AN ANALYTIC PROBLEM WHOSE SOLUTION FOLLOWS FROM A SIMPLE ALGEBRAIC IDENTITY

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1. Introduction. It is convenient to describe the point of view of this paper in terms of a very simple example. The unique solution of

$$\frac{dy}{dx} = \lambda\varphi(x)y \ , \qquad y(0) = 1 \ , \tag{1.1}$$

where $\varphi(x)$ is a continuous function and λ is a parameter, is given by

$$y = \exp\left\{\lambda\int_0^x \varphi(\xi)\,d\xi\right\} \ . \tag{1.2}$$

For any continuous function $\varphi(x)$ define

$$\varphi^+ = \varphi^+(x) = \int_0^x \varphi(\xi)d\xi \ . \tag{1.3}$$

After integrating both sides of the equation in (1.1) and using the notation of (1.3), we find that

$$y = 1 + \lambda(\varphi y)^+ \tag{1.4}$$

has the solution

$$y = \exp(\lambda\varphi^+) = 1 + \lambda\varphi^+ + \lambda^2\varphi^{+2}/2! + \lambda^3\varphi^{+3}/3! + \cdots \tag{1.5}$$

By the method of successive substitutions it is also possible to give a unique solution to (1.4) in the form

$$y = 1 + \lambda\varphi^+ + \lambda^2(\varphi\varphi^+)^+ + \lambda^3(\varphi(\varphi\varphi^+)^+)^+ + \cdots \tag{1.6}$$

Equating coefficients in (1.5) and (1.6) we arrive at the well-known *identities in* φ

$$\begin{aligned} (\varphi\varphi^+)^+ &= \varphi^{+2}/2! \\ (\varphi(\varphi\varphi^+)^+)^+ &= \varphi^{+3}/3! \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{1.7}$$

We now wish to focus on the following fact: *All of the identities in* (1.7) *are a consequence of the first identity and the linear property*

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of the operator$_+$. For example, putting $\varphi = \psi + \psi\psi^+$ and using only the first identity and the linearity of $+$

$$\begin{aligned}\psi^{+3}/2 &= \varphi^{+2}/2 - \psi^{+2}/2 - (\psi\psi^+)^{+2}/2 \\ &= (\varphi\varphi^+)^+ - (\psi\psi^+)^+ - (\psi\psi^+(\psi\psi^+)^+)^+ \\ &= (\psi(\psi\psi^+)^+)^+ + (\psi\psi^{+2}) = 3(\psi(\psi\psi^+)^+)^+ .\end{aligned}$$

The fact in general is a special case of our Lemma 1. We observe that the identities in (1.7) are necessary and sufficient for the simplification of (1.6) to (1.5).

In this paper we are interested in certain sets of operator identities like (1.7) which allow a striking simplification of the form of the solution of a linear equation found by the method of successive substitution. In every case the whole set of identities follows from the first identity and the linear property of the operator. Our main theorems are as follows:

THEOREM 1. *Let A be a commutative Banach algebra of elements φ on which a bounded, linear operator $+$ of norm N is defined taking A into A. Furthermore, let*

$$2(\varphi\varphi^+)^+ = (\theta\varphi^2)^+ + \varphi^{+2} \tag{1.8}$$

be satisfied for every φ in A, where θ is a fixed element of A. Then the equation

$$\psi = 1 + \lambda(\varphi\psi)^+ \tag{1.9}$$

has a unique solution in A for $|\lambda| \cdot \|\varphi\| \max(\|\theta\|, N) < 1$ given by

$$\psi = \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\theta^{k-1}\varphi^k)^+\right\} . \tag{1.10}$$

Formula (1.10) arises out of a formal manipulation of the coefficients in a certain power series. For this reason we state an alternative form of Theorem 1 which emphasizes the algebraic character of our result. We use notation more natural to algebra.

THEOREM 1*[1]. *Let A be a commutative algebra over a field of characteristic zero, let T be a group endomorphism of A into A, and for every a in A let*

$$2(a(aT))T = (a^2b)T + (aT)^2 , \tag{1.8'}$$

where b is a fixed element in A. Define $a_0 = 1$, $a_n = (aa_{n-1})T$. Then

[1] The author is indebted to the referee for suggesting this elegant reformulation of Theorem 1.

$$(1.10')\qquad \sum_{n=0}^{\infty} a_n x^n = \exp\left\{\sum_{k=1}^{\infty} \frac{x^k}{k}(b^{k-1}a^k)T\right\} \text{ in } A\langle x\rangle .$$

One of the most interesting special cases of (1.10) in the literature was given by Frank Spitzer [4], Other special cases of (1.10) of interest in probability theory were given by E. Sparre Andersen [1, 2]. Our proof of Theorem 1 is most similar to the proof of the case of (1.10) given by Spitzer. A combinatorial lemma is proved and then applied to prove Theorem 1. The combinatorial lemma behind (1.10) is actually a consequence of a simple "algebraic" condition similar to (1.8).

Before the combinatorial lemma can be stated more notation must be introduced. Let R be a commutative ring of elements φ on which a linear mapping $+$ taking R into R is defined. Furthermore, for any two elements φ_1 and φ_2 in R let

$$(1.11)\qquad (\varphi_1\varphi_2^+)^+ + (\varphi_2\varphi_1^+)^+ = (\theta\varphi_1\varphi_2)^+ + \varphi_1^+\varphi_2^+ ,$$

where θ is some fixed element in R. For any fixed set of elements $\psi_1, \psi_2, \cdots, \psi_n$ in R and any permutation $P = (i_1 i_2 \cdots i_{m_1})(i_{m_1+1} \cdots i_{m_2}) \cdots (i_{m_k+1} \cdots i_n)$ of the integers $1, 2, \cdots, n$ written as a product of cycles including 1-cycles with no integer in more than one cycle, we define

$$(1.12)\qquad \psi_P = (\theta^{m_1-1}\psi_{i_1}\psi_{i_2} \cdots \psi_{i_{m_1}})^+(\theta^{m_2-m_1-1}\psi_{i_{m_1+1}} \cdots \psi_{i_{m_2}})^+ \cdots (\theta^{n-m_k-1}\psi_{i_{m_k+1}} \cdots \psi_{i_n})^+$$

Lemma 1. *Let $\psi_1, \psi_2, \cdots, \psi_n$ be fixed elements in R. Then,*

$$(1.13)\qquad \sum_{(\sigma)} (\psi_{i_1}(\psi_{i_2} \cdots (\psi_{i_{n-1}}\psi_{i_n}^+)^+ \cdots)^+)^+ = \sum_{(P)} \psi_P ,$$

where the summation on the left in (1.13) *extends over all permutations* $\sigma: i_1 i_2 \cdots i_n$ *of the integers* $1, 2, \cdots, n$ *and where the summation on the right in* (1.13) *extends over all permutations* P.

We note that (1.11) is the special case of (1.13) for $n = 2$. It is a simple exercise to show that if R is a ring of the type described above with a linear mapping $+$ satisfying (1.11), then $\varphi^- = \theta\varphi - \varphi^+$ defines another linear mapping taking R into R for which (1.11) is true.

In the next theorem we consider a slightly more general equation than (1.9). It is interesting to note that the results of Theorem 1 and Theorem 2 do not in general overlap.

THEOREM 2. *Let A be a commutative Banach algebra with an operator $+$ satisfying the conditions of Theorem* 1. *Define*

$$p(\lambda) = \exp\left\{\sum_{k=1}^{\infty}\frac{\lambda^k}{k}(\theta^{k-1}\varphi^k)^+\right\}, \tag{1.14}$$
$$q(\lambda) = \exp\left\{\sum_{k=1}^{\infty}\frac{\lambda^k}{k}(\theta^{k-1}\varphi^k)^-\right\}.$$

Then for $|\lambda|\cdot\|\varphi\|\max(\|\theta\|, N) < 1$

(a) *the equation* $\psi = \Phi^+ + \lambda(\varphi\psi)^+$, *where* φ, Φ *are in* A, *has the unique solution* $\psi = p(\Phi q)^+$,

(b) *the equation* $\psi = \Phi + \lambda(\varphi\psi)^+$, *where* φ, Φ *are in* A, *has the unique solution* $\psi = \Phi + \lambda p(\varphi\Phi q)^+$,

and

(c) *The equation* $\psi = 1 + \lambda u(\varphi\psi)^+ + \lambda(\varphi\psi)^-$, *where* $|u| < 1$ *and where* $\varphi^- = \theta\varphi - \varphi^+$, *has unique solution* $\psi = p(\lambda u)q(\lambda)$.

In the next section proofs of the theorems and the lemma are given. In § 3 we give three examples to illustrate the theorems.

2. Combinatorial lemmas and proofs. In this section A and R will denote, respectively, a commutative Banach algebra and a commutative ring of elements φ on which a linear mapping $+$ (which is a bounded operator in the case of the Banach algebra A) taking A into A or R into R is defined satisfying, respectively, (1.8) or (1.11). As mentioned in the introduction $\varphi^- = \theta\varphi - \varphi^+$ defines a linear mapping— which also satisfies (1.8) or (1.11) as the case may be. In terms of the— mapping we can give a slight but very convenient rewriting of (1.11). For any φ, ψ in R

$$(\varphi\psi^+)^+ = \varphi^+\psi^+ + (\psi\varphi^-)^+ \tag{2.1}$$

LEMMA 2. *Let* $\psi_1, \psi_2, \cdots, \psi_n$ *be in* R *and define*

$$p_m = (\psi_1(\psi_2(\psi_3 \cdots (\psi_{m-1}\psi_m^+)^+ \cdots)^+)^+)^+, \qquad p_0 = 1,$$
$$q_{n,m} = (\psi_n(\psi_{n-1} \cdots (\psi_{m+2}\psi_{m+1}^-)^- \cdots)^-)^-, \qquad q_{n,n} = 1,$$
$$r_{n,m} = (\psi_n(\psi_{n-1} \cdots (\psi_{m+2}\psi_{m+1}^-)^- \cdots)^-)^+, \qquad r_{n,n} = 0.$$

Then,

$$p_n = \sum_{m=0}^{n} p_m r_{n,m} = \sum_{m=0}^{n-1} p_m r_{n,m}, \tag{2.2}$$

$$\theta^n \prod_{m=1}^{n} \psi_m = \sum_{m=0}^{n} p_m q_{n,m}. \tag{2.3}$$

Proof. First, we prove (2.2) by induction on n. If $n = 1$ $p_n = \psi_1^+$, $r_{n,n} = 0$, $r_{n,0} = \psi_1^+$, and $p_0 = 1$. Relation (2.2) is clearly satisfied in this case. Assume that (2.2) has been demonstrated for all sets of elements $\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_n$ with $n < N$. Consider the set of elements $\psi_1, \psi_2, \cdots, \psi_N$

in R. We apply (2.2) to the set $\tilde{\psi}_1 = \psi_1, \tilde{\psi}_2 = \psi_2, \cdots, \tilde{\psi}_{N-2} = \psi_{N-2}$, and $\tilde{\psi}_{N-1} = \psi_{N-1}\psi_N^+$ and find that

$$(2.4) \qquad p_N = \tilde{p}_{N-1} = \sum_{m=0}^{N-2} \tilde{p}_m \tilde{r}_{N-1,m}$$

For all $m < N-1$ relation (2.1) implies

$$(2.5) \qquad \begin{aligned} \tilde{r}_{N-1,m} &= (\psi_{N-1}\psi_N^+(\psi_{N-2}\cdots(\psi_{m+2}\psi_{m+1}^-)^-\cdots)^-)^+ \\ &= \psi_N^+(\psi_{N-1}(\psi_{N-2}\cdots(\psi_{m+2}\psi_{m+1}^-)^-\cdots)^-)^+ \\ &\quad + (\psi_N(\psi_{N-1}(\psi_{N-2}\cdots(\psi_{m+2}\psi_{m+1}^-)^-\cdots)^-)^-)^+ \\ &= \psi_N^+ r_{N-1,m} + r_{N,m} \,. \end{aligned}$$

Putting (2.4) and (2.5) together

$$(2.6) \qquad p_N = \sum_{m=0}^{N-2} p_m \psi_N^+ r_{N-1,m} + \sum_{m=0}^{N-2} p_m r^N{}_{.m} \,.$$

From relation (2.2) applied to $\psi_1, \psi_2 \cdots, \psi_{N-1}$, one finds

$$p_{N-1} = \sum_{m-0}^{N-2} p_m r_{N-1,m} \,.$$

Thus, (2.6) becomes

$$p_N = \psi_N^+ p_{N-1} + \sum_{m=0}^{N-2} p_m r_{N,m} = \sum_{m=0}^{N-1} p_m r_{N,m} \,.$$

The proof of (2.2) follows by induction.

To prove (2.3) we first note that $q_{n,m} = \theta\psi_n q_{n-1,m} - r_{n,m}$ for all $n > m \geqq 0$. Thus

$$(2.7) \qquad \begin{aligned} \sum_{m=0}^{n} p_m q_{n,m} &= p_n + \sum_{m=0}^{n-1} p_m q_{n,m} \\ &= p_n + \theta\psi_n \sum_{m=0}^{n-1} p_m q_{n-1,m} - \sum_{m=0}^{n-1} p_m r_{n,m} \\ &= \theta\psi_n \sum_{m=0}^{n-1} p_m q_{n-1,m} \end{aligned}$$

Relation (2.3) follows from (2.7) by the obvious induction. This proves Lemma 2.

Proof of Lemma 1. We refer the reader to the notation introduced prior to the statement of Lemma 1. The proof is by induction on n. For the case $n = 1$, both sides of (1.13) equal ψ_1^+. Assume that (1.13) has been demonstrated for all $n = 1, 2, \cdots, N-1$, and let $\psi_1, \psi_2, \cdots, \psi_N$ be fixed elements in R. Let P' be a permulation in which the cycle containing the integer N is $(Ni_1i_2 \cdots i_k)$, $k \geqq 0$. We assume that in all permutations P the cycle containing N is written so that N appears first. For the time being $i_1, i_2, \cdots, i_k$ are *fixed* and $i_1, i_2, \cdots, i_{N-1}$ is a *fixed* permutation of $1, 2, \cdots, N-1$. There are many permutations P' containing the fixed cycle $(Ni_1i_2 \cdots i_k)$. In fact, there is one such permutation P' for every permutation (as a product of cycles or otherwise) of $i_{k+1}, i_{k+2}, \cdots, i_{N-1}$. Applying the induction hypothesis we find

$$(2.8) \quad \sum_{(P')} \psi_{P'} = \sum_{(\sigma')} (\psi_{j_{k+1}}(\psi_{j_{k+2}} \cdots (\psi_{j_{N-2}}\psi_{j_{N-1}}^+)^+ \cdots)^+)^+(\theta^k\psi_N\psi_{i_1} \cdots \psi_{i_k})^+$$

where P' is any permutation of $1, 2, \cdots, N$ containing the fixed cycle $(Ni_1i_2 \cdots i_k)$ and where σ': $j_{k+1}j_{k+2} \cdots j_{N-1}$ is any permutation of $i_{k+1}i_{k+2} \cdots i_{N-1}$. We concentrate for a moment on the factor $(\theta^k\psi_N\psi_{i_1} \cdots \psi_{i_k})^+$. Applying (2.3) to the elements $\psi_{i_1}, \psi_{i_2}, \cdots, \psi_{i_k}$, we deduce that

$$(2.9) \quad \begin{aligned} &(\theta^k\psi_N\psi_{i_1} \cdots \psi_{i_k})^+ \\ &\quad = \sum_{m=0}^{k} \{\psi_N(\psi_{i_1}(\psi_{i_2} \cdots \psi_{i_m}^+)^+ \cdots)^+(\psi_{i_k}(\psi_{i_{k-1}} \cdots \psi_{i_{m+1}}^-)^- \cdots)^-\}^+ \end{aligned}$$

Now, any permutation of $i_1, i_2, \cdots, i_k$ changes the cycle $(Ni_1 \cdots i_k)$. Any change in k or any change in the set of integers $i_1, i_2, \cdots, i_k$ also changes the cycle $(Ni_1i_2 \cdots i_k)$. Thus, the summation on the right in (1.13) is equal to the sum of all sums of the type on the left in (2.8) over all possible choices of $k = 0, 1, \cdots N-1$, over all sets of k integers, and over all permutations of these integers. Combining (2.8) and (2.9) this implies that the right side of (1.13) is equal to the sum of all terms of the form

$$(2.10) \quad \begin{aligned} &\{\psi_N(\psi_{i_1}(\psi_{i_2} \cdots \psi_{i_m}^+)^+ \cdots)^+(\psi_{i_k}(\psi_{i_{k-1}} \cdots \psi_{i_{m+1}}^-)^- \cdots)^-\}^+ \cdot \\ &\quad \cdot (\psi_{i_{k+1}}(\psi_{i_{k+2}} \cdots \psi_{i_N}^+)^+ \cdots)^+ \end{aligned}$$

over all permutations $i_1, i_2, \cdots, i_{N-1}$ of $1, 2, \cdots, N-1$ and over all integers m and k satisfying $0 \leq m \leq k \leq N-1$.

We finish the proof by showing that the left side of (1.13) is equal to the same sum of terms in (2.10). For any permutation $i_1i_2 \cdots i_N$ of $1, 2, \cdots, N$, there exists an integer m with $1 \leq m+1 \leq N$ such that $i_{N-m} = N$. To the term

$$(\psi_{i_1}(\psi_{i_2} \cdots (\psi_{i_{N-m-1}}(\psi_N(\psi_{i_{N-m+1}} \cdots \psi_{i_N}^+)^+ \cdots)^+)^+ \cdots)^+)^+$$

on the left side of (1.13) we apply (2.2) where

$$\begin{aligned} \tilde{\psi}_1 &= \psi_{i_1} \\ &\vdots \\ \tilde{\psi}_{N-m-1} &= \psi_{i_{N-m-1}} \\ \tilde{\psi}_{N-m} &= (\psi_N(\psi_{i_{N-m+1}}(\psi_{i_{N-m+2}} \cdots \psi_{i_N}^+)^+ \cdots)^+)^+ . \end{aligned}$$

This yields the equality

$$(\psi_{i_1}(\psi_{i_2}\cdots\psi_{i_N}^+)^+\cdots)^+=(\tilde{\psi}_1(\tilde{\psi}_2\cdots\tilde{\psi}_{N-m}^+)^+\cdots)^+$$

$$=\sum_{k=m}^{N}\{\psi_N(\psi_{i_{N-m+1}}(\cdots\psi_{i_N}^+)^+\cdots)^+(\psi_{i_{N-m-1}}\cdots(\psi_{i_{N-k+1}}\psi_{i_{N-k}}^-)^-\cdots)^-\}^+\cdot$$
$$\cdot(\psi_{i_1}(\psi_{i_2}\cdots(\psi_{i_{N-k-2}}\psi_{i_{N-k-1}}^+)^+\cdots)^+)^+ .$$

Thus, the sum on the left in (1.13) is equal to the sum of terms

$$\begin{aligned}&\{\psi_N(\psi_{i_{N-m+1}}(\cdots\psi_{i_N}^+)^+\cdots)^+(\psi_{i_{N-m-1}}\cdots(\psi_{i_{N-k+1}}\psi_{i_{N-k}}^-)^-\cdots)^-\}^+\cdot\\&\cdot(\psi_{i_1}\cdots(\psi_{i_{N-k-2}}\psi_{i_{N-k-1}}^+)^+\cdots)^+\end{aligned}\tag{2.11}$$

over all permutations $i_1, i_2, \cdots, i_{N-m-1}, i_{N-m+1}, \cdots, i_N$ of the integers $1, 2, \cdots, N-1$, and over all integers m and k which satisfy $0\le m\le k\le N-1$. It is easy to see that the terms of type (2.10) and those of type (2.11) are actually the same by means of the change of subscript

$$\begin{array}{lll} i_{N-m+1}\longrightarrow j_1 , & i_{N-m-1}\longrightarrow j_k , & i_1 \longrightarrow j_{k+1} ,\\ \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots\\ i_N \longrightarrow j_m , & i_{N-k}\longrightarrow j_{m+1}, & i_{N-k-1}\longrightarrow j_{N-1} .\end{array}$$

This completes the proof of Lemma 1.

Before proving Theorems 1, 1*, and 2, we observe a fact. The elements of the Banach algebra (or algebra) A satisfy condition (1.11). To see this one simply puts $\varphi=\varphi_1+\varphi_2$ into (1.8).

Proof of Theorem 1. It is known that for $|\lambda|\cdot\|\varphi\|N<1$,

$$\psi=1+\lambda\varphi^++\lambda^2(\varphi\varphi^+)^++\lambda^3(\varphi(\varphi\varphi^+)^+)^++\cdots$$

is a unique solution of the equation

$$\psi=1+\lambda(\varphi\psi)^+ .$$

By the remark preceding this proof we know that Lemma 1 applies. From Lemma 1 with $\varphi=\psi_1=\psi_2=\cdots=\psi_n$, we get the relation (see for example [1] or [4])

$$\begin{aligned}&n!\,(\varphi(\varphi\cdots(\varphi\varphi^+)^+\cdots)^+)^+\\&=\sum_{1\alpha_1+\cdots+n\alpha_n=n}n!\prod_{k=1}^{n}\left[\frac{(\theta^{k-1}\varphi^k)^+}{k}\right]^{\alpha_k}\frac{1}{\alpha_k!} .\end{aligned}\tag{2.12}$$

The summation in (2.12) extends over all sets of non-negative integers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which $1\alpha_1+2\alpha_2+\cdots+n\alpha_n=n$, Now if $|\lambda|\cdot\|\varphi\|\cdot\|\theta\|<1$, then

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\lambda^k}{k}(\theta^{k-1}\varphi^k)^+\right\}$$

can be expanded in a power series in λ. Furthermore, the coefficient of λ^n *is* exactly the right side of (2.12) divided by $n!$ This implies that

$$\begin{aligned}\psi &= 1 + \lambda\varphi^+ + \lambda^2(\varphi\varphi^+)^+ + \lambda^3(\varphi(\varphi\varphi^+)^+)^+ + \cdots \\ &= \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\theta^{k-1}\varphi^k)^+\right\} .\end{aligned}$$

and Theorem 1 is proved.

Proof of Theorem 1*. Once again by Lemma 1 we have (in the new notation)

$$n!\, a_n = \sum_{1\alpha_1+\cdots+n\alpha_n=n} n! \prod_{k=1}^{n} \left[\frac{(b^{k-1}a^k)T}{k}\right]^{\alpha_k} \frac{1}{\alpha^k!} , \tag{2.13}$$

where the summation extends over all sets of non-negative integers $\alpha_1, \cdots, \alpha_n$ for which $1\alpha_1 + \cdots + n\alpha_n = n$. Relation (2.13) is equivalent to (1.10′), proving Theorem 1*.

Proof of Theorem 2. Consider first the equation

$$\psi = \Phi^+ + \lambda(\varphi\psi)^+ ,$$

where φ and Φ are elements of A. For $|\lambda| \cdot \|\varphi\| \cdot N < 1$ a unique solution of the above equation is

$$\psi = \Phi^+ + \lambda(\varphi\Phi^+)^+ + \lambda^2(\varphi(\varphi\Phi^+)^+)^+ + \lambda^3(\varphi(\varphi(\varphi\Phi^+)^+)^+)^+ + \cdots \tag{2.14}$$

We denote by p_m and q_m, respectively, the coefficients of λ^m in p and q as defined in (1.14). By Theorem 1 we see that

$$\begin{aligned}p_m &= (\underbrace{\varphi(\varphi \cdots (\varphi\varphi}_{m}^+)^+ \cdots)^+)^+ \\ q_m &= (\underbrace{\varphi(\varphi \cdots (\varphi\varphi}_{m}^-)^- \cdots)^-)^- .\end{aligned}$$

We now apply (2.2), where $\tilde{\psi}_1 = \cdots = \tilde{\psi}_n = \varphi$ and $\tilde{\psi}_{n+1} = \Phi$. In the notation of Lemma 2, the coefficients of λ^n in (2.14) are

$$\tilde{p}_{n+1} = \sum_{m=0}^{n} \tilde{p}_m \tilde{r}_{n+1,m} = \sum_{m=0}^{n} p_m(\Phi q_{n-m})^+ . \tag{2.15}$$

Thus,

$$\psi = p\sum_{n=0}^{\infty} \lambda^n(\Phi q_n)^+ = p(\Phi q)^+ . \tag{2.16}$$

Next, consider the equation

$$\psi = \Phi + \lambda(\varphi\psi)^+ .$$

A unique solution for $|\lambda|\cdot\|\varphi\|\cdot N < 1$ is given in this case by

$$\psi = \Phi + \lambda(\varphi\Phi)^+ + \lambda^2(\varphi(\varphi\Phi)^+)^+ + \lambda^3(\varphi(\varphi(\varphi\Phi)^+)^+)^+ + \cdots \tag{2.17}$$

One sees easily that $\psi - \Phi$ in (2.17) is similar to (2.14) except that the Φ^+ in (2.14) has been replaced by $\lambda(\varphi\Phi)^+$ in (2.17). Thus, we have the indicated solution.

Finally, there is a unique solution to

$$\psi = 1 + \lambda u(\varphi\psi)^+ + \lambda(\varphi\psi)^- .$$

From Theorem 1 applied to both $+$ and to $-$,

$$p_u \equiv p(\lambda u) = 1 + \lambda u(\varphi p_u)^+,\ q = 1 + \lambda(\varphi q)^- . \tag{2.18}$$

From (1.11) it follows that for any two elements ψ_1, ψ_2 of A

$$(\psi_1\psi_2^-)^+ + (\psi_2\psi_1^+)^- = \psi_1^+\cdot\psi_2^- . \tag{2.19}$$

Thus, by (2.19) and (2.18)

$$\begin{aligned} p_u q &= 1 + \lambda u(\varphi p_u)^+ + \lambda(\varphi q)^- + \lambda^2 u(\varphi p_u)^+(\varphi q)^- \\ &= 1 + \lambda u\{\varphi p_u[1 + \lambda(\varphi q)^-]\}^+ + \lambda\{\varphi q[1 + \lambda u(\varphi p_u)^+]\}^- \\ &= 1 + \lambda u(\varphi p_u q)^+ + \lambda(\varphi p_u q)^- . \end{aligned}$$

3. Examples. In this section we illustrate the use of the previous results by means of three simple examples.

EXAMPLE 1. *Symmetric functions.* Let $x_1, x_2, \cdots, x_n, \cdots$ be a sequence of commuting symbols and let R be the commutative ring generated by the rationals and $x_1, x_2, x_3, \cdots$. Finally let A be the commutative algebra of all sequences $a_1 = (r_1, r_2, r_3, \cdots)$, $a_2 = (s_1, s_2, s_3, \cdots)$ etc., where r_i, s_i are in R, and for which addition and multiplication are defined by

$$\begin{aligned} ra_1 &= (rr_1, rr_2, rr_3, \cdots) \\ a_1 + a_2 &= (r_1 + s_1, r_2 + s_2, r_3 + s_3, \cdots) \\ a_1 a_2 &= (r_1 s_1, r_2 s_2, r_3 s_3, \cdots) . \end{aligned}$$

If $a_1 = (r_1, r_2, r_3, \cdots)$ we define T by

$$a_1 T = (0, r_1, r_1 + r_2, \cdots, \sum_{k=1}^{n-1} r_k, \cdots) .$$

It is an easy exercise to show that for any a in A, condition (1.8′) is satisfied where $b = (-1, -1, -1, \cdots)$. Consider in particular the element

$$a = (x_1, x_2, x_3, \cdots) \tag{3.1}$$

Set

$$\begin{aligned}\sigma_0^{(k)} &= 1\\ \sigma_1^{(k)} &= x_1 + x_2 + x_3 \cdots + x_k\\ \sigma_2^{(k)} &= x_1 x_2 + x_1x_3 + \cdots + x_1x_k + x_2x_3 + \cdots + x_{k-1}x_k\\ &\vdots\\ \sigma_k^{(k)} &= x_1x_2x_3 \cdots x_k\\ \sigma_n^{(k)} &= 0 \ (n \geq k+1)\ .\end{aligned}$$

It is easy to show by an inductive argument that for the a in (3.1)

$$a_n = (a(a \cdots \underbrace{(a(aT))T \cdots T}_{n}))T = (\sigma_n^{(0)}, \sigma_n^{(1)}, \sigma_n^{(2)}, \cdots, \sigma_n^{(n-1)}, \cdots) \tag{3.2}$$

Using Theorem 1*

$$\sum_{n=0}^{\infty} a_n x^n = \exp\left\{-\sum_{k=1}^{\infty}(-1)^k \frac{x^k}{k}(a^k T)\right\} \varepsilon A\langle x\rangle\ . \tag{3.3}$$

But, if we set

$$s_n^{(k)} = \sum_{m=1}^{n} x_m^k\ ,$$

then for the element in (3.1)

$$a^k T = (0, s_1^{(k)}, s_2^{(k)}, \cdots, s_{n-1}^{(k)}, \cdots)\ . \tag{3.4}$$

Equating $(n+1)$th components on both sides of (3.3)

$$\sum_{k=0}^{\infty} x^k \sigma_k^{(n)} = \exp\left\{-\sum_{k=1}^{\infty}(-1)^k \frac{x^k}{k} s_n^{(k)}\right\} \varepsilon R\langle x\rangle\ .$$

Example 2. *Distribution of* $\max(0, {}_1S, \cdots, S_n)$[2]. Let A be the Banach algebra of functions

$$\varphi = \int_{-\infty}^{\infty} e^{itx} dG(x) \text{ with } \|\varphi\| = \int_{-\infty}^{\infty} |dG(x)| < \infty\ , \tag{3.5}$$

with pointwise multiplication for product. Let

$$\varphi^+ \equiv \int_0^{\infty} e^{itx} dG(x) + G(0) - G(-\infty)\ .$$

Condition (1.8) is satisfied in this case with $\theta = 1$. Thus

$$\psi = 1 + \lambda(\varphi\psi)^+ \tag{3.6}$$

has the unique solution

[2] The author is indebted to E. Sparre Andersen for this example. A similar general Banach algebra approach to this example can also be found in Wendel [6].

$$\psi = \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+\right\}, \qquad (|\lambda| < 1/\|\varphi\|). \tag{3.7}$$

Let $\{X_k\}$ be a sequence of independent, identically distributed random variables with characteristic function

$$\varphi = \int_{-\infty}^{\infty} e^{itx} dF(x) \tag{3.8}$$

and let $S_0 = 0$, $S_k = X_1 + \cdots + X_k$. Define $M_n = \max(0, S_1, \cdots, S_n)$ and let

$$\begin{aligned} \varphi_n &= \int_0^{\infty} e^{itx} d_x P\{M_n < x\}, && (n \geq 0), \\ \psi &= \sum_{n=0}^{\infty} \varphi_n \lambda^n, && (|\lambda| < 1). \end{aligned} \tag{3.9}$$

We now introduce $M_{n,1} = \max(0, S_2 - S_1, S_3 - S_1, \cdots, S_{n+1} - S_1)$. Since the X_k's are identically distributed, $M_{n,1}$ also has the characteristic function φ_n given in (3.9). Moreover, we note that

$$M_{n+1} = \max(0, X_1 + M_{n,1}). \tag{3.10}$$

If $\tilde{\varphi}$ is the characteristic function of any random variable $\tilde{X}$, then $\tilde{\varphi}^+$ is the characteristic function of $\max(0, \tilde{X})$. In the notation of (3.8) and (3.9) and using (3.10),

$$\varphi_{n+1} = (\varphi\varphi_n)^+. \tag{3.11}$$

Thus, the ψ of (3.9) satisfies (3.6) with φ given by (3.8). This means

$$\begin{aligned} &\sum_{n=0}^{\infty} \lambda^n \int_0^{\infty} e^{itx} d_x P\{M_n < x\} \\ &= \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k}\left[\int_0^{\infty} e^{itx} d_x P\{S_k < x\} + P\{S_k < 0\}\right]\right\}. \end{aligned} \tag{3.12}$$

Equation (3.12) is the original Spitzer's identity in [**4**]. A connection between this example and the Wiener-Hopf equation can be found in [**5**].

Example 3. *Number of positive partial sums.* Let A be as defined in example 2. Set

$$\varphi^+ = \int_{0^+}^{\infty} e^{itx} dG(x). \tag{3.13}$$

Condition (1.8) is satisfied where $\theta = 1$, and the norm of $+$ is $N = 1$.

Let $\{X_k\}$ be a sequence of independent, identically distributed random variables with characteristic function

$$\varphi = \int_{-\infty}^{\infty} e^{itx} dF(x) \tag{3.14}$$

and let $S_0 = 0, S_k = X_1 + \cdots + X_k$. Let N_n denote the number of positive partial sums among $S_0, S_1, \cdots, S_n$, and set

$$(3.15)\qquad \begin{aligned} \psi_{nm} &= \int_{-\infty}^{\infty} e^{itx} d_x P\{N_n = m, S_n < x\} , \qquad (m \leq n) , \\ \psi &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \psi_{nm} u^m \lambda^n , \qquad (|u|, |\lambda| < 1) . \end{aligned}$$

Now, for the φ in (3.14)

$$(3.16)\qquad \begin{aligned} (\varphi\psi_{nm})^+ &= \int_{0^+}^{\infty} e^{itx} d_x P\{N_{n+1} = m + 1, S_{n+1} < x\} = \psi^+_{n+1,m+1} \\ (\varphi\psi_{nm})^- &= \int_{-\infty}^{0+} e^{itx} d_x P\{N_{n+1} = m, S_{n+1} < x\} = \psi^-_{n+1,m} . \end{aligned}$$

Thus, by (3.16)

$$\psi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \psi^+_{nm} u^m \lambda^n + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi^-_{nm} u^m \lambda^n = 1 + u\lambda(\varphi\psi)^+ + \lambda(\varphi\psi)^- .$$

By Theorem 2 part (c), we have the generating function ψ for the number of positive partial sums $S_k = X_1 + \cdots + X_k$

$$\psi = \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k}\left[\int_{-\infty}^{0+} e^{itx} dP\{S_k < x\} + u^k \int_{0^+}^{\infty} e^{itx} dP\{S_k < x\}\right]\right\} .$$

This example was considered previously by the author in [3] and by Andersen in [1].

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