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An Analytical Solution in the Complex Plane for the Luminosity Distance in Flat Cosmology

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Abstract

We present an analytical solution for the luminosity distance in spatially flat cosmology with pressureless matter and the cosmological constant. The complex analytical solution is made of a real part and a negligible imaginary part. The real part of the luminosity distance allows finding the two parameters H_0 and Ω_M. A simple expression for the distance modulus for SNs of type Ia is reported in the framework of the minimax approximation.

Keywords

Cosmology, Observational Cosmology, Distances, Redshifts, Radial Velocities, Spatial Distribution of Galaxies

1. Introduction

The luminosity distance in flat cosmology has been recently investigated using different approaches. A fitting formula which has a maximum relative error of 4% in the case of common cosmological parameters has been introduced by [1]. An approximate solution in terms of Padé approximants has been presented by [2]. The integral of the luminosity distance has been found in terms of elliptical integrals of the first kind by [3].

2. Flat Cosmology

Following Equation (2.1) in [2], the luminosity distance d_L is

$$d_L\left(z;c,H_0,\Omega_M\right)=\frac{c}{H_0}(1+z)\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}a}{\sqrt{\Omega_M a+\left(1-\Omega_M\right)a^4}}, \qquad (1)$$

where H_0 is the Hubble constant expressed in $\mathrm{km\cdot s^{-1}\cdot Mpc^{-1}}$, c is the speed of light expressed in $\mathrm{km\cdot s^{-1}}$, z is the redshift, a is the scale-factor, and Ω_M is

$$\Omega_{\mathrm{M}}=\frac{8\pi G\rho_0}{3H_0^2}, \tag{2}$$

where G is the Newtonian gravitational constant and ρ_0 is the mass density at the present time. We now introduce the indefinite integral

$$\Phi(a)=\int\frac{\mathrm{d}a}{\sqrt{\Omega_{\mathrm{M}}a+(1-\Omega_{\mathrm{M}})a^4}}. \tag{3}$$

The solution is in terms of F, the Legendre integral or incomplete elliptic integral of the first kind

$$\Phi(a)=\frac{-4F(b_1,b_2)b_3b_4b_6b_1b_5}{b_7b_8\sqrt{b_9}b_{10}}, \tag{4}$$

where the incomplete elliptic integral of the first kind is

$$F(x,k)=\int_0^x\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \tag{5}$$

see formula (19.2.4) in [4], and

$$b_1=\sqrt{-\frac{a(\Omega_{\mathrm{M}}-1)\left(i\sqrt{3}+3\right)}{\left(-\Omega_{\mathrm{M}}a+\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+a\right)\left(i\sqrt{3}+1\right)}}, \tag{6}$$

$$b_2=\sqrt{\frac{\left(i\sqrt{3}+1\right)\left(i\sqrt{3}-3\right)}{\left(i\sqrt{3}+3\right)\left(i\sqrt{3}-1\right)}}, \tag{7}$$

$$b_3=\sqrt{\frac{i\sqrt{3}\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+2\Omega_{\mathrm{M}}a+\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}-2a}{\left(-\Omega_{\mathrm{M}}a+\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+a\right)\left(i\sqrt{3}+1\right)}}, \tag{8}$$

$$b_4=\sqrt{\frac{-i\sqrt{3}\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+2\Omega_{\mathrm{M}}a+\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}-2a}{\left(-\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+a(\Omega_{\mathrm{M}}-1)\right)\left(i\sqrt{3}-1\right)}}, \tag{9}$$

$$b_5=i\sqrt{3}+1, \tag{10}$$

$$b_6=\left(-\Omega_{\mathrm{M}}a+\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}+a\right)^2, \tag{11}$$

$$b_7=\sqrt[3]{\Omega_{\mathrm{M}}(\Omega_{\mathrm{M}}-1)^2}, \tag{12}$$

$$b_8=i\sqrt{3}+3, \tag{13}$$

$$b_9=\left(-4a^4+4a\right)\Omega_{\mathrm{M}}+4a^4, \tag{14}$$

$$b_{10}=\Omega_{\mathrm{M}}-1, \tag{15}$$

with $i^2=-1$. The incomplete elliptic integral $F(x,k)$ of complex arguments is evaluated according to Equation (17.4.11) in [5] or Section 19.7 (ii) in [4]. The luminosity distance is

$$d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)=\Re\left(\frac{c}{H_0}(1+z)\left(\Phi(1)-\Phi\left(\frac{1}{1+z}\right)\right)\right), \tag{16}$$

where $\Re$ means the real part.

The distance modulus is

$$(m-M)=25+5\log_{10}\left(d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)\right). \tag{17}$$

An approximation can be found when the argument of the integral (1) is expanded about $a=1$ in a Taylor series of order 10. The resulting Taylor approximation of order 10 to the luminosity distance, $d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)_{10}$, is

$$\begin{aligned}&d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)_{10}=\\&\frac{c(1+z)}{H_0}\left(\frac{1}{2}\left(\frac{3}{2}\Omega_{\mathrm{M}}-2\right)\left(1-(1+z)^{-2}\right)+3-3(1+z)^{-1}\right.\\&\left.-\frac{3}{2}\Omega_{\mathrm{M}}\left(1-(1+z)^{-1}\right)\right)+\cdots\end{aligned} \tag{18}$$

where we have reported the first few terms of the series. The goodness of the Taylor approximation is evaluated through the percentage error, δ, which is

$$\delta=\frac{\left|d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)-d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)_{10}\right|}{d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)}\times 100. \tag{19}$$

As an example when $H_0=70\ \mathrm{km\cdot s^{-1}\cdot Mpc^{-1}}$, $\Omega_{\mathrm{M}}=0.3$, $c=299792.458\ \mathrm{km\cdot s^{-1}}$ and $z=4$, we obtain $\delta=0.61\%$. As an example with the above parameters, d_{L} has its angle in the complex plane, θ, very small: $\theta\approx 10^{-11}$, which means that the solution is real for practical purposes. In the last years the Hubble Space Telescope (HST) has allowed the determination of the cosmological parameters through the modulus of the distance for SNs of type Ia, see [6]-[10]. At the moment of writing the two unknown parameters, H_0 and Ω_{M}, can be derived from two catalogs for the distance modulus of SNs of type Ia: 580 SNe in the Union 2.1 compilation, see [11] with data at http://supernova.lbl.gov/Union/, and 740 SNe in the joint light-curve analysis (JLA), see [12] with data at http://supernovae.in2p3.fr/sdss_snls_jla/ReadMe.html. This kind of analysis is not new and has been used, for example, by [13].

The best fit for the distance modulus of SNs is obtained adopting the Levenberg-Marquardt method (subroutine MRQMIN in [14]). The statistical parameters here adopted are the merit function or chi-square, χ^2, the reduced chi-square, χ^2_{red} and the maximum probability of obtaining a better fitting, Q, see Section 2.3 in [15] for more details. **Table 1** reports H_0 and Ω_{M} for the two catalogs of SNs and **Figure 1** and **Figure 2** display the best fits.

The Taylor approximation of order 10 to the distance modulus, $d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)_{10}$, is

$$(m-M)_{10}=25+5\log_{10}\left(d_{\mathrm{L}}\left(z;c,H_0,\Omega_{\mathrm{M}}\right)_{10}\right). \tag{20}$$

Table 1. Numerical values of χ^2, χ^2_{red} and Q where k stands for the number of parameters.

Compilation	SNs	k	Parameters	χ^2	χ^2_{red}	Q
Union 2.1	580	2	$H_0 = 70 \pm 0.34$; $\Omega_M = 0.277 \pm 0.019$	562.22	0.972	0.673
JLA	740	2	$H_0 = 69.83 \pm 0.31$; $\Omega_M = 0.287 \pm 0.018$	627.82	0.85	0.998

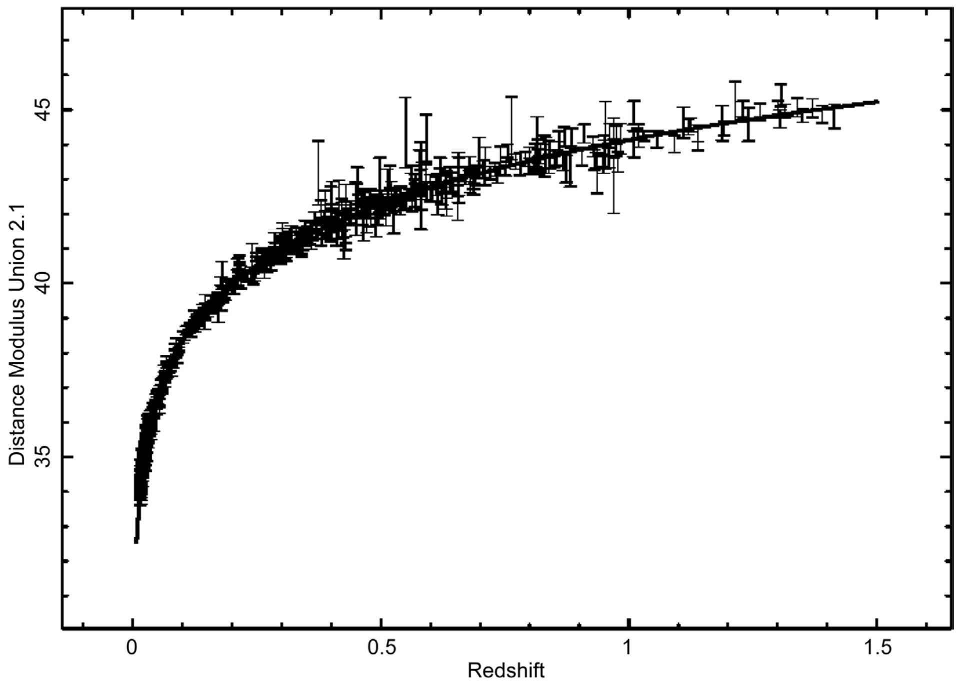

Figure 1. Hubble diagram for the Union 2.1 compilation. The solid line represents the best fit for the exact distance modulus in flat cosmology as represented by Equation (7), parameters as in first line of **Table 1**.

The above equation takes a simple expression when the minimax rational approximation is used, see [4] [16] [17]; here we have used a polynomial of degree 3 for the numerator and degree 2 for the denominator. With the parameters of **Table 1** for the Union 2.1 compilation over the range in $z \in [0,4]$, we obtain the following minimax approximation

$$(m-M)_{3,2,10} = \frac{0.413991 + 6.080622z + 5.501967z^2 + 0.029254z^3}{0.012154 + 0.148352z + 0.112017z^2} \tag{21}$$

the maximum error being 0.002956.

3. Conclusion

We have presented an analytical approximation for the luminosity distance in terms of elliptical integrals with complex argument. The fit of the distance modulus of SNs of type Ia allows finding the pair H_0 and Ω_M for the Union 2.1 and JLA compilations.

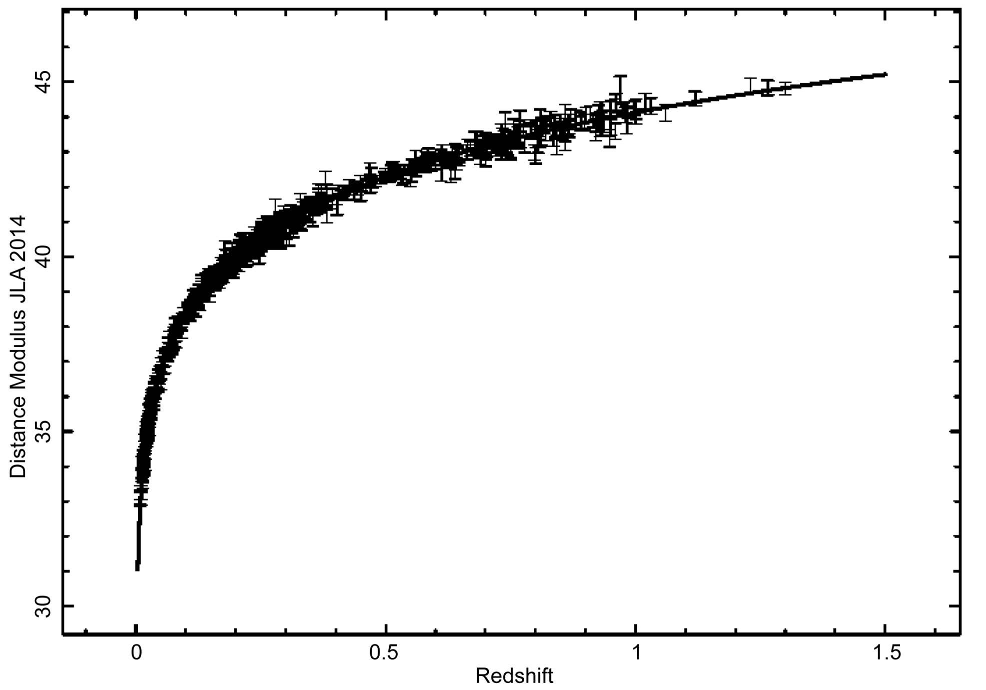

Figure 2. Hubble diagram for the JLA compilation. The solid line represents the best fit for the exact distance modulus in flat cosmology as represented by Equation (7), parameters as in second line of Table 1.

A simple expression for the distance modulus relative to the Union 2.1 compilation is given through the minimax approximation applied to a Taylor expansion of the luminosity distance of order 10.

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