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An analytic characterization of symbols of operators on white noise functionals

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Introduction.

In the recent years Hida's white noise calculus [8] has been established as a Schwartz type distribution theory on Gaussian space by many authors, for instance, Kubo and Takenaka [20], Kubo and Yokoi [21], Kuo [22], Lee [23], Potthoff and Streit [30], Yan [32], Yokoi [33], see also Berezansky and Kondrat'ev [2] where a similar framework is proposed. Applications of white noise calculus to quantum physics have been discussed also actively, see e.g., [1], [6], [12], [15] and references cited therein. Meanwhile, we have started a systematic investigation of operators on white noise functionals with a new viewpoint of harmonic analysis. In [11] we formulated integral kernel operators and observed that an infinitesimal generator of rotations on white noise functionals bears interesting analogies to the finite dimensional case. In [10] we characterized Kuo's Fourier transform as a unique operator which intertwines differential operators and coordinate multiplications. Furthermore, in [28] we determined all rotation-invariant operators on white noise functionals and obtained a group-theoretical characterization of infinite dimensional Laplacians (Gross Laplacian and the number operator).

In this paper we continue a further study of operators on white noise functionals and discuss in detail how to construct such an operator from a given behavior on the exponential vectors. We are going to explain the essence to some extent. Let T be a topological space with a Borel measure ν and let $E\equiv\mathcal{S}_A(T)\subset H\equiv L^2(T, \nu)\subset E^*$ be a Gelfand triple constructed in the standard manner from an operator A on H. We think of T being a time parameter space including multi-time parameter case where quantum field theory may be formulated. Using the method of second quantization, we obtain a Gelfand triple:

$$(E) \subset (L^2) \equiv L^2(E^*, \mu) \subset (E)^*,$$

where μ is the Gaussian measure on E^*. Then, (L^2) is (a realization of) Fock space, (E) the space of test white noise functionals and $(E)^*$ the space of generalized

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white noise functionals. Let $\mathcal{L}((E), (E)^*)$ denote the space of continuous operators from (E) into $(E)^*$, which are studied in detail.

For $\xi \in E$ an exponential vector (coherent state) is by definition a function ϕ_ξ on E^* given as

$$\phi_\xi(x) = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad x \in E^*.$$

Since $\{\phi_\xi ; \xi \in E\}$ is linearly independent and spans a dense subspace of (E), it is worthwhile to study the behavior of an operator $\Xi \in \mathcal{L}((E), (E)^*)$ on the exponential vectors. We thus consider the *symbol* of Ξ:

$$\hat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \qquad \xi, \eta \in E,$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the canonical bilinear form on $(E)^* \times (E)$. This terminology was introduced by Krée and Rączka [**19**] in order to characterize a certain class of Fock space operators, see also [**4**], [**16**] and [**17**]. We are then interested in analytic properties of $\Theta = \hat{\Xi}$. In fact, it satisfies the following two properties:

(i) for any $\xi, \xi_1, \eta, \eta_1 \in E$, the function $s, t \mapsto \Theta(s\xi_1 + \xi, t\eta_1 + \eta)$, $s, t \in \boldsymbol{R}$, admits an entire holomorphic extension to $\boldsymbol{C} \times \boldsymbol{C}$;

(ii) there exist constant numbers $C \geqq 0$, $K \geqq 0$ and $p \in \boldsymbol{R}$ such that

$$|\Theta(z\xi, w\eta)| \leqq C \exp K(|z|^2 |\xi|_p^2 + |w|^2 |\eta|_p^2)$$

for all $\xi, \eta \in E$ and $z, w \in \boldsymbol{C}$, where $\Theta(z\xi, w\eta)$ denotes the entire holomorphic extension of $s, t \mapsto \Theta(s\xi, t\eta)$ to $\boldsymbol{C} \times \boldsymbol{C}$.

Here $|\xi|_p = |A^p \xi|_0$, $|\cdot|_0$ being the norm of the Hilbert space H, is one of the defining norms of E, see Sections 1 and 2 for further notations.

The bulk of this paper is devoted to construction of an operator $\Xi \in \mathcal{L}((E), (E)^*)$ from a given function Θ satisfying the conditions (i) and (ii) above. To state the main assertion we need some more notation. As is well known, the most fundamental operators on Fock space are annihilation and creation operators. In white noise calculus they are realized as a family of operators ∂_t and ∂_t^*, $t \in T$, which in fact belong to $\mathcal{L}((E), (E))$ and $\mathcal{L}((E)^*, (E)^*)$, respectively. We then consider an operator expressed in a formal integral:

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

which will be called an *integral kernel operator*. Mathematical treatment of such operators expressed in terms of creation and annihilation operators (with normal ordering) has been discussed by many authors, see e.g., Berezin [**3**] and Bogolubov et al. [**5**] where further references can be found. Among others, being based on theory of nuclear spaces, Krée [**18**] has developed a very general framework of Fock space operators which is partly similar to ours, see also a note by Meyer [27]. Nevertheless, here are some advantages of white noise approach. Namely, $\Xi_{l,m}(\kappa)$ is defined as an operator in $\mathcal{L}((E), (E)^*)$ for an arbitrary distribution $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. Moreover, the concrete structure of white noise functionals leads us to the following

MAIN THEOREM. *Assume that a* $\mathbf{C}$*-valued function* Θ *on* $E \times E$ *satisfies two conditions* (i) *and* (ii) *above. Then, there exists a unique family of kernel distributions* $\{\kappa_{l,m}\}_{l,m=0}^{\infty}$, $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ *being symmetric independently with respect to the first* l *and the last* m *variables, such that*

$$\Theta(\xi, \eta) = \sum_{l,m=0}^{\infty} \langle\langle \Xi_{l,m}(\kappa_{l,m})\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E.$$

Moreover, the series

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E),$$

converges in $(E)^*$, $\Xi \in \mathcal{L}((E), (E)^*)$ *and* $\hat{\Xi}(\xi, \eta) = \Theta(\xi, \eta)$ *for all* $\xi, \eta \in E$.

The proof will be divided into two steps. Modelled after the argument by Potthoff and Streit [30], we obtain the kernel distributions individually. Then, using a precise norm estimate of $\Xi_{l,m}(\kappa)$ obtained in [11], we prove the convergence.

The main theorem has interesting applications. We derive an operator version of the characterization theorem for generalized white noise functionals due to Potthoff and Streit [30], see Corollary 5.2. More important is that every $\Xi \in \mathcal{L}((E), (E)^*)$ admits a *Fock expansion*, i.e., it is expressible uniquely as a sum of integral kernel operators, see Theorem 6.1. While, in many practical problems (usually unbounded) operators on Fock space are defined only on the exponential vectors owing to the fact that the exponential vectors are linearly independent. Our characterization theorem, therefore, gives us a simple criterion for checking when such operators are defined on white noise functionals. In his quite recent paper [13] Huang initiated a study of quantum probability, in particular, quantum Itô formula [14] within the framework of white noise calculus. It is highly expected that our discussion will play an important role in this direction as well, see also Example 7.8.

The paper is organized as follows: In Section 1 we reformulate a well known construction of a Gelfand triple. Section 2 is devoted to a review of construction of white noise functionals. In Section 3 we define an integral kernel operator. In Section 4 we introduce a symbol of an operator and observe characteristic properties. Section 5 is devoted to the proof of the main theorem. In Section 6 we discuss Fock expansion. Finally Section 7 contains examples of Fock expansion including integral-sum kernel operators which have played an interesting role in quantum probability theory, [24], [25] and [26].

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1. Preliminaries.

We assemble some general notations used throughout.

For a real vector space $\mathfrak{X}$ we denote its complexification by $\mathfrak{X}_C$. If $\mathfrak{X}$ is a topological vector space, the dual space $\mathfrak{X}^*$ is always assumed to carry the strong dual topology. For two topological vector spaces $\mathfrak{X}$ and $\mathfrak{Y}$ let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ stand for the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are nuclear spaces, we denote simply by $\mathfrak{X}\otimes\mathfrak{Y}$ the completion of the algebraic tensor product $\mathfrak{X}\otimes_{\mathrm{alg}}\mathfrak{Y}$ with respect to the π-topology. In that case the π-topology coincides with the ε-topology, see e.g., [31]. If H and K are Hilbert spaces, we denote by $H\otimes K$ the completed Hilbert space tensor product (hence $H\otimes K$ is again a Hilbert space). We thus use the same symbol for different meanings, however, no confusion will occur. Let $\mathfrak{X}$ be a Hilbert space or a nuclear space. We denote by $\mathfrak{X}^{\hat{\otimes} n}\subset\mathfrak{X}^{\otimes n}$ the closed subspace of symmetric tensor products. We also use the symbol $(\mathfrak{X}^{\otimes n})^*_{\mathrm{sym}}$ for the same meaning in case of dual spaces.

Let H be a real separable Hilbert space with norm $|\cdot|_0$ and inner product $\langle\cdot, \cdot\rangle$. We shall be mostly concerned with a Gelfand triple $E\subset H\subset E^*$ constructed from a standard operator on H. Here an operator A is called *standard* if the domain $\mathrm{Dom}(A)\subset H$ contains a complete orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H such that

(S1) $Ae_j=\lambda_j e_j$ with $\lambda_j>0$;

(S2) $\sum_{j=0}^{\infty}\lambda_j^{-2r}<\infty$ for some $r>0$.

Given a standard operator A on H, we shall construct a Gelfand triple. For $p\in\boldsymbol{R}$ let E_p be the completion of $\mathrm{Dom}(A^p)$ with respect to the norm:

$$|\xi|_p=|A^p\xi|_0, \qquad \xi\in\mathrm{Dom}(A^p).$$

(Here we understand that $\mathrm{Dom}(A^p)=H$ for $p<0$.) We thus obtain a chain of Hilbert spaces $\{E_p\}_{p\in\boldsymbol{R}}$ with natural inclusion relations:

$$\cdots\subset E_p\subset\cdots\subset E_q\subset\cdots\subset H=E_0\subset\cdots\subset E_{-q}\subset\cdots\subset E_{-p}\subset\cdots, \quad 0\leqq q\leqq p,$$

where E_p is densely and continuously imbedded in E_q whenever $-\infty<q\leqq p<\infty$. The inner product $\langle\cdot, \cdot\rangle$ of H is naturally extended to a bilinear form on

$E_{-p}\times E_p$, $p\geqq 0$, and through this bilinear form we identify E_{-p} with the dual space E_p^*. By definition,

$$|\xi|_p^2=\sum_{j=0}^{\infty}\lambda_j^{2p}\langle\xi, e_j\rangle^2, \qquad \xi\in E_p, \qquad p\in \boldsymbol{R}. \tag{1-1}$$

PROPOSITION 1.1. *Equipped with the Hilbertian norms* $|\cdot|_p$, $E=\bigcap_{p\geq 0}E_p$ *becomes a nuclear Fréchet space. The dual space* E^* *with the strong dual topology is isomorphic to the inductive limit space:* $E^*\cong\bigcup_{p\geq 0}E_{-p}$. *Moreover,* $E\subset H\subset E^*$ *is a Gelfand triple.*

The above mentioned construction of a Gelfand triple is well known (e.g., [2], [7]) and is called *standard* in this paper. We denote the canonical bilinear form on $E^*\times E$ by $\langle\cdot, \cdot\rangle$ again.

Let Ω be a topological space with a Borel measure ν. If A is a standard operator on $H=L^2(\Omega, \nu; \boldsymbol{R})$, the Gelfand triple constructed in the standard manner is written as

$$\mathcal{S}_A(\Omega)\subset L^2(\Omega, \nu; \boldsymbol{R})\subset \mathcal{S}_A^*(\Omega). \tag{1-2}$$

By construction each $\xi\in\mathcal{S}_A(\Omega)$ is a function on Ω determined up to ν-null functions. In this connection, suggested by [20], we formulate three hypotheses:

(H1) For each $\xi\in\mathcal{S}_A(\Omega)$ there exists a unique continuous function $\tilde{\xi}$ on Ω such that $\xi(\omega)=\tilde{\xi}(\omega)$ for ν-a.e. $\omega\in\Omega$.

When (H1) is satisfied, we always identify $\mathcal{S}_A(\Omega)$ with a space of continuous functions on Ω and we do not use the symbol $\tilde{\xi}$. Under (H1) we consider two hypotheses:

(H2) For each $\omega\in\Omega$, the evaluation map $\delta_\omega:\xi\mapsto\xi(\omega)$, $\xi\in\mathcal{S}_A(\Omega)$, is continuous, namely, $\delta_\omega\in\mathcal{S}_A^*(\Omega)$.

(H3) The map $\omega\mapsto\delta_\omega\in\mathcal{S}_A^*(\Omega)$, $\omega\in\Omega$, is continuous.

We end this section with the following.

PROPOSITION 1.2. *For* $i=1, 2$ *let* Ω_i *be a topological space with a Borel measure* ν_i. *Let* A_i *be a standard operator on* $H_i=L^2(\Omega_i, \nu_i; \boldsymbol{R})$ *with domain* $\mathrm{Dom}(A_i)$. *Then* $A_1\otimes A_2$ *becomes a standard operator on* $H_1\otimes H_2$ *with domain* $\mathrm{Dom}(A_1)\otimes_{\mathrm{alg}}\mathrm{Dom}(A_2)$ *and*

$$\mathcal{S}_{A_1\otimes A_2}(\Omega_1\times\Omega_2)=\mathcal{S}_{A_1}(\Omega_1)\otimes\mathcal{S}_{A_2}(\Omega_2)$$

under the identification: $L^2(\Omega_1\times\Omega_2, \nu_1\times\nu_2; \boldsymbol{R})=L^2(\Omega_1, \nu_1; \boldsymbol{R})\otimes L^2(\Omega_2, \nu_2; \boldsymbol{R})$.

The proof is easy and omitted. It is also proved that the properties (H1)-(H3) are preserved under forming tensor products. The detailed discussion will appear elsewhere.

2. White noise functionals.

Let T be a topological space equipped with a Borel measure ν and let A be a standard operator on $H=L^2(T, \nu; \boldsymbol{R})$ satisfying the conditions:

(A1) $Ae_j=\lambda_j e_j$ with $\lambda_j \in \boldsymbol{R}$;

(A2) $1<\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$;

(A3) $\sum_{j=0}^{\infty} \lambda_j^{-2}<\infty$.

Then, applying the standard construction (Proposition 1.1), we obtain a Gelfand triple:

$$\mathcal{S}_A(T) \subset L^2(T, \nu; \boldsymbol{R}) \subset \mathcal{S}_A^*(T),$$

which is from now on denoted by

$$E \subset H \subset E^* \tag{2-1}$$

for simplicity. We further assume that $E=\mathcal{S}_A(T)$ satisfies the hypotheses (H1)-(H3) and fix this setup hereafter.

It follows from (A1) and (A3) that A^{-1} is extended to a Hilbert-Schmidt operator on H. The following two constant numbers are frequently used:

$$\delta=\|A^{-1}\|_{HS}=\left(\sum_{j=0}^{\infty} \lambda_j^{-2}\right)^{1/2}, \tag{2-2}$$

$$\rho=\|A^{-1}\|=\lambda_0^{-1}. \tag{2-3}$$

Note also

$$0<\rho<1, \qquad \rho<\delta,$$

and

$$|\xi|_p \leqq \rho|\xi|_{p+1}, \qquad \xi \in E, \qquad p \in \boldsymbol{R}.$$

The fact that $0<\rho<1$, which follows from (A2), is indispensable to our discussion. The norm of $E_p^{\otimes n}$ will be denoted also by $|\cdot|_p$. Then by definition

$$|f|_p \leqq \rho^n|f|_{p+1}, \qquad f \in E^{\otimes n}, \qquad p \in \boldsymbol{R}.$$

The canonical bilinear form on $(E^{\otimes n})^* \times E^{\otimes n}$ is denoted by $\langle\cdot, \cdot\rangle$ again.

Let μ be the Gaussian measure on E^* which is uniquely determined by the characteristic functional:

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right)=\int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E.$$

The probability space (E^*, μ) is called *Gaussian space*. The norm and the inner product of $L^2(E^*, \mu; \boldsymbol{R})$ are denoted by $\|\cdot\|_0$ and $\langle\!\langle\cdot, \cdot\rangle\!\rangle$, respectively.

We next recall the Wiener-Itô decomposition of $L^2(E^*, \mu; \boldsymbol{R})$. For that purpose we define $:x^{\otimes n}: \in (E^{\otimes n})^*$ for $x \in E^*$ and $n \geqq 0$ inductively as follows:

$$\begin{cases} :x^{\otimes 0}:=1 \\ :x^{\otimes 1}:=x \\ :x^{\otimes n}:=x\hat{\otimes}:x^{\otimes(n-1)}:-(n-1)\tau\hat{\otimes}:x^{\otimes(n-2)}:, & n\geqq 2, \end{cases}$$

where $\tau\in(E\otimes E)^*$ is defined by

(2-4) $$\langle\tau,\ \xi\otimes\eta\rangle=\langle\xi,\ \eta\rangle,\qquad \xi,\ \eta\in E.$$

By definition $:x^{\otimes n}:\in(E^{\otimes n})^*_{\mathrm{sym}}$. With these notation we come to a variant statement of Wiener-Itô decomposition theorem.

PROPOSITION 2.1. *For each $\phi\in L^2(E^*,\ \mu;\ \boldsymbol{R})$ there exists a sequence $f_n\in H^{\hat{\otimes}n}$, $n=0,\ 1,\ 2,\ \cdots$, such that*

(2-5) $$\phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ f_n\rangle,\qquad x\in E^*,$$

where the right hand side is an orthogonal direct sum of functions in $L^2(E^*,\ \mu;\ \boldsymbol{R})$. Moreover,

$$\|\phi\|_0^2=\sum_{n=0}^{\infty}n!\ |f_n|_0^2.$$

We then need the second quantized operator $\Gamma(A)$, where A is the operator used for construction of $E=\mathcal{S}_A(T)$. Let $\mathrm{Dom}(\Gamma(A))$ be the subspace of $\phi\in L^2(E^*,\ \mu;\ \boldsymbol{R})$ given as in (2-5) such that

(i) $f_n=0$ except finitely many n;

(ii) $f_n\in\mathrm{Dom}(A)\otimes_{\mathrm{alg}}\cdots\otimes_{\mathrm{alg}}\mathrm{Dom}(A)$ (n-times).

Then, for such $\phi\in\mathrm{Dom}(\Gamma(A))$ we define

$$(\Gamma(A)\phi)(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ A^{\otimes n}f_n\rangle.$$

LEMMA 2.2. *$\Gamma(A)$ is a standard operator on $L^2(E^*,\ \mu;\ \boldsymbol{R})$.*

PROOF. It is known that

$$\phi_{\boldsymbol{n}}(x)=(n_0!n_1!\cdots)^{-1/2}\langle :x^{\otimes n}:,\ e_0^{\otimes n_0}\hat{\otimes}e_1^{\otimes n_1}\hat{\otimes}\cdots\rangle,$$

where $\boldsymbol{n}=(n_0,\ n_1,\ \cdots)$, $|\boldsymbol{n}|=n_0+n_1+\cdots=n<\infty$, form a complete orthonormal basis for $L^2(E^*,\ \mu;\ \boldsymbol{R})$. Obviously, these are eigenfunctions of $\Gamma(A)$ with eigenvalues $\lambda_0^{n_0}\lambda_1^{n_1}\cdots$. On the other hand, it follows from (A3) that

$$\sum_{n=0}^{\infty}\sum_{n_0+n_1+\cdots=n}(\lambda_0^{n_0}\lambda_1^{n_1}\cdots)^{-2}=\prod_{j=0}^{\infty}\sum_{n_j=0}^{\infty}\lambda_j^{-2n_j}=\prod_{j=0}^{\infty}(1-\lambda_j^{-2})^{-1}<\infty,$$

and consequently, $\Gamma(A)$ is standard. (Q.E.D.)

Then application of the standard construction leads us to a Gelfand triple:

$$\mathcal{S}_{\Gamma(A)}(E^*) \subset L^2(E^*, \mu; \boldsymbol{R}) \subset \mathcal{S}^*_{\Gamma(A)}(E^*).$$

Its complexification is denoted by

$$(E) \subset (L^2) \subset (E)^*. \tag{2-6}$$

An element of (E) (resp. $(E)^*$) is called a *test* (resp. *generalized*) *white noise functional*. It will be crucial to our discussion in Section 4 that (E) is a nuclear Fréchet space. The canonical $\boldsymbol{C}$-bilinear form on $(E)^*\times(E)$ is also denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the norm of (L^2) is denoted by $\|\cdot\|_0$. According to our convention we put

$$\|\phi\|_p = \|\Gamma(A)^p\phi\|_0, \qquad \phi \in \mathrm{Dom}(\Gamma(A)^p), \qquad p \in \boldsymbol{R}.$$

Then, by definition $\mathcal{S}_{\Gamma(A)}(E^*)$ (and therefore (E) as well) is equipped with the Hilbertian norms $\|\cdot\|_p$, $p\geqq 0$. A simple application of Kubo-Yokoi's continuous version theorem [**21**] yields the following

PROPOSITION 2.3. *According to Proposition* 2.1 *let* $\phi\in(L^2)$ *be given as*

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \quad x \in E^*, \quad f_n \in H_{\boldsymbol{C}}^{\hat{\otimes} n}, \quad n = 0, 1, 2, \cdots. \tag{2-7}$$

Then $\phi\in(E)$ *if and only if*

(i) $f_n\in E_{\boldsymbol{C}}^{\hat{\otimes} n}$ *for all* $n=0, 1, 2, \cdots$;

(ii) $\sum_{n=0}^{\infty} n!|f_n|_p^2<\infty$ *for all* $p\geqq 0$.

In that case,

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2$$

for all $p\geqq 0$ *and, for each* $x\in E^*$ *the right hand side of* (2-7) *converges absolutely and defines a continuous function on* E^* *which coincides with* $\phi(x)$ *for* μ-*a.e.* $x\in E^*$.

In other words, $\mathcal{S}_{\Gamma(A)}(E^*)$ (and therefore (E) as well) satisfies the hypothesis (H1) introduced in Section 1. According to our convention there (E) is regarded as a space of continuous functions on E^* and we agree that every $\phi\in(E)$ is pointwisely defined by an absolutely convergent series as in (2-7). It is proved in [**21**] that $\mathcal{S}_{\Gamma(A)}(E^*)$ satisfies (H2). Moreover, (H3) can be proved with further detailed consideration.

For generalized functionals we only mention the following

PROPOSITION 2.4. *For* $\Phi\in(E)^*$ *there exists a unique sequence* $\{F_n\}_{n=0}^{\infty}$ *satisfying*

(i) $F_n\in(E_{\boldsymbol{C}}^{\otimes n})^*_{\mathrm{sym}}$ *for all* $n=0, 1, 2, \cdots$;

(ii) $\sum_{n=0}^{\infty} n!|F_n|_{-p}^2<\infty$ *for some* $p\geqq 0$;

such that

(2-8) $$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle,$$

for all $\phi\in(E)$ *given as in* (2-7). *Conversely, if* $\{F_n\}_{n=0}^{\infty}$ *satisfies* (i) *and* (ii), *then the formula* (2-8) *defines a generalized functional* $\Phi\in(E)^*$.

It is therefore convenient to adopt a formal notation:

(2-9) $$\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, F_n\rangle.$$

However, we do not go into a discussion about convergence of the right hand side.

3. Integral kernel operators.

We begin with recalling a differential operator ∂_t which plays a fundamental role in the white noise calculus. For $t\in T$ and $f\in E_{\mathbf{C}}^{\hat{\otimes}(n+1)}$ we define $\delta_t\hat{\otimes}_1 f\in E_{\mathbf{C}}^{\hat{\otimes}n}$ by

$$\delta_t\hat{\otimes}_1 f(t_1, \cdots, t_n) = f(t, t_1, \cdots, t_n), \qquad t_1, \cdots, t_n \in T.$$

According to Proposition 2.3 let $\phi\in(E)$ be given as

(3-1) $$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \quad x\in E^*, \quad f_n\in E_{\mathbf{C}}^{\hat{\otimes}n}, \quad n=0, 1, 2, \cdots.$$

For $y\in E^*$ put

(3-2) $$D_y\phi(x) = \sum_{n=1}^{\infty} n\langle :x^{\otimes(n-1)}:, y\hat{\otimes}_1 f_n\rangle, \qquad x\in E^*.$$

It then follows from Proposition 2.3 that $D_y\phi\in(E)$ and $D_y\in\mathcal{L}((E),(E)^*)$. Moreover, by a direct calculation we have

$$\|D_y\phi\|_p \leqq \left(\frac{\rho^{-2q}}{-2qe\log\rho}\right)^{1/2} |y|_{-(p+q)}\|\phi\|_{p+q}, \qquad \phi\in(E),$$

for any $p\geqq 0$ and $q>0$ with $|y|_{-(p+q)}<\infty$. It is also noted that D_y is a derivation, namely,

$$D_y(\phi\psi) = (D_y\phi)\psi + \phi(D_y\psi), \qquad \phi, \psi\in(E).$$

In fact, it is known that

(3-3) $$D_y\phi(x) = \lim_{\theta\to 0}\frac{\phi(x+\theta y)-\phi(x)}{\theta}, \quad x\in E^*, \quad \phi\in(E).$$

The operator defined by

$$\partial_t = D_{\delta_t}, \qquad t\in T,$$

where $\delta_t\in E^*$ denotes the Dirac function at $t\in T$ (see also (H2)), is called *Hida's* *differential operator.*

We introduce an operator in $\mathcal{L}((E),(E)^*)$ which is expressed in a formal integral:

(3-4) $$\int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m)\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. The rigorous definition is outlined below, for detailed proofs see [11]. For $\phi, \psi \in (E)$ we put

(3-5) $$\eta_{\phi,\psi}(s_1, \cdots, s_l, t_1, \cdots, t_m) = \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \phi, \psi \rangle\rangle.$$

Then $\eta_{\phi,\psi}$ becomes a function on T^{l+m} satisfying

(3-6) $$|\eta_{\phi,\psi}|_p \leq \rho^{-p}(l^l m^m)^{1/2}\left(\frac{\rho^{-p}}{-2pe\log\rho}\right)^{(l+m)/2} \|\phi\|_p \|\psi\|_p, \quad p > 0.$$

In particular, we observe that $\eta_{\phi,\psi} \in E_{\mathbf{C}}^{\otimes(l+m)}$. Hence, for any $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ there exists a continuous linear operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

(3-7) $$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\rangle = \langle \kappa, \eta_{\phi,\psi} \rangle, \qquad \phi, \psi \in (E).$$

Moreover, for any $p>0$ with $|\kappa|_{-p}<\infty$ it holds that

(3-8) $$\|\Xi_{l,m}(\kappa)\phi\|_{-p} \leq \rho^{-p}(l^l m^m)^{1/2}\left(\frac{\rho^{-p}}{-2pe\log\rho}\right)^{(l+m)/2} |\kappa|_{-p} \|\phi\|_p.$$

In view of (3-7) we also employ a formal integral expression as in (3-4) for $\Xi_{l,m}(\kappa)$. Such an operator is called an *integral kernel operator* with *kernel distribution* κ.

Since $[\partial_s, \partial_t]=0$ for $s, t \in T$, it is natural to consider "partially symmetrized" kernel distributions. For $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ we define $s_{l,m}(\kappa) \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ by

$$\langle s_{l,m}(\kappa), \xi_1 \otimes \cdots \otimes \xi_l \otimes \eta_1 \otimes \cdots \otimes \eta_m \rangle = \frac{1}{l!m!} \sum_{\substack{\sigma \in \mathfrak{S}_l \\ \tau \in \mathfrak{S}_m}} \langle \kappa, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(l)} \otimes \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(m)} \rangle,$$

where $\xi_1, \cdots, \xi_l, \eta_1, \cdots, \eta_m \in E$. We put

$$(E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)} = \{\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^* ;\ s_{l,m}(\kappa)=\kappa\}.$$

PROPOSITION 3.1. *Let* $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. *Then* $\Xi_{l,m}(\kappa)=\Xi_{l,m}(s_{l,m}(\kappa))$. *Moreover,* $\Xi_{l,m}(\kappa)=0$ *if and only if* $s_{l,m}(\kappa)=0$.

For operators in $\mathcal{L}((E),(E))$, which is a subclass of $\mathcal{L}((E),(E)^*)$, we only mention the following result proved in [11].

PROPOSITION 3.2. *Let* $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. *Then* $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E))$ *if and only if* $\kappa \in (E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*$.

Here we note that $\mathcal{L}(E_C^{\otimes m}, E_C^{\otimes l}) \cong (E_C^{\otimes l}) \otimes (E_C^{\otimes m})^*$, see e.g., [31]. For example, $\tau \in (E \otimes E)^*$ defined as in (2-4) belongs to $E \otimes E^*$ and the corresponding operator is the identity in $\mathcal{L}(E, E)$.

4. Symbol of an operator on white noise functionals.

First we recall exponential vectors (coherent states). For $\xi \in E_C$ an *exponential vector* ϕ_ξ is a function on E^* defined by

$$\phi_\xi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad x \in E^* . \tag{4-1}$$

As is easily checked, $\phi_\xi \in (E)$ and

$$\|\phi_\xi\|_p = \exp\left(\frac{1}{2}|\xi|_p^2\right), \qquad p \in \boldsymbol{R} . \tag{4-2}$$

We then define the *symbol* of $\Xi \in \mathcal{L}((E), (E)^*)$ as a function on $E_C \times E_C$ given by

$$\hat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \qquad \xi, \eta \in E_C . \tag{4-3}$$

This definition is suggested by Berezin [4] and Krée and Rączka [19]. The symbol of an integral kernel operator is given in the following

LEMMA 4.1. *For* $\kappa \in (E_C^{\otimes(l+m)})^*$,

$$\widehat{\Xi_{l,m}(\kappa)}(\xi, \eta) = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_C . \tag{4-4}$$

PROOF. The action of $\Xi_{l,m}(\kappa)$ on exponential vectors is described explicitly. Namely, for $\xi \in E_C$ it holds that

$$\Xi_{l,m}(\kappa)\phi_\xi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes(n+l)}:, (\kappa \otimes_m \xi^{\otimes m}) \otimes \xi^{\otimes n} \rangle ,$$

where $\otimes_m$ is a contraction of tensor products determined as a unique extension of

$$\eta^{\otimes(l+m)} \otimes_m \xi^{\otimes m} = \langle \eta, \xi \rangle^m \eta^{\otimes l}, \qquad \xi, \eta \in E_C .$$

Then (4-4) follows by a simple calculation. (Q.E.D.)

For the uniqueness of the symbol we have the following

LEMMA 4.2. *Let* $\alpha, \beta \in \boldsymbol{C}$ *be non-zero and let* $\Xi \in \mathcal{L}((E), (E)^*)$. *If* $\hat{\Xi}(\alpha\xi, \beta\eta) = 0$ *for all* $\xi, \eta \in E$, *then* $\Xi = 0$.

PROOF. We need only to note that $\{\phi_{\alpha\xi} ; \xi \in E\}$ spans a dense subspace of (E) for any non-zero complex number $\alpha \in \boldsymbol{C}$. (Q.E.D.)

We next prepare the following

LEMMA 4.3. *For each* $\Xi \in \mathcal{L}((E), (E)^*)$ *there exist* $C \geqq 0$ *and* $p \geqq 0$ *such that*

$$\|\Xi\phi\|_{-p} \leqq C\|\phi\|_p, \qquad \phi \in (E).$$

PROOF. Since (E) is a nuclear space, it follows from the kernel theorem that

$$\mathcal{L}((E), (E)^*) \cong ((E) \otimes (E))^* .$$

Namely, for a given $\Xi \in \mathcal{L}((E), (E)^*)$ there exists $\Omega \in ((E) \otimes (E))^*$ such that

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \langle\langle \Omega, \phi \otimes \psi \rangle\rangle, \qquad \phi, \psi \in (E). \tag{4-5}$$

On the other hand, it follows from Propositions 1.1 and 1.2 that

$$(\mathcal{S}_{\Gamma(A)}(E^*) \otimes \mathcal{S}_{\Gamma(A)}(E^*))^* = \mathcal{S}^*_{\Gamma(A) \otimes \Gamma(A)}(E^* \times E^*) = \bigcup_{p \geq 0} \mathcal{S}_{-p}(E^* \times E^*),$$

where $\mathcal{S}_{-p}(E^* \times E^*)$ is the completion of $L^2(E^* \times E^*, \mu \times \mu; \boldsymbol{R})$ with respect to the norm

$$\|\Omega\|_{-p} = \|(\Gamma(A) \otimes \Gamma(A))^{-p}\Omega\|_0, \qquad \Omega \in L^2(E^* \times E^*, \mu \times \mu; \boldsymbol{R}).$$

Since (E) is simply the complexification of $\mathcal{S}_{\Gamma(A)}(E^*)$, we have

$$((E) \otimes (E))^* = \bigcup_{p \geq 0} \mathcal{S}_{-p}(E^* \times E^*)_C .$$

Therefore, for each $\Omega \in ((E) \otimes (E))^*$ there exists $p \geqq 0$ such that $\|\Omega\|_{-p} < \infty$. With this $p \geqq 0$, (4-5) is estimated as follows:

$$|\langle\langle \Xi\phi, \psi \rangle\rangle| = |\langle\langle \Omega, \phi \otimes \psi \rangle\rangle| \leqq \|\Omega\|_{-p} \|\phi \otimes \psi\|_p = \|\Omega\|_{-p} \|\phi\|_p \|\psi\|_p .$$

Consequently,

$$\|\Xi\phi\|_{-p} \leqq \|\Omega\|_{-p} \|\phi\|_p, \qquad \phi \in (E),$$

which proves the assertion. (Q.E.D.)

The *S-transform* of $\Phi \in (E)^*$ is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in E_C .$$

This definition is equivalent to the original one due to Kubo and Takenaka [**20**].

LEMMA 4.4. *Let* $\Phi \in (E)^*$ *and* $\xi, \xi_1 \in E_C$. *Then, the function*

$$z \longmapsto S\Phi(z\xi_1 + \xi) = \langle\langle \Phi, \phi_{z\xi_1 + \xi} \rangle\rangle, \qquad z \in \boldsymbol{C},$$

is an entire holomorphic function on $\boldsymbol{C}$.

The proof is straightforward, see also [**30**]. With these results we may prove important properties of the symbol of an operator $\Xi \in \mathcal{L}((E), (E)^*)$.

THEOREM 4.5. *Let* $\Xi\in\mathcal{L}((E),(E)^*)$. *Then:*

(1) *For any* $\xi, \xi_1, \eta, \eta_1\in E_C$, *the function* $z, w\mapsto\hat{\Xi}(z\xi_1+\xi, w\eta_1+\eta)$, $z, w\in \boldsymbol{C}$, *is an entire holomorphic function on* $\boldsymbol{C}\times\boldsymbol{C}$.

(2) *There exist constant numbers* $C\geqq 0$, $K\geqq 0$ *and* $p\in\boldsymbol{R}$ *such that*

$$|\hat{\Xi}(z\xi, w\eta)|\leqq C\exp K(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2)$$

for all $\xi, \eta\in E_C$ *and* $z, w\in\boldsymbol{C}$.

PROOF. (1) Since the symbol is expressed in terms of S-transform:

$$\hat{\Xi}(\xi, \eta)=S(\Xi\phi_\xi)(\eta)=S(\Xi^*\phi_\eta)(\xi), \qquad \xi, \eta\in E_C,$$

the assertion is an immediate consequence of Lemma 4.4.

(2) In view of Lemma 4.3 we take $C\geqq 0$ and $p\geqq 0$ such that

$$\|\Xi\phi\|_{-p}\leqq C\|\phi\|_p, \qquad \phi\in(E).$$

Then, taking (4-2) into account, we obtain

$$\begin{aligned}|\hat{\Xi}(z\xi, w\eta)| &= |\langle\langle\Xi\phi_{z\xi}, \phi_{w\eta}\rangle\rangle| \\ &\leqq \|\Xi\phi_{z\xi}\|_{-p}\|\phi_{w\eta}\|_p \\ &\leqq C\|\phi_{z\xi}\|_p\|\phi_{w\eta}\|_p \\ &= C\exp\frac{1}{2}(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2),\end{aligned}$$

as desired. (Q.E.D.)

5. A characterization of symbols.

More important is that the converse of Theorem 4.5 is also true. Keeping some applications in mind, we prove the following

THEOREM 5.1. *Assume that a* $\boldsymbol{C}$-*valued function* Θ *on* $E\times E$ *satisfies the following two conditions:*

(i) *for any* $\xi, \xi_1, \eta, \eta_1\in E$, *the function* $s, t\mapsto\Theta(s\xi_1+\xi, t\eta_1+\eta)$, $s, t\in\boldsymbol{R}$, *admits an entire holomorphic extension to* $\boldsymbol{C}\times\boldsymbol{C}$;

(ii) *there exist constant numbers* $C\geqq 0$, $K\geqq 0$ *and* $p\in\boldsymbol{R}$ *such that*

$$|\Theta(z\xi, w\eta)|\leqq C\exp K(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2)$$

for all $\xi, \eta\in E$ *and* $z, w\in\boldsymbol{C}$, *where* $\Theta(z\xi, w\eta)$ *denotes the entire holomorphic extension of* $s, t\mapsto\Theta(s\xi, t\eta)$ *to* $\boldsymbol{C}\times\boldsymbol{C}$.

Then, there exists a unique family of kernel distributions $\kappa_{l,m}\in(E_C^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, $l, m=0, 1, 2, \cdots$, *such that*

$$\Theta(\xi, \eta)=\sum_{l, m=0}^{\infty}\langle\!\langle \Xi_{l, m}(\kappa_{l, m})\phi_{\xi}, \phi_{\eta}\rangle\!\rangle, \qquad \xi, \eta \in E.$$

Moreover, the series

$$\Xi\phi=\sum_{l, m=0}^{\infty} \Xi_{l, m}(\kappa_{l, m})\phi, \qquad \phi \in (E),$$

converges in $(E)^*$, $\Xi \in \mathcal{L}((E), (E)^*)$ *and* $\hat{\Xi}(\xi, \eta)=\Theta(\xi, \eta)$ *for all* $\xi, \eta \in E$.

PROOF. We put

$$\Psi(\xi, \eta)=e^{-\langle\xi, \eta\rangle}\Theta(\xi, \eta), \qquad \xi, \eta \in E. \tag{5-1}$$

By assumption (i), for fixed $\xi, \eta \in E$ the function $s, t \mapsto \Psi(s\xi, t\eta)$, $s, t \in \boldsymbol{R}$, admits an entire holomorphic extension which we denote simply by $\Psi(z\xi, w\eta)$, $z, w \in \boldsymbol{C}$. In order to obtain the Taylor expansion we introduce differential operators:

$$D_{\xi_1}^{(1)}\Psi(\xi, \eta)=\frac{d}{dz}\Big|_{z=0}\Psi(z\xi_1+\xi, \eta),$$

$$D_{\eta_1}^{(2)}\Psi(\xi, \eta)=\frac{d}{dw}\Big|_{w=0}\Psi(\xi, w\eta_1+\eta).$$

We then put

$$\kappa_{l, m}(\eta_1, \cdots, \eta_l, \xi_1, \cdots, \xi_m)=\frac{1}{l!m!}D_{\xi_1}^{(1)}\cdots D_{\xi_m}^{(1)}D_{\eta_1}^{(2)}\cdots D_{\eta_l}^{(2)}\Psi(0, 0) \tag{5-2}$$

and

$$A_{l, m}(\eta, \xi)=\kappa_{l, m}(\underbrace{\eta, \cdots, \eta}_{l\text{ times}}, \underbrace{\xi, \cdots, \xi}_{m\text{ times}}).$$

It follows from the differentiability of Ψ that $\kappa_{l, m}$ becomes an $(l+m)$-linear form on E. The Taylor expansion of $\Psi(z\xi, w\eta)$ is thus given by

$$\Psi(z\xi, w\eta)=\sum_{l, m=0}^{\infty}\frac{\partial^{l+m}}{\partial z^m \partial w^l}\Psi(z\xi, w\eta)\Big|_{z=w=0}\frac{z^m w^l}{m!l!} \tag{5-3}$$

$$=\sum_{l, m=0}^{\infty}A_{l, m}(\eta, \xi)z^m w^l.$$

While, with the help of the Cauchy integral formula we obtain

$$A_{l, m}(\eta, \xi)=\left(\frac{1}{2\pi i}\right)^2\int_{|z|=R_1}\int_{|w|=R_2}\frac{\Psi(z\xi, w\eta)}{z^{m+1}w^{l+1}}dzdw, \qquad R_1, R_2>0. \tag{5-4}$$

Since for $\xi, \eta \in E_C$ and for $z, w \in \boldsymbol{C}$

$$|zw\langle\xi, \eta\rangle| \leqq |z||\xi|_0|w||\eta|_0$$
$$\leqq \frac{1}{2}(|z|^2|\xi|_0^2+|w|^2|\eta|_0^2)$$
$$\leqq \frac{\rho^{2p}}{2}(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2),$$

we obtain

$$(5\text{-}5)\qquad |\exp(-zw\langle\xi,\ \eta\rangle)| \leqq \exp\frac{\rho^{2p}}{2}(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2).$$

On the other hand, by assumption (ii)

$$(5\text{-}6)\qquad |\Theta(z\xi,\ w\eta)| \leqq C\exp K(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2).$$

Multiplying (5-5) and (5-6), we obtain

$$(5\text{-}7)\qquad |\Psi(z\xi,\ w\eta)| \leqq C\exp K'(|z|^2|\xi|_p^2+|w|^2|\eta|_p^2),$$

where $K'=K+\rho^{2p}/2$. We then estimate (5-4) with the help of (5-7).

$$(5\text{-}8)\qquad \begin{aligned}|A_{l,m}(\eta,\ \xi)| &\leqq \sup\{|\Psi(z\xi,\ w\eta)|R_1^{-m}R_2^{-l}\ ;\ |z|=R_1,\ |w|=R_2\}\\ &\leqq CR_1^{-m}\exp(K'|\xi|_p^2R_1^2)R_2^{-l}\exp(K'|\eta|_p^2R_2^2),\end{aligned}$$

where R_1, $R_2>0$ are arbitrary. With an elementary fact:

$$\min_{R>0} R^{-m}\exp(\alpha R^2)=\left(\frac{2e\alpha}{m}\right)^{m/2},\qquad \alpha>0,\qquad m=1,\ 2,\ \cdots,$$

(5-8) becomes

$$(5\text{-}9)\qquad \begin{aligned}|A_{l,m}(\eta,\ \xi)| &\leqq C\left(\frac{2eK'|\xi|_p^2}{m}\right)^{m/2}\left(\frac{2eK'|\eta|_p^2}{l}\right)^{l/2}\\ &= C(l^lm^m)^{-1/2}(2eK')^{(l+m)/2}|\xi|_p^m|\eta|_p^l.\end{aligned}$$

This is also valid for $l=0$ or $m=0$ on the understanding that $0^0=1$. On the other hand, the polarization formula yields

$$(5\text{-}10)\qquad \begin{aligned}&\sup\left\{|\kappa(\eta_1,\ \cdots,\ \eta_l,\ \xi_1,\ \cdots,\ \xi_m)|\ ;\ \begin{matrix}|\eta_i|_p\leqq 1, & 1\leqq i\leqq l\\ |\xi_j|_p\leqq 1, & 1\leqq j\leqq m\end{matrix}\right\}\\ &\leqq \frac{l^l}{l!}\frac{m^m}{m!}\sup\{|A_{l,m}(\eta,\ \xi)|\ ;\ |\eta|_p\leqq 1,\ |\xi|_p\leqq 1\}.\end{aligned}$$

Combining (5-9), (5-10) and an obvious inequality

$$\frac{n^n}{n!}\leqq e^n,\qquad n=0,\ 1,\ 2,\ \cdots,$$

we obtain

$$(5\text{-}11)\qquad \begin{aligned}&\sup\left\{|\kappa(\eta_1,\ \cdots,\ \eta_l,\ \xi_1,\ \cdots,\ \xi_m)|\ ;\ \begin{matrix}|\eta_i|_p\leqq 1, & 1\leqq i\leqq l\\ |\xi_j|_p\leqq 1, & 1\leqq j\leqq m\end{matrix}\right\}\\ &\leqq C(l^lm^m)^{-1/2}(2e^3K')^{(l+m)/2}.\end{aligned}$$

We then compute $|\kappa_{l,m}|_{-(p+1)}$ by means of Fourier expansion, see (1-1).

$$|\kappa_{l,m}|^2_{-(p+1)} = \sum_{j_1,\cdots,j_{l+m}=0}^{\infty} |\kappa_{l,m}(e_{j_1}, \cdots, e_{j_{l+m}})|^2 \lambda_{j_1}^{-2(p+1)} \cdots \lambda_{j_{l+m}}^{-2(p+1)}$$

$$= \sum_{j_1,\cdots,j_{l+m}=0}^{\infty} |\kappa_{l,m}(\lambda_{j_1}^{-p} e_{j_1}, \cdots, \lambda_{j_{l+m}}^{-p} e_{j_{l+m}})|^2 \lambda_{j_1}^{-2} \cdots \lambda_{j_{l+m}}^{-2}.$$

Since $|\lambda_j^{-p} e_j|_p = 1$, in view of (5-11) we obtain

$$|\kappa_{l,m}|^2_{-(p+1)} \leqq C^2 (l^l m^m)^{-1} (2e^3 K')^{l+m} \Big(\sum_{j=0}^{\infty} \lambda_j^{-2} \Big)^{l+m}$$

$$= C^2 (l^l m^m)^{-1} (2e^3 \delta^2 K')^{l+m},$$

namely,

$$|\kappa_{l,m}|_{-(p+1)} \leqq C (l^l m^m)^{-1/2} (2e^3 \delta^2 K')^{(l+m)/2}, \tag{5-12}$$

where (2-2) is used. We have thus proved that $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$, and hence, from now on we write

$$\kappa_{l,m}(\eta_1, \cdots, \eta_l, \xi_1, \cdots, \xi_m) = \langle \kappa_{l,m}, \eta_1 \otimes \cdots \otimes \eta_l \otimes \xi_1 \otimes \cdots \otimes \xi_m \rangle.$$

It is obvious from (5-2) that $s_{l,m}(\kappa_{l,m}) = \kappa_{l,m}$. It follows from (4-2) that

$$\langle\!\langle \Xi_{l,m}(\kappa_{l,m}) \phi_\xi, \phi_\eta \rangle\!\rangle = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle} = A_{l,m}(\eta, \xi) e^{\langle \xi, \eta \rangle},$$

and therefore,

$$\sum_{l,m=0}^{\infty} \langle\!\langle \Xi_{l,m}(\kappa_{l,m}) \phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle \xi, \eta \rangle} \sum_{l,m=0}^{\infty} A_{l,m}(\xi, \eta).$$

In view of (5-1) and (5-3) for $z = w = 1$ we conclude that

$$\sum_{l,m=0}^{\infty} \langle\!\langle \Xi_{l,m}(\kappa_{l,m}) \phi_\xi, \phi_\eta \rangle\!\rangle = e^{\langle \xi, \eta \rangle} \Psi(\xi, \eta) = \Theta(\xi, \eta).$$

We next prove that $\sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \phi$ converges in $(E)^*$ for any $\phi \in (E)$. In fact, it follows from (3-8) that

$$\|\Xi_{l,m}(\kappa_{l,m}) \phi\|_{-(p+q+1)}$$

$$\leqq \rho^{-(p+q+1)} (l^l m^m)^{1/2} \Big(\frac{\rho^{-(p+q+1)}}{-2(p+q+1) e \log \rho} \Big)^{(l+m)/2} |\kappa_{l,m}|_{-(p+q+1)} \|\phi\|_{p+q+1}$$

$$\leqq \rho^{-(p+q+1)} (l^l m^m)^{1/2} \Big(\frac{\rho^{-(p+q+1)} \rho^{2q}}{-2(p+q+1) e \log \rho} \Big)^{(l+m)/2} |\kappa_{l,m}|_{-(p+1)} \|\phi\|_{p+q+1}.$$

In view of (5-12) we obtain

$$\|\Xi_{l,m}(\kappa_{l,m}) \phi\|_{-(p+q+1)} \leqq C \rho^{-(p+q+1)} \Big(\frac{K' e^2 \delta^2 \rho^{-(p+1)} \rho^q}{-(p+q+1) \log \rho} \Big)^{(l+m)/2} \|\phi\|_{p+q+1},$$

and therefore,

$$\sum_{l,m=0}^{\infty} \|\Xi_{l,m}(\kappa_{l,m}) \phi\|_{-(p+q+1)} \leqq C C_1 \rho^{-(p+q+1)} \|\phi\|_{p+q+1}, \qquad \phi \in (E), \tag{5-13}$$

where

$$C_1 = \sum_{l,m=0}^{\infty} \left(\frac{K' e^2 \delta^2 \rho^{-(p+1)} \rho^q}{-(p+q+1) \log \rho} \right)^{(l+m)/2}.$$

Obviously, $C_1 < \infty$ for a large $q \geqq 0$ because $0 < \rho < 1$. It then follows from (5-13) that

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E),$$

converges in $(E)_{-(p+q+1)}$ and $\Xi \in \mathcal{L}((E)_{p+q+1}, (E)_{-(p+q+1)})$. In particular, the series converges in $(E)^*$ and $\Xi \in \mathcal{L}((E), (E)^*)$. It is then obvious that $\hat{\Xi}(\xi, \eta) = \Theta(\xi, \eta)$ for all $\xi, \eta \in E$. (Q.E.D.)

Combining Theorems 4.5 and 5.1 we obtain a characterization of symbols which corresponds to an operator version of Potthoff-Streit characterization theorem for generalized white noise functionals [**30**].

COROLLARY 5.2. *Let Θ be a function on $E \times E$ with values in $\boldsymbol{C}$. Then, there exists a continuous operator $\Xi \in \mathcal{L}((E), (E)^*)$ with $\Theta(\xi, \eta) = \hat{\Xi}(\xi, \eta)$ for all $\xi, \eta \in E$ if and only if Θ satisfies the conditions* (i) *and* (ii) *in Theorem* 5.1.

6. Fock expansion.

As a remarkable application of Theorem 5.1 we prove

THEOREM 6.1. *For any $\Xi \in \mathcal{L}((E), (E)^*)$ there exists a unique family of kernel distributions $\kappa_{l,m} \in (E_{\boldsymbol{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, $l, m = 0, 1, 2, \cdots$, such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E), \tag{6-1}$$

where the right hand side converges in $(E)^*$.

PROOF. For a given $\Xi \in \mathcal{L}((E), (E)^*)$ we put $\Theta = \hat{\Xi}$. Then it follows from Theorem 4.5 that Θ satisfies (i) and (ii) in Theorem 5.1. Hence there is a unique family of kernel distributions $\kappa_{l,m} \in (E_{\boldsymbol{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$, $l, m = 0, 1, 2, \cdots$, such that

$$\Theta(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle = \sum_{l,m=0}^{\infty} \langle\langle \Xi_{l,m}(\kappa_{l,m})\phi_\xi, \phi_n \rangle\rangle, \qquad \xi, \eta \in E.$$

Moreover, we see from Theorem 5.1 that

$$\Xi'\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E),$$

converges in $(E)^*$, $\Xi' \in \mathcal{L}((E), (E)^*)$ and $\Xi'(\xi, \eta) = \Theta(\xi, \eta)$ for all $\xi, \eta \in E$. It is then sufficient to prove that $\Xi' = \Xi$. But this is already clear by Lemma 4.2

because

$$\hat{\Xi}(\xi, \eta) = \Theta(\xi, \eta) = \hat{\Xi}'(\xi, \eta), \qquad \xi, \eta \in E.$$

This completes the proof. (Q.E.D.)

The unique expression of $\Xi \in \mathcal{L}((E), (E)^*)$ given as in (6-1) is called the *Fock expansion* of Ξ and denoted by

$$\Xi = \sum_{l, m=0}^{\infty} \Xi_{l, m}(\kappa_{l, m}).$$

Since bounded operators on (L^2) belong to $\mathcal{L}((E), (E)^*)$, we have

COROLLARY 6.2. *Every bounded operator Ξ on (L^2) admits a Fock expansion.*

However, the convergence of the Fock expansion of a bounded operator on (L^2) can not be discussed only within the framework of Hilbert space. The next result also illustrates this remark.

PROPOSITION 6.3. *Let $\kappa \in (E_C^{\otimes(l+m)})^*$. If $\Xi_{l, m}(\kappa)$ admits an extension to a bounded operator on (L^2), then $s_{l, m}(\kappa)=0$ or $l=m=0$. Namely, except scalar operators no integral kernel operator admits an extension to a bounded operator on (L^2).*

PROOF. The action of $\Xi_{l, m}(\kappa)$ on exponential vectors ϕ_ξ, $\xi \in E_C$, is described during the proof of Lemma 4.1:

$$\Xi_{l, m}(\kappa)\phi_\xi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :x^{\otimes(n+l)}:, (\kappa \otimes_m \xi^{\otimes m}) \otimes \xi^{\otimes n} \rangle.$$

Hence,

$$\begin{aligned} \|\Xi_{l, m}(\kappa)\phi_\xi\|_0^2 &= \sum_{n=0}^{\infty} \frac{(n+l)!}{n!n!} |(\kappa \otimes_m \xi^{\otimes m}) \otimes \xi^{\otimes n}|_0^2 \\ &= |\kappa \otimes_m \xi^{\otimes m}|_0^2 \sum_{n=0}^{\infty} \frac{(n+l)!}{n!n!} |\xi|_0^{2n}. \end{aligned} \tag{6-2}$$

On the other hand, one may easily see that

$$P_l(t) \equiv e^{-t} \sum_{n=0}^{\infty} \frac{(n+l)!}{n!n!} t^n = \sum_{j=0}^{l} j! \binom{l}{j}^2 t^{l-j}. \tag{6-3}$$

With this notation, (6-2) becomes

$$\|\Xi_{l, m}(\kappa)\phi_\xi\|_0^2 = |\kappa \otimes_m \xi^{\otimes m}|_0^2 P_l(|\xi|_0^2) \exp(|\xi|_0^2). \tag{6-4}$$

We now assume that $\Xi_{l, m}(\kappa)$ admits an extension to a bounded operator on (L^2). Then, there exists some $C \geqq 0$ such that

$$\|\Xi_{l, m}(\kappa)\phi_\xi\|_0^2 \leqq C\|\phi_\xi\|_0^2 = C \exp(|\xi|_0^2), \qquad \xi \in E_C. \tag{6-5}$$

Combining (6-4) and (6-5), we obtain

(6-6) $$|\kappa\otimes_m \xi^{\otimes m}|_0^2 P_l(|\xi|_0^2) \leqq C, \qquad \xi \in E_C.$$

Suppose first that $l \neq 0$. Since $\lim_{t\to\infty} P_l(t) = \infty$ by (6-3), (6-6) is true only when

$$\kappa\otimes_m \xi^{\otimes m} = 0, \qquad \xi \in E_C.$$

Then, for any $\xi, \eta \in E_C$,

$$0 = \langle \kappa\otimes_m \xi^{\otimes m}, \eta^{\otimes l}\rangle = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle = \langle s_{l,m}(\kappa), \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle.$$

This implies that $s_{l,m}(\kappa)=0$, and therefore by Proposition 3.1 we obtain $\Xi_{l,m}(\kappa) = 0$.

In order to discuss the case of $m \neq 0$ we first note that

$$\Xi_{l,m}(\kappa)^* = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{t_1}^* \cdots \partial_{t_m}^* \partial_{s_1} \cdots \partial_{s_l} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

Then, carrying out a similar argument for $\Xi_{l,m}(\kappa)^*$ instead, we conclude that $\Xi_{l,m}(\kappa)^*=0$, that is, $\Xi_{l,m}(\kappa)=0$. Consequently, $\Xi_{l,m}(\kappa)=0$ unless $l=m=0$.

(Q.E.D.)

7. Examples of Fock expansion.

Let $\Xi \in \mathcal{L}((E), (E)^*)$ be given and let $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m})$ be its Fock expansion. It then follows from Theorem 5.1 that

$$e^{-\langle\xi,\eta\rangle}\hat{\Xi}(\xi,\eta) = \sum_{l,m=0}^\infty \langle \kappa_{l,m}, \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle, \qquad \xi, \eta \in E_C.$$

Hence, to find the kernel distributions of Ξ we need only to compute the Taylor expansion of $e^{-\langle z\xi, w\eta\rangle}\hat{\Xi}(z\xi, w\eta)$.

Below we assemble a few examples of integral kernel operators and Fock expansions. The proofs are omitted since the computations are carried out easily in the above mentioned manner.

EXAMPLE 7.1 (differential operators and translations). For $y \in E^*$ a differential operator D_y is defined by (3-2) or equivalently by (3-3). Then

$$D_y = \Xi_{0,1}(y), \qquad D_y^* = \Xi_{1,0}(y).$$

In particular, for $t \in T$ it holds that

$$\partial_t = \Xi_{0,1}(\delta_t), \qquad \partial_t^* = \Xi_{1,0}(\delta_t).$$

As is easily expected, D_y is related to a translation operator. For $y \in E^*$ we define

$$T_y\phi(x) = \phi(x+y), \qquad x \in E^*, \qquad \phi \in (E).$$

It then holds that

$$T_y = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi_{0,n}(y^{\otimes n}) = \sum_{n=0}^{\infty} \frac{1}{n!} D_y^n .$$

This illustrates that the Fock expansion includes Taylor expansion.

EXAMPLE 7.2 (multiplication operators). It is known (e.g., [**20**]) that the pointwise multiplication induces a continuous bilinear map from $(E)\times(E)$ into (E). Hence for $\Phi\in(E)^*$ we may define a product $\phi\Phi=\Phi\phi\in(E)^*$ for $\phi\in(E)$ by the formula:

$$\langle\langle \Phi\phi, \psi \rangle\rangle = \langle\langle \Phi, \phi\psi \rangle\rangle, \qquad \psi \in (E).$$

Moreover, $\Phi\in(E)^*$ is regarded as continuous linear operator from (E) into $(E)^*$, i.e., $\Phi\in\mathcal{L}((E), (E)^*)$. According to (2-9), we write

$$\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, F_n \rangle .$$

Then, as a multiplication operator,

$$\Phi = \sum_{l,m=0}^{\infty} \binom{l+m}{m} \Xi_{l,m}(F_{l+m}). \tag{7-1}$$

Here are special cases. For $t\in T$ put $x(t)=\langle :x^{\otimes 1}:, \delta_t \rangle$, which may be considered as coordinate system of white noise space. It then follows from (7-1) that

$$x(t) = \Xi_{1,0}(\delta_t)+\Xi_{0,1}(\delta_t) = \partial_t^*+\partial_t .$$

It is noteworthy that $\langle :x^{\otimes 2}:, \tau \rangle$ is a white noise analogue of the usual Euclidean norm, for the definition of τ see (2-4). Regarded as a multiplication operator,

$$\langle :x^{\otimes 2}:, \tau \rangle = \Xi_{2,0}(\tau)+2\Xi_{1,1}(\tau)+\Xi_{0,2}(\tau). \tag{7-2}$$

EXAMPLE 7.3 (Laplacians). The integral kernel operators with τ being the kernel distribution are of great importance. We put

$$\Delta_G = \Xi_{0,2}(\tau) = \int_{T\times T} \tau(s, t)\partial_s\partial_t ds dt ,$$

$$N = \Xi_{1,1}(\tau) = \int_{T\times T} \tau(s, t)\partial_s^*\partial_t ds dt .$$

These are respectively called the *Gross Laplacian* and the *number operator*. In fact, by Proposition 3.2 both are continuous operators from (E) into itself. Obviously, N^* is an extension of N and Δ_G^* is given as

$$\Delta_G^* = \Xi_{2,0}(\tau).$$

With these notations we obtain an alternative expression for (7-2):

$$\langle :x^{\otimes 2}:, \tau \rangle = \Delta_G^*+2N+\Delta_G ,$$

which is discussed in line with a systematic study of rotation-invariant operators, see [11] and [28].

EXAMPLE 7.4 (projection onto the n-th chaos). Let $(L^2)=\sum_{n=0}^{\infty}\oplus\mathcal{H}_n$ be the Wiener-Itô decomposition (see also Proposition 2.1) and let π_n be the projection onto the n-th chaos $\mathcal{H}_n$. It is easy to see that $\pi_n\in\mathcal{L}((E),(E))$ if restricted to (E). Then,

$$\pi_n=\sum_{l=n}^{\infty}\frac{(-1)^{l-n}}{(l-n)!n!}\Xi_{l,l}(\lambda_l),$$

where

$$\lambda_l=\sum_{i_1,\cdots,i_l=0}^{\infty}e_{i_1}\otimes\cdots\otimes e_{i_l}\otimes e_{i_1}\otimes\cdots\otimes e_{i_l}\in(E^{\otimes 2l})^*. \tag{7-3}$$

It is also interesting to note that

$$\Xi_{l,l}(\lambda_l)=N(N-1)\cdots(N-l+1).$$

EXAMPLE 7.5 (Fourier-Wiener transform). Let $\{\exp(i\theta N)\}_{\theta\in\boldsymbol{R}}$ be the one-parameter group of Fourier-Wiener transform, namely, it is a one-parameter group of unitary operators on (L^2) with the number operator N being the infinitesimal generator. Obviously, $\exp(i\theta N)\in\mathcal{L}((E),(E))$. We then obtain

$$\exp(i\theta N)=\sum_{l=0}^{\infty}\frac{(e^{i\theta}-1)^l}{l!}\Xi_{l,l}(\lambda_l),$$

where λ_l is defined as in (7-3).

EXAMPLE 7.6 (Weyl form of canonical commutation relation). We consider representations of the additive group E. For $\xi\in E$ and $\phi\in(E)$ put

$$P_\xi\phi(x)=\phi(x+\xi)\exp\left(-\frac{1}{2}\langle x,\xi\rangle-\frac{1}{4}\langle\xi,\xi\rangle\right),$$

$$Q_\xi\phi(x)=e^{i\langle x,\xi\rangle}\phi(x).$$

In fact, P_ξ and Q_ξ belong to $\mathcal{L}((E),(E))$ and are extended to unitary operators on (L^2). It is straightforward to see that

$$P_{\xi+\eta}=P_\xi P_\eta,\qquad Q_{\xi+\eta}=Q_\xi Q_\eta,\qquad P_\xi Q_\eta=e^{i\langle\xi,\eta\rangle}Q_\eta P_\xi,$$

for $\xi,\eta\in E$. The Fock expansions of P_ξ and Q_ξ are given as

$$P_\xi=e^{-\langle\xi,\xi\rangle/8}\sum_{l,m=0}^{\infty}\frac{(-1)^l}{l!m!}\left(\frac{1}{2}\right)^{l+m}\Xi_{l,m}(\xi^{\otimes(l+m)}),$$

$$Q_\xi=e^{-\langle\xi,\xi\rangle/2}\sum_{l,m=0}^{\infty}\frac{i^{l+m}}{l!m!}\Xi_{l,m}(\xi^{\otimes(l+m)}).$$

Moreover,

$$p_\xi \equiv \frac{d}{d\theta} P_{\theta\xi}\Big|_{\theta=0} = \frac{1}{2}(D_\xi - D_\xi^*),$$

$$q_\xi \equiv \frac{d}{d\theta} Q_{\theta\xi}\Big|_{\theta=0} = i(D_\xi + D_\xi^*).$$

These operators belong to $\mathcal{L}((E), (E))$ again and satisfy the canonical commutation relation.

EXAMPLE 7.7 (Kuo's Fourier transform). This is a white noise analogue of a usual Fourier transform on $\boldsymbol{R}^n$ introduced by Kuo. Here we omit the explicit definition for we have not prepared necessary notations. Instead we introduce its characteristic properties, for further details see [**10**].

First note that D_ξ and q_ξ are continuously extended to operators on $(E)^*$ whenever $\xi \in E$. The extensions are denoted by $\tilde{D}_\xi$ and $\tilde{q}_\xi$, respectively. Then, the Fourier transform $\mathfrak{F} \in \mathcal{L}((E)^*, (E)^*)$ is uniquely characterized up to constant factor by the following properties:

$$\text{(7-4)} \qquad \mathfrak{F}\tilde{D}_\xi = \tilde{q}_\xi \mathfrak{F}, \qquad \mathfrak{F}\tilde{q}_\xi = -\tilde{D}_\xi \mathfrak{F}, \qquad \xi \in E.$$

The constant factor is determined, for example, by the condition $\mathfrak{F}1 = \delta_0$. A formally written expression for (7-4) would be

$$\mathfrak{F}\partial_t = ix(t)\mathfrak{F}, \qquad \mathfrak{F}x(t) = i\partial_t \mathfrak{F}, \qquad t \in T.$$

Namely, $\mathfrak{F}$ possesses typical properties of the finite dimensional Fourier transform.

Moreover, the Fourier transform is imbedded in a one-parameter group of transformations called *Fourier-Mehler transforms* $\{\mathfrak{F}_\theta\}_{\theta \in \boldsymbol{R}} \subset \mathcal{L}((E)^*, (E)^*)$ in such a way that $\mathfrak{F}_{-\pi/2} = \mathfrak{F}$. Here we only record their Fock expansion regarded as an operator in $\mathcal{L}((E), (E)^*)$:

$$\mathfrak{F}_\theta = \sum_{l,m=0}^{\infty} \frac{1}{l!m!} \left(\frac{i}{2} e^{i\theta} \sin\theta\right)^l (e^{i\theta} - 1)^m \Xi_{2l+m,m}(\tau^{\otimes l} \otimes \lambda_m),$$

where λ_m is defined as in (7-3).

EXAMPLE 7.8 (integral-sum kernel operators). This example suggests a white noise approach to quantum probability theory. In order to describe integral-sum kernel operators we need another realization of Fock space. Let Ω_n be the collection of subsets $\sigma \subset T$ consisting of n points, $0 \leqq n < \infty$. Since the measure ν is smooth by assumption, we may identify Ω_n with the factor space $T^n/\mathfrak{S}_n$ up to ν-null sets. Let λ_n be the measure on Ω_n such that $n!\lambda_n$ is the image measure of ν^n under the canonical map $T^n \to T^n/\mathfrak{S}_n$. We then put

$$\Omega = \bigcup_{n=0}^{\infty} \Omega_n, \qquad \lambda = \sum_{n=0}^{\infty} \lambda_n.$$

Note that the correspondence $f \leftrightarrow \phi$ given by

$$\phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ f_n\rangle, \qquad f_n(t_1,\cdots,t_n)=\frac{1}{n!}f(\{t_1,\cdots,t_n\}),$$

yields a Hilbert space isomorphism $L^2(\Omega, \lambda; \boldsymbol{C})\cong(L^2)$.

Maassen [25] introduced an operator of the form:

$$\Xi f(\sigma)=\int_{\Omega}\sum_{\alpha_1\cup\alpha_2=\sigma}k(\alpha_1, \omega)f(\omega\cup\alpha_2)\lambda(d\omega), \qquad \sigma\in\Omega. \tag{7-5}$$

Under some regularity conditions, Ξ becomes an operator on $L^2(\Omega, \lambda)$ with dense domain. Then the Fock expansion $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})$ is given with the kernel distributions:

$$\kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m)=\frac{1}{l!m!}k(\{s_1,\cdots,s_l\},\ \{t_1,\cdots,t_m\}).$$

But many important operators are not expressible in the form (7-5). To cover a wider class of operators including the number operator Meyer [26] generalized (7-5) and introduced three-argument integral-sum kernel operators:

$$\Xi f(\sigma)=\int_{\Omega}\sum_{\alpha_1\cup\alpha_2\cup\alpha_3=\sigma}\rho(\alpha_1, \alpha_2, \omega)f(\omega\cup\alpha_2\cup\alpha_3)\lambda(d\omega), \qquad \sigma\in\Omega, \tag{7-6}$$

where ρ is a function on $\Omega\times\Omega\times\Omega$ with some regularity conditions. The Fock expansion of the operator (7-6) is given with the kernel distributions:

$$\kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m)=\frac{1}{l!m!}\sum_{j=0}^{l\wedge m}j!\binom{l}{j}\binom{m}{j}$$

$$\times\rho(\{s_{j+1},\cdots,s_l\},\ \{s_1,\cdots,s_j\},\ \{t_{j+1},\cdots,t_m\})\tau(s_1,t_1)\cdots\tau(s_j,t_j).$$

We thus easily understand that the number operator is expressible by means of Meyer's operators. A detailed study of such integral-sum kernel operators as in (7-5) and (7-6) is made by Lindsay [24] together with further generalization.

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