For the particular case of a gas,  $\beta = 1/T_{\infty}$  so that equation (46) becomes

$$\overline{\varphi} + \overline{\theta} = 0$$

The boundary conditions (eqs. (35) to (37)) now can be written

$$\overline{\psi}_{\overline{y}}(x,0) = \overline{\psi}_{x}(x,0) = 0 \tag{47}$$

$$\bar{\theta}(x,0) = \frac{1}{\beta T_{\infty}} \tag{48}$$

$$\overline{\psi_{\overline{y}}}(x,\infty) = \overline{\theta}(x,\infty) = 0 \tag{49}$$

If now it is assumed that  $\overline{\sigma}_x=0$  in equation (42) since consideration is here being given to a flat plate, and if  $\overline{\varphi}$  is eliminated from equation (42) by use of equation (46), there results the system of equations

(52)

$$\overline{\psi_{\overline{yyy}}} + \beta T_{\infty} \overline{\theta} = \overline{\psi_{\overline{y}}} \overline{\psi_{\overline{x}\overline{y}}} - \overline{\psi_{x}} \overline{\psi_{\overline{y}y}}$$
(50)

$$\vec{\theta}_{\overline{x}} = Pr(\vec{\psi}_{\overline{x}}\vec{\theta}_{\overline{x}} - \vec{\psi}_{\overline{x}}\vec{\theta}_{\overline{x}}) \tag{51}$$

Thus the problem has been reduced to the solution of the two simultaneous partial differential equations (eqs. (50) and (51)) subject to the boundary conditions (eqs. (47) to (49)).

Final simplification of the equations is made by application of the so-called similarity transformation of boundarylayer theory. Thus, let

 $\eta = \frac{\overline{y}}{1+\frac{1}{2}}$ 

and

$$(4x)^{-}$$

$$\overline{\psi} = (4x)^{\frac{1}{4}} F(\eta) \tag{53}$$

$$\overline{\theta} = \frac{H(\eta)}{\beta T_{\infty}} \tag{54}$$

Then equations (50) and (51) are reduced to the following ordinary differential equations:

$$F''' + 3FF'' - 2F'^2 + H = 0 \tag{55}$$

$$H^{\prime\prime} + 2Pr F H^{\prime} = 0 \tag{56}$$

where the primes denote differentiations with respect to  $\eta$ . The reciprocal one-fourth power similarity as given in equation (52) is characteristic of free-convection flows just as the reciprocal square-root type is characteristic of the forcedconvection flows. The boundary conditions become

$$F'(0) = F(0) = 0 \tag{57}$$

$$H(0) = 1$$
 (58)

$$F'(\infty) = H(\infty) = 0 \tag{59}$$

The use of a transformation like equation (52) essentially specifies an additional boundary condition, namely, that the conditions to be satisfied at  $y = \infty$  (or  $\eta = \infty$ ) should also be satisfied at x=0. It is for this reason that the flows previously discussed which would flow toward the edge (downward) are not amenable to this type of analysis, for such flows would violate this additional condition, which essentially states that the boundary-layer development starts at the edge of the plate.

#### SOLUTION OF THE BOUNDARY-VALUE PROBLEM

The solutions of the simplified equations (55) and (56), satisfying the boundary conditions as given by equations (57) to (59), were obtained by use of an IBM Card-Programmed Electronic Calculator. A detailed account of the procedure followed in the determination of the unknown functions is presented in appendix B by Dr. Lynn U. Albers. The functions F and H together with their derivatives are given in table I for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. Even though the Prandtl number for air is taken as 0.72 in this report, the solutions for Pr=0.733were also computed and are presented as a check with the Schmidt-Beckmann calculations wherein the value of Prandtl number of 0.733 was used. The particular values of the Prandtl numbers given were chosen to correspond to those for liquid metals, gases, liquids (such as water and oil), and very viscous liquids (such as glycerin or oils at very low temperatures).

#### RESULTS

#### VELOCITY AND TEMPERATURE DISTRIBUTIONS

By means of the various transformations made in the analysis it can easily be verified that

$$\frac{U}{2\sqrt{\beta(T_0 - T_{\infty})f_X X}} = \frac{\frac{U X}{\nu_{\infty}}}{2\sqrt{Gr_X}} = F'(\eta)$$
(60)

and

$$\left(\frac{T-T_{\infty}}{T_0-T_{\infty}}\right) = H(\eta) \tag{61}$$

where

$$\eta = \left[\frac{f_x(T_0 - T_{\infty})\beta}{4\nu_{\infty}^2 X}\right]^{\frac{1}{4}} Y = \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{Y}{X}$$
(62)

Equations (60) to (62) relate the physical quantities to the dimensionless functions F and H which are now known. The dimensionless velocity and temperature distributions as given by equations (60) and (61) are presented in figures 1 and 2, respectively, as functions of  $\eta$  for the various values of Prandtl number. The computations made here agree with those for Pr=0.733 as given in reference 2 up to the third significant figure. For Pr=10, 100, and 1000, the present results agree in general with those of reference 3. Since only curves are presented in reference 3, the precision of the agreement cannot be stated.

The maximum values of the dimensionless velocity distributions occur at larger values of the argument  $\eta$  as the Prandtl number decreases and the velocities decrease with increasing Pr. It should also be noted that the dynamic and thermal boundary-layer thicknesses can be estimated from the abscissas of figures 1 and 2, respectively, and that for  $Pr \gg 1$  the velocity boundary layer is much thicker than the thermal boundary layer.

The occurrence of  $f_x$  (or  $Gr_x$  as given by eq. (60)), which may be very large for flows generated by centrifugal forces, in the denominator of the ordinate implies that velocities of appreciable magnitude can be associated with such freeconvection flows. In particular, if  $f_x=10^6$  feet per second squared, which is a reasonable conservative figure for present-day rotating systems,  $\epsilon = 0.2$  (which is within the limits cf the theory presented herein), and arbitrarily X=0.25foot, then the maximum velocity attained at a Prandtl number of 0.72 is approximately 125 feet per second. This value of the maximum velocity could, of course, be doubled or even tripled under the proper conditions. One limitation to a calculation of this sort, as can be seen by comparison of the denominators of the left and middle terms of equation (60), should be kept in mind; namely, the limiting Grashof number for laminar flows. In lieu of a complete stability

analysis on this type of flow, this limiting value is taken to be 10°, as indicated in reference 10. Consideration of this limitation then implies (see eq. (60)) that for large laminar velocities either  $\nu_{\infty}$  must be large or X must be small.

### COMPARISON WITH EXPERIMENTS

Careful experiments of free-convection flows (as generated by gravitational forces) about vertical flat plates were made by Schmidt and Beckmann (ref. 2) in which velocity measurements at various points along the plate were made by means of a quartz-filament anemometer and the temperature measurements were obtained by means of manganese-constantan thermocouples. Eckert (ref. 1) performed similar experiments in which the measurements were made by means of a



FIGURE 1.-Dimensionless velocity distributions for various Prandtl numbers.

Zehnder-Mach interferometer. The results of both sets of experiments are in good agreement, but since the data presented in reference 2 by Schmidt and Beckmann appear in more detail, these data will be used for comparison with the theory.

The experiments of reference 2 were performed on two different (in that the edges were smoothed either symmetrically or not) 12- by 25-centimeter plates and on one 50- by 50-centimeter plate. It should here be pointed out that the results for the two smaller plates were almost identical and that the flow was entirely laminar except near the outer edge of the boundary layer where the slight turbulence of the room air disturbed the measurements somewhat. (This effect was also observed by Eckert.) Large periodic oscillations of the flow near the downstream edge of the larger plate were observed in addition to the slight turbulence near the



FIGURE 2.-Dimensionless temperature distributions for various Prandtl numbers.

outer edge of the boundary layer. Hence the data from the larger plate should not be expected to yield completely satisfactory agreement with the laminar theory as presented here.

Since the physical quantities can be expressed in terms of a single variable as in equations (60) and (61), it is to be expected that the data taken at the various points along the plates should all lie on a single line if the data are correlated according to equations (60) and (61). Thus for the smaller plates where  $(T_0-T_{\infty})=95.22^{\circ}$  R and  $T_{\infty}=518.68^{\circ}$  R, equations (60) to (62) become

$$\frac{U}{4.862\sqrt{X}} = F'(\eta) \tag{63}$$

$$\frac{T-518.68}{95.22} = H(\eta) \tag{64}$$

$$\eta = 88.26 \frac{Y}{X^{\frac{1}{2}}} \tag{65}$$

The velocity and temperature distributions are so plotted in figures 3 and 4, respectively, as are the curves computed theoretically for Pr=0.72. It can be seen that the agreement is in general very good for small values of  $\eta$  and somewhat less satisfactory though still rather good for the larger values of  $\eta$ . The scatter in the range of the larger values of  $\eta$  is believed to be caused by the previously discussed room



FIGURE 3.—Comparison of small plate experimental and theoretical velocity distributions for Prandtl number of 0.72.



FIGURE 4.—Comparison of small plate experimental and theoretical temperature distributions for Prandtl number of 0.72.

turbulence. It should also be noted that the points farthest away from the theoretical are those measured near the leading edge. These points should not, of course, be expected to agree too well with the theory since the boundary-layer assumptions made in the theoretical development imply that the distance along the plate is large as compared with the boundary-layer thickness. Hence, this assumption is invalid near the leading edge. Schmidt and Beckmann obtained closer agreement between the theory and the experiments for the temperature data and poorer agreement for the velocity data by basing the kinematic viscosity coefficient in equation (62) on the plate temperature rather than on the undisturbed stream temperature as was done here.

For the larger plate,  $(T_0 - T_{\infty}) = 83.7^{\circ}$  R and  $T_{\infty} = 527.14^{\circ}$  R, so that equations (60) to (62) become

$$\frac{U}{4.522\sqrt{X}} = F'(\eta) \tag{66}$$

$$\frac{T - 527.14}{83.7} = H(\eta) \tag{67}$$

$$\eta = 83.93 \frac{Y}{X^{\frac{1}{2}}}$$
 (68)

The velocity and temperature distributions for this experiment are plotted in figures 5 and 6, respectively, and again the theoretical curves for Pr=0.72 are included. In figure 5 it can be seen that for large  $\eta$  the agreement is rather poor, particularly for the data for both small and large values of X. The poor agreement for small values of X is again due to the theory limitation near the edge of the plate and for large values of X, to the fact that the flow was becoming turbulent there.

### FLOW AND HEAT-TRANSFER PARAMETERS

In addition to the velocity and temperature distributions, it is often desirable to compute other physically important quantities (such as shear stress, drag, heat-transfer rate, and heat-transfer coefficient) associated with the free-convection flow. To this end, two parameters, a flow parameter and a heat-transfer parameter, are derived in appendixes C and D, respectively.

The flow parameter

$$\frac{\tau}{(4Gr_X^3)^{\frac{1}{4}}(\nu_{\infty}\mu_0/X^2)} = F''(0)$$

is presented as a function of Prandtl number in figure 7. Thus, the various flow quantities for a given set of conditions can easily be computed by application of figure 7.

The local heat-transfer parameter

$$\frac{Nu}{(Gr_X/4)^{\frac{1}{4}}} = -H'(0)$$

as determined here is given as a function of Prandtl number in figure 8. A calculation of the local Nusselt number from this equation for Pr=0.72 and  $Gr_x=10^9$  yields a value of 63.5, which indicates that large heat-transfer coefficients can also be obtained with free-convection flows.



FIGURE 5.—Comparison of large plate experimental and theoretical velocity distributions for Prandtl number of 0.72.

On the basis of a simplified theory (that is, by use of integrated momentum and energy equations and assumed velocity and temperature distributions), Eckert (see p. 162 of ref. 7) obtained the approximate relation

$$\frac{Nu}{(Gr_X/4)^{\frac{1}{4}}} = \frac{0.718(Pr)^{\frac{5}{4}}}{(0.952 + Pr)^{\frac{1}{4}}}$$



FIGURE 6.—Comparison of large plate experimental and theoretical temperature distributions for Prandtl number of 0.72.







FIGURE 8.-Dimensionless heat-transfer parameter as function of Prandtl number.

The curve representing this equation is also presented in figure 8, and it closely approximates (to within about 10 percent) the curve determined by the more exact considerations of this report over the entire Prandtl number range. A semiempirical equation as given in reference 11 relating the average (over the length X) Nusselt number to the Prandtl and Grashof numbers which has been used in the heat-transfer calculations up to the present is

$$Nu_{as} = 0.548 [(Pr)Gr]^{\ddagger}$$

The constant 0.548 pertains specifically to air; for oil it should be 0.555 (see ref. 12) and for mercury, approximately 0.33 (see ref. 13). In order to obtain local values from the average ones given by the last equation, it is merely necessary to multiply the average values by 0.75. (The determination of this reduction factor of 0.75 is discussed in appendix D.) Thus in terms of the local quantities the semiempirical relation becomes

or

$$\frac{Nu}{(Gr_x/4)^{\frac{1}{4}}} = 0.581 (Pr)^{\frac{1}{4}}$$

 $Nu = 0.411 [(Pr)Gr_x]^{\frac{1}{4}}$ 

The curve given by this equation is also presented in figure 8 and the agreement with the theoretical curve is very good for Prandtl numbers near unity, not so good for large Prandtl numbers, and very poor for the small Prandtl numbers. Of course, changes of the constants in the semiempirical relation as previously discussed for the large or small Prandtl number cases (oil and mercury, respectively, for example) would cause the semiempirical curve to approximate the theoretical curve more closely. The values of the heat-transfer parameter obtained experimentally for mercury (Pr=0.03), air (Pr=0.72), water (Pr=7), and oil (Pr=75.5, 115, 190, 224. 275, 318, 368, and 442) are here reduced by the factor 0.75 from the average values reported. The value for mercury is an average taken of four readings from a curve, since this experiment was the only one not reported in tabular form. From figure 8 it can be seen that all of the experimental values except those for the oil experiments are in very good agreement with the theoretically computed values. The data from the oil experiments, though not so good, show reasonable agreement (maximum error of approximately 20 percent) with the theoretical curve and good agreement, as is to be expected, with the semiempirical curve. The difference between the theoretical values and the oil experiment results can possibly be due to the fact that the viscosity changes in oil are large even for small temperature differences or due to the end effects in the measurements.

#### CONCLUSIONS

An analysis was made of the free-convection flow about a flat plate oriented in a direction parallel to that of the generating body force under the prime assumption that the relative temperature difference is small. It was found that the Grashof number was the principal factor determining the type of flow and that for large Grashof numbers the flow was of the boundary-layer type. The theoretical development was then continued to consider only the cases of large Grashof number because these are of most importance in aeronautics.

Velocity and temperature profiles for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000 were computed on the basis of a constant body force and plate temperature and agreement with experiments where the fluid was air. (Prandtl number of 0.72) was good. It was also demonstrated that velocities and Nusselt numbers of the order of magnitude of those obtained in forced-convection could be obtained in free-convection flows.

A flow parameter and a heat-transfer parameter which are functions of the Prandtl number alone were derived. Calculations of the important physical quantities such as shear stress, heat-transfer rate, and the like can be computed from these parameters. Values of the heat-transfer parameter obtained from an approximate theoretical development and from experiments compared with values computed from the present development showed good agreement over a wide range of Prandtl number (0.01 to 1000). It is shown that the commonly used semiempirical relation for the heattransfer coefficient will yield good results only in restricted Prandtl number ranges.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS CLEVELAND, OHIO, October 3, 1951

# APPENDIX A

#### SYMBOLS

The follow	ing notation is used in this report:	G.	Grasho
$A_{ij}^{(n)}, B_{j}^{(n)},$	coefficients in numerical differentiation and		
$C^{(n)}, D_i^{(n)}$	integration formulas	$Gr_X$	Grasho
C <sub>p</sub>	specific heat at constant pressure	g	gravita
F	dimensionless velocity function	H	dimens
fi	components of body force per unit mass,	h	heat-tra
	i=1, 2, 3	K	isother
$f_x$	negative of X-component of body force per unit mass		— <i>р</i>

rashof number,  $\frac{\rho_{\infty}^2 f_X l^3 \epsilon}{\mu_{\infty}^2}$ 

Grashof number based on X gravitational force per unit mass (or acceleration due to gravity) dimensionless temperature function heat-transfer coefficient isothermal compressibility coefficient,

$$-\rho \left[\frac{\partial (1/\rho)}{\partial P}\right]_{T}$$

k	thermal-conductivity coefficient	β	coefficient of volumetric expansion,
l	characteristic length		$\rho \left  \frac{O(1/\rho)}{2T} \right $
<i>m</i> , <i>n</i>	arbitrary exponents		L OI Jp
N	a number, defined following equation (26)		Lanlagian operator
Nu	Nusselt number, $hX/k$		relative temperature difference $\mathcal{B}(T_{-}-T_{-})$
Num	average Nusselt number	е 77	similarity variable
P	Dressure	θ	dimensionless temperature function
* P#	Prendtl number	ĸ	step size used in numerical calculations
P	ran constant	μ	absolute viscosity
n i		ν	kinematic viscosity
r, s, t	arbitrary exponents	ρ	density
T	absolute temperature	σ	dimensionless pressure function
$U_i$	velocity components, $i=1, 2, 3$	τ	shear stress
$u_i$	dimensionless velocity components, $i=1, 2, 3$	φ	dimensionless density function
u	dimensionless velocity component in x-direc-	ψ	stream function
	tion	Subscripts:	
Ð	dimensionless velocity component in <i>u</i> -direc-	i, j	Cartesian tensor and summation subscripts
-	tion	8	denotes evaluation at static conditions ( $\epsilon=0$ )
X.	Cartesian coordinates, $i=1, 2, 3$	. 0	denotes evaluation at plate surface
	dimensionless Cartesian coordinates $i=1, 2, 3$	8	denotes evaluation at undisturbed conditions
~ f	dimensionless Cartesian coordinates, 1-1, 2, 5	Subscript not	ation is used to denote partial differentiation.
x 	dimensiomess Cartesian coordinate	Superscripts:	
<b>)</b> ′	Cartesian coordinate	Primes denot	e ordinary differentiation.
y	dimensionless Cartesian coordinate	Bars (as $\overline{\sigma}$ or	$\overline{y}$ ) denote transformed dimensionless quantities.

## **APPENDIX B**

### NUMERICAL SOLUTION OF SIMPLIFIED BOUNDARY-VALUE PROBLEM

By LYNN U. ALBERS

The method is presented herein by which solutions to the boundary-value problem

 $F''' + 3FF'' - 2F'^2 + H = 0 \tag{B1}$ 

 $H'' + 3PrFH' = 0 \tag{B2}$ 

$$F(0) = F'(0) = 0$$
  
 $F'(\infty) = H(\infty) = 0$   
 $H(0) = 1$ 

were obtained for the cases of Pr equal to 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. This discussion will enable the results to be clearly evaluated and will perhaps serve as a guide in the numerical solution of similar problems.

Each of the cases of the problem has a solution for a particular set of values for F''(0) and H'(0), hereinafter called eigenvalues. The basic approach to the problem was to estimate the eigenvalues and to integrate out from zero, obtaining functions which satisfied equations (B1) and (B2) at each step. The integration was continued until the functions F' and H behaved in a fashion inconsistent with the boundary values at infinity; for example, when they became negative or diverged to infinity. Improved estimates of the eigenvalues were then made on the basis of the results of preceding runs and the process was repeated successively until a solution was obtained.

Modifications required to overcome specific obstacles will be discussed after sufficient details of the basic procedure have been given. Then an evaluation of the accuracy of the numerical results will be made.

The integration process consists of two parts, a starting phase and an extension phase. The starting phase begins with an estimate of the eigenvalues F''(0) and H'(0) and a decision on the step size  $\kappa$  to be used. It continues with an iterative process of alternately computing F''' and H'' at the first four points and integrating them by five-point formulas. This process and that in the extension phases are so arranged that the differential equations are satisfied at each integral multiple of the step size.

The extension phase used preceding data to integrate step by step beyond the fourth point. Diagrams of both phases will be given after a few preliminary explanations.

All integration formulas used are based on the same idea. If a function, for example, F''', is known at five points, there is a unique fourth-degree polynomial which agrees with it at these five points. Moreover, if the successive antiderivatives (integrals) F'', F', and F of F''' are known at one point, there are unique fifth-, sixth-, and seventh-degree polynomials which are successive antiderivatives of this fourth-degree polynomial and which agree with F'', F', and F, respectively, at the one point. It is then a simple algebra problem to deduce from the values of F''' at five points and

F, F', and F'' at a single point the values of any of these four polynomials at any point. These results will approximate the functions F, F', F'', and F''' to a degree dependent on step size, the relative positions of the points in question, and the magnitude of the fifth derivative of F''' in the neighborhood of these points.

The preceding algebra problem can be presolved in all situations that arise in the starting and extension phases of the present problem and specific integration formulas may be deduced. These formulas are discussed in the next paragraph.

Let F''' be denoted at five successive points by  $F_0'''$ ,  $F_1^{\prime\prime\prime}, F_2^{\prime\prime\prime}, F_3^{\prime\prime\prime}$ , and  $F_4^{\prime\prime\prime}$ . In the starting phase, these points are 0,  $\kappa$ ,  $2\kappa$ ,  $3\kappa$ , and  $4\kappa$ , and  $F_0$ ,  $F_0'$ , and  $F_0''$  are also known. Then the five sets of formulas required in the starting phase are.

$$F_{i}'' = F_{0}'' + \frac{\kappa}{D_{i}^{(1)}} \sum_{j=0}^{4} A_{ij}^{(1)} F_{j}''' \qquad i=1, 2, 3, 4$$
(B3)

$$H_{i}' = H_{0}' + \frac{\kappa}{D_{i}^{(0)}} \sum_{j=0}^{4} A_{ij}^{(0)} H_{j}'' \qquad i=1, 2, 3, 4 \quad (B4)$$

$$F_{i}' = F_{0}' + i\kappa F_{0}'' + \frac{\kappa^{2}}{D_{i}^{(2)}} \sum_{j=0}^{4} A_{ij}^{(2)} F_{j}''' \qquad i=1, 2, 3, 4$$
(B5)

$$H_{i} = H_{0} + i\kappa H_{0}' + \frac{\kappa^{2}}{D_{i}^{(2)}} \sum_{j=0}^{4} A_{ij}^{(2)} H_{j}'' \qquad i = 1, 2, 3, 4$$
(B6)

$$F_{i} = F_{0} + i\kappa F_{0}' + \frac{i^{2}\kappa^{2}}{2} F_{0}'' + \frac{\kappa^{3}}{D_{i}^{(3)}} \sum_{j=0}^{4} A_{ij}^{(3)} F_{j}'''$$

$$i = 1, 2, 3, 4 \quad (B7)$$

where the superscripts on the  $A_{ij}^{(n)}$  and  $D_i^{(n)}$  refer to the order of integration.

The constants  $A_{ii}^{(n)}$  and  $D_i^{(n)}$  may be read from the following tables:

For  $A_{ij}^{(1)}$  and  $D_{i}^{(1)}$ :

Aij(l)									
1	i 0 1 2 3 4								
1 23 3 4	251 29 27 14	646 124 102 64	-264 24 72 24 24	106 4 42 64	19 1 3 14	720 90 80 45			

For  $A_{ij}^{(2)}$  and  $D_i^{(2)}$ :

r.	<i>i j</i> 0 1 2 3 4								
1 2 3 4	367 53 441 56	540 144 1404 192	$-282 \\ -30 \\ 162 \\ 48$	116 16 180 64	-21 -3 -27 0	1440 90 480 45			

For  $A_{ij}^{(3)}$  and  $D_{i}^{(3)}$ :

•	·							
1	j 0 1 2 3 4							
1 2 3 4	1017 331 1431 744	1070 664 3726 2176	618 240 486 96	258 104 450 384	-47 -19 -81 -40	10080 630 1120 315		

It is now possible to diagram the steps of the starting phase of the integration. If each bar above a function denotes an improved estimate of it, and the first estimates of  $F_1^{\prime\prime\prime}, F_2^{\prime\prime\prime}$ ,  $F_3'''$ , and  $F_4'''$  are all equal to  $F_0'''$ , and similarly for the  $H^{\prime\prime}$ , then the starting phase diagrams are

(1)  $(F_0, F_0', F_0'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_1, F_1', F_1''$ (This diagram means that the values in parentheses are used with appropriate integration formulas from (B3) to (B7) to obtain F, F', and F'' at  $\eta = \kappa$ .) (2)  $(H_0, H_0', H_0'', H_1'', H_2'', H_3'', H_4'') \rightarrow H_1, H_1'$ (3)  $(F_1, F_1', F_1'', H_1, H_1') \rightarrow \overline{F_1}''', \overline{H_1}''$ 

(The preceding diagram means that the values in parentheses are substituted in the differential equations (B1) and (B2) to obtain F''' and H'' at  $\eta = \kappa$ .)

- (4)  $(F_0, F_0', F_0'', F_0'', \overline{F_1''}, F_2'', F_3'', F_4'') \rightarrow F_2, F_2', F_3''$ (5)  $(H_0, H_0', H_0'', \overline{H_1}'', H_2'', H_3'', H_4'') \rightarrow H_2, H_2'$ (b)  $(H_0, H_0, H_0, H_1, H_2, H_3, H_4) \to H_2, H_2$ (c)  $(F_2, F_2', F_2'', H_2, H_2') \to \overline{F_2''}, \overline{H_2''}$ (c)  $(F_0, F_0', F_0'', F_0''', \overline{F_1'''}, \overline{F_2''}, F_3''', F_4''') \to F_3, F_3', F_3''$ (e)  $(H_0, H_0', H_0'', \overline{H_1''}, \overline{H_2''}, H_3'', H_4'') \to H_3, H_3'$ (f)  $(F_3, F_3', F_3'', H_3, H_3') \to \overline{F_3'''}, \overline{H_3''}$  $\begin{array}{c} (3) & (F_{3}, F_{3}, F_{3}, F_{3}, H_{3}, H_{3}, F_{3}) \rightarrow F_{3}, H_{3} \\ (10) & (F_{0}, F_{0}', F_{0}'', F_{0}''', \overline{F}_{1}''', \overline{F}_{2}''', \overline{F}_{3}''', F_{4}''') \rightarrow F_{4}, F_{4}', F_{4}'' \\ (11) & (H_{0}, H_{0}', H_{0}'', \overline{H}_{1}'', \overline{H}_{2}'', \overline{H}_{3}'', H_{4}'') \rightarrow H_{4}, H_{4}' \\ (12) & (F_{4}, F_{4}', F_{4}'', H_{4}, F_{4}') \rightarrow \overline{F}_{4}''', \overline{H}_{4}'' \\ \end{array}$

It may be noted here that all four values of F''' and H''have been improved, and further improvement will require iteration of steps 1 to 12. The start of the second iteration is diagrammed as follows:

(13)	$(F_0, F_0', F_0'', F_0'')$	$G^{\prime\prime\prime},\overline{F}_1^{\prime\prime\prime},\overline{F}_2^{\prime\prime}$	$(',\overline{F}_{3}''',\overline{F}_{4}''')$	$\rightarrow \overline{F}_1, \overline{F}_1', \overline{F}_1''$
(14)	$(H_0, H_0', H_0'',$	$\overline{H}_1^{\prime\prime}, \overline{H}_2^{\prime\prime}, \overline{I}_2$	<u></u> <i>H</i> ₃′′, <i>H</i> ₄′′)→ <i>E</i>	$\overline{H}_1, \overline{H}_1'$
(15)	$(\overline{F}_1, \overline{F}_1', \overline{F}_1'',$	$\overline{H}_1, \overline{H}_1') \rightarrow \overline{\overline{F}}_1$	$\mathbf{U}'', \overline{H}_{1}''$	
(16)	$(F_0, F_0', F_0'', F_0'')$	$\mathbb{S}^{\prime\prime\prime}, \overline{F}_{1}^{\prime\prime\prime}, \overline{F}_{2}^{\prime\prime}$	$(',\overline{F}_{3}''',\overline{F}_{4}''')$	$\rightarrow \overline{F}_2, \overline{F}_2', \overline{F}_2''$
	•	•	•	•

Successive sets of 12 steps are performed until the values of  $F_1^{\prime\prime\prime}$  and  $H_i^{\prime\prime}$  no longer change.

On the IBM Card-Programmed Electronic Calculator, a deck of punched cards 2 inches thick sufficed to perform steps 1 to 12. Three runs of this starter deck at 3 minutes per run accomplished complete convergence in most cases. At the end of the starting process there have been computed and stored  $F_4$ ,  $F_4'$ ,  $F_4''$ ,  $H_4$ , and  $H_4'$ , and final estimates of  $F_1'''$ , F<sub>2</sub>", F<sub>3</sub>", F<sub>4</sub>", H<sub>1</sub>", H<sub>2</sub>", H<sub>3</sub>", and H<sub>4</sub>".

The extension phase has now been reached. It used a different set of integration formulas based on the same general ideas as equations (B3) to (B7). If  $F_0^{\prime\prime\prime}$ ,  $F_1^{\prime\prime\prime}$ ,  $F_2^{\prime\prime\prime}$ ,  $F_{3}^{\prime\prime\prime}$ , and  $F_{4}^{\prime\prime\prime}$  now designate  $F^{\prime\prime\prime}$  at any five successive points, and the subscript 5 denotes the next point,

$$F_{b}'' = F_{4}'' + \frac{\kappa}{C^{(1)}} \sum_{j=0}^{4} B_{j}^{(1)} F_{j}'''$$
(B8)

$$H_{5}' = H_{4}' + \frac{\kappa}{C^{(1)}} \sum_{j=0}^{4} B_{j}^{(1)} H_{j}''$$
(B9)

$$F_{5}' = F_{4}' + \kappa F_{4}'' + \frac{\kappa^{2}}{C^{(2)}} \sum_{j=0}^{4} B_{j}^{(2)} F_{j}'''$$
(B10)

$$H_{5} = H_{4} + \kappa H_{4}' + \frac{\kappa^{2}}{C^{(2)}} \sum_{j=0}^{4} B_{j}^{(2)} H_{j}''$$
(B11)

$$F_5 = F_4 + \kappa F_4' + \frac{\kappa^2}{2} F_4'' + \frac{\kappa^3}{C^{(3)}} \sum_{j=0}^4 B_j^{(3)} F''' \qquad (B12)$$

where the  $B_{j}^{(n)}$  and  $C^{(n)}$  are given in the following table:

$B_{i}^{(a)}$								
n	n j 0 1 2 3 4							
1 2 3	251 135 410	-1274 -692 -2116	2616 1446 4476	-2774 1596 5084	1901 1427 5674	720 1440 20160		

The extension phase may then be diagrammed simply as follows:

(1)  $(F_4, F_4', F_4'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_5, F_5', F_5''$ 

(2)  $(H_4, H_4', H_0'', H_1'', H_2'', H_3'', H_4'') \rightarrow H_5, H_5'$ 

(3)  $(F_5, F_5', F_5'', H_5, H_5') \rightarrow F_5''', H_5''$ 

The values of the functions at the next point are computed in similar manner, where the latest sets of five values of F''' and H'' are used. This process advances step by step toward infinity.

The extended deck of punched cards was about 3 inches thick and took a little over 3 minutes per run. For Pr=0.72, a step size of 0.1 was used, the starting phase took 10 minutes. and the extension phase, about 30 minutes. When it is realized that about 11,000 operations were performed in the 40 minutes per run, it may be seen that solution of the present problem would have been prohibitively difficult on desk-type calculators. Simplifications in method would have sacrificed accuracy or required smaller step size.

In two-point boundary-value problems where one point is infinity, some problems of judgment are involved as to where infinity is, and as to when a satisfactory approximation to a solution has been obtained. In most cases this question was settled for the present problem by calling a run satisfactory when it fell between two runs for which F' and H did not differ at important points in the fourth decimal place, and for which F' and H flattened out at zero, correct to four decimal places.

Certain difficulties were met in the attempt to use the basic procedure previously discussed. These necessitated certain modifications.

For Pr=2, 10, 100, and 1000, H would settle down to zero at an early stage; but while F' was still coming down,  $H^{\prime\prime}$  would begin to oscillate and these oscillations increased and fed back into all other functions. This trouble was avoided by the following modifications: It is a consequence of equation (B2) that

$$H'(\eta) = H'(0) \exp\left(-3Pr\int_0^{\eta} F(t)dt\right)$$
$$= H'(\eta - \kappa) \exp\left(-3Pr\int_{\eta - \kappa}^{\eta} F(t)dt\right) \quad (B13)$$

The extension phase was modified to require the additional integration formulas

$$\int_{\eta-\kappa}^{\eta} F(t) dt = \frac{\kappa}{720} \sum_{i=1}^{5} A_i F_i$$
(B14)

$$H_{\delta} = H_{i} + \frac{\kappa}{720} \sum_{i=1}^{5} A_{i} H_{i}'$$
(B15)

where  $A_1 = -19$ ,  $A_2 = 106$ ,  $A_3 = -264$ ,  $A_4 = 646$ , and  $A_5 = 251$ .

These formulas were used along with equations (B8) to (B10) according to the following diagram:

(1)  $(F_4, F_4', F_4'', F_0''', F_1''', F_2''', F_3''', F_4''') \rightarrow F_5, F_5', F_5''$ (2)  $(F_1, F_2, F_3, F_4, F_5, H_4') \rightarrow H'$  by means of (B14) and (B13)

(3)  $(H_1', H_2', H_3', H_4', H_5', H_4) \rightarrow H_5$  by means of (B15)

(4)  $(F_5, F_5', F_5'', H_5) \rightarrow F_5'''$ The value  $F_0'''$  is discarded and F''' at the last five points is used to repeat the whole process again and again ad infinitum. As long as F stays positive, H' is guaranteed to approach zero and H will flatten out to some value and not oscillate.

For Pr=0.01, 0.72, 0.733, and 1, the F''' began to oscillate at an advanced point and these oscillations grew and fed into the other functions. For all cases but Pr=0.01, the oscillations appeared very late, near the end of the run, and a suitable halving of step size when oscillation was detected in the fourth differences of F''' was sufficient to avoid the difficulty. But for the 0.01 case, oscillations of F''' appeared early in the run, namely, soon after the peak in F. These oscillations were found to be step-size connected, so that reduction of the step to 0.02 avoided them. Even then oscillations in F''' would begin to appear every 25 steps or so, and these were smoothed out regularly by repeated runs of a deck similar to the starter deck. Each run under these conditions took about 16 hours, making this the most difficult case to solve.

## APPENDIX C

## DERIVATION OF FLOW PARAMETER

By definition the shear stress is given by

$$\tau = \mu_0 \left(\frac{\partial U}{\partial Y}\right)_0 \tag{C1}$$

To express  $(\partial U/\partial Y)_0$  in terms of the known function  $F(\eta)$ , use can be made of equations (60) and (62). Then

$$\frac{\partial U}{\partial Y} = (4Gr_X^3)^{\frac{1}{4}} \frac{\nu_{\infty}}{X^2}$$

Substitution of this expression into equation (C1) yields the flow parameter

$$\frac{\tau}{(4Gr_{x}^{3})^{\frac{1}{4}}(\nu_{\infty}\mu_{0}/X^{2})}=F^{\prime\prime}(0)$$

Note that from the general derivation, the flow parameter contains the viscosity evaluated at two different points. Recall, however, that the analysis has shown that to a first approximation the variation of viscosity with temperature can be neglected. Thus the viscosity can be taken as constant in the entire flow field.

## APPENDIX D

#### DERIVATION OF HEAT-TRANSFER PARAMETER

The local Nusselt number is defined as

$$Nu = \frac{hX}{k} = \frac{-X}{(T_0 - T_\infty)} \left(\frac{\partial T}{\partial Y}\right)_0 \tag{D1}$$

To express  $(\partial T/\partial Y)_0$  in terms of the known function  $H(\eta)$ , use is made of equations (61) and (62). Thus

$$\frac{\partial T}{\partial Y} = \frac{(T_0 - T_\infty)}{X} \left(\frac{Gr_X}{4}\right)^{\frac{1}{2}} H'(\eta)$$

Substitution of this expression into equation (D1) yields the heat-transfer parameter

$$\frac{Nu}{(Gr_x/4)^{\frac{1}{4}}} = -H'(0) \tag{D2}$$

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The heat-transfer parameter as given by equation (D2) is, as was previously stated, a local parameter. It is often desired to compute the average (over the length X) value of this parameter. To this end, the Nusselt number (as given in equation (D1)) must be defined in terms of an average heat-transfer coefficient and the quantity thus obtained must then be integrated over the length X and divided by X. This procedure yields the result

$$\dot{Nu} = \frac{3}{4} (Nu)_{av}$$

It is from this last equation that the 0.75 reduction factor previously discussed was obtained.

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## FREE-CONVECTION FLOW AND HEAT TRANSFER ABOUT A FLAT PLATE

# TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS

(a) Prandtl number, 0.01

η	F	F'	F"	Н	H'	7	F	F'	<b>F</b> "	H	H'
0	0.0000	0.0000	0.9862	1.0000	-0.0812	4,16	1,9319	0.4037	0.0615	0.6741	-0.0723
.1	.0048	. 0936	. 8868	. 9919	0812	4.18	1.9400	.4024	- 0614	. 6726	0722
.2	.0184	.1774	. 7891	. 9838	0812	4.20	1.9480	. 4012	0612	. 6712	0721
.3	. 0399	. 2516	. 6942	, 9756	0812	4.24	1.9640	. 3988	0609	. 6683	0720
.4	.0684	.3164	. 6030	. 9675	0811	4.28	1.9799	. 3964	0605	. 6654	0718
.5	.1029	. 3723	. 5163	. 9594	0811	4.32	1.9957	. 3939	0602	. 6626	0716
.6	.1426	. 4198	. 4849	. 9513	0811	4.86	2.0114	.3915	0599	.6597	0714
1.7	. 1866	. 4595	. 3595	.9432	0811	4.40	2.0270	.3891	0595	. 6568	0718
.8	. 2342	. 4919	. 2906	. 9351	0810	4.50	2.0667	. 3832	0587	.6497	0708
1.9	.2848	. 5178	. 2286	.9270	0809	4.60	2.1037	.3774	05/9	. 6427	0/04
1.0	- 3376	. 6379	.1784	.9189	0509	4.70	2.1411	.3/10	00/2	6007	- 0000
1 11	. 3924	.002/	.1203	.9109	06005	4.00	0 0142	3000	0558	8919	0000
1.2	.4480	- 0031	.0839	. 9028	0007	4.80	0 0701	2549	0540	6140	
1.3	. 001/ re10	509/	.0109	.0011	0800	£ 10	0.09852	2/04		6091	- 0681
1 12	.0010	5790	0020	9798		5 20	2 2000	3440	- 0594	6013	- 0676
1.0	.0184	5705	0000	9708	0003	5 40	2.3200	3335	- 0520	5978	- 0687
1 1.2	7398	5604	- 0220	8826		5 60	2 4534	3232	- 0508	5748	- 0657
1 1 8	7004	5650	- 0493	8548	- 0798	5.80	2,5170	3132	- 0493	. 5615	0648
1 1 9	8466	5598	- 0579	.8466	0796	8.00	2, 5787	. 8035	0479	. 5487	0638
1 50	9023	5535	- 0641	.8387	0794	6.20	2,6384	. 2940	0467	5360	0628
21	.9573	5469	0685	.8307	- 0792	6.40	2,6963	. 2849	0454	. 5236	0618
22	1.0117	. 5399	0714	. 8228	0789	6.60	2.7524	. 2759	0442	. 5113	0608
2.3	1.0653	. 5326	0732	. 8150	0787	6.80	2.8067	. 2671	0430	. 4993	0598
24	1.1182	. 5253	0742	.8071	0784	7.00	2.8593	. 2586	0419	. 4874	0588
2.5	1.1703	. 5178	0746	. 7993	0782	7.40	2.9594	. 2423	0397	. 4643	0568
2.6	1.2217	. 5104	0745	. 7915	0779	7.80	3.0532	. 2269	0376	. 4420	0547
2.7	1.2724	. 5029	0741	. 7837	0776	8.20	3.1411	. 2123	0356	. 4205	0527
2.8	1.3223	. 4956	0735	. 7760	0773	8.60	3.2232	. 1984	0337	. 3998	0508
2.9	1.3715	. 4882	0728	. 7682	0770	9.00	3.2999	. 1853	0319	. 3799	0489
3.0	1.4200	.4810	0719	. 7606	0768	9.40	3.3715	.1729	0302	.8607	0470
3.1	1.4677	.4739	0711	.7529	0763	9.80	3.4383	1612	0285	.3423	0451
3.2	1.6148	. 4668	0701	. 7453	0760	10.20	3.5005	. 1601	02/0	.3240	0432
3.3	1.5611	.4598	0692	.7377	0750	10.60	3.0084	1390	0201	- 3009	- 0207
3.4	1.6007	.4030	0083	.7302	0/50	11.00	3.0123	1907		. 2010	
3.40	1.0233	.4490	00/8	7201	0781	11.90	3.0042	1114	0215	2610	0
3.0	1.0017	. 1102	0660	7190	- 0747	12 20	3 7514	1031	- 0210	2468	- 0348
2.00	1 6060	4305		7152	- 0745	12 60	3 7011	0952	- 0192	2332	- 0332
2 45	1 7170	4369		7115	- 0743	13.00	3 8276	0877	-0181	2202	- 0317
37	1 7206	4320	- 0655	7078	- 0741	13 60	3 8771 .	.0773	0165	2018	0296
3 75	1 7811	4206	- 0851	7041	- 0739	14 20	3 9203	.0679	0151	. 1847	0276
3.8	1.7825	4264	- 0646	7004	- 0738	14.80	3,9587	0592	0138	.1687	0257
4.85	1,8308	4232	0642	6967	0736	15.40	3,9918	. 0513	0126	. 1538	0239
3.9	1.8249	.4200	0638	. 6930	0734	16.00	4.0204	.0441	0114	.1399	0223
3, 95	1.8458	. 4168	0633	. 6893	0732	17.00	4.0591	. 0335	0097	. 1189	0197
4.00	1.8665	. 4136	0629	. 6857	0729	18.00	4.0880	.0246	0082	.1003	0175
4.05	1.8871	. 4105	0625	. 6821	0727	19.00	4.1088	.0171	0069	. 0839	0154
4.10	1.9076	. 4074	0620	. 6784	0725	20.00	4.1226	.0108	0057	.0694	0137
4.12	1.9157	. 4061	0619	. 6770	0725	21.00	4.1308	.0057	0046	.0565	0121
4.14	1.9238	.4049	0617	. 6755	0724	22.00	4.1343	.0015	0037	- 0462	0107
1		1		I	1	1	1	1	1	1	1

(b) Prandtl number, 0.72

η	F	F'	F"	H	H'
7 0	F 0.0000 .0032 .0122 .0231 .0440 .0651 .0887 .1141 .1407 .1681 .2231 .2501 .2764 .3017 .3260 .3490 .3707 .3260 .3490 .4100 .4277 .4440 .4554 .5332 .5401 .5464 .5520 .5735 .5808 .5961 .5961	F*           0.0000           0.0527           11692           1602           22460           2708           27759           2778           27104           1983           2203           1697           1596           0992           0815           0735           06801           0533           0427           0437           0430           02039           02122           0364           0037           0036<	F** 0. 6780 .5785 .4800 .2479 .1817 .1224 .0700 .0246 .0143 .0704 .0468 .0734 .0468 .0734 .0468 .1224 .1302 .1347 .1363 .1234 .1365 .1331 .1290 .1238 .1178 .1365 .1238 .1178 .1365 .1238 .1178 .0903 .0934 .0707 .0903 .0642 .0682 .0682 .0682 .0642 .05851 .0682 .0682 .0682 .0682 .0048 .0048 .0048 .0048 .00585 .0048 .00585 .0048 .00585 .0048 .00585 .00058 .0	H 1,0000 9495 8991 9495 9455 9455 9455 9455 9455 9455 9455 9455 9455 9455 9455 9455 9455 9455 94555 9455 9455 9455 9455 9455 9455 9455 9455 9455	H' -0.5046 5045 5037 5016 4979 4979 4991 4840 4735 4607 4467 4467 4284 4284 4284 3676 38591 3576 2556 2556 2554 2141 1688 1688 1688 1688 1688 1688 1688 0657 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0518 0037 0022 0003 0003 0003
	1	1	1	1	1

-	(c) Prandtl number, 0.733										
	η	F	F	F''	H	H'					
	0 12345678901234567890123468024680246802468048260 111111112222245678901234680246802468048260 444655668048260	0.0000 .0032 .0132 .0260 .0438 .0649 .0884 .1137 .1402 .2420 .2752 .2490 .2752 .3244 .4079 .4254 .4079 .4254 .4079 .4254 .4079 .4254 .4079 .4254 .4079 .4254 .5139 .5045 .5139 .5534 .5543 .5544 .5543 .5544 .5554 .5543 .5544 .55545 .5554 .5554 .5554 .5554 .5554 .5554 .5554 .5554 .5554 .555	- 0.0000 .0625 .1155 .1297 .1955 .2238 .2461 .2695 .2745 .2745 .2745 .2662 .2485 .23485 .2588 .2485 .2588 .2485 .2588 .0599 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .00559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0559 .0055 .0051 .0052 .0051 .0052 .0051 .0052 .0051 .0052 .0051 .0052 .0051 .0051 .0052 .0051 .0051 .0051 .0052 .0051 .0052 .0051 .0051 .0052 .0051 .0055 .0051 .0051 .0051 .0051 .0051 .0051 .0051 .0051 .0055 .0051 .0051 .0055 .0051 .0051 .0055 .0051 .0055 .0051 .0055 .0051 .0055 .0051 .0055 .0051 .0055 .0051 .0055 .00	0. 6741 .5767 .8990 .2465 .1804 .2465 .2037 .0737 .0737 .0737 .0737 .0946 .1106 .1106 .1222 .0965 .06955 .00022 .0003 .00022 .0005	1.0000 9492 8894 8478 7477 7477 8855 6503 6033 6033 6578 4718 33581 3246 2955 2957 2957 1913 1913 1913 1913 1913 1913 1913 191	-0.5050 -5079 -5079 -5079 -5079 -5079 -5079 -41853 -41853 -4170 -44783 -44783 -44783 -44783 -44783 -3902 -3302					

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# TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS-Continued

(d) Prandtl number, 1

(e) Prandtl number, 2

, 7	F	F	<b>F''</b>	H	Н'
$\begin{array}{c} 0\\ .1\\ .2\\ .3\\ .4\\ .6\\ .8\\ .90\\ 1.1\\ 1.1\\ .1\\ .1\\ .1\\ .1\\ .1\\ .1\\ .1\\ $	0.0000 .0030 .0115 .0246 .0413 .0610 .0629 .1064 .1308 .1588 .2058 .2058 .2058 .2075 .3177 .3543 .3543 .3543 .3543 .3543 .3597 .3543 .3577 .3543 .3597 .3543 .4726 .5016 .5016 .5016 .5017 .5183 .5189 .5194 .5194 .5194 .5194	0.0000 0.0533 10923 15033 12033 12039 22777 2406 2481 22177 2406 2481 2217 2202 2203 2095 2095 2095 2005 20	0.6421 .5450 .3894 .2016 .2016 .0033 0263 0263 0263 0263 02657 0773 0775 0775 0775 1283 	1.0000 .9433 .8367 .7742 .7742 .7745 .6645 .6116 .5602 .5109 .4533 .4192 .3381 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .3381 .2684 .1922 .2685 .0722 .0339 .0221 .0033 .003	-0. 5671 5667 5667 5667 5627 5572 5437 5221 6438 4330 4330 4350 43
F		(f) Prandt	l number,	10	
7	F	F'	F"	H	H'
0 12345678901234567890123456789012346802468804820000 111111111122222222222333334680246804820000	0.0000 .0019 .00171 .0147 .02411 .0346 .0458 .0573 .0687 .0687 .0908 .0112 .1111 .1205 .1593 .1656 .11714 .1769 .1820 .1656 .11714 .1769 .1820 .2027 .2060 .2019 .2146 .2253 .2286 .2215 .2283 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2215 .2286 .2216 .2216 .2286 .2216 .2286 .2216 .2286 .2218 .2286 .2286 .2286 .2488 .2488 .2489 .2489 .2489	0.0000 .0371 .0654 .0861 .1091 .1137 .1137 .1137 .1137 .1137 .1137 .1066 .0903 .0853 .0908 .0853 .0908 .0853 .0908 .0567 .0533 .0491 .0491 .0495 .0496 .0343 .0491 .0495 .0496 .0343 .0319 .0277 .0226 .0226 .0227 .0228 .0227 .0228 .0227 .0228 .0178 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0153 .0057 .0153 .0153 .0153 .0153 .0153 .0057 .0153 .0057 .0053 .0153 .0153 .0057 .0153 .0153 .0057 .0053 .0057 .0053 .0057 .0153 .0055 .0055 .0057 .0153 .0153 .0055 .0057 .0055 .0057 .0053 .0057 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0053 .0057 .0053 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0053 .0057 .0055 .0055	0.4192 -3251 -3253 -1723 -1723 -1723 -0279 -0011 -0221 -0357 -0462 -0518 -0552 -0462 -0552 -0462 -0555 -0462 -0552 -0462 -0555 -0462 -0552 -0462 -0552 -0462 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0455 -0555 -0257 -0255 -0257 -0255 -0055	1,0000 -8832 -6700 -6534 -6534 -6434 -6434 -6434 -1519 -1090 -0763 -0522 -0349 -0228 -0146 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0052 -0056 -0050 -0000	$\begin{array}{c} -1.1694\\ -1.1671\\ -1.1621\\ -1.1621\\ -1.1852\\9640\\8545\\9640\\8545\\9640\\8545\\9601\\7322\\6061\\7322\\6061\\7323\\2045\\7323\\2045\\7323\\2045\\7373\\2045\\0073\\0073\\0073\\0073\\0073\\0073\\0073\\0073\\0005\\0033\\0002\\0013\\0002\\0013\\0002\\0003\\0000\\0000\\ .000$

(e) I fandel humber, *									
7	F	F	F*'	Н	<i>H</i> ′				
7 0 123.45.67.78.90 111.1.1.1.1.1.1.2.2.12.3.4 1.1.1.1.1.1.2.2.12.3.45.67.8.90 1.1.2.3.4.5.67.8.90 2.2.3.4.5.68.0.2.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4	F 0,0000 0027 0101 0215 0358 0525 0709 0904 1104 1307 1508 1705 1895 2077 2260 2412 2464 2706 2412 2464 2706 2836 3166 3068 3166 3340 3453 3543 3555 3555 3555 3555 3555 3555 3555 3555 3555 3555 3	<i>F</i> * 0.0000 0.0523 0.952 0.952 0.952 0.952 0.952 0.1297 1.565 1.763 1.1901 1.1985 1.1901 1.1985 1.1804 1.1938 1.864 1.864 1.864 1.864 1.864 1.865 1.357 1.157 1.157 1.157 1.157 1.157 1.157 1.157 0.057 0.057 0.0188 0.026 0.027 0.0188 0.028 0.	F" 0, 5713 4749 .3861 .4749 .3861 .0052 .00602 .0185 .00602 .0185 .00602 .0185 .00602 .0185 .00602 .0185 .00602 .0105 .00824 .0942 .1020 .1063 .1073 .1073 .1073 .1073 .1073 .00849 .0966 .09849 .09661 .09664 .00575 .00572 .00575 .00572 .00575 .00575 .00575 .00575 .00575 .00575 .00575	H 1.0000 9224 8.5699 7.858 7.858 7.858 5.5183 4.564 4.564 3.979 2.268 3.476 4.564 3.979 2.268 2.2999 2.268 3.476 4.564 1.220 1.647 1.220 1.032 0.052 0.0151 0.0151 0.0151 0.0105 0.0055 0.0025 0.0025 0.0077 0.0017 0.0007 0.0007 0.0005 0.0055 0.0055 0.0055 0.0055 0.0055 0.0055 0.0077 0.0077 0.0007 0.0007 0.0007 0.0005 0.0055 0.0055 0.0077 0.0077 0.0007 0.0007 0.0007 0.0007 0.0007 0.0005 0.0055 0.0055 0.0077 0.0077 0.0005 0.0007 0.0007 0.0007 0.0007 0.0005 0.0007 0.0005 0.0005 0.0005 0.0007 0.0007 0.0005 0.0005 0.0005 0.0007 0.0007 0.0007 0.0005 0.0005 0.0005 0.0005 0.0007 0.0007 0.0007 0.0005 0.00	<i>H</i> ′′ -0.7165 -7.7161 -7.7161 -7.7135 -7.7069691969196523652364136522568256825682568256825682568256825682568256825682568256825685656856568				
4.6 4.8 5.0 5.4 5.8 6.4 7.0 8.0 9.0 10.0 11.0	. 4004 . 4015 . 4023 . 4035 . 4043 . 4049 . 4053 . 4058 . 4058 . 4059 . 4059	. 0039 .0047 .0037 .0024 .0015 .0008 .0005 .0002 .0001 .0001 .0001	0068 0054 0042 0028 0016 0008 0004 0001 .0000 .0000	.0002 .0001 .0001 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000	0004 0002 0001 .0000 .0000 .0000 .0000 .0000 .0000 .0000				
	ú	g). Prandtl	number, 1	00					
η	F	F'	F″	H	H'				
0.000 .025 .050 .075 .100 .125 .150	0.0000 .0001 .0003 .0006 .0011 .0017 .0023	0.0000 .0060 .0114 .0162 .0205 .0244 .0277	0. 2517 . 2274 . 2044 . 1828 . 1626 . 1438 . 1263	1,0000 .9462 .8905 .8369 .7815 .7276 .6744	$\begin{array}{r} -2.191 \\ -2.191 \\ -2.188 \\ -2.180 \\ -2.166 \\ -2.144 \\ -2.113 \end{array}$				

ų	F	F'	F″	H	H'
0.000	0.0000	0.0000	0.2517	1.0000	-2.191
.025	.0001	.0060	. 2274	9452	-2, 191
.050	.0003	.0114	. 2044	. 8905	-2, 188
.075	.0006	.0162	. 1828	. 8359	-2.180
.100	.0011	. 0205	.1626	.7815	-2.166
• .125	.0017	.0244	.1438	.7276	-2.144
.150	.0023	.0277	. 1263	. 6744	-2.113
.175	.0031	1.0307	.1101	.0221	-2.0/1
200	0047	0255	0916	5913	-1 054
250	0056	0373	0602	4733	
. 975	.0066	1389	0580	4273	-1.786
.300	.0076	.0402	.0479	3836	-1.704
.325	.0086	.0413	.0389	.3422	-1.604
.350	.0096	.0422	.0309	. 3034	-1.498
.375	. 0107	.0429	. 0238	. 2674	-1.388
.400	.0118	.0434	.0176	. 2341	-1.276
. 425	. 0129	. 0438	.0123	. 2036	-1. 163
. 450	.0140	.0440	.0076	. 1769	-1.052
.475	.0151	.0442	.0036	. 1510	9430
	.0162	.0442	.0002	.1287	8393
. 620	.0173	.0442	0026	. 1089	7404
. 000	0105	.0441	0000	.0910	01/8
.070	0208	0437	- 0087	0694	- 4937
85	0227	.0.07	- 0111	0427	- 3408
.70	. 0249	.0426	0126	.0280	- 2440
.75	.0270	.0420	0135	.0178	-, 1657
. 80	.0291	.0413	0140	.0111	1088
.85	.0311	.0406	0142	.0067	0693
.90	.0331	. 0399	0142	.0039	0428
. 95	.0351	. 0392	0141	. 0022	0256
1.00	.0371	. 0385	0140	.0012	0149
	.0408	.0371	0136	.0004	0046
1.20	.0440	.0307	0132	.0001	0013
1.00	.0460	.0399	- 0120	.0000	0003
1.50	0546	0320	_ 0110		0001
1 80	0578		- 0116		.0000
1.70	.0608	. 0297	0112	.0000	.0000
ĩ.80	.0637	.0286	0108	.0000	. 0000
1.90	.0665	.0275	0104	.0000	.0000
2.0	.0692	. 0265	0101	.0000	.0000
2.1	.0718	. 0255	0097	.0000	.0000
2.2	.0743	. 0245	0094	.0000	.0000
23	.0767	.0236	0091	.0000	.0000
2.4	.0790	.0227	0088	.0000	.0000
2.0	-0844	.0210	0082	.0000	.0000
2.8	.08/1	.0190	0070	.0000	.0000
3,0		.0120	00/1	.0000	.0000

## FREE-CONVECTION FLOW AND HEAT TRANSFER ABOUT A FLAT PLATE

# TABLE I.—FUNCTIONS F AND H AND DERIVATIVES FOR VARIOUS PRANDTL NUMBERS—Concluded

(g) Prandtl number, 100-Concluded

(h) Prandtl number, 1000

η	F	F	F"	Ħ	H'
3.2 3.4 3.8 4.4 4.8 5.6 0.6 6.6 7.2 8.0 9.0 11.0 11.0 11.0 11.0 11.0	0.0947 .0979 .1035 .1031 .1105 .1143 .1203 .1254 .1276 .1297 .1315 .1326 .1336 .1336	0.0166 .0164 .0142 .0131 .0103 .0088 .0074 .0053 .0041 .0032 .0022 .0014 .0005 .0004 .0005 .0000	-0.0086 -0056 -0057 -0057 -0057 -0057 -0057 -0017 -0017 -0018 -0018 -0017 -0005 -0007 -0005 -0007 -000	0.0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000 .0000	0.0000 .00000 .00000 .00000 .0000 .0000 .0000 .0000 .0000 .0000 .0000

ŋ	F	F'	F"'	H	H'
0. .025 .050 .075	0.0000 .0000 .0002 .0003	0.0000 .0033 .0061 .0083	0. 1450 . 1212 . 0999 . 0811	1.0000 .9009 .8021 .7046	-3.966 -3.962 -3.933 -3.881
.100 .125 .150 .175 .200 .225	.0006 .0008 .0012 .0015 .0018 .0022	.0102 .0116 .0127 .0135 .0142 .0146	.0047 .0506 .0387 .0289 .0209 .0209 .0146	.5186 .4332 .3549 .2847 .2236	-3, 731 -3, 538 -3, 283 -2, 975 -2, 628 -2, 261
.250 .275 .300 .825 .350 .375	.0026 .0029 .0033 .0037 .0041	.0149 .0151 .0152 .0153 .0153 .0153	.0096 .0059 .0032 .0012 0008 0012	.1717 .1288 .0944 .0675 .0471	-1.893 -1.541 -1.220 9331 7012 5093
.400 .425 .450 .475 .500	.0018 .0052 .0056 .0060 .0063	.0152 .0152 .0153 .0151 .0150 .0150	0012 0019 0023 0026 0027 0028	.0213 .0138 .0087 .0053 .0053	
. 525 . 550 . 575 . 600 . 625 . 800	.0067 .0071 .0075 .0078 .0082 .0107	.0149 .0148 .0147 .0147 .0146 .0146	0029 0029 0029 0029 0029 0029	.0019 .0011 .0006 .0003 .0002 .0000	0413 0245 0142 0050 0044 . 0000
1.000 1.40 1.80 2.20 2.60	.0135 .0187 .0235 .0279 .0320	.0138 .0125 .0115 .0108 .0098	0027 0025 0023 0022 0020	.0000 .0000 .0000 .0000 .0000	.0000 .0000 .0000 .0000 .0000
3.6 4.2 5.0 5.8 7.0	.0505 .0409 .0454 .0505 .0549 .0503	.0030 .0080 .0070 .0060 .0050 .0039	0013 0017 0015 0012 0011 0008	.0000 .0000 .0000 .0000 .0000	.0000 .0000 .0000 .0000 .0000
8.0 10.0 12.0 14.0 16.0	.0638 .0691 .0727 .0752 .0771	.0032 .0022 .0015 .0011 .0008 .0007	0007 0004 0003 0002 0001	.0000 .0000 .0000 .0000 .0000 .0000	.0000 .0000 .0000 .0000 .0000
20.0 22.0 23.6	.0798 .0809 .0816	.0005 .0005	.0000 .0000 .0000	.0000 .0000 .0000	.0000 .0000 .0000

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