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An Analysis of the Discontinuous Galerkin Method for a Scalar Hyperbolic Equation

By C. Johnson and J. Pitkäranta

Abstract. We prove L_p stability and error estimates for the discontinuous Galerkin method when applied to a scalar linear hyperbolic equation on a convex polygonal plane domain. Using finite element analysis techniques, we obtain L_2 estimates that are valid on an arbitrary locally regular triangulation of the domain and for an arbitrary degree of polynomials. L_p estimates for $p \neq 2$ are restricted to either a uniform or piecewise uniform triangulation and to polynomials of not higher than first degree. The latter estimates are proved by combining finite difference and finite element analysis techniques.

1. Introduction. In this note we prove stability and error estimates for the discontinuous Galerkin method applied to the scalar linear hyperbolic model problem

$$(1.1) \qquad \begin{cases} u_\beta + au = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_-, \end{cases}$$

where Ω is a bounded convex plane domain, $\beta = (\beta_1, \beta_2)$ is a constant unit vector, $u_\beta = \beta \cdot \nabla u$, a is a bounded measurable function on Ω, and Γ_- denotes the "inflow" part of the boundary $\partial\Omega$: $\Gamma_- = \{x \in \partial\Omega\colon \nu(x) \cdot \beta < 0\}$, where $\nu(x)$ is the outward unit normal to $\partial\Omega$ at x. Let us recall the definition of the discontinuous Galerkin method for (1.1) (cf., [9]). Given a finite element partitioning $\mathscr{C}^h = \{T\}$ of Ω, let $P_k(T)$ denote the space of polynomials of degree $\leqslant k$ on $T \in \mathscr{C}^h$, and seek a function u_h defined on Ω such that for all $T \in \mathscr{C}^h$, $u_{h|T} \in P_k(T)$ and

$$(1.2) \qquad \int_T (u_{h\beta} + au_h) v\,dx + \int_{\partial T_-} |\nu \cdot \beta| (u_h^+ - u_h^-) v\,ds = 0, \qquad v \in P_k(T),$$

where ν denotes the outward unit normal to ∂T, $\partial T_- = \{x \in \partial T\colon \nu(x) \cdot \beta < 0\}$, and $u_h^\pm(x) = \lim_{\varepsilon \to 0^\pm} u_h(x + \varepsilon\beta)$, with $u_h^-(x) = g(x)$ if $x \in \Gamma_-$. As will be seen below, u_h is uniquely determined by (1.2) and it is possible to compute u_h successively on each $T \in \mathscr{C}_h$ starting at the inflow boundary Γ_- where u_h^- is given [9]. Thus, (1.2) is an essentially explicit scheme for (1.1).

The subsets T in (1.2) are usually triangles or quadrilaterals with possibly curved sides on $\partial\Omega$. Here, we assume for simplicity that Ω is a polygon and that all the subdomains in $\mathscr{C}^h$ are triangles. In this form, the scheme (1.2) has been used successfully for solving the neutron transport equation approximately, cf., [11].

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To be able here to easily present our results and compare with previous work, let us for simplicity assume that Ω is the unit square $I \times I$ with $I = (0,1)$, $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$, and that $\mathscr{C}_h$ is a uniform triangulation of Ω with nodes (ih, jh), $0 \leqslant i, j \leqslant N = h^{-1}$, where h is the mesh length. Given a piecewise smooth function v on Ω write $v^n(\cdot) = v^-(\cdot, nh)$. We shall under various assumptions prove error estimates of the form

$$\|u - u_h\|_{L_p(\Omega)} \leqslant Ch^{k+1/2}|u|_{W_p^{k+1}(\Omega)}, \qquad 1 \leqslant p \leqslant \infty, \tag{1.3a}$$

$$\begin{aligned} &\|u - u_h\|_{L_p(\Omega)} + \max_n \|u^n - u_h^n\|_{L_p(I)} \\ &\quad \leqslant Ch^{k+\min(1/2, 1-1/p)}|u|_{W_p^{k+1}(\Omega)}, \qquad 1 \leqslant p \leqslant \infty, \end{aligned} \tag{1.3b}$$

$$\max_n \|u^n - u_h^n\|_{L_p(I)} \leqslant Ch^{k+1/2}\left[|u|_{W_p^{k+1}(\Omega)} + \max_n |u^n|_{W_p^{k+1}(I)}\right], \qquad 1 \leqslant p \leqslant 2. \tag{1.3c}$$

For $p = 2$ we prove analogues of (1.3a, b) on general meshes and arbitrary $k \geqslant 0$ using finite element techniques. For $p \neq 2$ and $k = 0$ or $k = 1$ we prove (1.3b, c) with $1 \leqslant p \leqslant \infty$ and (1.3a) with $2 \leqslant p \leqslant \infty$ on piecewise uniform meshes using a combination of techniques from Fourier and finite element analysis and finally in a similar way we prove (1.3a) for $1 \leqslant p \leqslant 2$ and $k = 0, 1$ on uniform meshes.

Notice that if $1 \leqslant p \leqslant 2$, then (1.3b) is an optimal estimate in the sense that the exponent of h cannot be increased while keeping the norm on u, nor can the regularity requirements on u be weakened while keeping the exponent of h. On the other hand, (1.3a) is not optimal in this sense since for the interpolation error $u - \tilde{u}_h$, where $\tilde{u}_{h|T} \in P_k(T)$, $T \in \mathscr{C}_h$, is a suitable interpolant of $u|_T$, we have

$$\|u - \tilde{u}_h\|_{L_p(\Omega)} \leqslant Ch^{k+1}|u|_{W_p^{k+1}(\Omega)}. \tag{1.4}$$

Most likely, the estimate (1.3a) cannot be improved in the above sense for the method (1.2) and it is an open problem if there are other methods for (1.1) which allow such an improvement.

Estimates of the form (1.3a–c) for $p = 2$ on general meshes were proved in [6], [10] for the so-called streamline diffusion method. In fact, the discontinuous Galerkin method and the streamline diffusion method have very similar properties when applied to (1.1), and the analysis of the two methods is also similar. In particular, it is possible to prove localization results and local error estimates for the discontinuous Galerkin method which are analogous to those presented in [6], [10]. Let us also mention that the L_2-analysis of both the streamline diffusion method and the discontinuous Galerkin method can be extended to Friedrichs systems, see [6], [8].

The error estimates (1.3) for $p = 2$ (and the localization results mentioned above) are based on a stability estimate for (1.2) of the form

$$|u_h|_{h,\beta} + \|u_h\|_{L_2(\Omega)} \leqslant C\left[\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma_-)}\right] \tag{1.5}$$

where $|\cdot|_{h,\beta}$ is a mesh-dependent seminorm which controls the derivative $u_{h\beta}$ and the jumps of $\nu \cdot \beta u_h$ across the interelement boundaries. The stability estimate (1.5) is a discrete (weak) counterpart of the stability inequality

$$\|u_\beta\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)} \leqslant C\left[\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma_-)}\right],$$

which obviously holds for the continuous problem.

Lesaint and Raviart [9], who gave the first analysis of the discontinuous Galerkin method, proved the following estimate for (1.2) on general meshes for $k \geqslant 0$,

$$\|u - u_h\|_{L_2(\Omega)} \leqslant Ch^k |u|_{W_2^{k+1}(\Omega)}. \tag{1.6}$$

Notice that in (1.6) the gap, i.e., the difference between the number of derivatives of u and the exponent of h, is equal to one. Results of this type are typical in the usual finite element analysis of linear hyperbolic problems which is based on weaker stability estimates of the form

$$\|u_h\|_{L_2(\Omega)} \leqslant C\big(\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma_-)}\big).$$

The estimate (1.3a) with $p = 2$, for which the gap is only $\frac{1}{2}$, was used in a crucial way in [7] where an L_2-analysis of a fully discrete scheme for neutron transport in cylindrical geometry was given. The estimate (1.3a) with $p \neq 2$ may be used to generalize this analysis to L_p, $p \neq 2$. Of particular interest (for eigenvalue problems) would then be the case $p = 1$. Unfortunately, we have been able to prove (1.3a) for $p = 1$ only on a uniform mesh.

The estimates (1.3b, c) are of interest when we consider (1.1) as a model for a linear hyperbolic initial-boundary value problem with x_2 representing a time variable and where the approximate solution u_h is computed successively on the strips $S_n = \{x \in \Omega\colon (n-1)h < x_2 \leqslant nh\}$, $n = 1, \ldots, N$, so that $\|u^n - u_h^n\|_{L_p(I)}$ is the error on each time level $x_2 = nh$. For conventional finite element methods (not including the streamline diffusion method) for (1.1) with piecewise polynomials of degree k, the typical result for $p = 2$ reads

$$\max_n \|u^n - u_h^n\|_{L_2(I)} \leqslant Ch^k |u|_{W_2^{k+1}(\Omega)},$$

with a loss of a factor $h^{1/2}$ as compared with (1.3b). For (dissipative) finite difference methods a typical result for (1.1) (again with gap = 1) obtained by Fourier methods in the case of a uniform mesh reads

$$\max_n \|u^n - u_h^n\|_{L_p} \leqslant Ch^m \|g\|_{W_p^{m+1}(\mathbf{R})}, \tag{1.7}$$

where now $\Omega = \{x \in \mathbf{R}^2\colon 0 < x_2 < 1,\ -\infty < x_1 < \infty\}$, $f = 0$, $a = 0$, the initial data g is given at $x_2 = 0$, and the order of accuracy of the difference scheme is m. One can verify for $k = 0$ and $k = 1$ that the discontinuous Galerkin method in these cases corresponds to such a dissipative finite difference method of order $m = 2k + 1$ (with a special choice of initial data for $k = 1$), and thus we may by interpolation from (1.7) obtain the result

$$\max_n \|u^n - u_h^n\|_{L_p} \leqslant Ch^{k+1/2} \|g\|_{W_p^{k+1}(\mathbf{R})},$$

which is the same as (1.3b, c) in the present situation.

The plan of the present paper is as follows. In Section 2 we introduce the notation, prove a basic local stability estimate for the scheme (1.2), and carry out the error analysis in L_2. Sections 3 and 4 are devoted to the L_p stability and error analysis. First, in Section 3, we carry out a Fourier analysis of a finite difference scheme associated with (1.2) in the case where $k = 1$, $a = f = 0$, Ω is a half-plane, and $\mathscr{C}^h$ is a uniform triangulation of Ω. Finally, in Section 4, we prove L_p error estimates using the results of Section 3.

2. Preliminaries and L_2-Analysis. For Ω a convex polygonal plane domain, let $\mathscr{E}$ be a given family of triangulations of Ω indexed by a parameter h such that if $\mathscr{C}^h \in \mathscr{E}$ then $h = \max_{T \in \mathscr{C}^h} h_T$, where h_T denotes the diameter of T. For convenience, we assume the geometry of the triangulations to be such that if $T_1, T_2 \in \mathscr{C}^h \in \mathscr{E}$, $T_1 \neq T_2$, and $\partial T_1 \cap \partial T_2$ is nonempty then $\partial T_1 \cap \partial T_2$, is either a common side or a common vertex of T_1 and T_2. In the analysis below we will further assume that the triangulations are either locally quasi-uniform (Section 2), uniform or piecewise uniform (Sections 3 and 4). The family $\mathscr{E}$ is called *locally quasi-uniform* if there is a positive constant κ such that if $T \in \mathscr{C}^h \in \mathscr{E}$, then the angles of T are bounded from below by κ. In a *uniform* triangulation $\mathscr{C}^h$, all the triangles are identical up to translation and rotation. Finally, $\mathscr{E}$ is called a *piecewise uniform* triangulation *generated by a triangulation* $\mathscr{C}$, if any $\mathscr{C}^h$ is a refinement of $\mathscr{C}$ such that the restriction of $\mathscr{C}^h$ to any $T \in \mathscr{C}$ defines a uniform triangulation of T.

In what follows, we use the spaces $L_p(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$, $m \geqslant 1$, $1 \leqslant p \leqslant \infty$, in their usual meaning for Ω a domain in $\mathbf{R}^2$. The norm in $L_p(\Omega)$ is denoted by $\|\cdot\|_{p,\Omega}$ if $p \neq 2$ and by $\|\cdot\|_\Omega$ if $p = 2$. Similarly, if Γ is a piecewise smooth curve or a union of such curves, $\|\cdot\|_{p,\Gamma}$ denotes the norm in $L_p(\Gamma)$, with the subindex p omitted if $p = 2$. Some further mesh-dependent norms will be introduced later on. Below, we denote by C or c, a positive constant which may take different values on different occurrences. The constant may depend on the above parameters κ and k but not on other parameters, unless indicated explicitly.

As is shown in [9], one can always solve (2.4) successively, triangle by triangle, with u_h^- given on ∂T_- either by the previously computed values or by the boundary condition. Thus, u_h is determined uniquely if (1.2) can be solved locally in each $T \in \mathscr{C}^h$ for given f and u_h^-. That this is the case for h small enough is established by the following result:

LEMMA 2.1. *Assume that* $f \in L_p(\Omega)$ *and* $g \in L_p(\Gamma_-)$, $1 \leqslant p \leqslant \infty$ *in* (1.1). *Then there is a positive constant* h_0 (*depending on* κ *and* k) *such that if* $h_T\|a\|_{\infty,T} < h_0$ *for all* $T \in \mathscr{C}^h$, *then* u_h *is determined uniquely by* (1.2) *and one has for each* $T \in \mathscr{C}^h$ *the local stability estimates*

$$\begin{aligned}\|u_h\|_{p,T} + h_T^{1/p}\|u_h^+\|_{p,\partial T_-} + h_T^{1/p}\|u_h^-\|_{p,\partial T_+}\\ \leqslant C\left\{h_T\|f\|_{p,T} + h_T^{1/p}\|\nu \cdot \beta u_h^-\|_{p,\partial T_-}\right\}\end{aligned}$$

and

$$\|u_{h\beta}\|_{p,T} \leqslant C\left\{h_T^{-1+1/p}\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,\partial T_-} + \|u_h\|_{p,T} + \|f\|_{p,T}\right\}.$$

Proof. Let us first note that, by a scaling argument and by the equivalence of norms in a finite-dimensional space, we have for any $w \in P_k(T)$, $T \in \mathscr{C}^h$, the inequalities

$$\begin{aligned}&C^{-1}h_T^{1/2-2/p}\|w\|_{p,T} + h^{1/2-1/p}\|w\|_{p,\partial T}\\ &\leqslant \left\{h_T\int_T w_\beta^2\,dx + \int_{\partial T}|\nu \cdot \beta|w^2\,ds\right\}^{1/2} \leqslant Ch_T^{1/2-2/p}\|w\|_{p,T},\end{aligned} \tag{2.1}$$

where $1 \leqslant p \leqslant \infty$ and C depends only on κ and k. Now consider a given $T \in \mathscr{C}^h$, denote $u_{h|T}$ by ϕ_h, $\phi_h \in P_k(T)$, and set

$$\ell(\phi_h, v) = \int_T (\phi_{h\beta} + a\phi_h) v \, dx + \int_{\partial T_-} \phi_h |\nu \cdot \beta| v \, ds.$$

Choosing $v = \phi_h + \gamma h_T \phi_{h\beta}$, where $\gamma \in (0,1]$ is defined below, we have

$$\|v\|_{p,T} \leqslant C\|\phi_h\|_{p,T}, \qquad 1 \leqslant p \leqslant \infty, \tag{2.2}$$

and

$$\ell(\phi_h, v) = \frac{1}{2}\int_{\partial T} \phi_h^2 |\nu \cdot \beta| \, ds + \gamma h_T \int_T \phi_{h\beta}^2 \, dx$$
$$+ \gamma h_T \int_{\partial T_-} |\nu \cdot \beta| \phi_h \phi_{h\beta} \, ds + \int_T a\phi_h(\phi_h + \gamma h_T \phi_{h\beta}) \, dx.$$

Using here (2.1) and (2.2) and recalling that $\gamma \leqslant 1$ we obtain

$$\ell(\phi_h, v) \geqslant \frac{\gamma}{4}\left(1 - C\gamma - Ch_T\|a\|_{\infty,T}\right)\left\{ h_T \int_T \phi_{h\beta}^2 \, dx + \int_{\partial T} \phi_h^2 |\nu \cdot \beta| \, ds \right\}.$$

Choosing here $\gamma = \min\{1, 1/2C\}$ and assuming that $Ch_T\|a\|_{\infty,T} \leqslant \frac{1}{4}$, say, we conclude, recalling (2.1) and (1.2) that for some constant C,

$$C^{-1}\left\{ h_T^{1-4/p}\|\phi_h\|_{p,T}^2 + h^{1-2/p}\|\phi_h\|_{p,\partial T}^2 \right\}$$
$$\leqslant \ell(\phi_h, v) = \int_T fv \, dx + \int_{\partial T_-} u_h^- v |\nu \cdot \beta| \, ds.$$

Applying on the right-hand side the Hölder inequality, and recalling (2.1) and (2.2), the first inequality in the lemma follows. The second inequality can now be proved analogously, by choosing $v = \phi_{h\beta}$ in (1.2). We omit the details. □

In what follows, we associate with each $\mathscr{C}^h \in \mathscr{E}$ a finite element space V_h defined by

$$V_h = \left\{ v \in L_2(\Omega) \colon v|_T \in P_k(T), \, T \in \mathscr{C}^h \right\}. \tag{2.3}$$

Further, we set

$$\Gamma_h = \left(\bigcup_{T \in \mathscr{C}^h} \partial T \right) \setminus \partial\Omega.$$

If $T \subset \Omega$ and $S \subset \Gamma_h \cup \partial\Omega$, we use the abbreviations

$$(v, w)_T = \int_T vw \, dx, \qquad \langle v, w \rangle_S = \int_S vw |\nu \cdot \beta| \, ds,$$

where ν is a normal to S. In this section we drop the subindex T if $T = \Omega$. For a piecewise continuous function v, we set $v^{\pm}(x) = \lim_{\varepsilon \to 0^{\pm}} v(x \pm \varepsilon\beta)$ for $x \in \Gamma_h$.

By summing over $T \in \mathscr{C}^h$ in (1.2), we obtain the following equivalent formulation of the discontinuous Galerkin method. Given k and $\mathscr{C}^h$, find $u_h \in V_h$ such that

$$\mathscr{B}(u_h, v) = (f, v) + \langle g, v \rangle_{\Gamma_-}, \qquad v \in V_h, \tag{2.4}$$

where

$$\mathscr{B}(w, v) = \sum_{T \in \mathscr{C}^h} (w_\beta + aw, v)_T + \langle w^+ - w^-, v^+ \rangle_{\Gamma_h} + \langle w, v \rangle_{\Gamma_-}. \tag{2.5}$$

Using the partial integration formula

$$(w_\beta, v)_T = \langle w, v\rangle_{\partial T_+} - \langle w, v\rangle_{\partial T_-} - (w, v_\beta)_T$$

we can also write

$$\mathscr{B}(w,v) = \sum_{T\in\mathscr{C}^h} (w, -v_\beta + av)_T + \langle w^-, v^- - v^+\rangle_{\Gamma_h} + \langle w, v\rangle_{\Gamma_+}. \tag{2.6}$$

Note that we can replace u_h by the exact solution u in (2.4), i.e., we have the consistency relation

$$\mathscr{B}(u - u_h, v) = 0, \qquad v \in V_h. \tag{2.7}$$

In the error analysis below, we also make use of the following dual variational problem: Given $\varphi \in L_1(\Omega)$, find $\phi_h \in V_h$ such that

$$\mathscr{B}(w, \phi_h) = (\varphi, w), \qquad w \in V_h. \tag{2.8}$$

In view of (2.6), ϕ_h is simply the discontinuous Galerkin solution to the problem

$$\begin{cases} -\phi_\beta + a\phi = \varphi & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma_+. \end{cases} \tag{2.9}$$

Let us now assume that $\mathscr{E}$ is a locally quasi-uniform family of triangulations and let us associate with each $\mathscr{C}^h \in \mathscr{E}$ a seminorm $|\cdot|_{h,\beta}$ and a norm $|||\cdot|||_{h,\beta}$ defined by

$$|v|_{h,\beta}^2 = \sum_{T\in\mathscr{C}^h} h_T\|v_\beta\|_T^2 + \langle\langle v^+ - v^-\rangle\rangle_{\Gamma_h}^2 + \langle\langle v\rangle\rangle_{\Gamma}^2,$$

$$|||v|||_{h,\beta}^2 = \|v\|^2 + |v|_{h,\beta}^2,$$

where $\langle\langle v\rangle\rangle_S^2 = \langle v, v\rangle_S$. Then we can prove the following result, which is one of the main results of this paper.

THEOREM 2.1. *Let $\mathscr{E}$ be a locally quasi-uniform family of triangulations of* Ω, *and let* V_h *be defined by* (2.3) *for each* $\mathscr{C}^h \in \mathscr{E}$ *and for some given* $k \geqslant 0$. *Then, there is a constant* h_0 *depending on* κ *and* k *such that if* u *is the solution to* (1.1) *and* $h^{1/2}\|a\|_{\infty,\Omega} < h_0$, *the solution* u_h *to* (2.4) *satisfies the stability estimate*

$$|||u_h|||_{h,\beta} \leqslant C\left(\|f\|_\Omega + \left\||\nu\cdot\beta|^{1/2} g\right\|_{\Gamma_-}\right)$$

and the error estimate

$$|||u - u_h|||_{h,\beta} \leqslant Ch^{k+1/2}|u|_{H^{k+1}(\Omega)}, \tag{2.10}$$

where the constants depend on $\|a\|_{\infty,\Omega}$ *and* $\operatorname{diam}(\Omega)$.

Remark. Note that from the error estimate (2.10) it follows, in particular, that

$$\langle\langle u - u_h\rangle\rangle_\Gamma \leqslant Ch^{k+1/2}|u|_{H^{k+1}(\Omega)}$$

or, more generally: we have for any $\Omega' \subset \Omega$ with boundary Γ' such that $\Gamma'_- = \Gamma_-$ that

$$\langle\langle u - u_h\rangle\rangle_{\Gamma'} \leqslant Ch^{k+1/2}|u|_{H^{k+1}(\Omega')}.$$

This means that the estimate (2.10) is optimal. (See the Introduction.)

Proof. Following [10] we introduce the weight function

$$\psi(x) = e^{-\gamma(x-x_0)\cdot\beta},$$

where x_0 is a point on $\partial\Omega$ chosen so that $\|\psi\|_{\infty,\Omega} = 1$. For $\chi \in L_2(\Omega)$, let $\tilde{\chi}$ denote the L_2-projection of χ into V_h, i.e., $\tilde{\chi}$ is defined by

$$(\chi - \tilde{\chi}, v)_T = 0, \qquad v \in P_k(T),\ T \in \mathscr{C}^h.$$

By (2.4), we have

$$\mathscr{B}(u_h, \psi u_h) = \mathscr{B}\big(u_h, \psi u_h - \widetilde{\psi u_h}\big) + \big(f, \widetilde{\psi u_h}\big) + \big\langle g, \widetilde{\psi u_h}\big\rangle_{\Gamma_-}. \tag{2.11}$$

On the left-hand side we obtain by straightforward computation

$$\begin{aligned}\mathscr{B}(u_h, \psi u_h) = {} & \big(u_h, \big(\psi a - \tfrac{1}{2}\psi_\beta\big)u_h\big) \\ & + \tfrac{1}{2}\langle u_h^+ - u_h^-, \psi(u_h^+ - u_h^-)\rangle_{\Gamma_h} + \tfrac{1}{2}\langle u_h, \psi u_h\rangle_\Gamma.\end{aligned}$$

Choosing $\gamma = 2\|a\|_\infty + 1$ and noting that

$$-\psi_\beta(x) = \gamma\psi(x) \geqslant \gamma e^{-\gamma\,\mathrm{diam}(\Omega)},$$

we obtain

$$\mathscr{B}(u_h, \psi u_h) \geqslant C\Big\{\|u\|^2 + \tfrac{1}{2}\langle\langle u_h^+ - u_h^-\rangle\rangle_{\Gamma_h}^2 + \tfrac{1}{2}\langle\langle u_h\rangle\rangle_\Gamma^2\Big\}, \tag{2.12}$$

where C depends on γ and $\mathrm{diam}\,\Omega$. To estimate the right-hand side of (2.11), we use the standard estimates

$$\|\chi - \tilde{\chi}\|_T \leqslant Ch_T^{k+1}|\chi|_{H^{k+1}(T)}, \tag{2.13a}$$

$$\|\chi - \tilde{\chi}\|_{\partial T} \leqslant Ch_T^{k+1/2}|\chi|_{H^{k+1}(T)}. \tag{2.13b}$$

Since $|u_h|_{H^{k+1}(T)} = 0$, $\|\psi\|_{W^{l,\infty}(T)} \leqslant C\gamma^l$ and $\|u_h\|_{H^l(T)} \leqslant Ch_T^{-l}\|u_h\|_T$, $0 \leqslant l \leqslant k+1$, we obtain

$$\big\|\psi u_h - \widetilde{\psi u_h}\big\|_T + h_T^{1/2}\big\|\psi u_h - \widetilde{\psi u_h}\big\|_{\partial T} \leqslant C\max\big\{\gamma h_T, (\gamma h_T)^{k+1}\big\}\|u_h\|_T.$$

Now setting $v = \psi u_h - \widetilde{\psi u_h}$ and recalling that $(v, w)_T = 0$ for $w \in P_k(T)$, $T \in \mathscr{C}^h$, we conclude from (2.5) that

$$\mathscr{B}(u_h, v) = \langle u_h^+ - u_h^-, v^+\rangle_{\Gamma_h} + \langle u_h, v\rangle_{\Gamma_-} + (au_h, v).$$

By the above interpolation error estimates, if $\gamma h_T \leqslant 1$, we have

$$\big\langle\big\langle v^\pm\big\rangle\big\rangle_{\Gamma_h} + \langle\langle v\rangle\rangle_\Gamma \leqslant C\gamma h^{1/2}\|u_h\|, \qquad \|v\| \leqslant C\gamma h\|u_h\|,$$

so that

$$\big|\mathscr{B}\big(u_h, \psi u_h - \widetilde{\psi u_h}\big)\big| \leqslant \tfrac{1}{4}\Big(\|u_h\|^2 + \langle\langle u_h^+ - u_h^-\rangle\rangle_{\Gamma_h}^2 + \langle\langle u_h\rangle\rangle_\Gamma^2\Big) + C\gamma^2 h\|u_h\|^2.$$

Assuming now that $\gamma h^{1/2} = (2\|a\|_\infty + 1)h^{1/2}$ is small enough, and combining (2.11), (2.12) and Lemma 2.1, we obtain

$$C|||u_h|||_{h,\beta}^2 \leqslant \mathscr{B}\big(u_h, \widetilde{\psi u_h}\big) \leqslant C'\Big(\|f\| + \big\|\,|\nu\cdot\beta|^{1/2}g\big\|_{\Gamma_-}\Big)|||u_h|||_{h,\beta},$$

which proves the asserted stability estimate.

The error estimate follows in a similar fashion noting that

$$\begin{aligned}C|||u_h - \tilde{u}|||_{h,\beta}^2 & \leqslant \mathscr{B}\big(u_h - \tilde{u}, \widetilde{\psi(u_h - \tilde{u})}\big) = \mathscr{B}\big(u - \tilde{u}, \widetilde{\psi(u_h - \tilde{u})}\big) \\ & \leqslant Ch^{k+1/2}|u|_{K^{k+1}(\Omega)}|||u_h - \tilde{u}|||_{h,\beta}.\end{aligned}$$

Here the equality is a consequence of (2.7) and the last inequality follows from (2.6) and (2.13). □

3. Finite Difference Analysis. In this section we apply the discontinuous Galerkin method to the simplified model problem: Find the function $u = u(x,t)$ defined on $\mathbf{R} \times \mathbf{R}^+$ such that

$$(3.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \gamma \dfrac{\partial u}{\partial x} = 0, & (x,t) \in \mathbf{R} \times \mathbf{R}^+, \\ u(0,x) = g(x) & x \in \mathbf{R}. \end{cases}$$

This is obviously a special case of problem (1.1) where Ω is a half-plane, $a = f = 0$ and $\gamma = \beta_2/\beta_1$ is a constant. We assume below that $0 \leqslant \gamma \leqslant 1$. The finite element partitioning $\mathscr{C}^h$ of $\Omega = \mathbf{R} \times \mathbf{R}^+$ is defined in terms of the grid points

$$x_i = ih, \qquad i = 0, \pm 1, \pm 2, \ldots,$$
$$t_n = nh, \qquad n = 0, 1, 2, \ldots.$$

We set

$$\mathscr{C}^h = \{T_{ni}^+, T_{ni}^-,\ n = 0, 1, \ldots,\ i = 0, \pm 1, \ldots\},$$

where T_{ni}^+ is a triangle with vertices at (x_{i-1}, t_{n-1}), (x_i, t_{n-1}) and (x_i, t_n) and T_{ni}^- has its vertices at (x_{i-1}, t_{n-1}), (x_{i-1}, t_n) and (x_i, t_n) (see Figure 1). Now if $u_h^-(\cdot, t_{n-1})$ is given for some $n \geqslant 1$ we can solve (1.2) for u_h on any $T = T_{ni}^+$ and thereafter on any $T = T_{ni}^-$, thus obtaining $u_h^-(\cdot, t_n)$. Denoting by W_h the space

$$W_h = \{v \in L_2(\mathbf{R}) \colon v_{|I_i} \in P_k(I_i),\ I_i = (x_{i-1}, x_i),\ i = 0, \pm 1, \ldots\},$$

we can write the discrete solution algorithm formally as

$$(3.2)\qquad \begin{aligned} u_h^-(\cdot, t_n) &= G_h u_h^-(\cdot, t_{n-1}), \qquad n \geqslant 1, \\ u_h^-(\cdot, 0) &= \pi_h g, \end{aligned}$$

where $G_h \colon W_h \to W_h$ is a linear operator independent of n, and π_h denotes the L_2-projection into W_h. We have here utilized the fact that the numerical scheme is unchanged if g is replaced by $\pi_h g$, a property that is obvious from (1.2).

To see the structure of the operator G_h more closely, consider the triangles $T_{n,i-1}^+, T_{ni}^+$ and T_{ni}^- for some given n, i (see Figure 1). Since $\gamma \in [0,1]$, we can solve (1.2) for $T = T_{ni}^+$ once $u_h^-(x, t_{n-1})$ is known for $x \in (x_{i-1}, x_i)$. In fact, we obtain for all i,

$$(3.3)\qquad u_h(x,t) = u_h^-(x - \gamma(t - t_{n-1}), t_{n-1}), \qquad (x,t) \in T_{ni}^+.$$

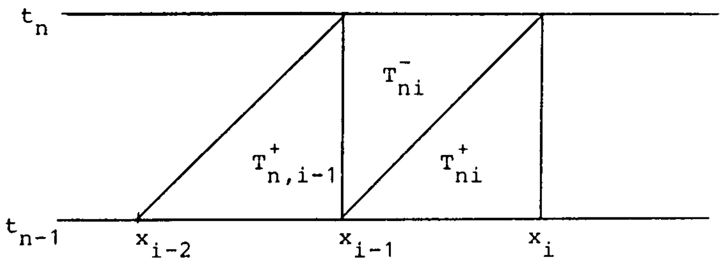

FIGURE 1

Using (3.3) we can now determine u_h on T_{ni}^- from

$$(3.4)\qquad \int_{T_{ni}^-} \left(\frac{\partial u_h}{\partial t} + \gamma \frac{\partial u_h}{\partial x} \right) v\,dx + \int_{(\partial T_{ni}^-)_-} (u_h^+ - u_h^-) v \nu \cdot \beta\,ds = 0, \qquad v \in P_k(T_{ni}^-).$$

Let us choose here $v(x,t) = w(x - \gamma t)$, where w is a polynomial of degree $\leqslant k$ on $\mathbf{R}$. Then $\partial v/\partial t + \gamma\,\partial v/\partial x = 0$ and we obtain by partial integration in (3.4) that

$$\int_{(\partial T_{ni}^{-})_{-}} u_h^{-} v\, \nu \cdot \beta\, ds = \int_{(\partial T_{ni}^{-})_{+}} u_h^{-} v\, \nu \cdot \beta\, ds.$$

Noting that $\nu \cdot \beta\, ds = \alpha\, dx$ on ∂T_{ni}^{+} with α a constant, and recalling (3.3), we further obtain

$$\int_{x_{i-1}}^{x_i} \left[u_h^{-}(x, t_n) - u_h^{-}(x - \gamma h, t_{n-1})\right] w(x - \gamma t_n)\, dx = 0.$$

Since w is here an arbitrary polynomial of degree $\leqslant k$, we conclude that G_h is defined by

$$G_h = \pi_h G, \tag{3.5}$$

where G denotes the corresponding exact solution operator:

$$(Gv)(x) = v(x - \gamma h). \tag{3.6}$$

We may interpret (3.2) as a finite difference scheme by introducing the notation

$$\begin{aligned} U^n(x) &= \left[U_1^n(x), \ldots, U_k^n(x)\right]^T, \\ U_1^n(x) &= u_h^{-}(x, t_n), \\ U_2^n(x) &= h \frac{\partial u_h^{-}}{\partial x}(x, t_n), \\ &\ \vdots \\ U_{k+1}^n(x) &= h^k \frac{\partial^k u_h^{-}}{\partial x^k}(x, t_n), \end{aligned}$$

where $x \in \{x_{i+1/2} = (i + 1/2)h, i = 0, \pm 1, \ldots\}$. Obviously, $U^n(x_{i+1/2})$ defines $u_h^{-}(\cdot, t_n)$ uniquely on the interval (x_i, x_{i+1}) and vice versa. Since $u_h^{-}(\cdot, t_n)$ is determined uniquely on (x_{i-1}, x_i) when $u_h^{-}(\cdot, t_{n-1})$ is known on (x_{i-2}, x_i), (3.2) corresponds to a difference scheme of the form

$$U^n(x) = A_1 U^{n-1}(x) + A_2 U^{n-1}(x - h) \equiv (EU^{n-1})(x), \qquad x \in \mathbf{R}, \tag{3.7}$$

where A_1 and A_2 are $(k+1) \times (k+1)$ matrices, and we have made the usual extension of the scheme to all $x \in \mathbf{R}$. The matrices depend only on the parameters γ and k, so the operator E defined by (3.7) is translation-invariant.

In the case $k = 0$ we have

$$A_1 = 1 - \gamma, \qquad A_2 = \gamma,$$

i.e., (3.7) reduces to the ordinary upwind scheme in this case. If $k = 1$, one has

$$A_1 = (1 - \gamma)\begin{pmatrix} 1 & -\frac{1}{2}\gamma \\ 6\gamma & 1 - 2\gamma - 2\gamma^2 \end{pmatrix}, \qquad A_2 = \gamma \begin{pmatrix} 1 & \frac{1}{2}(1 - \gamma) \\ -6(1 - \gamma) & -3 + 6\gamma - 2\gamma^2 \end{pmatrix}.$$

Below we confine our attention to the case $k = 1$. We let S_n, $n \geqslant 0$, be defined by

$$S_n = \bigcup_i (\partial T_{ni}^{+})_{+} = \bigcup_i (\partial T_{ni}^{-})_{-}.$$

Also, we let $\beta = (1, \gamma)$, so that

$$|\nu \cdot \beta(x)| = \begin{cases} \gamma, & x \in (\partial T_{ni}^{-})_{-} \cap (\partial T_{n,i-1}^{+})_{+}, \\ 1-\gamma, & x \in (\partial T_{ni}^{-})_{-} \cap (\partial T_{ni}^{+})_{+}. \end{cases}$$

We also introduce the parameter ρ defined by

$$\rho = \gamma(1-\gamma).$$

The main purpose of this section is to prove the following

THEOREM 3.1. *Let $k = 1$ and let u_h be defined on $\mathbf{R} \times \mathbf{R}^+$ by (3.2) through (3.6). Then there is a constant C such that for all n, p, $n \geqslant 0$, $1 \leqslant p \leqslant \infty$,*

$$\|u_h^-(\cdot, t_n)\|_{p,\mathbf{R}} \leqslant C\|g\|_{p,\mathbf{R}} \tag{3.8}$$

and

$$\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,S_n} = C(\rho/n)^{1/2}\|g\|_{p,\mathbf{R}}. \tag{3.9}$$

The proof of Theorem 3.1 is based on Fourier analysis of the finite difference scheme (3.7) and is split below into several lemmas. We begin by introducing the Fourier transform

$$\hat{U}^n(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} U^n(x)\, dx, \qquad \xi \in \mathbf{R},$$

which allows (3.7) to be written as

$$\hat{U}^n(\xi) = \hat{E}(h\xi)\hat{U}^{n-1}(\xi),$$

where $\hat{E}(\theta) = A_1 + e^{-i\theta}A_2$. Below we use a vector norm $|\cdot|$ defined by

$$|V| = \left(|V_1|^2 + \frac{1}{12}|V_2|^2\right)^{1/2},$$

where V_1 and V_2 may be real- or complex-valued. The reason for using this norm, as will be seen in Lemma 3.2 below, is the obvious relation

$$|V| = \|v\|_{L_2(0,1)}, \qquad v(x) = V_1 + V_2(x - \tfrac{1}{2}),\ x \in (0,1).$$

If A is a 2×2 matrix, we set

$$|A| = \sup_{V \neq 0} \frac{|AV|}{|V|}.$$

Further, if $V = (V_1, V_2)^T$ is a vector-valued function defined on $S \subset \mathbf{R}$, we denote by $\|V\|_{p,S}$ or by $\|V\|_{L_p(S)}$ a norm of V defined by

$$\|V\|_{p,S} = \|V\|_{L_p(S)} = \||V(\cdot)|\|_{L_p(S)}.$$

Similarly, if A is a 2×2 matrix with coefficients real- or complex-valued functions defined on $S \subset \mathbf{R}$, we set

$$\|A\|_{p,S} = \|A\|_{L_p(S)} = \||A(\cdot)|\|_{L_p(S)}.$$

Let $[L_p(\mathbf{R})]^2$ denote the space of vector-valued functions $V = (V_1, V_2)^T$ defined on $\mathbf{R}$ such that $V_i \in L_p(\mathbf{R})$, $i = 1, 2$, and let E be a linear translation-invariant operator defined on the whole space $[L_p(\mathbf{R})]^2$ for some p, $1 \leqslant p \leqslant \infty$. Denote by $\|E\|_p$ the norm of E defined as

$$\|E\|_p = \sup_{\substack{V \in [L_p(\mathbf{R})]^2 \\ V \neq 0}} \frac{\|EV\|_{p,\mathbf{R}}}{\|V\|_{p,\mathbf{R}}}.$$

Assume further that the Fourier transform of EV, $V \in [L_p(\mathbf{R})]^2$, as defined above, is given by

$$(\widehat{EV})(\xi) = \hat{E}(h\xi)\hat{V}(\xi), \qquad \xi \in \mathbf{R},$$

where $h > 0$ and $\hat{E}(\theta)$ is 2π-periodic. Note that the operator E, defined by (3.7), satisfies these assumptions for any p.

The operator norm $\|E\|_p$ can be estimated using the following lemma, which states the well-known Carlson-Beurling inequality (cf. [4]). In the lemma, χ denotes a smooth function defined on $\mathbf{R}$ such that $\chi(\theta) = 0$ for $|\theta| > 3\pi/2$ and $\chi(\theta) = 1$ for $|\theta| \leqslant \pi$.

Lemma 3.1. *Under the above assumptions write* $\tilde{E}(\theta) = e^{-i\alpha\theta}\hat{E}(\theta)$, *where* $\alpha \in \mathbf{R}$ *is arbitrary. Then* $\|E\|_p = M_p(\tilde{E}(\theta))$, *where the latter can be estimated as*

$$M_p(\tilde{E}(\theta)) \leqslant C\left\{\|\chi\tilde{E}\|_{L_2(\mathbf{R})}\left\|\frac{d}{d\theta}(\chi\tilde{E})\right\|_{L_2(\mathbf{R})}\right\}^{1/2}.$$

In order to prove the stability estimate (3.8), we obviously need an estimate for the translation-invariant operator E^n: $U^0 \to U^n$ defined by (3.7). We prepare the situation so that Lemma 3.1 can finally be used. An essential step is then to estimate the growth rate of $\hat{E}(\theta)^n = (A_1 + e^{-i\theta}A_2)^n$ as n increases. If $\hat{E}(\theta)$ has two linearly independent eigenfunctions associated with the eigenvalues λ_1 and λ_2, we can use the usual splitting

$$\hat{E}^n(\theta) = D^{-1}(\theta)\Lambda(\theta)^n D(\theta), \qquad n = 1, 2, \ldots, \tag{3.10}$$

where $\Lambda(\theta) = \operatorname{diag}\{\lambda_1, \lambda_2\}$. However, since $\hat{E}(\theta)$ is nonsymmetric and depends on both γ and θ, it is difficult to prove the existence of the diagonalizing matrix D in (3.10) in general. Therefore, we use below the splitting (3.10) only for small $|\theta|$ and near the points $(\theta, \gamma) = (\pm\pi, 0)$ and $(\theta, \gamma) = (\pm\pi, 1)$; for the remaining values of the parameters, as it turns out, it suffices to estimate the matrix norm of $\hat{E}(\theta)$.

The essential properties of $\hat{E}(\theta)$ to be required in the subsequent analysis are established by the following three lemmas.

Lemma 3.2. *There is a positive constant* κ *such that* $\hat{E}^n(\theta)$ *admits the representation* (3.10) *in the range* $(\theta, \gamma) \in D_\kappa = [-\kappa, \kappa] \times [0, 1]$. *The eigenvalues of* $\hat{E}(\theta)$ *are smooth functions of* γ *and* θ *in* D_κ *and satisfy*

$$\lambda_1 = \exp\left\{-i\gamma\theta - a(\gamma)\rho\theta^4 + O\left(\rho|\theta|^5\right)\right\},$$

$$\lambda_2 = \exp\left\{-c\rho + O\left(\rho|\theta|\right)\right\},$$

where $a(\gamma)$ *is strictly positive for* $\gamma \in [0, 1]$ *and* c *is a positive constant. Moreover,* $|D(\theta)|$ *and* $|D^{-1}(\theta)|$ *are bounded by a constant for all* $(\theta, \gamma) \in D_\kappa$.

Proof. For a given $\gamma \in (0, 1)$, let $\lambda(\theta)$ be an eigenvalue of $\hat{E}(\theta)$ and write

$$\lambda(\theta) = e^{-i\gamma\theta} + \rho\xi = (1 - \phi)^\gamma + \rho\xi = 1 - \gamma\phi + \rho[\psi(\gamma, \phi) + \xi],$$

where we have introduced the new variable $\phi = 1 - e^{-i\theta}$. For any $\gamma \in [0, 1]$, ψ admits the series representation

$$\psi(\gamma, \phi) = -\sum_{k=2}^{\infty} \frac{\Gamma(k - \gamma)}{\Gamma(k + 1)\Gamma(2 - \gamma)}\phi^k,$$

which converges in the open unit disc $|\phi| < 1$. Thus ψ is an analytic function of ϕ on the open unit disc and so $\psi = \psi(\gamma, \phi(\theta))$ is a smooth function of θ on the interval $[-c, c]$ for any $\gamma \in [0,1]$ and $c < \pi/3$.

Denoting by I the identity matrix, we obtain, using the above notation,

$$\hat{E}(\theta) - \lambda(\theta)I = \rho\begin{pmatrix} \psi(\gamma,\phi) - \xi & \frac{1}{2}\phi \\ -6\phi & -6 + (4 - 2\gamma)\phi + \psi - \xi \end{pmatrix}.$$

Using the series representation of ψ, we further obtain, after a straightforward computation,

$$\operatorname{Det}(\hat{E}(\theta) - \lambda(\theta)I) = \xi^2 + a_1\xi + a_2,$$

where a_1 and a_2 are smooth functions of γ and ϕ behaving for small $|\phi|$ as

$$a_1 = 6 + O(|\phi|), \qquad a_2 = \frac{1}{12}(1 - \gamma + \gamma^2)\phi^4 + O\big(|\phi|^5\big).$$

Thus, the eigenvalues behave for small $|\phi|$ as

$$\lambda_1 = e^{-i\gamma\theta} - \rho\big[c(\gamma)\phi^4 + b_1(\gamma,\phi)\big], \qquad \lambda_2 = e^{-i\gamma\theta} - \rho\big[6 + b_2(\gamma,\phi)\big],$$

where $c(\gamma) = \frac{1}{72}(1 - \gamma + \gamma^2) > 0$ and b_1 and b_2 are smooth functions of γ and ϕ in the range $\gamma \in [0,1]$, $|\phi| \leqslant \kappa$, κ small enough, with $b_1 = O(|\phi|^5)$, $b_2 = O(|\phi|)$. Recalling that $\phi = i\theta + O(\theta^2)$, we obtain the asserted representation of the eigenvalues for small $|\phi|$.

The eigenvectors corresponding to the eigenvalues λ_1 and λ_2 are given respectively by

$$V^1 = \begin{pmatrix} 1 \\ i\theta \end{pmatrix} + O(\theta^2), \qquad V^2 = \begin{pmatrix} \frac{1}{12}i\theta \\ 1 \end{pmatrix} + O(\theta^2),$$

so that

$$D(\theta) = \begin{pmatrix} 1 & \frac{1}{12}i\theta \\ i\theta & 1 \end{pmatrix} + O(\theta^2). \tag{3.11}$$

This proves the assertions concerning D, so the proof is complete. □

LEMMA 3.3. *There is a positive constant* κ *such that* $\hat{E}(\theta)$ *allows the representation* (3.10) *in the range*

$$D_\kappa = \big\{(\theta,\gamma) \in [-\pi,\pi] \times [0,1]\colon \gamma(1-\gamma) \leqslant \kappa,\ \pi - |\theta| \leqslant \kappa\big\}.$$

The eigenvalues of $\hat{E}(\theta)$ *are smooth functions of* γ *and* θ *in* D_κ *satisfying*

$$|\lambda_{1,2}| \leqslant e^{-c\rho}, \qquad (\gamma,\theta) \in D_\kappa,$$

where c *is a positive constant. Moreover,* $|D(\theta)|$ *and* $|D^{-1}(\theta)|$ *are bounded by a constant for all* $(\theta,\gamma) \in D_\kappa$.

Proof. By a straightforward computation, the eigenvalues of $\hat{E}(\theta)$ are given for small γ by

$$\lambda_{1,2} = 1 - \big[2 + e^{-i\theta} + \gamma a_1(\gamma,\theta)\big]\gamma + \big\{4 + 10e^{-i\theta} - 5e^{-2i\theta} + \gamma a_2(\gamma,\theta)\big\}^{1/2}\gamma,$$

where a_1 and a_2 are smooth functions of γ and θ. By inspection, if κ is small enough, the eigenvalues are smooth functions of γ and θ in the range $\kappa \leqslant |\theta| \leqslant \pi - \kappa$ and satisfy $|\lambda_{1,2}| \leqslant e^{-c\gamma}$, where c is a positive constant. Moreover, near the points

$(\gamma, \theta) = (0, \pm\pi)$ the matrix D in (3.10) is given by

$$D = \begin{pmatrix} 1 & 1 \\ 1 - \sqrt{11}\,i & 1 + \sqrt{11}\,i \end{pmatrix} + O(\gamma) + O(|e^{i\theta} + 1|),$$

which shows that $|D|$ and $|D^{-1}|$ are bounded in the range $\gamma \leqslant \kappa$, $\pi - |\theta| \leqslant \kappa$, κ small enough.

The case where γ is close to unity is handled similarly. □

LEMMA 3.4. *Let* $0 < \kappa \leqslant \frac{1}{2}$. *Then there is a positive constant* c *depending only on* κ *such that*

$$|\hat{E}(\theta)| \leqslant e^{-c\rho}, \quad \text{if } \kappa \leqslant |\theta| \leqslant \pi - \kappa \text{ and } \gamma \in [0,1],$$

and

$$|\hat{E}(\theta)| \leqslant e^{-c} \quad \text{if } \kappa \leqslant |\theta| \leqslant \pi \text{ and } \max\{\gamma, 1-\gamma\} \geqslant \kappa.$$

Proof. We need first some notation. For V a vector, set $W = \hat{E}(\theta)V$ and define the functions v, w and z on $[0,1]$ as

$$v(x) = V_1 + V_2(x - \tfrac{1}{2}), \qquad w(x) = W_1 + W_2(x - \tfrac{1}{2}),$$

$$z(x) = \begin{cases} e^{-i\theta} v(x + 1 - \gamma), & 0 \leqslant x \leqslant \gamma, \\ v(x - \gamma), & \gamma < x \leqslant 1. \end{cases}$$

It follows from the definition of $\hat{E}(\theta)$ (see in particular (3.5) and (3.6)) that $w = \pi z$, where π denotes the L_2-projection into the space of polynomials (with complex coefficients) of degree $\leqslant 1$ on $[0,1]$. Note that $\|z\|_{L_2[0,1]} = \|v\|_{L_2[0,1]}$ and that z is a polynomial only if either $\rho = 0$ or if $e^{-i\theta} = 1$ and $V_2 = 0$.

Assume now that $\kappa \leqslant |\theta| \leqslant \pi - \kappa$. Then, since $\operatorname{Im}(e^{-i\theta})$ is strictly nonzero, it is easy to verify that for any polynomial p of degree $\leqslant 1$ one has the inequality

$$\max\left\{ \|z - p\|_{L_\infty[0,\gamma]}, \|z - p\|_{L_\infty[1-\gamma,1]} \right\} \geqslant C\|v\|_{L_\infty[0,1]},$$

where C is a positive constant depending only on κ. Setting $p = \pi z$, it follows easily that

$$\|\pi z - z\|^2_{L_2[0,1]} \geqslant C \min\{\gamma, 1-\gamma\} \|v\|^2_{L_2[0,1]} \geqslant C\rho \|v\|^2_{L_2[0,1]}.$$

By a similar reasoning, if $|\theta| \geqslant \kappa$ and $\rho \geqslant \kappa$, $\kappa > 0$, there is a positive constant depending on κ such that

$$\|\pi z - z\|^2_{L_2[0,1]} \geqslant C\|v\|^2_{L_2[0,1]}.$$

Combining the last two inequalities we have

$$|\hat{E}(\theta)V|^2 = \|w\|^2_{L_2[0,1]} = \|z\|^2_{L_2[0,1]} - \|z - \pi z\|^2_{L_2[0,1]}$$
$$\leqslant q\|v\|^2_{L_2[0,1]} = q|V|^2,$$

where $q = 1 - C(\kappa)\rho$ if $0 < \kappa \leqslant |\theta| \leqslant \pi - \kappa$, and $q = q(\kappa) < 1$ if $0 < \kappa \leqslant |\theta| \leqslant \pi$ and $\rho \geqslant \kappa$. Since this is valid for any vector V, the proof is complete. □

Remark. From the proof of Lemma 3.4 one might think that the inequality $|\hat{E}(\theta)| \leqslant e^{-c\rho}$ holds whenever $|\theta| \geqslant \kappa$, independently of γ. That this is not true is seen by choosing $v(x) = x - \frac{1}{2}$ and $\theta = \pm\pi$. Then

$$\|\pi z - z\|^2_{L_2[0,1]} \leqslant \|v - z\|^2_{L_2[0,1]} = O(\rho^3),$$

which shows that one can only have (and in fact has!) the estimate

$$|\hat{E}(\theta)| \leqslant e^{-c\rho^3}, \qquad |\theta| \geqslant \kappa,\ \gamma \in [0,1].$$

This is why Lemma 3.3 is needed (see below). □

In the next lemma we combine the estimates of Lemmas 3.2 through 3.4.

Lemma 3.5. *Let* $\tilde{E}(\theta) = e^{i\gamma\theta}\hat{E}(\theta)$. *Then there are positive constants* C *and* c *such that for all* $n \geqslant 1$ *and* $\theta \in [-\pi, \pi]$,

$$|\tilde{E}(\theta)^n| \leqslant Ce^{-c\rho n\theta^4},$$

$$\left|\frac{d}{d\theta}\tilde{E}(\theta)^n\right| \leqslant C\left(1 + \rho n|\theta|^3\right)e^{-c\rho n\theta^4} + C(1 + \rho n)e^{-c\rho n}.$$

Proof. Choose $\kappa > 0$ so that the assertions of Lemma 3.2 and Lemma 3.3 hold. Using the representation (3.10) together with Lemma 3.2 and Lemma 3.3, we then obtain

$$|\tilde{E}(\theta)^n| \leqslant C\sum_{i=1}^{2} |\lambda_i(\theta)|^n,$$

$$\left|\frac{d}{d\theta}\tilde{E}(\theta)^n\right| \leqslant C\sum_{i=1}^{2}\left(|\lambda_i(\theta)|^n + \left|\frac{d}{d\theta}\left[e^{i\gamma\theta}\lambda_i(\theta)\right]^n\right|\right),$$

where $\lambda_{1,2}$ are the eigenvalues of $\hat{E}(\theta)$. From these estimates and from Lemma 3.2 and Lemma 3.3 the asserted estimates follow if $\kappa \leqslant |\theta| \leqslant \pi - \kappa$ or if $\pi - \kappa < |\theta| \leqslant \pi$ and $\max\{\gamma, 1 - \gamma\} \geqslant \kappa$. In the remaining cases, we use Lemma 3.4 to obtain

$$\begin{aligned}\left|\frac{d}{d\theta}\tilde{E}(\theta)^n\right| &\leqslant n\left|\frac{d}{d\theta}\tilde{E}(\theta)\right| |\tilde{E}(\theta)|^{n-1} \\ &= n\left|\frac{d}{d\theta}\left[e^{i\gamma\theta}A_1 + e^{i(\gamma-1)\theta}A_2\right]\right| |\tilde{E}(\theta)|^{n-1} \\ &\leqslant C\rho n|\tilde{E}(\theta)|^{n-1} \leqslant C'\rho ne^{-c\rho n}.\end{aligned}$$

This completes the proof. □

We are now ready to apply the Carlson-Beurling inequality for proving (3.8). Choosing $\alpha = \gamma n$ in Lemma 3.1 we conclude that

$$\|U^n\|_{p,\mathbf{R}} \leqslant M_p(\tilde{E}^n)\|U^0\|_{p,\mathbf{R}}, \qquad 1 \leqslant p \leqslant \infty, \tag{3.12}$$

where $M_p(\tilde{E}^n)$ is estimated as

$$M_p(\tilde{E}^n) \leqslant C\left\{\|\chi\tilde{E}^n\|_{L_2(\mathbf{R})}\left\|\frac{d}{d\theta}\chi\tilde{E}^n\right\|_{L_2(\mathbf{R})}\right\}^{1/2}.$$

Using the estimates of Lemma 3.5 it follows that

$$\|\chi\tilde{E}^n\|^2_{L_2(\mathbf{R})} \leqslant C\int_0^{3\pi/2} e^{-c\rho n\theta^4}\,d\theta \leqslant C_1\min\left\{1, (\rho n)^{-1/4}\right\}$$

and

$$\begin{aligned}\left\|\frac{d}{d\theta}\chi\tilde{E}^n\right\|^2_{L_2(\mathbf{R})} &\leqslant C\int_0^{3\pi/2}\left\{\left[1 + (\rho n)^2|\theta|^6\right]e^{-c\rho n\theta^4} + \left[1 + (\rho n)^2\right]e^{-c\rho n}\right\}d\theta \\ &\leqslant C_1\left[1 + (\rho n)^{1/4}\right].\end{aligned}$$

Thus, $M_p(\tilde{E}^n)$ is bounded by a constant independent of ρ and n. The estimate (3.8) now follows by combining (3.12) with the obvious estimates

$$\|u_h^-(\cdot, t_n)\|_{p,\mathbf{R}} \leqslant C\|U^n\|_{p,\mathbf{R}}, \qquad \|U^0\|_{p,\mathbf{R}} \leqslant C\|g\|_{p,\mathbf{R}}.$$

It remains to prove (3.9). To this end, we will estimate the vector-valued function W^n defined by

$$W^n = (W_1^n, W_2^n)^T, \qquad W_1^n(x) = U_1^n(x) - U_1^n(x-h) - U_2^n(x),$$
$$W_2^n(x) = U_2^n(x) - U_2^n(x-h), \qquad x \in \mathbf{R}.$$

Here $U^n = (U_1^n, U_2^n)^T$ is defined by (3.7). Let us first show that it suffices to estimate W^n.

LEMMA 3.6. *There is a constant C independent of γ and n such that*

$$\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,S_n} \leqslant C\rho\|W^n\|_{p,\mathbf{R}}.$$

Proof. Let v_i be a polynomial of degree $\leqslant 1$ on $\mathbf{R}^2$ such that $(u_h - v_i)_{|T_{ni}^+} = 0$ (see Figure 1). In the subdomain $\omega_n = \{(x,t)\colon x \in \mathbf{R},\ t_{n-1} < t < t_n\}$ we may interpret $w_h = u_h - v_i$ as the discontinuous Galerkin solution to the problem

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial u}{\partial x} = 0 \quad \text{in } \omega_n, \qquad u(\cdot, t_{n-1}) = (u_h^- - v_i)(\cdot, t_{n-1})$$

on the triangulation $\mathscr{C}_n^h = \{T \in \mathscr{C}^h\colon T \subset \omega_n\}$. Since $u_h - v_i$ vanishes on T_{ni}^+, we obtain by applying Lemma 2.1 and (3.3) that

$$\begin{aligned}\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,(\partial T_{ni}^-)_-} &= \|\nu \cdot \beta[(u_h - v_i)^+ - (u_h - v_i)^-]\|_{p,(\partial T_{ni}^-)_-} \\ &\leqslant C\gamma\|u_h^- - v_i\|_{p,(\partial T_{ni}^-)_- \cap (\partial T_{n,i-1}^+)_+} \\ &\leqslant C_1\gamma\|(u_h^- - v_i)(\cdot, t_{n-1})\|_{L_p(x_{i-2}, x_{i-1})}\end{aligned}$$

and similarly,

$$\begin{aligned}\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,(\partial T_{ni}^-)_-} &= \|\nu \cdot \beta[(u_h - v_{i-1})^+ - (u_h - v_{i-1})^-]\|_{p,(\partial T_{ni}^-)_-} \\ &\leqslant C(1-\gamma)\|(u_h^- - v_{i-1})(\cdot, t_{n-1})\|_{L_p(x_{i-1}, x_i)}.\end{aligned}$$

On the other hand, it follows from the definition of W^n that W^n vanishes on the interval (x_{i-1}, x_i) if and only if $u_h^-(\cdot, t_{n-1})$ is a polynomial of degree 1 on the interval (x_{i-2}, x_i), i.e., if $u_h^-(x, t_{n-1}) = v_i(x) = v_{i-1}(x)$ for $x \in (x_{i-2}, x_i)$. From this it follows easily that

$$\|u_h^-(\cdot, t_n) - v_i\|_{L_p(x_{i-2}, x_{i-1})} \leqslant C\|W^n\|_{L_p(x_{i-1}, x_i)}$$

and

$$\|u_h^-(\cdot, t_n) - v_{i-1}\|_{L_p(x_{i-1}, x_i)} \leqslant C\|W^n\|_{L_p(x_{i-1}, x_i)}.$$

Combining these inequalities we obtain

$$\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,(\partial T_{ni}^+)_-} \leqslant C\rho\|W^n\|_{L_p(x_{i-1}, x_i)}.$$

This proves the assertion for $p = \infty$, and for $p < \infty$ it remains only to sum over i. □

Let us now complete the proof by estimating $\|W^n\|_p$. Note first that the Fourier transform of W^n is given by

$$\hat{W}^n(\xi) = B(h\xi)\hat{U}^n(\xi) = (B\hat{E}^n)(h\xi)\hat{U}^0(\xi),$$

where

$$B(\theta) = \begin{pmatrix} 1 - e^{-i\theta} & -1 \\ 0 & 1 - e^{-i\theta} \end{pmatrix}.$$

Assume first that $|\theta|$ is small enough. Then using the diagonalization (3.10) of $\hat{E}(\theta)$ and recalling (3.11) we obtain after a simple computation

$$B\hat{E}^n = \lambda_1^n H_1 + \lambda_2^n H_2,$$

where $H_1 = O(\theta^2)$. Therefore, using Lemma 3.2, we conclude that for $|\theta| < \kappa$, κ small enough,

$$|B\tilde{E}^n(\theta)| \leqslant C(\theta^2 e^{-c\rho n\theta^4} + e^{-c\rho n}),$$
$$\left|\frac{d}{d\theta}B\tilde{E}^n(\theta)\right| \leqslant C\Big[\big(|\theta| + \rho n|\theta|^5\big)e^{-c\rho n\theta^4} + (1 + \rho n)e^{-c\rho n}\Big].$$

Using a similar argument as in the proof of Lemma 3.5, we conclude that these estimates remain valid also for $\kappa \leqslant |\theta| \leqslant \pi$. We can now apply the Carlson-Beurling inequality to obtain

$$\|W\|_{p,\mathbf{R}} \leqslant M_p(B\tilde{E}^n)\|U^0\|_{p,\mathbf{R}},$$

where $M_p(B\tilde{E}^n)$ is estimated as in Lemma 3.1. An easy computation shows that

$$M_p(B\tilde{E}^n) \leqslant C(\rho n)^{-1/2}.$$

Combining the last two estimates with that given in Lemma 3.6 we now end up with (3.9), and the proof of Theorem 3.1 is complete.

We conclude this section by stating a stability estimate for the discontinuous Galerkin method applied to the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \gamma\frac{\partial u}{\partial x} = 0, \qquad (x,t) \in \Omega_n, \\ &u = g, \qquad (x,t) \in (\partial\Omega_n)_-, \end{aligned} \tag{3.13}$$

where $\gamma \in [0,1]$ and Ω_n, $n \geqslant 1$, is the triangle

$$\Omega_n = \{(x,t);\, 0 < t < t_n = nh,\, 0 < x < t\}.$$

Let the triangulation $\mathscr{C}^h$ of Ω_n be defined as the restriction of the above triangulation of the half-plane to Ω_n, let $k = 1$ and let u_h be defined according to (1.2) on each $T \in \mathscr{C}^h$. Then we have in analogy with (3.8) the following stability estimate.

THEOREM 3.2. *There is a constant C such that for all $n \geqslant 1$ and $1 \leqslant p \leqslant \infty$,*

$$\|u_h^-(\cdot, t_n)\|_{L_p(0,t_n)} + \|u_h\|_{p,\Omega_n} \leqslant C\|\nu \cdot \beta g\|_{p,(\partial\Omega_n)_-}.$$

The proof is based on the following localization result.

LEMMA 3.7. *Write the solutions of* (3.7) *as*

$$U^n(x) = \sum_{j=-\infty}^{+\infty} B_{nj}U^0(x - jh), \qquad x \in \mathbf{R}.$$

Then the matrices B_{nj} *satisfy*

$$|B_{nj}| \leqslant C\min\left\{1, (\rho n)^{-1/4}, \left[1+(\rho n)^{1/4}\right](j-\gamma n)^{-2}\right\}.$$

Proof. Using the Fourier transform $\hat{E}(\theta)$ of G_h we have

$$B_{nj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{E}(\theta)^n e^{ij\theta}\,d\theta.$$

By Lemma 3.1,

$$|B_{nj}| \leqslant \frac{C}{2\pi}\int_{-\pi}^{\pi} e^{-c\rho n\theta^4}\,d\theta \leqslant C\min\left\{1, (\rho n)^{-1/4}\right\}.$$

On the other hand, integrating by parts and using the periodicity of $\hat{E}(\theta)$, we have

$$\begin{aligned} B_{nj} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{E}(\theta)^n e^{i(j-n\gamma)\theta}\,d\theta \\ &= -\frac{1}{2\pi(j-n\gamma)^2}\int_{-\pi}^{\pi} \frac{d^2}{d\theta^2}\left[\tilde{E}(\theta)^n\right] e^{i(j-n\gamma)\theta}\,d\theta. \end{aligned}$$

From the argument used in the proof of Lemma 3.5, it is easy to see that

$$\begin{aligned} \left|\frac{d^2}{d\theta^2}\left[\tilde{E}(\theta)^n\right]\right| \leqslant C\Big[&\left(1+\rho n\theta^2+(\rho n)^2\theta^6\right)e^{-c\rho n\theta^4} \\ &+\left(1+\rho n+(\rho n)^2 e^{-c\rho n}\right)\Big], \qquad 0 \leqslant |\theta| \leqslant \pi. \end{aligned}$$

Using this estimate we obtain

$$|B_{nj}| \leqslant C\left[1+(\rho n)^{1/4}\right](j-\gamma n)^{-2},$$

and so the assertion is proved. □

Remark. Localization estimates similar to those given by Lemma 3.7 are presented, e.g., in [3], [5] for scalar difference schemes. When compared with these estimates, Lemma 3.7 indicates that on a uniform mesh, the discontinuous Galerkin scheme with $k = 1$ behaves like a finite difference scheme which is accurate of order three and dissipative of order four. □

Let us now prove Theorem 3.2. For $1 \leqslant m \leqslant n$, define

$$g_m(x,t) = \begin{cases} g(x,t), & \text{if } (x,t) \in (\partial\Omega_n)_- \text{ and } t_{m-1} < t < t_m, \\ 0, & \text{elsewhere on } (\partial\Omega_n)_-. \end{cases}$$

Then

$$u_h^-(x,t_n) = \sum_{m=1}^{n} u_{hm}^-(x,t_n), \qquad 0 < x < t_n,$$

where u_{hm} denotes the approximate solution of (3.13) with g replaced by g_m. Now let m be given $1 \leqslant m \leqslant n$, and define for $m \leqslant r \leqslant n$ the vector-valued function U^{r-m} on $\mathbf{R}$ by

$$U^{r-m}(x) = \left[u_{hm}^-(x_{i+1/2}, t_r),\ h\frac{\partial u_{hm}}{\partial x}(x_{i+1/2}, t_r)\right]^T, \qquad x_i \leqslant x < x_{i+1},$$

where we set $u_{hm} = 0$ outside Ω_n. Then $U^{r-m} = E^{r-m}U^0$, where E is as in (3.7) and U^0 is nonvanishing on the intervals (x_0, x_1) and (x_{m-1}, x_m) only. Moreover, applying Lemma 2.1, we have the estimates

$$\|U^0\|_{L_\infty(x_0,x_1)} \leqslant C\gamma\|g(0,\cdot)\|_{L_\infty(t_{m-1},t_m)} \leqslant C\gamma\|g\|_{\infty,(\partial\Omega_n)_-},$$

and, similarly,

$$\|U^0\|_{L_\infty(x_{m-1},x_m)} \leqslant C(1-\gamma)\|g\|_{\infty,(\partial\Omega_n)_-}.$$

On the other hand, by Lemma 3.7 we have

$$q\left|U^{n-m}(x_{i+1/2})\right| \leqslant q_1(n-m,i)\left|U^0(x_{1/2})\right| + q_2(n-m,i)\left|U^0(x_{m-1/2})\right|, \qquad 0 \leqslant i \leqslant n-1,$$

where q_1 and q_2 satisfy

$$q_1(r,i) \leqslant C\min\left\{1, (\rho r)^{-1/4}, \left[1+(\rho r)^{1/4}\right](i-\gamma r)^{-2}\right\},$$

$$q_2(r,i) \leqslant C\min\left\{1, (\rho r)^{-1/4}, \left[1+(\rho r)^{1/4}\right](m-1-i-\gamma r)^{-2}\right\}.$$

Combining the above inequalities and summing over m we now obtain

$$\begin{aligned}&\|u_h^-(\cdot,t_n)\|_{L_\infty(x_i,x_{i+1})}\\ &\leqslant C\|g\|_{\infty,(\partial\Omega_n)_-}\left\{\gamma\sum_{r=0}^{\infty} q_1(r,i) + (1-\gamma)\sum_{r=0}^{\infty} q_2(r,i)\right\}.\end{aligned} \tag{3.14}$$

Using the above estimate for $q_1(r,i)$ we may estimate the first sum on the right side of (3.14) as

$$\gamma\sum_{m=0}^{\infty} q_1(r,i) \leqslant C\gamma\sum_{r:|i-\gamma r|\leqslant d}\min\left\{1,(\rho r)^{-1/4}\right\} + C\gamma\sum_{r:|i-\gamma r|>d}\left[1+(\rho r)^{1/4}\right](i-\gamma r)^{-2}.$$

Choosing here

$$d = \max\left\{1, (\rho i/\gamma)^{1/4}\right\},$$

it is easy to see that both sums on the right side are bounded by an absolute constant. Upon estimating the second sum on the right side of (3.14) in a similar manner we now obtain

$$\|u_h^-(\cdot,t_n)\|_{L_\infty(0,x_n)} \leqslant C\|g\|_{\infty,(\partial\Omega_n)_-}, \qquad n \geqslant 1.$$

Combining this estimate with the local estimates given in Lemma 2.1, the assertion follows in the case $p = \infty$. The case $p = 1$ can be handled in a similar manner, and finally the remaining cases can be treated by interpolation [1]. We omit these details. □

Remark 3.1. In the above analysis we have confined ourselves to the case $k = 1$. The case $k = 0$ is more elementary, and one can easily verify that the estimates stated in Theorems 3.1 and 3.2 are valid also in this case. Note that if $k = 0$, (3.8) merely states the well-known L_p-stability of the upwind finite difference scheme, cf. [4]. So far we have not been able to carry out the full L_p-stability analysis of the scheme (3.7) when $k \geqslant 2$. □

Remark 3.2. Using the stability estimate (3.8) one can perform an ordinary finite difference error analysis of the discontinuous Galerkin method when applied to problem (3.1). As an example, let us estimate the error of $u_h(\cdot, t_n)$ for some choices of initial data assuming that $k = 1$ and $n \leqslant Ch^{-1}$.

Case 1. Let us first seek a scheme of maximal order of accuracy by choosing the initial data as

$$U^0(x) = \left(g(x), c_1\Delta_h g(x) + c_2\Delta_h^2 g(x)\right),$$

where $\Delta_h g(x) = g(x) - g(x-h)$ and c_1 and c_2 are constants to be defined shortly. We have for the discrete solution U

$$\hat{U}^n(\xi) = [\hat{E}(h\xi)]^n\hat{U}^0(\xi).$$

Correspondingly, we have for the exact solution, defining

$$W^n(x) = \left(u(x, t_n), c_1\Delta_h u(x, t_n) + c_2\Delta_h^2 u(x, t_n)\right),$$

that

$$\hat{W}^n(\xi) = e^{-i\gamma nh\xi}\hat{W}^0(\xi) = e^{-i\gamma nh\xi}\hat{U}^0(\xi).$$

Let us now choose the constants c_1 and c_2 so that $\hat{U}^0(\xi) = \hat{g}(\xi)[V(\xi) + O(|h\xi|^3)]$, where $V(\xi) = (1, V_2(\xi))$ is the eigenvector of $\hat{E}(h\xi)$ corresponding to the eigenvalue $\lambda_1(\xi) = e^{-i\gamma h\xi} + O((h\xi)^4)$. From the proof of Lemma 3.2 we obtain

$$c_1 = 1, \qquad c_2 = \tfrac{1}{3}(2 - \gamma).$$

With this choice we have, for $nh \leqslant C$,

$$\begin{aligned}\hat{W}^n(\xi) - \hat{U}^n(\xi) &= e^{-i\gamma nh\xi}\hat{W}^0(\xi) - \lambda_1(\xi)^n V(\xi) + O\left(|h\xi|^3\right)\\ &= \left[e^{-i\gamma nh\xi} - \lambda_1(\xi)^n\right]\hat{W}^0(\xi) + O\left(|h\xi|^3\right)\\ &= O(h^3\xi^4)\hat{W}^0(\xi) + O\left(|h\xi|^3\right).\end{aligned}$$

This leads to the error estimate

$$\left|u(x_{i+1/2}, t_n) - u_h^-(x_{i+1/2}, t_n)\right| \leqslant Ch^3\|g\|_{W^{4,\infty}(\mathbf{R})},$$

so with a proper choice of initial data, the scheme is accurate of order three. Obviously, the high accuracy can only be achieved at discrete points; if the error is measured in the norm $\|\cdot\|_{p,\mathbf{R}}$, a finite difference analysis only gives

$$\|(u - u_h^-)(\cdot, t_n)\|_{p,\mathbf{R}} \leqslant Ch\|g\|_{W^{2,p}(\mathbf{R})},$$

which is an estimate typical for first-order schemes. We see below in Section 4 that the latter estimate can be improved by using the improved stability estimate (3.9).

Case 2. Let the initial data be chosen as

$$U^0(x) = \left(\pi_h g(x), h\frac{\partial}{\partial x}(\pi_h g)(x)\right),$$

which is the choice made by the usual discontinuous Galerkin scheme. Then

$$\|U^0 - W^0\|_{p,\mathbf{R}} \leqslant Ch^2\|g\|_{W^{3,p}(\mathbf{R})},$$

which gives

$$\left|(u - u_h^-)(x_{i+1/2}, t_n)\right| \leqslant Ch^2\|g\|_{W^{3,\infty}(\mathbf{R})}.$$

To sum up, we have for $k = 1$ and $n \leqslant Ch^{-1}$,

$$\left|(u - u_h^-)(x_{i+1/2}, t_n)\right| \leqslant Ch^s \|g\|_{W^{s+1,\infty}(\mathbf{R})},$$

where $s = 3$ with a special choice of initial data and $s = 2$ with the usual choice of data. Moreover, for (essentially) all choices of initial data we have

$$\|(u - u_h^-)(\cdot, t_n)\|_{p,\mathbf{R}} \leqslant Ch \|g\|_{W^{2,p}(\mathbf{R})}, \qquad 1 \leqslant p \leqslant \infty. \qquad \square$$

4. L_p-Error Estimates. We now apply the results of the previous section to derive L_p-error estimates for the scheme (1.2) in the case where $\mathscr{C}^h$ is a uniform or piecewise-uniform triangulation of Ω and either $k = 0$ or $k = 1$. In the case of a uniform triangulation we will only need the estimates of Theorem 3.1, whereas if the triangulation is only piecewise uniform, also Theorem 3.2 will be required (in the case $p = \infty$). The results below are thus valid for any value of k for which the estimates of Theorem 3.1 and 3.2 can be proved.

We will need stability estimates analogous to (3.8) and (3.9) for the discontinuous Galerkin method applied to the problem

$$\begin{cases} u_\beta + au = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega_-, \end{cases} \tag{4.1}$$

where $a \in L_\infty(\Omega)$. To generalize the situation stepwise, let us first assume that Ω is still the half-plane, with $u_\beta = \partial u/\partial t + \gamma \partial u/\partial x$ and $\gamma \in [0, 1]$ as above and with the triangulation $\mathscr{C}^h$ defined as in the previous section. For each h we further use the notation

$$\begin{aligned} \omega_n &= \{(x, t)\colon x \in \mathbf{R},\ t_{n-1} < t < t_n\}, \\ \Omega_n &= \{(x, t)\colon x \in \mathbf{R},\ 0 < t < t_n\}, \\ \Gamma_n &= \bigcup_{m=1}^{n} S_m, \end{aligned} \tag{4.2}$$

where $t_n = nh$ and S_m is defined in Section 3. We now prove

THEOREM 4.1. *Let $\mathscr{C}^h$ be as above and let u_h be the approximate solution of* (4.1) *defined by* (2.4) *with either $k = 0$ or $k = 1$ and with $h\|a\|_{\infty,\Omega}$ sufficiently small. Then we have for all $n \geqslant 1$ and $1 \leqslant p \leqslant \infty$ the estimates*

$$\begin{aligned} &\|u_h^-(\cdot, t_n)\|_{p,\mathbf{R}} + \|u_h\|_{p,\Omega_n} \\ &\qquad \leqslant CMq^n (hn)^{1-1/p} \left(\|f\|_{p,\Omega_n} + \|g\|_{p,\mathbf{R}}\right), \end{aligned}$$

and

$$\begin{aligned} &\|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,\Gamma_n} \\ &\qquad \leqslant CM^2 q^n \left[h^{1-1/p} n^{1/2} (hn)^{1-1/p} \|f\|_{p,\Omega_n} + n^{\max\{0, 1/p - 1/2\}} \|g\|_{p,\mathbf{R}}\right], \end{aligned}$$

where $M = 1 + \|a\|_{\infty,\Omega_n}$, $q = 1 + CMh$ and C is an absolute constant.

Proof. Let us derive first some local estimates for u_h on ω_n. Applying repeatedly Lemma 2.1 to the triangles $T^+_{n,i-1}$, T^+_{ni} and T^-_{ni} (see Figure 1), we see that

$$\begin{aligned} &\|u_h\|_{p,K_i} + h^{1/p} \|u_h^-(\cdot, t_n)\|_{L_p(x_{i-1}, x_i)} + h^{1/p} \|\nu \cdot \beta(u_h^+ - u_h^-)\|_{p,(T^-_{ni})_-} \\ &\qquad \leqslant Ch \|f\|_{p,K_i} + Ch^{1/p} \|u_h^-(\cdot, t_{n-1})\|_{L_p(x_{i-2}, x_i)}, \end{aligned}$$

where $K_i = T^+_{n,i-1} \cup T^+_{ni} \cup T^-_{ni}$. For $p = \infty$ this implies immediately that

$$\|u_h\|_{p,\omega_n} + h^{1/p}\|u_h^-(\cdot,t_n)\|_{p,\mathbf{R}} + h^{1/p}\|\nu\cdot\beta(u_h^+ - u_h^-)\|_{p,S_n} \leqslant Ch\|f\|_{\omega_n} + Ch^{1/p}\|u_h^-(\cdot,t_{n-1})\|_{p,\mathbf{R}}, \tag{4.3}$$

and for $1 \leqslant p < \infty$ the same estimate readily follows by summation over i.

It is easy to sharpen the estimate for $u_h^-(\cdot,t_n)$ in (4.3) by writing

$$u_h^-(\cdot,t_n) = G_h u_h^-(\cdot,t_{n-1}) + v_n(\cdot,t_n), \qquad n \geqslant 1, \tag{4.4}$$

where G_h is defined by (3.4) and (3.5). By linearity, v_n is the discontinuous Galerkin solution on ω_n to the problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + \gamma\dfrac{\partial w}{\partial x} = f_n, & (x,t) \in \omega_n, \\ u(\cdot,t_{n-1}) = 0, \end{cases}$$

where $f_n = f - au_h$. Thus, by (4.3), v_n satisfies

$$\|v_n(\cdot,t_n)\|_{p,\mathbf{R}} \leqslant Ch^{1-1/p}\|f_n\|_{p,\omega_n} \leqslant C_1 M\big(h\|u_h^-(\cdot,t_{n-1})\|_{p,\mathbf{R}} + h^{1-1/p}\|f\|_{p,\omega_n}\big). \tag{4.5}$$

Upon solving for u_h in (4.4), we get

$$u_h^-(\cdot,t_n) = \sum_{j=0}^{n} G_h^{n-1} w_j, \qquad w_j = v_j(\cdot,t_j), \qquad j \geqslant 1,\ w_0 = g.$$

Applying here (4.5) and recalling that, by Theorem 3.1,

$$\|G_h^n g\|_{p,\mathbf{R}} \leqslant C\|g\|_{p,\mathbf{R}}, \qquad g \in L_p(\mathbf{R}),\ n \geqslant 1,\ 1 \leqslant p \leqslant \infty,$$

we see that for any $n \geqslant 1$,

$$\|u_h^-(\cdot,t_n)\|_{p,\mathbf{R}} \leqslant CMh\sum_{j=0}^{n-1}\|u_h^-(\cdot,t_n)\|_{p,\mathbf{R}} + CM(hn)^{1-1/p}\|f\|_{p,\Omega_n}.$$

The iteration of this inequality now gives

$$\|u_h^-(\cdot,t_n)\|_{p,\mathbf{R}} \leqslant C(1+CMh)^n\big[M(hn)^{1-1/p}\|f\|_{p,\Omega_n} + \|g\|_{p,\mathbf{R}}\big],$$

which proves the first estimate in the theorem.

In the remaining part of the proof we use the splitting

$$u_h = \sum_{m=0}^{n} u_{hm}, \qquad (x,t) \in \Omega_n, \tag{4.6}$$

where for $m \geqslant 1$, u_{hm} denotes the discontinuous Galerkin solution to the problem

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = \begin{cases} f - au_h, & (x,t) \in \omega_m, \\ 0, & \text{elsewhere on } \Omega, \end{cases}$$
$$u(\cdot,0) = 0,$$

and u_{h0} is the discontinuous Galerkin solution to $\partial w/\partial t + \gamma\partial w/\partial x = 0$ in Ω, $w(\cdot,0) = g$. Since u_{hm} obviously vanishes for $t < t_{m-1}$, we have by (4.3) that for $m \geqslant 1$

$$\|\nu\cdot\beta(u_{hm}^+ - u_{hm}^-)\|_{p,S_m} + \|u_{hm}^-(\cdot,t_n)\|_{p,\mathbf{R}} \leqslant Ch^{1-1/p}\|f - au_h\|_{p,\omega_m}. \tag{4.7}$$

Now Theorem 3.1 implies that for $j > m \geqslant 1$,

$$\|\nu \cdot \beta(u_{hm}^{+} - u_{hm}^{-})\|_{p,S_j} \leqslant C(j-m)^{-1/2} h^{1-1/p} \|f - au_h\|_{p,\omega_m},$$

and

$$\|\nu \cdot \beta(u_{h0}^{+} - u_{h0}^{-})\|_{p,S_j} \leqslant Cj^{-1/2}\|g\|_{p,\mathbf{R}}.$$

Combining this with (4.7) and summing over m, recalling (4.6), we have

$$\|\nu \cdot \beta(u_h^{+} - u_h^{-})\|_{\infty,\Gamma_n} \leqslant C\left[hn^{1/2}\|f - au_h\|_{\infty,\Omega_n} + \|g\|_{\infty,\mathbf{R}}\right],$$

and

$$\sum_{j=1}^{n} \|\nu \cdot \beta(u_h^{+} - u_h^{-})\|_{1,S_j} = \|\nu \cdot \beta(u_h^{+} - u_h^{-})\|_{1,\Gamma_n}$$
$$\leqslant Cn^{1/2}\left[\|f - au_h\|_{\Omega_n} + \|g\|_{1,\mathbf{R}}\right].$$

When combined with the estimate already proved for $\|u_h\|_{p,\Omega_n}$, these inequalities prove the second estimate of the theorem in the cases $p = \infty$ and $p = 1$. The remaining values of p can be treated similarly, and so the proof is complete. □

Let us now return to the original situation of Eq. (1.1) where Ω is a bounded convex polygonal domain. We assume first that Ω allows a uniform triangulation, which obviously is possible only in specific cases. To simplify the notation, let us reduce the situation to that of (4.1) by introducing the affine mapping F: $\mathbf{R}^2 \to \mathbf{R}^2$ with the following properties: (i) if $\tilde{\beta} = F(\beta)$ then $\tilde{\beta}_2/\tilde{\beta}_1 \in [0,1]$, (ii) if $T \in \mathscr{C}^h$ then for some n, $0 \leqslant n \leqslant Ch^{-1}\operatorname{diam}\Omega$ and for some i, $F(T)$ has two of its vertices at the points (ih, nh) and $((i-1)h, (n-1)h)$ and the third vertex either at $((i-1)h, nh)$ or at $(ih, (n-1)h)$. It is obvious that such a mapping exists and is nonsingular. Moreover, if we write $\tilde{v}(F(x)) = v(x)$, $x \in \Omega$ and $\tilde{\Omega} = F(\Omega)$, it is easy to see that $\tilde{u}_h$ is the discontinuous Galerkin solution of the transformed problem

$$\begin{aligned} \tilde{\beta} \cdot \nabla \tilde{u} + \tilde{a}\tilde{u} &= \tilde{f} \quad \text{in } \tilde{\Omega}, \\ \tilde{u} &= \tilde{g} \quad \text{on } \partial\tilde{\Omega}_{-}, \end{aligned}$$

on the triangulation $\{\tilde{T} = F(T)\colon T \in \mathscr{C}^h\}$. Note also that $C^{-1} \leqslant |\tilde{\beta}| \leqslant C$ and that the error estimates to be proved below are invariant under the transformation F. Thus, we may as well consider the transformed problem. Below we suppress the tildes for simplicity, i.e., we write Ω instead of $\tilde{\Omega}$, u_h instead of $\tilde{u}_h$, etc. Moreover, we use the coordinates (x, t) instead of (x_1, x_2) and denote by $\mathscr{M}^h$ the uniform triangulation of the half-plane $\{(x,t)\colon x \in \mathbf{R},\ t > 0\}$ such that $\mathscr{C}^h = \{T \in \mathscr{M}^h\colon T \subset \Omega\}$, i.e., $\mathscr{M}_h$ coincides with the triangulation $\mathscr{C}^h$ referred to in Theorem 4.1. We also use the notation

$$D_j = \left\{x \in \mathbf{R}\colon (x, t_j) \in \Omega\right\}.$$

We can now prove

THEOREM 4.2. *Let the domain* Ω *and the triangulation* $\mathscr{C}^h = \mathscr{M}^h_{|\Omega}$ *be as above, let* u *be the solution of* (1.1) *and let* u_h *be defined by* (1.2) *with either* $k = 0$ *or* $k = 1$. *Assume further that* $h\|a\|_{\infty,\Omega}$ *is small enough and that* $h^{1/2}\|a\|_{\infty,\Omega} \leqslant C$. *Then we have the error estimates*

$$\|u - u_h\|_{p,\Omega} \leqslant CM^2 h^{k+1/2} |u|_{W^{k+1,p}(\Omega)}, \qquad 1 \leqslant p \leqslant \infty,$$

and

$$\max_{j:\, D_j \neq \phi} \|(u - u_h)(\cdot, t_j)\|_{p, D_j}$$
$$\leqslant CM^2 \begin{cases} h^{k+1/2}|u|_{W^{k+1,p}(\Omega)}, & p \geqslant 2, \\ h^{k+1-1/p}|u|_{W^{k+1,p}(\Omega)}, & 1 \leqslant p \leqslant 2, \end{cases}$$

where $M = 1 + \|a\|_{\infty,\Omega}$ and C is a constant depending only on diam(Ω).

Proof. Let n be sufficiently large so that $\Omega \subset \Omega_n = \{(x, t)\colon\ x \in \mathbf{R},\ 0 < t < nh\}$ and let $\varphi \in L_1(\Omega_n)$ be such that φ vanishes outside Ω. Define the function ϕ_h on Ω_n so that $\phi_{h|T} \in P_k(T)$, $T \in \mathscr{M}_h$, and

$$\text{(4.8)} \qquad \begin{aligned} &\int_T v(-\phi_{h\beta} + a\phi_h)\,dx\,dt + \int_{\partial T_+} v^-(\phi_h^- - \phi_h^+)\nu \cdot \beta\,ds \\ &\quad = \int_T \varphi v\,dx\,dt, \qquad T \in \mathscr{M}_h,\ T \subset \Omega_n, \end{aligned}$$

where $\phi_h^+ = 0$ on $(\partial\Omega_n)_+$ and $a = 0$ outside Ω. Via the simple coordinate transformation $t \to nh - t$, ϕ_h becomes the discontinuous Galerkin solution of the problem (4.1) with $\Omega = \Omega_n$ and $f(x, t) = \varphi(x, nh - t)$ and $g = 0$. Thus, by Theorem 4.1 and since $n \leqslant C\,\mathrm{diam}(\Omega)h^{-1}$,

$$\text{(4.9a)} \qquad \|\phi_h\|_{p,\Omega_n} \leqslant CM\|\varphi\|_{p,\Omega},$$

$$\text{(4.9b)} \qquad \|\nu \cdot \beta(\phi_h^+ - \phi_h^-)\|_{p,\Gamma_n} \leqslant CM^2h^{1/2-1/p}\|\phi\|_{p,\Omega}, \qquad \varphi_{|\Omega} \in L_p(\Omega).$$

Let us now sum over $T \subset \Omega$ in (4.8) to obtain

$$\mathscr{B}(v, \phi_h) = (\varphi, v), \qquad v \in V_h,$$

where V_h is as in (2.3) and $\mathscr{B}$ is defined by (2.5) or equivalently, by (2.6). Choosing here $v = u_h - \tilde{u}$, where $\tilde{u} \in V_h$ is an interpolant of u to be defined below and recalling (2.7), we obtain

$$\text{(4.10)} \qquad (u_h - \tilde{u}, \varphi) = \mathscr{B}(u - \tilde{u}, \phi_h).$$

Let us define the interpolant by requiring

$$\text{(4.11a)} \qquad \int_T (u - \tilde{u})v\,dx\,dt = 0, \qquad v \in P_{k-1}(T)\ (\text{if } k = 1),$$

$$\text{(4.11b)} \qquad \int_l (u - \tilde{u})v\,dx = 0, \qquad v \in P_k(l),\ T \in \mathscr{C}^h,$$

where l is the side of T which is parallel to the x-axis. It is easy to see that $\tilde{u}$ is uniquely defined so far as $u \in W^{1,1}$. Applying the Bramble-Hilbert lemma [2] we obtain by standard reasoning the interpolation error estimates

$$\|u - \tilde{u}\|_{p,T} \leqslant Ch^{k+1}|u|_{W^{k+1,p}(T)},$$
$$\|u - \tilde{u}\|_{p,\partial T} \leqslant Ch^{k+1-1/p}|u|_{W^{k+1,p}(T)}, \qquad 1 \leqslant p \leqslant \infty,$$

where C depends only on κ and k.

Now by (2.6) and (4.11a),

$$\mathscr{B}(u-\tilde{u},\phi_h)=\int_{(\Gamma_n\cap\bar{\Omega})\setminus\partial\Omega_-}(u-\tilde{u})^-(\phi_h^- -\phi_h^+)\nu\cdot\beta\,ds$$
$$+\int_\Omega a(u-\tilde{u})\phi_h\,dx\,dt,$$

where Γ_n is as in (4.2). Applying on the right side the Hölder inequality, (4.9) and (4.11), and recalling (4.10), we have

$$\int_\Omega(u_h-\tilde{u})\varphi\,dx\,dt\leqslant CM^2\left(h^{k+1/2}+\|a\|_{\infty,\Omega}h^{k+1}\right)|u|_{W^{2,p}(\Omega)}\|\varphi\|_{q,\Omega},$$

where $1/p+1/q=1$. Since this is valid for any φ such that $\varphi_{|\Omega}\in L_q(\Omega)$, $1\leqslant q\leqslant\infty$, and since $\|a\|_{\infty,\Omega}h^{1/2}\leqslant C$, by our assumptions, it follows that

$$\|u_h-\tilde{u}\|_{p,\Omega}\leqslant CM^2h^{k+1/2}|u|_{W^{2,p}(\Omega)},\qquad 1\leqslant p\leqslant\infty.$$

Recalling the estimate for $\|u-\tilde{u}\|_{p,\Omega}$ and using the triangle inequality one obtains the asserted estimate for $\|u-u_h\|_{p,\Omega}$.

The remaining estimates are proved in a similar manner by first introducing a function $\varphi\in L_1(\mathbf{R})$, replacing the right-hand side of (4.8) by

$$\int_{\partial T\cap\{t=jh\}}\varphi v\,dx,\qquad T\in\mathscr{M}^h,\ T\subset\Omega_n,$$

and then proceeding as above. We omit the details. □

We consider finally a more practical situation where the triangulation $\mathscr{C}^h$ in (2.1) is only piecewise uniform.

THEOREM 4.3. *Let* $\mathscr{C}^h$ *be a piecewise-uniform triangulation of* Ω *generated by* $\mathscr{C}$ *and let otherwise the assumptions of Theorem* 4.2 *hold. Then if* $u\in W^{k+1,p}(\Omega)$, $1\leqslant p\leqslant\infty$, *we have the error estimate*

$$\|u-u_h\|_{p,\Omega}\leqslant\begin{cases}Ch^{k+1-1/p}|u|_{W^{k+1,p}(\Omega)}, & \text{if } 1\leqslant p\leqslant 2,\\ Ch^{k+1/2}|u|_{W^{k+1,p}(\Omega)}, & \text{if } 2\leqslant p\leqslant\infty,\end{cases}$$

where C depends on $\mathscr{C}$ *and on* $\|a\|_{\infty,\Omega}$.

Proof. For $K\in\mathscr{C}$, let u_h^K be the local discontinuous Galerkin approximation to u on K defined by (1.2) for $T\in\mathscr{C}^h$, $T\subset K$, with u_h^- replaced by u on ∂K_-. Then $u_{h|_K}=u_h^K+v_h^K$, where v_h^K is the discontinuous Galerkin solution to the problem

$$\begin{aligned}\varphi_\beta+a\varphi&=0\quad\text{in } K,\\ \varphi&=u_h^- -u\quad\text{on }\partial K_-.\end{aligned}\tag{4.12}$$

Applying Theorem 4.2, we have

$$\begin{aligned}\|u-u_h\|_{p,K}&\leqslant\|u-u_h^K\|_{p,K}+\|v_h^K\|_{p,K}\\ &\leqslant Ch^{k+1/2}|u|_{W^{k+1,p}(\Omega)}+\|v_h^K\|_{p,K},\qquad 1\leqslant p\leqslant\infty.\end{aligned}\tag{4.13}$$

Assume for a moment the further estimates

$$\|v_h^K\|_{p,K}\leqslant C\|\nu\cdot\beta(u-u_h^-)\|_{p,\partial K_-},\qquad K\in\mathscr{C},\tag{4.14}$$

and

$$\|\nu \cdot \beta(u - u_h^-)\|_{p,\partial K_+} \leqslant Ch^{k+1/2}|u|_{W^{k+1,p}(K)}, \qquad K \in \mathscr{C}. \tag{4.15}$$

Then the repeated use of (4.13) through (4.15) gives

$$\|u - u_h\|_{p,\Omega} \leqslant Ch^{k+1/2}|u|_{W^{k+1,p}(\Omega)}. \tag{4.16}$$

Let us now consider the validity of (4.14) and (4.15). Assume first that ∂K_- contains two sides of K and denote by w_h^K the approximate solution of (4.12) in the case $a = 0$. Then it follows from Theorem 3.2, via an affine transformation, that

$$\|w_h^K\|_{p,K} \leqslant C\|\nu \cdot \beta(u - u_h^-)\|_{p,\partial K_-}, \qquad 1 \leqslant p \leqslant \infty.$$

Further, applying Theorem 4.1, we easily see that

$$\|v_h^K - w_h^K\|_{p,K} \leqslant C\|aw_h^K\|_{p,K}, \qquad 1 \leqslant p \leqslant \infty,$$

so combining these estimates we conclude that (4.14) is valid for any p if ∂K_- contains two sides of K. Finally, if ∂K_- consists of one side of K only, the same result can be read directly from Theorem 4.2.

If $p = 2$, the estimate (4.15) is valid by Theorem 2.1, and for $p = \infty$ (4.15) follows from Theorem 4.2. Thus, by interpolation [1], (4.15) and (4.16) are true in the range $2 \leqslant p \leqslant \infty$. In the range $1 \leqslant p < 2$, however, (4.15) cannot hold, since it would in this case violate the approximation properties of piecewise-polynomial spaces. Thus, we need a different reasoning if $p < 2$.

Consider the case $p = 1$. Let $\varphi \in L_\infty(\Omega)$ and $\phi_h \in V_h$ be defined by (2.8). By Theorem 3.2 and by Theorem 4.1, we have

$$\|\phi_h\|_{\infty,K} \leqslant C\left(\|\varphi\|_{\infty,K} + \|\phi_h^+\|_{\infty,\partial K_+}\right), \qquad K \in \mathscr{C},$$

and so, by iteration,

$$\|\phi_h\|_{\infty,\Omega} \leqslant C\|\varphi\|_{\infty,\Omega}. \tag{4.17}$$

Using now an argument from the proof of Theorem 4.2, with (4.9) replaced by the weaker estimate (4.17), it follows that

$$\|u - u_h\|_{1,\Omega} \leqslant Ch^k|u|_{W^{k+1,1}(\Omega)}.$$

Upon interpolating between this estimate and the estimate (4.16) with $p = 2$, and recalling that (4.16) holds for $p \geqslant 2$, the proof is finally complete. □

Remark. In the proof of Theorems 4.2 and 4.3 we have used the increased stability of the discontinuous Galerkin method as established by the fundamental inequality (3.9). It is possible to state this stability inequality in a form analogous to that given in Theorem 2.1. For example, if $g = 0$ in (1.1) and the triangulation $\mathscr{C}^h$ is uniform, we have (in the cases $k = 0$ and $k = 1$) that

$$|u_h|_{h,\beta,p} \leqslant Ch^{-1/2}\|f\|_{p,\Omega}, \qquad 1 \leqslant p \leqslant \infty,$$

where

$$|v|_{h,\beta,p} = \left\{ \sum_{T \in \mathscr{C}^h} \|v_\beta\|_{p,T}^p + h^{-p+1}\|\nu \cdot \beta(v^+ - v^-)\|_{p,\Gamma_h}^p \right\}^{1/p}.$$

For a piecewise-uniform triangulation, the estimate remains valid in the range $1 \leqslant p \leqslant 2$, but ceases to be valid in the range $2 < p \leqslant \infty$, as can be shown by a counterexample. In particular, if $p = \infty$ one can show that the estimate

$$|u_h|_{h,\beta,\infty} \leqslant Ch^{-1}\|f\|_{\infty,\Omega}$$

is optimal. Since this follows already from the L_∞-stability estimate using an inverse inequality, we see that in this case the smoothing property of the discontinuous Galerkin method is lost. □

Department of Mathematics
Chalmers University of Technology
S-41296 Göteborg, Sweden

Institute of Mathematics
Helsinki University of Technology
SF-02150 Espoo 15, Finland

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