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An Analytic Approach to a General Class of G/G/s Queueing Systems

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Abstract

We solve the queueing system (QS) $C_k/C_m/s$, where C_k is the class of Coxian probability density functions (pdfs) of order k, which is a subset of the pdfs that have rational Laplace transform (R). We formulate the model as a continuous-time, infinite-space Markov chain by generalizing the method of stages. By using a generating function technique, we solve an infinite system of partial difference equations and find closed form expressions for the system-size, general-time, pre-arrival, post-departure probability distributions and the usual performance measures. In particular, we prove that the probability of n customers being in the system, when it is "saturated" ($n \geq s$) is a linear combination of exactly $\binom{s+m-1}{s}$ geometric terms. The closed form expressions involve a solution of a system of nonlinear equations that involves only the Laplace transforms of the interarrival and service time distributions. We conjecture that this result holds for the more general model $G/R/s$. Following these theoretical results we propose an exact algorithm for finding the system-size distribution and system's performance measures, which has an algorithmic complexity of $O(k^3\binom{s+m-1}{s}^3)$. We examine special cases and apply this method for solving numerically the QS $C_2/C_2/s$ and $E_k/C_2/s$.

Subject classification: 705 Markovian queues, 693 Multichannel queues.

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Introduction

Throughout this century queueing theory has been studied thoroughly, but still many problems remain unsolved, in spite of the effort and intelligence devoted to them. Among these problems, the analysis of the G/G/s queueing system(QS) has survived the "attacks" of many excellent mathematicians, obviously due to its inherent complexity.

In this paper we generalize the method of stages, introduced earlier in the century by Erlang and combine it with a generating function technique to achieve the solution of the general class of problems $C_n/C_m/s$, where C_n is the Coxian class of probability density functions (pdf's) with rational Laplace transform.

A brief critical presentation of alternative solution methods

When it comes to exact solutions of multi-server QS's, the more one departs from the assumption of exponentiallity, the more thorny problems become (especially if this happens for the service time pdf or, worse, for both the service and arrival-time pdf). Thus the only solution approaches that have up-to-now been established as supposedly "general-purpose" are, in chronological order of appearance:

1. The method of succesive exponential stages

2. Complex variable theory

3. Imbedded Markov chain

4. Inclusion of supplementary variables

We believe that the following very short commentary on the application of these solution approaches to multi-server models assesses fairly their inherent potential to produce theoretical and, especially, exact numerical results (which should usually be their ultimate goal).

The Imbedded Markov chain method

In the past decade it was mainly Neuts and a number of other authors who, by exploiting the phase-type (PH) distribution and by developing the powerful but still "esoteric" formalism of the associated matrix-analytic methods and algorithmic approaches (as presented in Neuts (1981), (1984)) have managed, by "bridging" this method with the method of stages, to considerably extend its potential. Yet, for multi-server systems with non-exponential service times the relevant solutions (although providing qualitative insight) lead to major dimensionality problems. For the $E_k/C_m/s$ model this approach requires the solution of a non-linear matrix equation involving matrices of order m^s, so that (even for C_2) Bellman's dimensionality curse precludes the derivation of any exact results (except for very small values of s).

The inclusion of supplementary variables method

Despite the initial great expectations concerning its capabilities, the method has been used for multi-server systems in a limited number of analytic investigations (not to mention numerical implementations). In the last decade Ishikawa (1979) tried to tackle the $G/E_m/s$ system through this method, but he restricted his attention to the $G/E_3/3$ system. In parallel, Hokstad (1978) started his investigation of the $M/G/s$ system by writing the associated equations, which he managed to "solve" in Hokstad (1979) but only for the $M/R/2$ case, where R is the class of distributions with rational Laplace transforms. Then he specialized to the $M/C_2/2$ (and thence to the $M/E_2/2$ and $M/H_2/2$) producing some limited results. In his next (Hokstad (1980)) paper he uses a "refined and modified" version of his previous analysis, which he claims to be in principle applicable to the general $M/C_m/s$ model (but with complicated results). He was compelled to restrict attention to the $M/C_2/s$ and (after various simplifications) to present very limited results for $s = 3$. Finally, in Hokstad (1986) he gave bounds for the mean queue length of the

$M/C_2/s$ QS. Both the above approaches are complicated, while Hokstad's general algorithm (according to van Hoorn (1983), p.29) is numerically unstable and not reliable for higher s values.

The complex variable theory method

Pollaczek's method (Pollaczek (1961)) is by its structure limited to First-Come-First-Serve (FCFS). The method was exploited by de Smit, whose venture started with some abstract results for the $G/G/s$ QS (de Smit (1973a)), which were then specialized to G/M/s (de Smit (1973b)), and continued with a method of approach for $G/H_m/s$ (de Smit (1983a),(1983c)). Yet, this approach needs deep arguments from complex variable theory, leading to a numerical solution in the $G/H_2/s$ case, but with a high computational complexity which is proportional to s^6, to be compared with $O(s^3)$ of our method.

The method of succesive exponential stages

Erlang's ingenious method of stages has for some time been neglected, obviously because of researchers' concern about the fact that it does lead to systems of equations which usually are complicated (but certainly not looking more formidable than those of the previous methods) and become almost intractable if one attempts to tackle them directly by various seemingly powerful techniques. For example in the $E_k/E_m/s$ case this intractability is apparent both

1. In many of the earlier attempts at an exact solution via multidimensional generating function techniques (see, e.g., the annotated, but not totally complete, bibliography by Ovuworie (1980))

2. In Yu's (1977) ambitious theoretical treatise, via an intricate partitioning of the system-states and an exploitation of the cyclic structure (exhibited by the corresponding transitions) through the use of polynomial matrices.

Concerning in particular the latter approach, (see, e.g., Hillier and Lo (1971)), the computations necessary for the derivation of numerical results involve the numerical expansion of such matrices, which is still an enormous task even for moderate values of k, m and s.

The above mentioned attitude is reflected in the comments of Kleinrock (1975), p.146-7. But on the contrary, and more in line with Cohen (1982), p.346, we believe that this method with a proper solution strategy, like the one presented in this paper, is so powerful that it is difficult to overemphasize its importance for the exact solution of intrinsically difficult multi-server queueing systems.

On the class of Coxian distributions C_n

The general Coxian class C_n was introduced in Cox's (1955) pioneering paper and is clearly presented in Kleinrock (1975). It is remarkable that even if we permit transitions from a stage with rate μ_i to a stage with rate μ_j in figure 1 we do not obtain a new class of distributions. We can still formulate this situation with a C_n distribution with different transition rates. The salient feature of the class C_n is its high versatility (see e.g. Neuts (1981), Whitt (1982)) based on its ability to:

1. Generalize well-known distributions such as the exponential, the hyperexponential and all the forms (i.e., special, general, weighted, compound, etc.) of the Erlangian.

2. Be dense in the set of all probability distributions concentrated on $(0, \infty)$ and thus to be able to approximate a general pdf.

3. Permit coefficients of variation V_s^2 greater than 1.

Figure 1: The C_m class of distributions

The important feature we exploit in this paper is the decomposition of the C_n pdf into exponential-stages, so that we can use a powerful modification of the method of stages.

Our result

Our result generalizes the well known $G/G/1$ theory to multi-server systems in a natural probabilistic way. Our approach, in comparison to the methods used earlier in the evolution of queueing theory is easier to grasp and easier to implement. More importantly, it leads to an algorithm of relatively low order of complexity. On the other hand, in comparison to the purely numerical methods of Allen, Andersen and Seneta (1977), Takahashi and Takami (1976) (which was recently specialized by Groenevelt, van Hoorn and Tijms (1984) for the solution of the much simpler models $M/H_2/s$, $M/E_{1,2}/s$, $M/E_{1,3}/s$) and Seelen (1986) the present approach offers full qualitative insight by providing closed-form expressions, which apart from their computational value, are also of theoretical interest. Furthermore, our solution strategy leads to an exact waiting-time analysis under FCFS, which is going to be presented in a forthcoming paper.

From an algorithmic point of view we propose an $O(k^3\binom{s+m-1}{s}^3)$ algorithm for the calculation of the performance measures and the probability distributions of this QS, which for a given m is polynomial in the number of servers. In other words, we prove that for a given service time pdf the solution of the related queueing problem is polynomial in the number of servers.

To test properly the potential and reliability of this method, we prepared computer programs for the numerical solution of the QS's $E_k/C_2/s$, $C_2/C_2/s$. These exact results are in full agreement with all others available in the literature and can be exploited for the always desirable sensitivity analysis and comparative evaluation in the continuously active areas of

1. approximations for multi-server models (e.g. those on the $M/G/s$ QS recently reviewed by Tijms, van Hoorn and Federguen (1981), Sze (1984)).

2. inequalities, bounds and stochastic order relationships

on both of which there is a rapidly increasing literature.

In the next section we formulate the model as a continuous time Markov chain using the method of stages. In section 2 we apply a generating function technique for solving the difference equations that describe the system. In this section, which is central to the analysis, we combine results of the complex-variable-theory method developed by Pollaczek (1961) and de Smit (1983a) with results of the present paper to prove what we call the "separability property": The probability of n customers ($n \geq s$) being in the system is a linear combination of $\binom{s+m-1}{s}$ geometric terms. In section 2.5 we examine as special cases the QS's $C_n/C_m/1$, $C_n/M/s$, $E_k/E_m/s$ and $E_n/C_2/s$.

The derivations of closed form expressions for the system size probability distributions and the usual performance measures are outlined in section 3. In section 4 we include some computational and complexity considerations, while the final section contains some concluding remarks and open problems.

1 Formulation of the model as a continuous-time Markov chain

We shall examine, henceforth, an s identical-single-waiting-line QS with interarrival and service time distributions of Coxian type of order k and m respectively. There are no restrictions in the queue discipline except that no server can be idle if the queue is not empty.

1.1 Probabilistic interpretation of the QS

To analyse the model we conceive of the arrival process as an arrival timing channel (ATC) consisting of k consecutive stages with rates $\lambda_1, \lambda_2, ..., \lambda_k$ and with probabilities $p_1, p_2, ..., p_k \triangleq 1$ of entering the QS after the completion of the 1st, 2nd, ...kth stage. We remark that as soon as a customer in the ATC enters the QS a new customer arrives at stage 1 of the ATC. For the service time distribution we consider as above a service timing channel (STC) consisting of m consecutive stages with rates $\mu_1, \mu_2, ..., \mu_m$ and with probabilities $q_1, q_2, ..., q_m \triangleq 1$ of leaving the system.

1.2 Notation

For the steady-state we introduce the random variables

1. $N \triangleq$ The number of customers in the system.
2. $N^- \triangleq$ The number of customers seen by an arriving customer just before his arrival.
3. $N^+ \triangleq$ The number of customers seen by a departing customer just after his departure.
4. $R_a \triangleq$ The number of the ATC stage currently occupied by the arriving customer.
5. $R_j \triangleq$ The number of customers being served at the jth STC stage ($j = 1, 2, ..., m$).
6. $R_j^- \triangleq$ The number of customers being served at the jth STC stage ($j = 1, 2, ..., m$), just before the arrival of an entering customer.
7. $T_q \triangleq$ The waiting time of an arriving customer.

For simplicity of notation we introduce the vectors of random variables

$$\vec{R} \triangleq (R_1, ..., R_m), \quad \vec{R^-} \triangleq (R_1^-, ..., R_m^-)$$

and also we will use the notation:

$$\vec{\delta_j} \triangleq (0,...,0,1,0...,0) \qquad a(s,m) \triangleq \binom{s+m-1}{s}$$

$|\vec{i}| = s$ meaning that $\sum_{j=1}^{m} i_j = s$.

With the above definitions the system can be formulated as a continuous time Markov chain with infinite state space:

$$\{(N, R_a, R_1, ..., R_m), N = 0, 1, ..., \quad R_a = 1, 2, ...k, \quad \sum_{j=1}^{m} R_j = min(N, s)\}$$

where the states with $N < s$ (i.e., the states with at least one server free) and $N \geq s$ (or all servers busy) will be termed "unsaturated" and "saturated" respectively.

We now introduce the following set of probabilities, some of which will be used in later sections.

$P_{n,l,\vec{i}} \triangleq Pr\{N = n, R_a = l, \vec{R} = \vec{i}\}$

$P^-_{n,\vec{i}} \triangleq Pr\{N = n, \vec{R}^- = \vec{i}\}$

$P_n \triangleq Pr\{N = n\}$

$P^-_n \triangleq Pr\{N^- = n\}$

$P^+_n \triangleq Pr\{N^+ = n\}$

We also define:

$f^*_{T_a}(\theta)$, $f^*_{T_s}(\theta) \triangleq$ The Laplace transform of the interarrival and service time distributions respectively.

$\frac{1}{\lambda} \triangleq$ The mean interarrival time.

$\frac{1}{\mu} \triangleq$ The mean service time.

$\rho \triangleq \frac{\lambda}{s\mu} =$ The traffic intensity.

1.3 The equations

After drawing the rather complicated state-transition diagram in figure 2 (for the case $l = 2, \ldots, k$, the case $l = 1$ is similar) we write down the following system of equations:

1. $l = 1,\ |\vec{i}| = \min\{n, s\}$

$$P_{n,1,\vec{i}}\{\lambda_1 + \sum_{j=1}^{m} i_j\mu_j\} = \sum_{l=1}^{k} p_l\lambda_l P_{n-1,l,\vec{i}} + q_1\mu_1 i_1 P_{n+1,1,\vec{i}} +$$

$$\sum_{j=2}^{m} q_j\mu_j(i_j + 1)P_{n+1,1,\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1 - q_j)\mu_j(i_j + 1)P_{n,1,\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} \qquad (1)$$

2. $l = 2, \ldots, k,\ |\vec{i}| = \min\{n, s\}$

$$P_{n,l,\vec{i}}\{\lambda_l + \sum_{j=1}^{m} i_j\mu_j\} = (1 - p_{l-1})\lambda_{l-1}P_{n,l-1,\vec{i}} + q_1\mu_1 i_1 P_{n+1,1,\vec{i}} +$$

$$\sum_{j=2}^{m} q_j\mu_j(i_j + 1)P_{n+1,1,\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1 - q_j)\mu_j(i_j + 1)P_{n,l,\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} \qquad (2)$$

Figure 2: The state-transition diagram for $l = 2, \ldots, k$

2 Analysis of the equations

2.1 Separation of variables technique

Initially we consider the infinite number of equations (1), (2) for $n \geq s$. Using the separation of variables technique, we assume that the saturated probabilities are of the form:

$$P_{n,l,\vec{i}} = D_l\ R_{\vec{i}}\ w^n \quad n \geq s$$

with the obvious conditions

$$R_{\vec{i}} = 0 \ \ for \ \ \sum_{j=1}^{m} i_j \neq s$$

Then from equations (1),(2) for $n \geq s$ we get:

1. $l = 1,\ |\vec{i}| = s$

$$D_1 R_{\vec{i}}\{\lambda_1 + \sum_{j=1}^{m} i_j\mu_j\} = \frac{1}{w}\sum_{l=1}^{k} p_l\lambda_l D_l R_{\vec{i}} + wq_1\mu_1 i_1 D_1 R_{\vec{i}} +$$

$$w\sum_{j=2}^{m} q_j\mu_j(i_j+1)D_1 R_{\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1-q_j)\mu_j(i_j+1)D_1 R_{\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} \qquad (3)$$

2. $l = 2,\ldots,k,\ \ |\vec{i}| = s$

$$D_l R_{\vec{i}}\{\lambda_l + \sum_{j=1}^{m} i_j\mu_j\} = (1-p_{l-1})\lambda_{l-1}D_{l-1}R_{\vec{i}} + wq_1\mu_1 i_1 D_l R_{\vec{i}} +$$

$$w\sum_{j=2}^{m} q_j\mu_j(i_j+1)D_l R_{\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1-q_j)\mu_j(i_j+1)D_l R_{\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} \qquad (4)$$

Then (4) can be written:

$$R_{\vec{i}}\{D_l\lambda_l - (1-p_{l-1})\lambda_{l-1}D_{l-1}\} = D_l\{wq_1\mu_1 i_1 R_{\vec{i}} +$$

$$w\sum_{j=2}^{m} q_j\mu_j(i_j+1)R_{\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1-q_j)\mu_j(i_j+1)R_{\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} - R_{\vec{i}}\sum_{j=1}^{m} i_j\mu_j\}$$

Using the usual "separation of variables technique" arguments we demand that

$$D_l\lambda_l - (1-p_{l-1})\lambda_{l-1}D_{l-1} = -xD_l \qquad l = 2,\ldots,k \qquad (5)$$

$$wq_1\mu_1 i_1 R_{\vec{i}} + w\sum_{j=2}^{m} q_j\mu_j(i_j+1)R_{\vec{i}+\vec{\delta_j}-\vec{\delta_1}} + \sum_{j=1}^{m}(1-q_j)\mu_j(i_j+1)R_{\vec{i}+\vec{\delta_j}-\vec{\delta}_{j+1}} -$$

$$R_{\vec{i}}\sum_{j=1}^{m} i_j\mu_j = -xR_{\vec{i}},\ |\vec{i}| = s \qquad (6)$$

for some constant x that depends on w. Solving (5) we find that

$$D_l = D_1 \prod_{r=1}^{l-1}\frac{(1-p_r)\lambda_r}{x+\lambda_{r+1}} \quad l = 2,\ldots,k \qquad (7)$$

Substituting (7) to (3) and using (6) we find the relation between x and w.

$$w = \sum_{l=1}^{k}\frac{p_l\lambda_l}{x+\lambda_l}\prod_{r=1}^{l-1}\frac{(1-p_r)\lambda_r}{x+\lambda_r} = f^*_{T_a}(x) \qquad (8)$$

(the product for $l = 1$ is defined to equal 1)

Equations (6) form a linear homogeneous system of $a(s,m)$ with $a(s,m)$ unknowns. The only way for a nontrivial solution to exist is for the determinant of the system to equal 0, i.e.

$$Det(w) = 0 \tag{9}$$

2.2 A generating function technique

From (9) we can in principle find the roots w inside the unit circle that satisfy (3), (4). Since the numerical determination of the roots w is of extreme computational complexity, because of the large dimensions of the determinant, we will use a method based on generating functions in order to find the roots w. For this purpose we introduce the m multivariable generating function:

$$U(\vec{z}) \triangleq \sum_{|\vec{i}|=s} R_{\vec{i}} z_1^{i_1} \dots z_m^{i_m}, \quad \vec{i} = (i_1, \dots, i_m), \quad \vec{z} = (z_1, \dots, z_m)$$

Multiplying (6) by $z_1^{i_1} \dots z_m^{i_m}$ we get the following partial differential equation of first order:

$$\sum_{j=1}^{m} \frac{\partial U(\vec{z})}{\partial z_j} (\mu_j z_j - w z_1 q_j \mu_j - (1 - q_j)\mu_j z_{j+1}) = x U(\vec{z}) \tag{10}$$

For the derivation of (10) we have used the identities:

$$z_j \frac{\partial U(\vec{z})}{\partial z_j} = \sum_{|\vec{i}|=s} R_{\vec{i}} i_j z_1^{i_1} \dots z_m^{i_m}$$

$$z_r \frac{\partial U(\vec{z})}{\partial z_j} = \sum_{|\vec{i}|=s} R_{\vec{i}+\vec{\delta}_j-\vec{\delta}_r} (i_j + 1) z_1^{i_1} \dots z_m^{i_m}, \quad r = 1, j+1$$

The new idea we introduce in this section, the exploitation of which is presented in section 2.3, is that the solution of (10) combined with the requirement that this solution must be a multivariable polynomial, will lead to the determination of the roots w.

2.3 The method of characteristics

Our goal is to solve (10) which is a linear partial differential equation (pde) of first order involving m variables. We will use a well known method, the method of characteristics, from the theory of pde's in order to solve it. Then we form the system of ordinary d.e's:

$$\frac{dz_1}{(1-wq_1)\mu_1 z_1 - (1-q_1)\mu_1 z_2} = \ldots = \frac{dz_j}{-wq_j\mu_j z_1 + \mu_j z_j - (1-q_j)\mu_j z_{j+1}} = \ldots =$$
$$= \frac{dz_m}{-w\mu_m z_1 + \mu_m z_m} = \frac{dU(\vec{z})}{xU(\vec{z})} = dt \tag{11}$$

which can be written:

$$\frac{d\vec{z}}{dt} = A_m \vec{z} \tag{12}$$

where A_m is the $m \times m$ matrix:

$$A_m = \begin{bmatrix} -wq_1\mu_1 + \mu_1 & -(1-q_1)\mu_1 & 0 & \ldots\ldots & 0 \\ -wq_2\mu_2 & \mu_2 & -(1-q_2)\mu_2 & \ldots\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -wq_1\mu_{m-1} & 0 & \ldots\ldots & \mu_{m-1} & -(1-q_{m-1})\mu_{m-1} \\ -wq_1\mu_m & 0 & \ldots\ldots & 0 & \mu_m \end{bmatrix}$$

In order to solve the system (12) we must find the eigenvalues of the matrix A_m . In the following proposition 1 we find the equation that the eigenvalues of matrix A_m satisfy.

Proposition 1

The eigenvalues of matrix A_m are the m roots $\theta_i(x)$ $\quad i = 1, \ldots, m$ of the following equation:

$$f^*_{T_a}(x) f^*_{T_s}(-\theta(x)) = 1 \tag{13}$$

Proof

We will prove (13) inductively using the following induction hypothesis:

Induction hypothesis: $|A_k - \theta I| = (1 - w f^*_{T_s,k}(-\theta)) \prod_{j=1}^{k} (\mu_j - \theta)$

where

$f^*_{T_s,k}(\theta) \triangleq$ the Laplace transform of the service time pdf with k poles

$f^*_{T_s,m}(\theta) \triangleq f^*_{T_s}(\theta)$.

For $k = 1$ we have

$$|A_1 - \theta I| = -w q_1 \mu_1 + \mu_1 - \theta$$

But since $q_1 = 1$, then

$$|A_1 - \theta I| = (\mu_1 - \theta)(1 - w f^*_{T_s,1}(-\theta))$$

Assume that the induction hypothesis is true for $k - 1$. Then expanding the determinant $|A_k - \theta I|$ we take

$$|A_k - \theta I| = (\mu_k - \theta)|A_{k-1} - \theta I| - w q_k \mu_k \prod_{j=1}^{k-1} (1 - q_j)\mu_j$$

Then, using the induction hypothesis:

$$|A_k - \theta I| = (1 - w f^*_{T_s,k-1}(-\theta)) \prod_{j=1}^{k} (\mu_j - \theta) - w q_k \mu_k \prod_{j=1}^{k-1} (1 - q_j)\mu_j$$

Therefore

$$|A_k - \theta I| = \prod_{j=1}^{k} (\mu_j - \theta)\{(1 - w f^*_{T_s,k-1}(-\theta)) - w \frac{q_k \mu_k}{\mu_k - \theta} \prod_{j=1}^{k-1} \frac{(1 - q_j)\mu_j}{\mu_j - \theta}\} =$$

$$(1 - w f^*_{T_s,k}(-\theta)) \prod_{j=1}^{k} (\mu_j - \theta)$$

Therefore the eigenvalues of A_m are the roots of the equation:

$$(1 - w f^*_{T_s}(-\theta)) = 0$$

which combined with (8) gives (13).$\bigtriangledown$

Therefore from (12) and (13) we find

$$z_i = \sum_{j=1}^{m} c_{i,j} e^{\theta_j(x)t} \qquad i = 1, \ldots, m \tag{14}$$

where $\vec{C_j} \triangleq [c_{1,j}, \ldots, c_{m,j}]^T$ is the eigenvector of matrix A_m corresponding to the eigenvalue $\theta_j(x)$. Also from (11)

$$U(\vec{z}(t)) = Ce^{tx} \tag{15}$$

Since $\vec{C_j}$ is an eigenvector of A_m its components $c_{2,j}, \ldots, c_{m,j}$ are multiples of $c_{1,j}$ and thus

$$c_{i,j} = a_{i,j} c_{1,j}, \;\; a_{1,j} \triangleq 1$$

where $a_{i,j}$ can be found explicitly. Simplifying t from the equations (14), (15) and defining $b_j \triangleq c_{1,j}/C^x$ (b_j are still undetermined) we find:

$$z_i = \sum_{j=1}^{m} b_j a_{i,j} U^{\frac{\theta_j(x)}{x}}(\vec{z}) \qquad i = 1, \ldots, m$$

We solve this system for the m coefficients b_j $(j = 1, \ldots, m)$ using Cramer's rule and we find that:

$$b_j = \frac{\begin{vmatrix} 1 & \dots & 1 & z_1 & 1 & \dots & 1 \\ a_{2,1} & \dots & a_{2,j-1} & z_2 & a_{2,j+1} & \dots & a_{2,m} \\ . & \dots & . & . & . & \dots & . \\ . & \dots & . & . & . & \dots & . \\ . & \dots & . & . & . & \dots & . \\ a_{m,1} & \dots & a_{m,j-1} & z_m & a_{m,j+1} & \dots & a_{m,m} \end{vmatrix}}{\begin{vmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ a_{2,1} & \dots & a_{2,j-1} & a_{2,j} & a_{2,j+1} & \dots & a_{2,m} \\ . & \dots & . & . & . & \dots & . \\ . & \dots & . & . & . & \dots & . \\ . & \dots & . & . & . & \dots & . \\ a_{m,1} & \dots & a_{m,j-1} & a_{m,j} & a_{m,j+1} & \dots & a_{m,m} \end{vmatrix}} U^{-\frac{\theta_j(x)}{x}}(\vec{z}) = (b_{1,j}z_1 + \ldots + b_{m,j}z_m)U^{-\frac{\theta_j(x)}{x}}(\vec{z}) \tag{16}$$

where $b_{i,j}$ can be computed analytically from $a_{i,j}$ by expanding the determinants in (16). From (16), after solving with respect to $U(\vec{z})$, we find that

$$U^{\frac{\theta_j(x)}{x}}(\vec{z}) = \frac{1}{b_j}(b_{1,j}z_1 + \ldots + b_{m,j}z_m) \qquad (j = 1, \ldots, m) \tag{17}$$

We want to find a general solution of (10), which satisfies the condition that $U(\vec{z})$ is a multivariable integer polynomial of $z_1, \ldots, z_m$ of degree s.

Raising (17) to some integer i_j and multiplying these m equations we find:

$$U^{\frac{\sum_{j=1}^{m} \theta_j(x) i_j}{x}}(\vec{z}) = K \prod_{j=1}^{m} (b_{1,j}z_1 + \ldots + b_{m,j}z_m)^{i_j} \tag{18}$$

where $K \triangleq \prod_{j=1}^{m} \frac{1}{b_j}$ is an undetermined constant.

Clearly (18) satisfies (10). In order for $U(\vec{z})$ to be a multivariable integer polynomial of $\vec{z}$ of degree s we demand:

$$\frac{\sum_{j=1}^{m} \theta_j(x) i_j}{x} = 1 \qquad \sum_{j=1}^{m} i_j = s \qquad i_j \in Z^+ \tag{19}$$

Therefore, if the above conditions hold, we have found a solution of (10) that satisfies the "polynomiality" condition. Hence the generating function $U(\vec{z})$ that corresponds to the combination $\vec{i} = (i_1, \ldots, i_m)$ is of the form :

$$U_{\vec{i}}(\vec{z}) = K_{\vec{i}} \prod_{j=1}^{m} (b_{1,j} z_1 + \ldots + b_{m,j} z_m)^{i_j} \tag{20}$$

Let us summarize what we have shown up to this point. Our goal was to find the roots of the determinantal equation (9). For this goal we have defined the auxiliary generating function $U(\vec{z})$ and we have shown that for every of the $a(s,m)$ combinations of the vector $\vec{i}$, such that $|\vec{i}| = s$ ($\sum_{j=1}^{m} i_j = s$), there exists a polynomial solution $U_{\vec{i}}(\vec{z})$ of the form (20), if there exists an $x(w_j)$ (and thus a $w_j = f^*_{T_a}(x)$) that satisfies (19). The functions $\theta_j(x)$ which appear in (19) are computed from equation (13).

In the next section we prove that for each of the $a(s,m)$ combinations of the vector $\vec{i}$, such that $|\vec{i}| = s$ there is at least one root w inside the unit cycle. Furthermore if $\rho < 1$, by using a result from the method of complex-variable-theory we prove that this root is unique and all the roots of (9) for $\rho < 1$ satisfy a nonlinear system of equations (see equations (21), (22) and (23)). Thus the use of the generating function will enable us to completely characterize the equation that each root w satisfies (note that each root w satisfies a different equation).

2.4 The basic separability theorem

From the results of the previous section we have to solve the following equation in order to find the roots w that satisfy (9):

$$\phi_{\vec{i}}(x) \triangleq i_1\theta_1(x) + i_2\theta_2(x) + \ldots + i_m\theta_m(x) = x, \quad i_1 + i_2 + \ldots + i_m = s \tag{21}$$

where $\theta_j(x) \quad (j = 1, \ldots, m)$ are the m roots of the equation:

$$f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x)) = 1 \tag{22}$$

Since $w = f^*_{T_a}(x)$ we are interested in the roots x that satisfy the following equation

$$|f^*_{T_a}(x)| < 1 \tag{23}$$

In order to investigate the number of roots of (21) we prove the following theorem.

Theorem 2

If $\rho < 1$, for every of the $a(s,m)$ combinations of $\vec{i}$, $|\vec{i}| = s$ equation (21) has at least one root x that satisfies (23).

Proof

First we prove that there are no roots x such that

$$Re\{x\} < 0$$

If $Re\{x\} < 0$ then from (21) there exists a j $(1 \leq j \leq m)$ such that $Re\{\theta_j(x)\} < 0$. Then

$$|f^*_{T_s}\{-\theta_j(x)\}| \triangleq |\int_0^\infty e^{\theta_j(x)t} f_{T_s}(t)dt| \leq \int_0^\infty |e^{\theta_j(x)t}| f_{T_s}(t)dt =$$
$$\int_0^\infty e^{Re\{\theta_j(x)\}t} f_{T_s}(t)dt < 1$$

Therefore, combining the above strict inequality and (22) we conclude that since $f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x)) = 1$ then $|f^*_{T_a}(x)| > 1$. Therefore the assumption $Re\{x\} < 0$ violates (23) and hence there are no roots x such that $Re\{x\} < 0$.

We now prove that for every combination $\vec{i}$ there exists a root x in the right half plane ($Re\{x\} \geq 0$) which satisfies the system of equations (21), (22) using the following fixed point theorem.

Fixed point theorem

Every continuous function $f(x)$ defined from a convex, bounded and closed region into itself has a fixed point x_0, i.e. there exists an x_0 such that $f(x_0) = x_0$.

We will prove that there exists an M such that $\phi_{\vec{i}}(x)$ defined in (21) has a fixed point in the region $D_M \triangleq \{x : Re\{x\} \geq 0\ ,\ |x| \leq M\}$, in other words that there

exists a root x in the right half plane.

Clearly D_M (see also figure 3) is convex, bounded and closed (compact).

Figure 3: Fixed point theorem in region D_M

Also all functions $\theta_j(x)$ defined from (22) are continuous, since they are roots of a polynomial equation, where all coefficients are continuous functions of x. Therefore $\phi_{\bar{i}}(x)$ is a continuous function, since it is a linear combination of continuous functions. In order to complete the proof that there is a fixed point it suffices to prove that there exists an M such that $\phi_{\bar{i}}(x)$ takes values in D_M. From (22), we have that for every $j = 1, \ldots, m \quad Re\{\theta_j(x)\} \geq 0$, because if $Re\{\theta_j(x)\} < 0$ then

$$|f^*_{T_s}\{-\theta_j(x)\}| \triangleq |\int_0^\infty e^{\theta_j(x)t} f_{T_s}(t)dt| < 1$$

Also

$$|f^*_{T_a}(x)| \triangleq |\int_0^\infty e^{-xt} f_{T_a}(t)dt| \leq 1$$

Then

$$|f^*_{T_a}(x)| f^*_{T_s}\{-\theta_j(x)\}| < 1$$

and therefore (22) cannot possibly be satisfied. Thus $Re\{\phi_{\bar{i}}(x)\} \geq 0$.

We now claim that there exists an M such that $|\theta_j(x)| \leq M_1 \triangleq \frac{M}{s}$.

If not, for all M there exists a y such that $|\theta_j(y)| \geq M_1$, that is $\theta_j(x)$ tends to infinity as $x \to y$. Since $\lim_{|x|\to\infty} f^*_{T_a}(x) = 0$, then from (22) in order for the product $f^*_{T_a}(x) f^*_{T_s}(-\theta_j(x))$ to be non-zero, $-\theta_j(x)$ must tend to a pole of $f^*_{T_s}(.)$. Thus

$$\lim_{|x|\to\infty} \theta_j(x) = \mu_j \quad , j = 1, \ldots, m$$

But since $\theta_j(x)$ is bounded at infinity, y is finite and

$$\lim_{x \to y} \theta_j(x) = \infty$$

which contradicts the continuity of $\theta_j(x)$. Therefore there exists an M such that $|\theta_j(x)| \leq \frac{M}{s}$ and therefore from (21) $\phi_{\vec{i}}(x) \leq M$, which proves that $\phi_{\vec{i}}(x)$ is from D_M into D_M and thus, since D_M is convex and compact, has a fixed point in D_M. Therefore we have shown that for every combination of $\vec{i}$ there exists a root of the system of equations (21) and (22). Furthermore, if $Re\{x\} \geq 0, x \neq 0$ then $|f^*_{T_a}(x)| < 1$. Yet, the solution $x = 0$ is not excluded. In fact, for $\vec{i} = (0, \ldots, 0, s)$ $x = 0$ is a solution to (21), (22). If there are two non-zero components i_k, i_l of the vector $\vec{i}$, we can easily check that $x = 0$ cannot be a solution to (21), (22).

Therefore, in order to prove the theorem, we are led to the investigation of the roots for the m combinations of $\vec{i}$ where $m-1$ components are 0 and one is equal to s. Using Rouche's theorem we prove that when $\rho < 1$ then these roots are unique and non zero. Then equation (21) becomes

$$f^*_{T_a}(x) f^*_{T_s}(-\frac{x}{s}) = 1$$

We are going to apply Rouche's theorem in the region of figure 4. Then

1. $Re\{x\} = 0 \ (x \neq 0)$

   We easily get that $|f^*_{T_a}(x)| < 1$ and $|f^*_{T_s}(-\frac{x}{s})| < 1$, from where

$$|f^*_{T_a}(x)| < \frac{1}{|f^*_{T_s}(-\frac{x}{s})|}$$

2. $Re\{x\} > 0, |x| \to \infty$

   Then $\lim_{|x| \to \infty} f^*_{T_a}(x) = 0$ and $\lim_{|x| \to \infty} f^*_{T_s}(-\frac{x}{s}) = 0$. Thus for $|x| = L$ for some big enough L and $Re\{x\} > 0$

$$|f^*_{T_a}(x)| < \frac{1}{|f^*_{T_s}(-\frac{x}{s})|}$$

3. $x \to 0^+$

Using Taylor expansion we find

$$f^*_{T_a}(x) = 1 - \frac{x}{\lambda} + o(x) \qquad f^*_{T_s}(-\frac{x}{s}) = 1 + \frac{x}{s\mu} + o(x)$$

Thus $|f^*_{T_a}(x)| = 1 - \frac{x}{\lambda} + o(x)$ and $|\frac{1}{f^*_{T_s}(-\frac{x}{s})}| = 1 - \frac{x}{s\mu} + o(x)$.
Hence $|f^*_{T_a}(x)| < |\frac{1}{f^*_{T_s}(-\frac{x}{s})}|$ if and only if $\rho \triangleq \frac{\lambda}{s\mu} < 1$

Figure 4: Rouche's theorem for $\vec{i} = (0,\ldots,0,s,0,\ldots,0)$

Since in order to apply Rouche's theorem the functions $f^*_{T_a}(x)$ and $\frac{1}{f^*_{T_s}(-\frac{x}{s})}$ should be analytic we must exclude from the region in figure 4 all the zeros of $f^*_{T_s}(-\frac{x}{s})$ (which coincide with the poles of $\frac{1}{f^*_{T_s}(-\frac{x}{s})}$), points where $\frac{1}{f^*_{T_s}(-\frac{x}{s})}$ is not analytic. But in the vicinity of the zeros of $f^*_{T_s}(-\frac{x}{s})$: $|f^*_{T_a}(x)| < |\frac{1}{f^*_{T_s}(-\frac{x}{s})}|$, i.e. the required strict inequality holds so we can apply Rouche's theorem in the region of figure 4 to find that (21) has m roots which obviously satisfy (23) since $Re\{x\} > 0$. Since we have proved that there are no roots for $Re\{x\} < 0$ we conclude that if $\rho < 1$ (21) has exactly m roots which satisfy (23), for the m combinations of $\vec{i}$ where $m-1$ components are 0 and one is equal to s. Combining the above result and the general proof that there exists a root for every combination of $\vec{i}$ we conclude that for every combination of the type $\vec{i} = (0,\ldots,0,s,0,\ldots,0)$ there exists a unique root if $\rho < 1$. As a result, we conclude that if $\rho < 1$, for every of the $a(s,m)$ combinations of $\vec{i}$, $|\vec{i}| = s$ equation (21) has at least one root x that satisfies (23). ∇

Up to this point we have established existence for every combination of $\vec{i}$ of a root x that satisfies (21), (22) and (23). Furtermore we have shown that for a particular type of combination $\vec{i}$ this root is unique, provided that $\rho < 1$. Under the condition that these roots are distinct and since there are $a(s,m)$ combinations of $i_1,\ldots,i_m$

, such that $\sum_{j=1}^{m} i_j = s$, we proved that there are at least $a(s,m)$ roots of equation (21). This condition is clearly "almost always" satisfied, in the sense the subset of distributions for which this condition does not hold has Lebesgue measure 0. We have, however, not been able to construct any example in which this condition is violated. We conjecture that this condition will always be satisfied. In fact, we were able to prove this for the special case $m = 2$. For $m = 1$ this condition holds from the well known $G/M/s$ theory (see also special case 3 in section 2.5).

We did not prove the uniqueness of the roots x in the general case using results of the present theory exclusively, but this is seen to hold by combining our result of theorem 2 and the results of Pollaczek (1961), who showed that the waiting-time distribution for the $G/C_m/s$ QS is a mixture of at most $a(s,m)$ exponential terms, which implies that there are at most $a(s,m)$ roots for (9). In Bertsimas (1986) it is shown that if there are t roots of (9) then the waiting-time distribution is a mixture of t exponential terms. Furthermore, de Smit (1983a) proved that, under some conditions, which do not seem to have any probabilistic meaning, and using an interesting matrix generalization of Rouche's theorem, if $\rho < 1$ there are $a(s,m)$ roots for the $G/H_m/s$ QS.

As a result of the above discussion we conclude, by combining our result of theorem 2 and the results of Pollaczek's and de Smit's, that there are exactly $a(s,m)$ roots of equation (9), which satisfy (23), provided that $\rho < 1$. Furthermore, we have found explicit equations that the roots w satisfy. It is remarkable that the equations for the roots w depend only on the Laplace transforms of the interarrival and service time distributions.

In order to prove the uniqueness of the roots x for every combination of $\vec{i}$, using results of the present theory exclusively, one might use Rouche's theorem, but the problem that arises is that the functions $\theta_j(x)$ defined from (22) might not be analytic. In section 2.5 we examine some special cases in which we were able to prove the uniqueness of the roots w.

Remarks:

1. Since we proved that there are $a(s,m)$ roots w that correspond to the $a(s,m)$ combinations of $\vec{i} = (i_1, ..., i_m)$ such that $\sum_{j=1}^{m} i_j = s$ we label these roots $w_j \quad (j = 1, ..., a(s,m))$. We denote by $D_{l,j}, R_{\vec{i},j}$ the coefficients corresponding to w_j . Also $U_j(\vec{z})$ denotes the generating function corresponding to w_j . Then from (20) $U_j(\vec{z})$ is equal:

$$U_j(\vec{z}) = K_j \prod_{r=1}^{m} (b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m)^{i_r} \tag{24}$$

From the definition of $U_j(\vec{z}) \triangleq \sum_{|\vec{i}|=s} R_{\vec{i},j} z_1^{i_1} \ldots z_m^{i_m}$ the coefficient of z_m^s is equal to $R_{(0,\ldots,s),j}$. From the expression (24) the coefficient of z_m^s is equal to $K_j b_{m,1}^{i_1}(w_j) \ldots b_{m,m}^{i_m}(w_j)$. Thus

$$U_j(\vec{z}) = R_{(0,\ldots,s),j} \prod_{r=1}^{m} (\frac{b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m}{b_{m,r}(w_j)})^{i_r} \tag{25}$$

2. The above analysis explains the title of section 2.4. We proved that there are $a(s,m)$ roots w_j, each of which satisfies a different equation, corresponding to the $a(s,m)$ combinations of $\vec{i}$, such that $|\vec{i}| = s$. This separability property of the equations from which the roots w_j can be computed is not only theoretically interesting, since the equations for the roots w_j involve only the Laplace transforms of the interarrival and service time distributions, but also computationally useful.

2.5 Some special cases

1. $C_n/C_m/1$.

   Since the only combinations of $\vec{i}$ for $s = 1$ are of the type $\vec{i} = (0,0,\ldots,0,1,0,\ldots,0)$ we have already proved that there are exactly $a(1,m) = m$ roots if $\rho < 1$.

If we permit complex transition rates (λ_i, μ_i) the proof is still valid, but the points where $\frac{1}{f^*_{T_s}(-x)}$ is not analytic are not necessarily on the real line anymore (see also remark 3 in section 2.6).

It is interesting for computational purposes to investigate when these m roots are real or complex. We assume first that the m poles of the service time distribution are dinstinct and there are no zeros of $f^*_{T_s}(x)$ which coincide with any pole of $f^*_{T_s}(-x)$.

If $g(x) \triangleq f^*_{T_s}(x) f^*_{T_s}(-x) - 1$, then $\lim_{x \to 0^+} g(x) = -\frac{x(1-\rho)}{\lambda\rho} < 0$ and $\lim_{x \to \infty} g(x) = -1$. Graphically $g(x)$ is presented in figure 5.

Figure 5: $g(x)$ with all m poles of $f^*_{T_s}(x)$ real

Therefore all the m roots are real. If there are zeros of $f^*_{T_s}(x)$ that coincide with poles of $f^*_{T_s}(-x)$ then there may be roots which are complex.

An interesting case is also $C_n/E_m/1$. Then $g(x)$ in this case is presented in figure 6.

Figure 6: $g(x)$ with only 1 pole of $f^*_{T_s}(x)$ real

If m is odd then there is only one real root (figure 6). If m is even then there are 2 real roots.

As a result, since the algorithmic complexity of the determination of the roots increases if the roots are complex, we can say that the algorithmic complexity increases when the service time distribution becomes more homogeneous (E_m, for example, in the sense that the rates of the stages are the same).

2. $E_k/E_m/s$.

In this case (22) becomes

$$(\frac{k\lambda}{k\lambda+x})^k(\frac{m\mu}{m\mu-\theta_j(x)})^m = 1 \Rightarrow \theta_j(x) = m\mu(1-e^{\frac{2\pi(j-1)i}{m}}(\frac{k\lambda}{k\lambda+x})^{\frac{k}{m}}) \quad j=1,...,m \tag{26}$$

Substituting (26) to (21) we get

$$sm\mu - m\mu(\frac{k\lambda}{k\lambda+x})^{\frac{k}{m}} \sum_{j=1}^{m} i_j e^{\frac{2\pi(j-1)i}{m}} = x$$

If we define $\epsilon_{\vec{i}} \triangleq \sum_{j=1}^{m} i_j e^{\frac{2\pi(j-1)i}{m}}$ $(|\epsilon_{\vec{i}}| < s)$ then we must solve the equation:

$$sm\mu - m\mu(\frac{k\lambda}{k\lambda+x})^{\frac{k}{m}}\epsilon_{\vec{i}} = x$$

Then on the boundary C_2 of D (figure 7) as $|x| \to \infty$

$$|m\mu(\frac{k\lambda}{k\lambda+x})^{\frac{k}{m}}\epsilon_{\vec{i}}| < s\ m\ \mu < |x - sm\mu|$$

Figure 7: Rouche's theorem for $E_k/E_m/s$ QS

In particular, for $x = 0$ and $\vec{i} = (s, 0, \ldots, 0)$ the above strict inequality becomes equality. Thus, in order to have the required strict inequality, so that we can apply Rouche's theorem we consider a small semicircle and use Taylor expansion to take as $x \to 0^+$ with $Re\{x\} > 0$

$$|m\mu(\frac{k\lambda}{k\lambda+x})^{\frac{k}{m}}\epsilon_{\vec{i}}| = |sm\mu(1 - \frac{x}{\lambda m} + o(x))| < |x - sm\mu| \quad if \quad \rho < 1$$

In the boundary C_1 $(x = i\alpha)$ one can verify after straightforward algebraic manipulations that the above strict inequality holds.

Thus if $\rho < 1$ we conclude from Rouche's theorem that for each $\epsilon_{\vec{i}}$ there is a unique root in D, which satisfies (23). Furthermore, since we have proved that there are no roots for $Re\{x\} < 0$ and the radius of C_2 of the closed domain D can get arbitrarily large we conclude that if $\rho < 1$ there is a unique root for every combination of $\vec{i}$ which satisfies (23). ∇

3. $C_n/M/s$.

   For $m = 1$ we find that (22) becomes

$$f^*_{T_a}(x) f^*_{T_s}(-\frac{x}{s}) = 1 \Rightarrow x = s\mu(1 - f^*_{T_a}(x))$$

   Then w is the unique real root of the equation

$$w = f^*_{T_a}(s\mu(1-w))$$

   which is the well known result from $G/M/s$ theory. At this point we remark that our result which was obtained for the class of C_n interarrival distributions is still valid for a general distribution. This observation comes to support the conjecture in section 3 that our solution is still valid even for the $G/R/s$ QS.

4. $E_n/C_2/s$.

   This QS, which was studied by Bertsimas and Papaconstantinou (1986), has the very attractive property that all the $a(s,2) = s+1$ roots are real and thus the algorithmic complexity of this QS is low. In fact an $O(s^3)$ real arithmetic algorithm was proposed.

2.6 General remarks

1. We now investigate under which conditions we can find an explicit equation for x, in the sense that (21) is an implicit equation involving the functions $\theta_j(x) \quad (j = 1, ..., m)$, which are not in general known explicitly. This property is algorithmically useful since an explicit equation for x can be easily solved by numerical means. We exploit the fundamental result in the theory of

polynomial equations. Since (21) is a polynomial equation for $\theta(x)$ (which has m roots, $\theta_1(x), ..., \theta_m(x)$), a closed form expression for $\theta_j(x)$ can only be found for $m \leq 4$. For $m \geq 5$ we cannot in general find an explicit formula. For example for $m = 2$ the $s+1$ roots $x(w_j)$ ($w_j = f^*_{T_s}(x(w_j))$) satisfy the following equation:

$$(s-2j)\sqrt{\Delta(f^*_{T_s}(x(w_j))} - s(\mu_1+\mu_2) + q_1 s \mu_1 f^*_{T_s}(x(w_j)) + 2x(w_j) = 0, \quad j = 0, ..., s$$

where

$$\Delta(y) \triangleq (\mu_2 - \mu_1 + y q_1 \mu_1)^2 + 4\mu_1\mu_2(1-q_1)y$$

2. The complexity of the problem increases extremely fast with the number of roots, which increase exponentially in s, m, when both s, m vary. For this reason, it is better to use low values for m ($m = 2, 3, 4$) to approximate a service time pdf avoiding the venture of determining exponentially many roots. In the opinion of the author, for all practical purposes the value of $m = 2$ is a tradeoff between accuracy and simplicity, which has the additional advantage of the rather unexpected real arithmetic.

3. In the attempt to investigate Cox's idea of introducing complex transition rates, to obtain complete generality in synthesizing any pdf with rational Laplace transform, we observe that the general proof that there are $a(s, m)$ roots of equation (21) did not depend on the assuption that $\mu_1, ..., \mu_m$ or $\lambda_1, ..., \lambda_m$ are real. Since, in order to have a valid pdf with complex poles we have to have $m > 2$, the simplest multi-server model involving complex transition rates is $C_3/C_3/s$ which has $\frac{(s+2)(s+1)}{2} = O(s^2)$ complex roots.

4. A very promising idea lies in the exploitation of an old and very widely used idea in the field of Electrical Engineering, namely transfer functions. By reducing the order of a transfer function (which corresponds in queueing theory

terms to the rational Laplace transform of the pdfs) we can find an excellent approximation of a large order pdf by a low order pdf. This decreases tremendously the algorithmic complexity of the problem.

2.7 The algorithm for the unsaturated probabilities

Returning to the assumed form of the probabilities $P_{n,l,\vec{i}}$, $n \geq s$, $l = 1\ldots,k$, $|\vec{i}| = s$ we observe that the most general solution must be

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} D_{l,j}\ R_{\vec{i},j}\ w_j^n \quad n \geq s,\ l = 1,\ldots,k,\ |\vec{i}| = s \tag{27}$$

where from (7)

$$D_{l,j} = D_{1,j} \prod_{r=1}^{l-1} \frac{(1-p_r)\lambda_r}{x(w_j)+\lambda_{r+1}} \quad l = 2,\ldots,k$$

and $R_{\vec{i},j}$ satisfy (6). For each j (corresponding to the root w_j) the coefficients $R_{\vec{i},j}$ satisfy a system of $a(s,m)$ linear homogeneous equations (6). Thus for a fixed j we can find the ratios $\frac{R_{\vec{i},j}}{R_{(0,\ldots,0,s),j}} \triangleq f(\vec{i}, w_j)$ recursively. Thus the unknowns are the coefficients $B_j \triangleq D_{1,j}R_{(0,\ldots,0,s),j}$, i.e. there are $a(s,m)$ unknowns. Therefore

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} B_j\ (\prod_{r=1}^{l-1} \frac{(1-p_r)\lambda_r}{x(w_j)+\lambda_{r+1}})\ f(\vec{i}, w_j)\ w_j^n \quad n \geq s,\ l = 1,\ldots,k,\ |\vec{i}| = s$$

Furthermore, from (25) we observe that the generating function of $f(\vec{i}, w_j)$ is

$$G_j(\vec{z}) = \frac{U_j(\vec{z})}{R_{(0,\ldots,s),j}} = \prod_{r=1}^{m} (\frac{b_{1,r}(w_j)z_1 + \ldots + b_{m,r}(w_j)z_m}{b_{m,r}(w_j)})^{i_r} \tag{28}$$

Up to this point the remaining unknowns are the coefficients B_j and the unsaturated probabilities $P_{n,l,\vec{i}}$, $n < s$, $l = 1,\ldots,k$, $|\vec{i}| = n$. There are two strategies for finding these unknowns.

Strategy A

1. Using the equation (2) for $n < s$ we express the unsaturated probabilities $P_{n,l,\vec{i}}$ as linear combinations of the coefficients B_j, i.e. finding recursively from (1),

(2) the coefficients $g(n,l,\vec{i},j)$ in the expansion:

$$P_{n,l,\vec{i}} = \sum_{j=1}^{a(s,m)} B_j\, g(n,l,\vec{i},j) \quad n < s,\ l = 1,\ldots,k,\ |\vec{i}| = n \tag{29}$$

Thus after this step only the coefficients $B_j(j = 1,\ldots,a(s,m))$ remain unknown.

2. Using the identities $P_{n-1,k,\vec{i}} = 0 \quad n < s, \ |\vec{i}| = n$ we find $\sum_{n=0}^{s}\binom{n+m-1}{n} = \binom{s+m}{s}$ linear homogeneous equations for B_j. Selecting $\binom{s+m-1}{s} - 1$ of them and using the normalization equation

$$\sum_{n,l,\vec{i}} P_{n,l,\vec{i}} = 1 \tag{30}$$

we find a linear non-homogeneous system of $a(s,m)$ equations with the $a(s,m)$ unknowns B_j.

Strategy B

There are $k\sum_{n=0}^{s-1}\binom{n+m-1}{n} = k\binom{s+m-1}{s-1}$ unsaturated probabilities $P_{n,l,\vec{i}}$ and $\binom{s+m-1}{s}$ unknown coefficients B_j. Using equations (1), (2) for $n < s$ we find $k\binom{s+m-1}{s-1}$ equations for $P_{n,l,\vec{i}}$. Also, the equations (1) for $l = 1$ and $n = s$ give another $\binom{s+m-1}{s}$ equations that involve the unknown quantities $P_{n,l,\vec{i}} \quad (n < s)$ and B_j. So, we have a linear homogeneous system of $k\binom{s+m-1}{s-1} + \binom{s+m-1}{s}$ equations with the same number of unknowns. Using the normalization equation (30) we find a linear non-homogeneous system, which can be solved by numerical methods.

3 The system-size probability distributions and the usual performance mearures

In this section we find closed-form expressions for the quantities P_n , $P^-_{n,\vec{i}}$, P^-_n , P^+_n (see section 1.2) for $n \geq s$. Also closed-form expressions are provided in section 3.2 for the usual performance measures.

3.1 System-size probability distributions

Concerning the distribution of N we state the following proposition.

Proposition 3

The general-time probabilities of the number of customers in the system have the following form:

$$P_n = \begin{cases} \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1}) \frac{\lambda_1 + x(w_j)}{x(w_j)} (1 - w_j) w_j^n & if \quad n \geq s \\ \sum_{j=1}^{a(s,m)} B_j \quad \sum_{l=1}^{k} \sum_{|\vec{i}|=n} g(n, l, \vec{i}, j) & if \quad n < s \end{cases} \tag{31}$$

where $G_j(\vec{1})$ is computed from (28).

Proof

In general

$$P_n = \sum_{l=1}^{k} \sum_{|\vec{i}|=\min(n,s)} P_{n,l,\vec{i}}$$

Then for $n \geq s$, using (27), we take:

$$P_n = \sum_{j=1}^{a(s,m)} (\sum_{l=1}^{k} D_{l,j}) (\sum_{|\vec{i}|=s} R_{\vec{i},j}) w_j^n$$

But $\sum_{|\vec{i}|=s} R_{\vec{i},j} = U_j(\vec{1}) = G_j(\vec{1}) R_{(0,\ldots,s),j}$ from the definition of the generating function $U_j(\vec{z})$ and using (28). Also from (7)

$$\sum_{l=1}^{k} D_{l,j} = D_{1,j} \quad \sum_{l=1}^{k} \prod_{r=1}^{l-1} \frac{(1-p_r)\lambda_r}{x(w_j) + \lambda_{r+1}} = D_{1,j} \frac{\lambda_1 + x(w_j)}{x(w_j)} (1 - w_j)$$

where we have used the identity that

$$w_j = f_{T_a}^{*}(x(w_j)) = \sum_{l=1}^{k} \frac{p_l \lambda_l}{x(w_j) + \lambda_l} \prod_{r=1}^{l-1} \frac{(1-p_r)\lambda_r}{x(w_j) + \lambda_r}$$

Therefore

$$P_n = \sum_{j=1}^{a(s,m)} D_{1,j} \frac{\lambda_1 + x(w_j)}{x(w_j)} U_j(\vec{1}) w_j^n (1 - w_j) \quad n \geq s \tag{32}$$

Using the definition of $B_j = D_{1,j} R_{(0,\ldots,0,s),j}$ (31) follows for $n \geq s$. For $n < s$, using (29), (31) follows easily. ∇

Conjecture

Although the method of stages we presented is not immediately extendable to distributions, which do not have rational Laplace transform, we believe that this separability property holds for the more general model $G/R/s$, but does not hold for $G/G/s$ where G for the service time pdf does not belong to the class R. The reason for this difference is that it is the structure of class R and its probabilistic interpretation that enable us to "separate" the equations for the roots w_j. Summarizing we conjecture that for the $G/R_m/s$ QS

$$P_n = \sum_{j=1}^{\binom{s+m-1}{s}} L_j w_j^n \quad n \geq s$$

where w_j are the $\binom{s+m-1}{s}$ roots of the following system of equations

$$\begin{aligned} & w = f_{T_a}^*(x) \quad (|w| < 1) \\ & \sum_{j=1}^{m} i_j \theta_j(x) = x \quad such \quad that \quad \sum_{j=1}^{m} i_j = s \\ & f_{T_a}^*(x) f_{T_s}^*(-\theta_i(x)) = 1 \quad (i = 1, ..., m) \nabla \end{aligned}$$

Concerning now the pre-arrival probabilities we prove the following proposition 4

Proposition 4

The pre-arrival probabilities $P_{n,\vec{i}}^-$, P_n^- and the post-departure probabilities P_n^+ for $n \geq s$ can be expressed as follows:

$$P_{n,\vec{i}}^- = \frac{1}{\lambda} \sum_{j=1}^{a(s,m)} B_j \, f(\vec{i}, w_j) \, (\lambda_1 + x(w_j)) w_j^{n+1} \quad n \geq s, \ |\vec{i}| = s \tag{33}$$

$$P_n^- = P_n^+ = \frac{1}{\lambda} \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1})(\lambda_1 + x(w_j)) w_j^{n+1} \quad n \geq s \tag{34}$$

Proof

If we define the event $AAO \triangleq$Arrival about to occur in $(t, t+\delta t)$ then we take

$$P_{n,\vec{i}}^- = Pr\{N = n, \vec{R} = \vec{i} | AAO\} = \frac{Pr\{N = n \cap \vec{R} = \vec{i} \cap AAO\}}{Pr\{AAO\}} =$$

$$= \frac{Pr\{\cup_{l=1}^{k}(N = n \cap \vec{R} = \vec{i} \cap R_a = l) \cap AAO\}}{Pr\{\cup_{l=1}^{k} R_a = l \cap AAO\}} =$$

$$\frac{\sum_{l=1}^{k} Pr\{AAO|N = n \cap \vec{R} = \vec{i} \cap R_a = l\} Pr\{N = n \cap \vec{R} = \vec{i} \cap R_a = l\}}{\sum_{l=1}^{k} Pr\{AAO|R_a = l\} Pr\{R_a = l\}} .$$

But since

$$Pr\{AAO|N = n \cap \vec{R} = \vec{i} \cap R_a = l\} = Pr\{AAO|R_a = l\} = \lambda_l p_l \delta t$$

and

$$Pr\{R_a = l\} = \frac{\frac{1}{\lambda_l} \sum_{r=l}^{k} \{p_r \prod_{m=1}^{r-1}(1 - p_m)\}}{\frac{1}{\lambda}} .$$

we take

$$P^{-}_{n,\vec{i}} = \frac{\sum_{l=1}^{k} \lambda_l p_l P_{n,l,\vec{i}}}{\sum_{l=1}^{k} \lambda_l p_l \frac{\lambda}{\lambda_l} \sum_{r=l}^{k} \{p_r \prod_{m=1}^{r-1}(1 - p_m)\}} = \frac{1}{\lambda} \sum_{l=1}^{k} \lambda_l p_l P_{n,l,\vec{i}} \tag{35}$$

since $\sum_{l=1}^{k} p_l \sum_{r=l}^{k} \{p_r \prod_{m=1}^{r-1}(1 - p_m)\} = 1$.

Therefore, using (27) and (35) we have:

$$P^{-}_{n,\vec{i}} = \frac{1}{\lambda} \sum_{j=1}^{a(s,m)} \{\sum_{l=1}^{k} \lambda_l p_l D_{l,j}\} \; R_{\vec{i},j} \; w_j^n \tag{36}$$

Also from (7)

$$\sum_{l=1}^{k} \lambda_l p_l D_{l,j} = \sum_{l=1}^{k} \lambda_l p_l D_{1,j} \prod_{r=1}^{l-1} \frac{(1 - p_r)\lambda_r}{x(w_j) + \lambda_{r+1}} = D_{1,j}(\lambda_1 + x(w_j)) w_j$$

Thus from (36), (33) follows.

Also

$$P_n^{-} = \sum_{|\vec{i}|=s} P^{-}_{n,\vec{i}} = \frac{1}{\lambda} \sum_{j=1}^{a(s,m)} B_j \; (\lambda_1 + x(w_j)) w_j^{n+1} \sum_{|\vec{i}|=s} f(\vec{i}, w_j)$$

From $\sum_{|\vec{i}|=s} f(\vec{i}, w_j) = G_j(\vec{1})$ and the general relation $P_n^{-} = P_n^{+}$ which holds for the $G/G/s$ QS (34) follows. ▽

3.2 System performance measures

1. Mean queue length

   If L_q is the length of the queue then

$$E\{L_q\} = \sum_{n=s}^{\infty} (n - s) P_n \tag{37}$$

We substitute (31) into (37) and find

$$E\{L_q\} = \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1}) \frac{\lambda_1 + x(w_j)}{x(w_j)} \frac{w_j^{s+1}}{1 - w_j} \tag{38}$$

2. Proportion of time all servers are busy, P_{busy}

$$P_{busy} = \sum_{n=s}^{\infty} P_n = \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1}) \frac{\lambda_1 + x(w_j)}{x(w_j)} w_j^s \tag{39}$$

3. Probability of non-zero waiting time

$$Pr\{T_q > 0\} = \sum_{n=s}^{\infty} P_n^- = \frac{1}{\lambda} \sum_{j=1}^{a(s,m)} B_j G_j(\vec{1})(\lambda_1 + x(w_j)) \frac{w_j^{s+1}}{1 - w_j} \tag{40}$$

4 Computational and complexity considerations

4.1 The algorithm

In order to extract numerical results from the formulae presented in sections 2, 3 we propose the following algorithm based on strategy A (see section 2.7).

1. Determination of the $\binom{s+m-1}{s}$ roots w_j of the system of equations (21), (22), (23), (8).
2. Determination of the coefficients $f(\vec{i}, w_j)$ from (6).
3. Determination of the $P_{n,l,\vec{i}}$ for $n < s$ as linear combinations of B_j.
4. Determination of the $\binom{s+m-1}{s}$ unknowns B_j as a solution of a linear system with $\binom{s+m-1}{s}$ equations.

4.2 Complexity considerations

In table I we show the computational complexity of each step of the algorithm with respect to memory and time requirements.

Table I: Computational requirements for the solution of the $C_k/C_m/s$

From table I we can make the following observations. Since from a computational point of view the "heaviest" part of the algorithm is the third step, the time complexity of this algorithm is of $O(k^3a^3(s,m))$. We consider the following cases:

1. For a fixed m, the algorithm is polynomial in the number of servers s since

$$O(k^3a^3(s,m)) = O(k^3s^{3(m-1)})$$

2. For a fixed s, the algorithm is still polynomial in m, since

$$O(k^3a^3(s,m)) = O(k^3m^{3s})$$

3. When both m, s vary then the algorithm becomes exponential, since

$$O(k^3a^3(s,m)) = O(k^3e^{s+m})$$

As a result, this algorithm has a polynomial time complexity when only one of the parameters m, s vary, but it is exponential when both parameters vary. We can also observe that this algorithm is always polynomial in k. The above results verify that the complexity of the analysis of the $C_k/C_m/s$ QS increases much faster with the service time than with the interarrival time pdf. That is, we expect for example that the derivation of numerical results for the $M/C_m/s$ QS is much harder than for the $C_m/M/s$ QS.

4.3 The numerical solution of the $C_2/C_2/s$ QS

To fully gauge the performance of the proposed algorithm we programmed it in FORTRAN, because of its inherent superiority in accuracy and speed and in BASIC because of its greater availability in microsystems. The first program has been run on a CYBER 171 and the second on a SPECTRUM 48K, in order to prove that even for such a "difficult" model exact numerical results can be obtained by a practitioner on a small personal computer.
The reasons we selected this model are :

1. It is representative of the general behavior of the algorithm for more general models.
2. It is in real arithmetic.
3. Its complexity $O(s^3)$ is not very high.
4. This model allows the determination of exact results when the coefficients of variation of the interarrival and of the service time pdf are both greater than 1.

Merely as an illustration of the stability and accuracy of the present algorithm, we present in table II a few typical results for the $C_2/C_2/15$ case with $\rho = 0.9$ and the conditions $\mu_1 = 2\mu$, $q_1 = 1 - \frac{1}{V_s^2}$, $\mu_2 = \mu_1(1-q_1)$, $\lambda_1 = 2\lambda$, $p_1 = 1 - \frac{1}{V_a^2}$, $\lambda_2 = \lambda_1(1-p_1)$ (two moment-fit; V_a^2, V_s^2 are the coefficients of variation of the interarrival and service distributions respectively).

Table II: $\mu E\{T_q\}$ as a function of V_a^2, V_s^2 for the $C_2/C_2/15$ QS

4.4 The numerical solution of the $E_k/C_2/s$ QS

This QS is solved by Bertsimas and Papaconstantinou (1986). In order to illustrate the dependence and the sensitivity of this algorithm on k we prepared another program for the analysis of the $E_k/C_2/s$ QS.

After careful and time-consuming tests of our computer programs we have produced extensive numerical results for a wide range of the parameters s, ρ, k, μ_1, μ_2 and q which complement or extend (being always in full agreement with) the results of Kuhn (1976) for the $E_{1,2}/M/s$, Sakasegawa (1978) for the $E_k/E_2/s$, Ishikawa (1979) for the $M/E_2/s$, Hillier and Yu (1981) for the $E_k/E_2/s$, Hokstad (1982) for the $M/H_2/s$, van Hoorn (1983) or Groenevelt, van Hoorn and Tijms (1984) for the $M/H_2/s$ and $M/E_{1,2}/s$ and finally of de Smit (1983b) for the $M/H_2/s$, $E_2/H_2/s$ and $E_5/H_2/s$ systems.

In table III we illustrate the dependence of the mean waiting-time $\mu\ E\{T_q\}$ (in units of service time) for the model $E_k/C_2/15$ ($\rho = 0.9$, $\mu_1 = \mu_2$ for $V_s^2 < 1$ and $q_1 = \frac{\mu_1}{\mu_1+\mu_2}$ for $V_s^2 > 1$). This selection of parameters coincides with that of Groenevelt, van Hoorn and Tijms (1983) but differs from de Smit's (1983b).

Table III: $\mu\ E\{T_q\}$ for the $E_k/C_2/15$ QS as a function of $V_a^2 = \frac{1}{k}$, V_s^2

This typical example demonstrates the well known for the $M/G/1$ QS deleterious effect of irregularity in service times. We observe that the numerical results as k increases converge quickly to the corresponding limiting (for $k \longrightarrow \infty$) $D/C_2/s$ and thus render any computations for k greater than a certain small value almost unnecessary.

Finally in figure 8 we present the computational times in CPU seconds on a CYBER 171 as functions of k and s.

Figure 8: Computational times in CPU seconds for the $E_k/C_2/s$ QS

5 Some concluding remarks

Several discussions of the major computational problems arising in the numerical solution of multi-server queues with non-exponential service time can be found in Yu (1977), in Tijms and van Hoorn (1981) and Neuts (1981).

For high s values the fact that

1. the monumental, ten-year computer-oriented project of Hillier and Yu (1981) was hinderd by severe computational feasibility limitations, and

2. the approaches of de Smit (1983b) Groenevelt, van Hoorn and Tijms (1984) lead to rather extreme levels of computational complexity,

seem to indicate that the key to success in making an intrinsically difficult QS solvable lies in the selection of a suitable solution strategy, like the one presented in this paper.

Strongly critical remarks have been made in the past both on the merits of queueing theory (see e.g. Neuts (1981), p.42 or van Hoorn (1983), p.101) and on unwarranted algorithmic claims in the applied literature on stochastic models. Being fully aware of the need for the probabilist to be very closely involved with the algorithmic analysis of a problem, we have undertaken ourselves the "painful" task of programming and testing our algorithms.

5.1 Open problems

In this paper we extended the method of stages to its complete generality solving the general class of multi-server Markovian queues. In our opinion, the major problems that are of importance in the field of theoretical and applied queueing theory are the following:

1. Find the waiting-time distribution under different queue disciplines, e.g. LCFS (Last Come First Served), SIRO (Service In Random Order). This problem

seems extremely hard, since very limited results are known. Even for $M/M/s$ the expressions for the pdf of the waiting-time of an arriving customer are of extraordinary complexity.

2. Find the steady-state probability distribution for the class of non-Markovian multi-server queues, thus settling the conjecture in section 3.1.

3. Perform busy-period analysis.

4. Perform transient analysis.

Concerning possible extensions of the present approach we believe that the following problems are tractable:

1. Extend this theory to solutions of multi-server QS where priorites are allowed.

2. Generalize this theory to the domain of transient behavior of queueing systems.

3. In the field of computational queueing theory develop a general purpose algorithm to approximate a general distribution not generally in the class R with one that belongs in this class preferably of low order. Then by exploiting the results of this paper we will have an approximate solution of the $G/G/s$ QS .

4. Introduce complex transition rates to obtain complete generality in synthesizing any pdf with rational Laplace transform.

We end this paper by expressing the hope that the solution strategy we presented might be the Ariadne's thread to a general but tractable calculus of rational distributions for multi-server queues.

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Step of algorithm	Time requirement	Memory requirement
(1)	$O(a(s,m))$	$O(a(s,m))$
(2)	$O(a^2(s,m))$	$O(a^2(s,m))$
(3)	$O(k^3\ a^3(s,m))$	$O(ka^2(s,m))$
(4)	$O(a^3(s,m))$	$O(a^2(s,m))$

Table I: Computational requirements for the solution of the $C_k/C_m/s$

$V_a^2 \backslash V_s^2$	0.5	0.8	2.0	5.0	10.0	50.0
0.5	0.1816	0.2375	0.4489	0.9498	1.7542	7.9025
0.8	0.2561	0.3133	0.5283	1.0334	1.8416	8.0838
2.0	0.5677	0.6259	0.8472	1.3662	2.1889	8.5139
5.0	1.3939	1.4490	1.6620	2.1953	3.0519	9.5035
10.0	2.8251	2.8676	3.0575	3.5764	4.4513	11.0235
50.0	14.5984	14.6074	14.6636	14.9542	15.6819	22.7259

Table II: $\mu E\{T_q\}$ as a function of V_a^2, V_s^2 for the $C_2/C_2/15$ QS

$k \backslash V_s^2$	0.5	0.8	2.0	5.0	10.0
1	0.3067	0.3647	0.5784	1.0810	1.8902
5	0.1124	0.1661	0.3685	0.8558	1.6507
10	0.0909	0.1431	0.3431	0.8279	1.6207
15	0.0839	0.1356	0.3347	0.8186	1.6107
20	0.0805	0.1319	0.3305	0.8140	1.6057

Table III: $\mu\ E\{T_q\}$ for the $E_k/C_2/15$ QS as a function of $V_a^2 = \frac{1}{k}$, V_s^2.

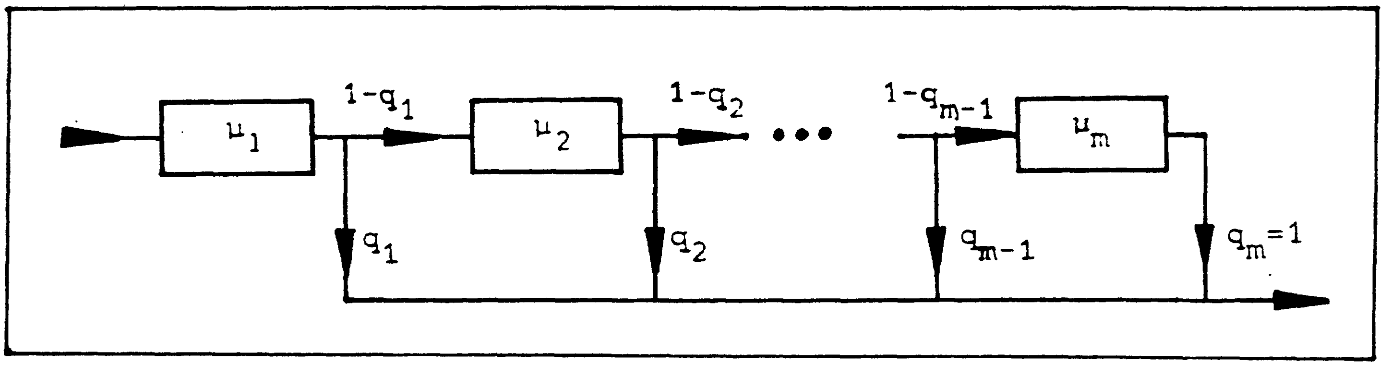

Figure 1: The C_m class of distributions

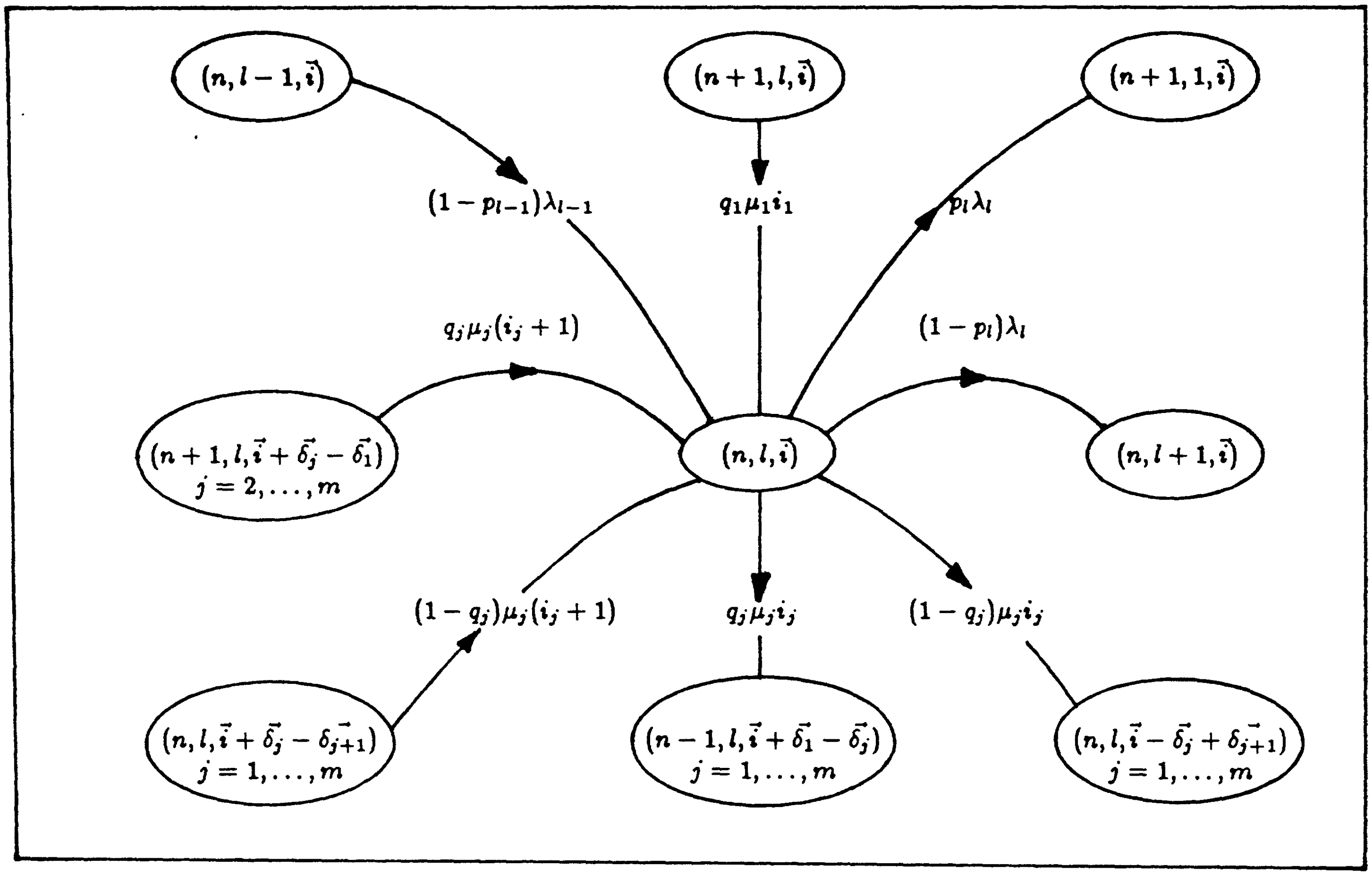

Figure 2: The state-transition diagram for $l = 2, \ldots, k$

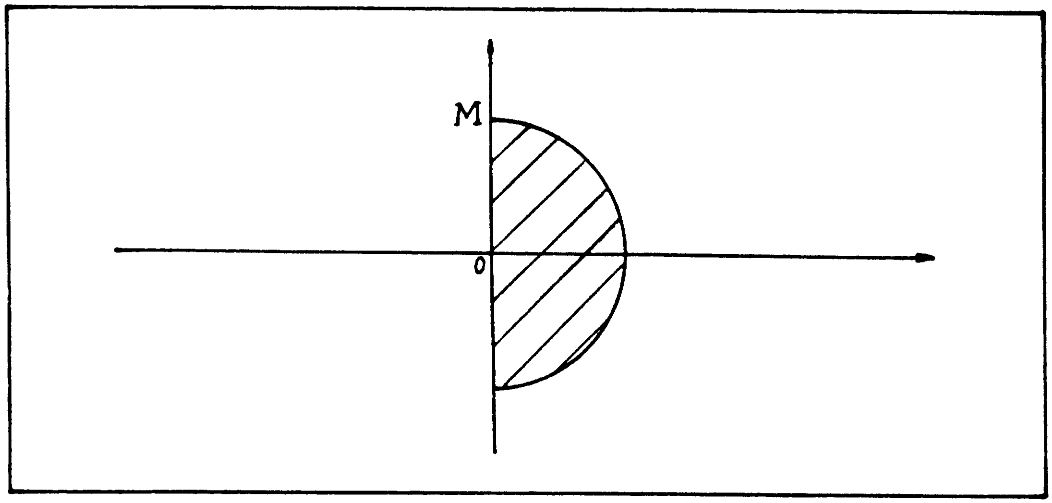

Figure 3: Fixed point theorem in region D_M

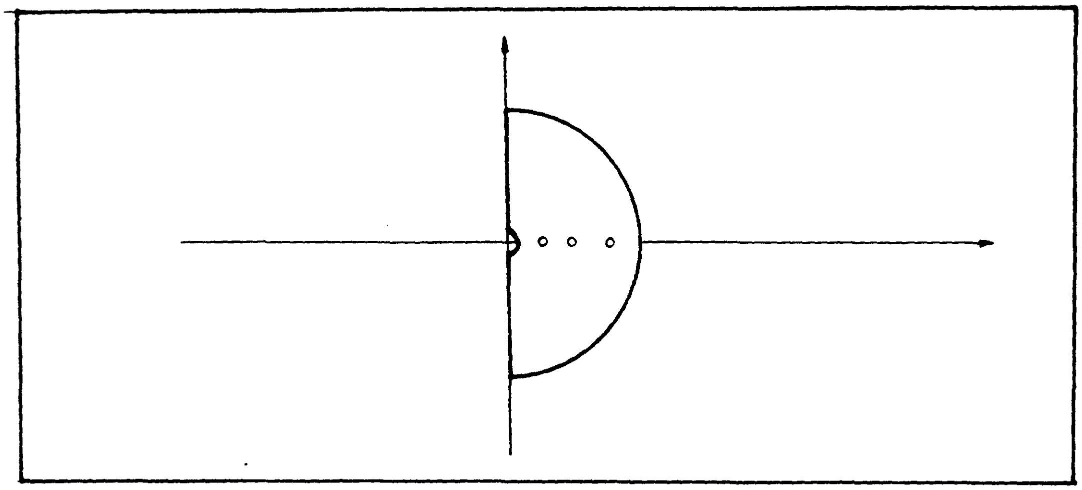

Figure 4: Rouche's theorem for $\vec{\imath} = (0, \ldots, 0, s, 0, \ldots, 0)$

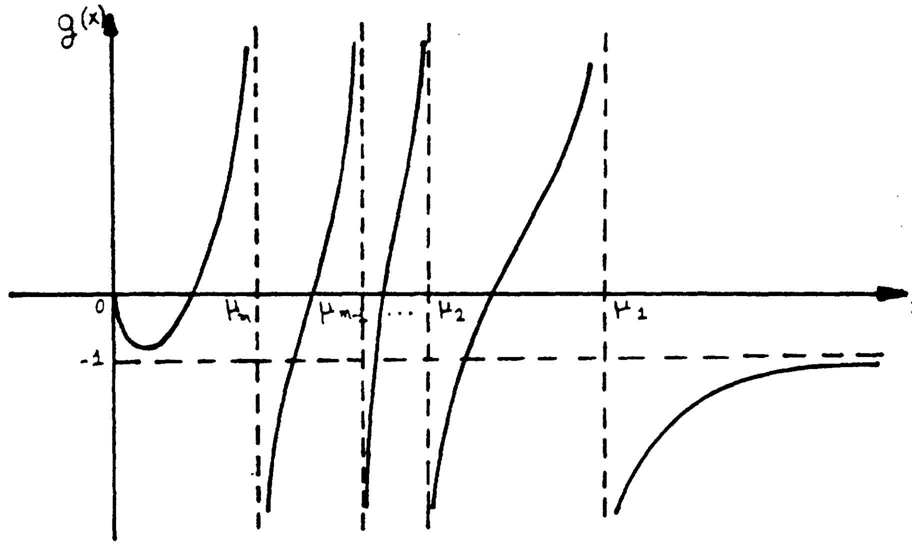

Figure 5: $g(x)$ with all m poles of $f^*_{T_s}(x)$ real

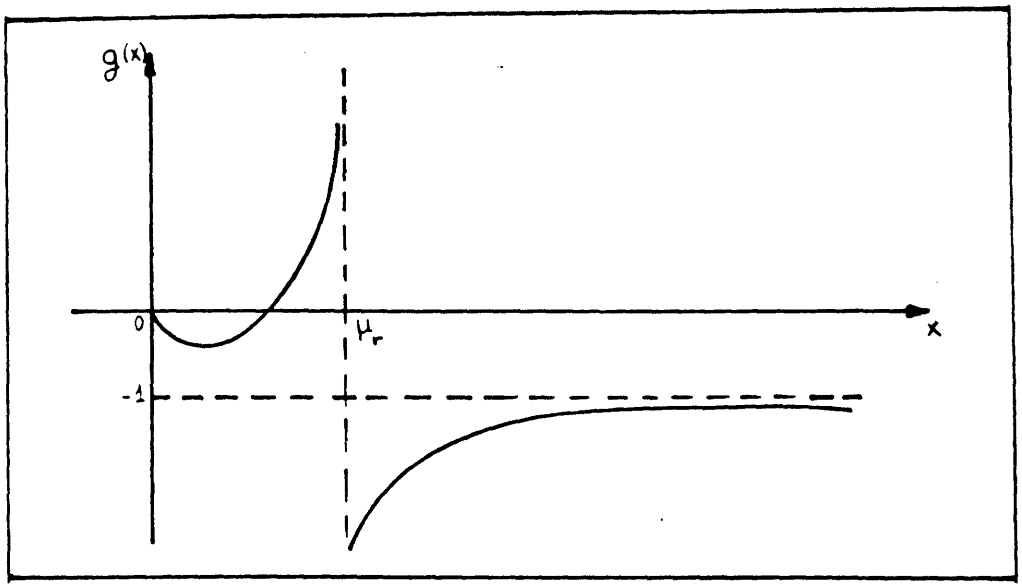

Figure 6: $g(x)$ with only 1 pole of $f^*_{T_s}(x)$ real

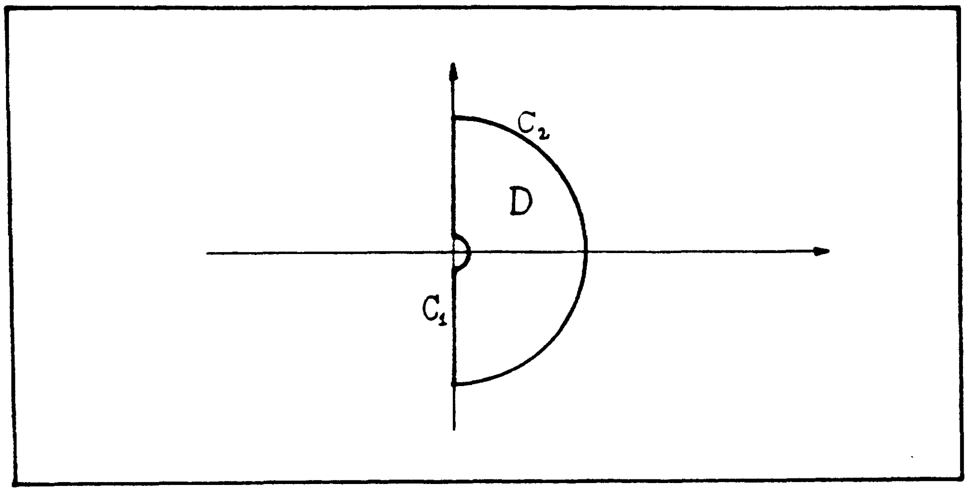

Figure 7: Rouche's theorem for $E_k/E_m/s$ QS

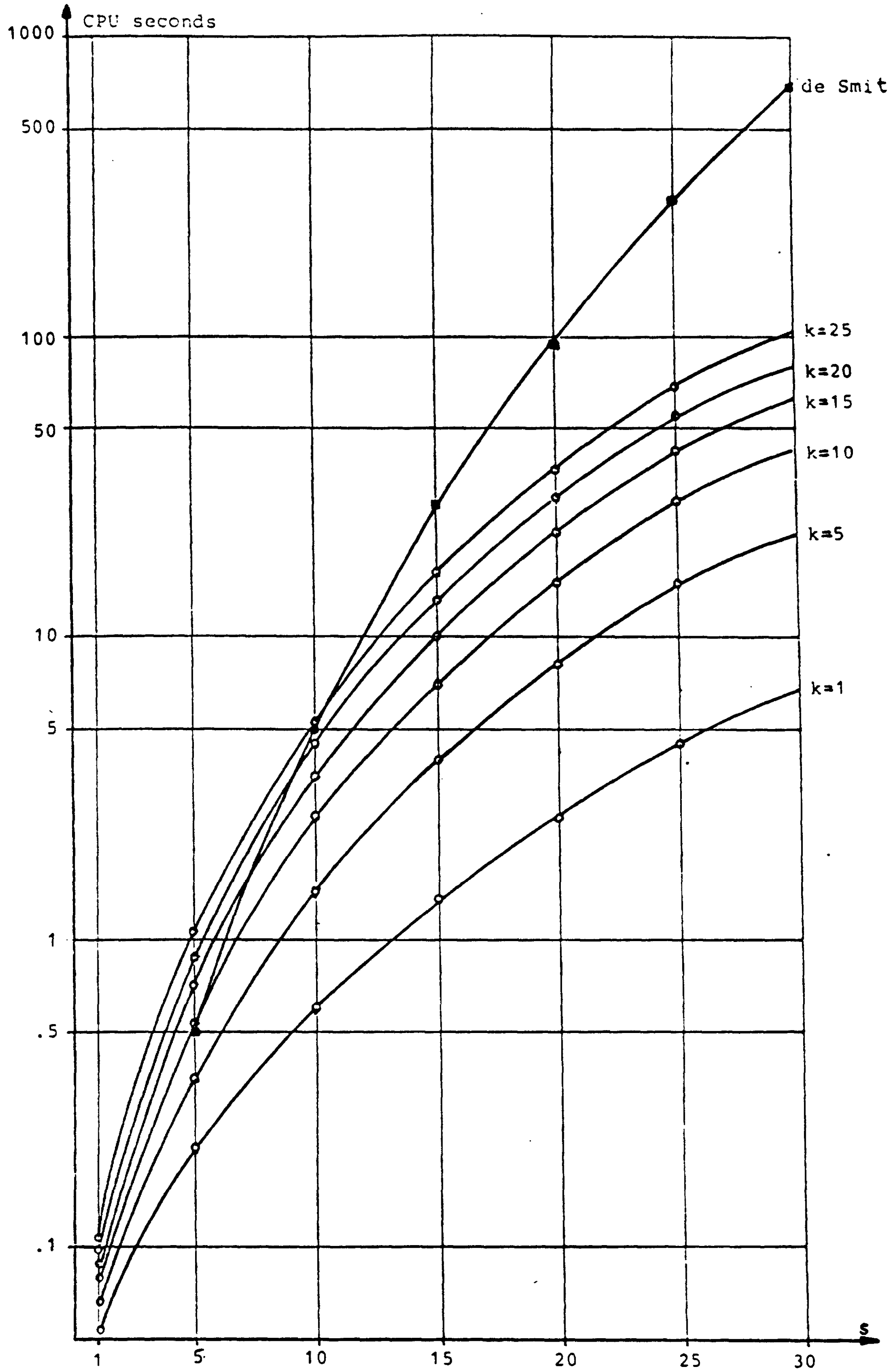

Figure 8: Computational times in CPU seconds for the $E_k/C_2/s$ QS