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# An analytical solution to the problem of time-fractional heat conduction in a composite sphere

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**Abstract.** An analytical solution to the problem of time-fractional heat conduction in a sphere consisting of an inner solid sphere and concentric spherical layers is presented. In the heat conduction equation, the Caputo time-derivative of fractional order and the Robin boundary condition at the outer surface of the sphere are assumed. The spherical layers are characterized by different material properties and perfect thermal contact is assumed between the layers. The analytical solution to the problem of heat conduction in the sphere for time-dependent surrounding temperature and for time-space-dependent volumetric heat source is derived. Numerical examples are presented to show the effect of the harmonically varying intensity of the heat source and the harmonically varying surrounding temperature on the temperature in the sphere for different orders of the Caputo time-derivative.

**Key words:** heat conduction, multilayered sphere, Caputo fractional derivative.

## 1. Introduction

Heat conduction problems in layered slabs, layered cylinders, and layered spheres modelled according to Fourier's law by a parabolic differential equation have been considered by many authors, for example in [1–3], where analytical solutions to the problems in the form of eigenfunction expansions were presented. Heat conduction in layered bodies in spherical coordinates was recently investigated in [4–10]. An analytical solution to the problem of heat conduction in a multilayered sphere with time-dependent boundary conditions was derived by Lu and Viljanen in [4]. The solution was obtained using the Laplace transform wherein an approximate inverse Laplace transform was determined by using a residue theorem. An exact solution of the radial heat conduction problem in a hollow multilayered sphere was presented by Siedlecka in [5]. The considerations concern heat conduction modeled by the parabolic differential equation. The solution was obtained using the Green's function method. An analytical series solution for a two-dimensional, transient boundary-value problem for multilayered heat conduction in spherical coordinates has been presented by Jain et al. in [6]. In the formulation of the problem, time-independent volumetric heat sources in the concentric layers were assumed. The obtained solution can be used to determine the temperature distribution in full sphere, hemisphere, spherical wedge, and spherical cone. A similar approach was also applied in [7] to one-dimensional heat conduction problems for nuclear applications. The steady-state temperature distribution in the functionally graded sphere, subjected to temperature gradient and internal pressure, was investigated by Bayat et al. in [8]. Temperature distribution was used to determine the thermal stresses in the sphere. An analytical solution to the problem for thermal and mechanical properties of the sphere was obtained in the form of power functions of the radial direction. Thermal stresses in a sphere of a functionally graded material were also considered by Pawar et al. [9]. Transient temperature distribution was determined by assuming that the material properties of the sphere were exponential functions of the radial direction.

The parabolic differential equation of heat conduction, derived under the framework of the classical theory of heat conduction is based on the local Fourier law. Non-local generalizations of the Fourier law lead to non-classical theories, in which the parabolic equation is replaced by a time-fractional and/or space-fractional heat conduction differential equation [10]. In these fractional differential equations, different kinds of derivatives of fractional order are used (the Riemann-Liouville derivative, the Caputo derivative and the Grűnwald-Letnikov derivative). Moreover, the boundary conditions may also include the fractional derivatives. The fundamentals of fractional calculus and of the theory of fractional differential equations are given in [11–15]. Some applications of fractional order calculus to modelling of real-world phenomena are presented in [16–18].

Heat conduction problems formulated under the framework of the non-classical theories in the spherical coordinates with fractional Caputo or Riemann-Liouville derivatives were studied in [19, 20]. An approximate analytical solution of time-fractional heat conduction in a composite medium consisting of an infinite matrix and a spherical inclusion is presented by Povstenko in [19]. The perfect thermal contact was realized by the conditions of equality of temperatures and heat fluxes at the boundary surfaces, wherein the heat fluxes are expressed by a Riemann-Liouville fractional derivative. An analytical solution to the problem of time-fractional heat conduction in a multilayered slab was presented by Siedlecka and Kukla in [20]. Ning and Jiang [21] use the Laplace transform and the method of variable separation to determine an analytical

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solution of the time-fractional equation for three-dimensional heat conduction in spherical coordinates.

From a mathematical point of view, the heat conduction equation and the diffusion equation are identical, which means that the same methods can be used to determine their solutions. In [22], Povstenko presents a solution of the diffusion-wave time-fractional equation with a source term. The solution is expressed by fundamental solutions, which are also derived. The Neumann problem for time-fractional differential equation in a sphere was considered by Povstenko in [23]. The presented results of numerical computations show the solutions as functions of distance from the center of the sphere for various orders of the time-fractional derivative. The fractional diffusion problems considered in [24, 25] were solved using the Green's function method. Lucena et al. [24] considered two diffusion problems – the first with inhomogeneous time-dependent boundary conditions and the second for diffusion with external force. Radial changes in the diffusion coefficient and the external force were assumed. In [25], a fractional diffusion equation with a spatial time-dependent coefficient and with external force was investigated. A numerical solution to the problem was obtained using a finite difference method. In [26], Abbas applied fractional order theory to study thermoelastic diffusion in an infinite medium with a spherical cavity using the Laplace transformation and the eigenvalue approach.

In this paper, time-fractional heat conduction in a multilayered solid sphere is studied. A space-time dependent volumetric inner heat source in the sphere, time-dependent ambient temperature, and perfect thermal contact at boundaries of the layers are assumed. An exact solution to the radial heat conduction equation with the Caputo time-derivative in the form of an eigenfunction expansion is presented.

## 2. Formulation of the problem

Consider the radial heat conduction in a solid sphere consisting of an inner solid sphere and $n-1$ concentric layers. The cross-section of the sphere is shown in Fig. 1. The time-fractional differential equation in spherical coordinates governing the temperature $T_i(r, t)$ in the $i$-th layer is given in [19]

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT_i)+\frac{1}{\lambda_i}q_i(r,t)=\frac{1}{a_i}\frac{\partial^\alpha T_i}{\partial t^\alpha},\qquad (1)$$

$$0<\alpha\le 2,\quad r\in[r_{i-1},r_i],\quad i=1,\dots,n,$$

where $\lambda_i$ is the constant thermal conductivity, $a_i$ is the constant thermal diffusivity, $q_i(r, t)$ is the volumetric energy generation, $r_i$ is the outer radius of the $i$-th layer ($r_0 = 0$, $r_n = b$), and $\alpha$ denotes the fractional order of a Caputo derivative with respect to time $t$. The Caputo fractional derivative is defined in [27]

$$\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{d^m f(\tau)}{d\tau^m}d\tau,\qquad (2)$$

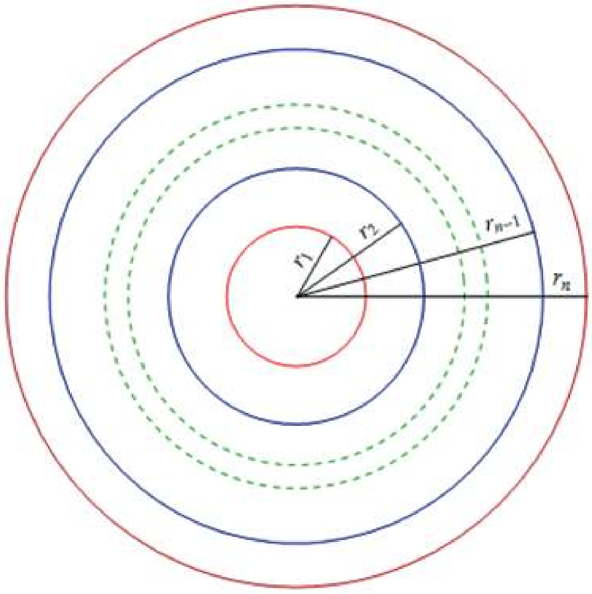


Fig. 1. Cross-section of a solid multilayered sphere

where $m-1<\alpha<m$, $m\in N=\{1,2,\dots\}$. The geometric and physical interpretation of the fractional derivatives are given in [28].

The condition at the center, the mathematical boundary condition on the outer surface of the sphere, and the mathematical conditions of perfect thermal contact at interfaces are [19]:

$$|T_1(0,t)|<\infty,\qquad (3)$$

$$T_i(r_i,t)=T_{i+1}(r_i,t),\quad i=1,\dots,n-1,\qquad (4)$$

$$\lambda_i\frac{\partial T_i}{\partial r}(r_i,t)=\lambda_{i+1}\frac{\partial T_{i+1}}{\partial r}(r_i,t),\quad i=1,\dots,n-1,\qquad (5)$$

$$\lambda_n\frac{\partial T_n}{\partial r}(r_n,t)=a_\infty\left(T_\infty(t)-T_n(r_n,t)\right),\qquad (6)$$

where $a_\infty$ is the heat transfer coefficient and $T_\infty$ is the ambient temperature. We assume the initial temperature in each layer as:

$$T_i(r,0)=f_i(r),\quad r\in[r_{i-1},r_i],\quad i=1,\dots,n.\qquad (7a)$$

If fractional order $\alpha$ is in the interval (1, 2], then the initial condition for the derivative $\partial T_i/\partial t$ is also required [22]. We assume that the condition is

$$\left.\frac{\partial T_i}{\partial t}\right|_{t=0}=g_i(r),\quad r\in[r_{i-1},r_i],\quad i=1,\dots,n.\qquad (7b)$$

In order to transform the non-homogeneous boundary condition (6) into a homogeneous one, we assume the temperature $T_i(r, t)$ in the form of a sum:



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$$T_i(r,t) = \theta_i(r,t) + T_\infty(t), \quad i = 1,2,...,n, \tag{8}$$

where $\theta_i(r, t)$ are the newly-searched functions. Next, to obtain a differential equation with constant coefficients, we introduce the functions:

$$V_i(r,t) = r\theta_i(r,t), \quad i = 1,...,n. \tag{9}$$

Combining the transformations (8, 9), we obtain the relationship between the functions $T_i(r, t)$ and $V_i(r, t)$ in the form of:

$$T_i(r,t) = \frac{1}{r}V_i(r,t) + T_\infty(t), \quad i = 1,..,n. \tag{10}$$

In (10) we have: $r \in (0, r_1]$ for $i = 1$ and $r \in [r_{i-1}, r_i]$ for $i = 2, \ldots, n$.

Taking into account functions $T_i(r, t)$, given by (10), into (1) and conditions (3–7), the formulation of the problem for functions $V_i(r, t)$ is received. The fractional differential equation with constant coefficients and the homogeneous boundary conditions are:

$$\frac{\partial^2 V_i}{\partial r^2} + \frac{1}{\lambda_i} q_i^*(r,t) = \frac{1}{a_i}\frac{\partial^\alpha V_i}{\partial t^\alpha}, \quad r \in [r_{i-1}, r_i], \ i = 1,...,n, \tag{11}$$

$$V_1(0,t) = 0, \tag{12}$$

$$V_i(r_i,t) = V_{i+1}(r_i,t), \quad i = 1,...,n-1, \tag{13}$$

$$\lambda_i \frac{\partial V_i(r_i,t)}{\partial r} + r_i^{-1}(\lambda_{i+1} - \lambda_i)V_i(r_i,t) = = \lambda_{i+1}\frac{\partial V_{i+1}(r_i,t)}{\partial r}, \quad i = 1,...,n-1, \tag{14}$$

$$\frac{\partial V_n(r_n,t)}{\partial r} = \left(\frac{1}{r_n} - \frac{a_\infty}{\lambda_n}\right)V_n(r_n,t), \tag{15}$$

where $q_i^*(r, t) = r\left(q_i(r, t) - \frac{\lambda_i}{a_i}\frac{d^\alpha T_\infty(t)}{dt^\alpha}\right)$. The initial conditions are:

– for $\alpha$ in the interval (0,2]:

$$V_i(r,0) = f_i^*(r), \quad r \in [r_{i-1}, r_i], \quad i = 1,...,n, \tag{16a}$$

– for $\alpha$ in the interval (1, 2]:

$$\left.\frac{\partial V_i}{\partial t}\right|_{t=0} = g_i^*(r), \quad r \in [r_{i-1}, r_i], \quad i = 1,...,n, \tag{16b}$$

where:

$$f_i^*(r) = r\left(f_i(r) - T_\infty(0)\right),$$

$$g_i^*(r) = r\left(g_i(r) - T_\infty'(0)\right),$$

and $T_\infty'(0) = \left.\frac{dT_\infty}{dt}\right|_{t=0}$. Equations (11–16) constitute the complete formulation of the initial-boundary value problem.

## 3. Solution of the problem

We seek the solution to problem (11–16) in the form of a series

$$V_i(r,t) = \sum_{k=1}^{\infty} \Lambda_k(t)\Phi_{i,k}(r), \quad i = 1,...,n, \tag{17}$$

where the function $\Phi_{i,k}$ is the $k$-th solution of the following eigenproblem:

$$\frac{d^2\Phi_i(r)}{dr^2} + \frac{\beta^2}{a_i}\Phi_i(r) = 0, \quad r \in [r_{i-1}, r_i], \quad i = 1,...,n, \tag{18}$$

$$\Phi_1(0) = 0, \tag{19}$$

$$\Phi_i(r_i) = \Phi_{i+1}(r_i), \quad i = 1,...,n-1, \tag{20}$$

$$\lambda_i \frac{d\Phi_i(r_i)}{dr} + r_i^{-1}(\lambda_{i+1} - \lambda_i)\Phi_i(r_i) = \lambda_{i+1}\frac{d\Phi_{i+1}(r_i)}{dr}, \quad i = 1,...,n-1, \tag{21}$$

$$\frac{d\Phi_n(r_n)}{dr} = \left(\frac{1}{r_n} - \frac{\alpha_\infty}{\lambda_n}\right)\Phi_n(r_n). \tag{22}$$

The values of parameter $\beta$ will be chosen in a way that non-zero solutions of problem (18–22) exist.

Functions $\Phi_i(r)$, satisfying differential equation (18), is assume in the form of:

$$\Phi_i(r) = A_i \cos\mu_i(r - r_{i-1}) + B_i \sin\mu_i(r - r_{i-1}), \quad \text{for } i = 1,...,n, \tag{23}$$

where $A_i$, $B_i$ are unknown constants and $\mu_i = \beta/\sqrt{a_i}$. Substituting the functions (23) into conditions (19–22), we obtain a system of $2n$ linear equations with respect to the constants $A_i$, $B_i$, $i = 1, \ldots, n$. The equation system can be written in a matrix form:

$$\mathbf{Cd} = \mathbf{0}, \tag{24}$$

where

$\mathbf{d} = [A_1 \quad B_1 \quad A_2 \quad B_2 \quad \ldots \quad A_{n-1} \quad B_{n-1} \quad A_n \quad B_n]^T$

and $\mathbf{C} = [c_{ij}]_{2n\times 2n}$. The non-zero elements of matrix $\mathbf{C}$ are:

$c_{11} = 1$, $c_{2i,2i-1} = \cos\mu_i R_i$, $c_{2i,2i} = \sin\mu_i R_i$,

$c_{2i,2i+1} = -1$, $c_{2i+1,2i-1} = -\sin\mu_i R_i - \frac{1}{\mu_i r_i}\cos\mu_i R_i$,

$c_{2i+1,2i} = \cos\mu_i R_i - \frac{1}{\mu_i r_i}\sin\mu_i R_i$, $c_{2i+1,2i+1} = \frac{\lambda_{i+1}}{\lambda_i \mu_i r_i}$,

$c_{2i+1,2i+2} = -\frac{\lambda_{i+1}\mu_{i+1}}{\lambda_i \mu_i}$ for $i = 1,...,n-1$



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$$c_{2n,2n-1} = \sin\mu_n R_n + \left(\frac{1}{\mu_n r_n} - \frac{a_\infty}{\lambda_n \mu_n}\right)\cos\mu_n R_n \quad \text{and}$$

$$c_{2n,2n} = -\cos\mu_n R_n + \left(\frac{1}{\mu_n r_n} - \frac{a_\infty}{\lambda_n \mu_n}\right)\sin\mu_n R_n,$$

where $R_i = r_i - r_{i-1}$.

The non-zero solution of (24) exists for those values of $\beta$, for which the determinant of the matrix $\mathbf{C}$ vanishes, i.e.:

$$\det\mathbf{C} = 0. \tag{25}$$

Equation (25) is then solved numerically with respect to $\beta$. For determined values $\beta_k$, $k = 1, 2, \ldots$, the corresponding functions

$$\tilde{\Phi}_{i,k}(r) = \begin{cases} \Phi_{i,k}(r), & r_{i-1} \le r \le r_i \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

are appointed. The functions $\Phi_{i,k}$ are given by equation (23) for $\mu_i = \mu_{i,k} = \beta_k/\sqrt{a_i}$. The coefficients $A_i$, $B_i$ occurring in (23) for each $\beta_k$ are determined by solving an equation system, which is obtained by assuming $\beta = \beta_k$, $B_n = 1$ in (24).

It can be shown that the functions $\Phi_{i,k}$ satisfy the orthogonality condition as:

$$\sum_{i=1}^{n}\frac{\lambda_i}{a_i}\int_{r_{i-1}}^{r_i}\Phi_{i,k}(r)\Phi_{i,k'}(r)dr = \begin{cases} 0 & \text{for} \quad k' \ne k \\ N_k & \text{for} \quad k' = k \end{cases}, \tag{27}$$

where: $$N_k = \frac{1}{4\beta_k}\cdot\sum_{i=1}^{n}\frac{\lambda_i}{\sqrt{a_i}}\Big[\left(A_i^2 - B_i^2\right)\sin 2\mu_{i,k}R_i + 4A_iB_i\sin^2\mu_{i,k}R_i + 2\left(A_i^2 + B_i^2\right)\mu_{i,k}R_i\Big]$$

Substituting function $V_i(r, t)$, given by (17), into (11) and by using the orthogonality condition (27), the equation for the function $\Lambda_k(t)$ is obtained:

$$\frac{d^\alpha \Lambda_k(t)}{dt^\alpha} + \beta_k^2\Lambda_k(t) = \frac{1}{N_k}\sum_{i=1}^{n}\int_{r_{i-1}}^{r_i} q_i^*(r,t)\Phi_{i,k}(r)dr. \tag{28}$$

This equation is complemented by the initial conditions, which are obtained on the basis of (16, 17) and the orthogonality condition (27). These initial conditions are:

– for $\alpha$ in the interval (0,2]:

$$\Lambda_k(0) = \frac{1}{N_k}\sum_{i=1}^{n}\frac{\lambda_i}{a_i}\int_{r_{i-1}}^{r_i} f_i^*(r)\Phi_{i,k}(r)dr, \tag{29}$$

– for $\alpha$ in the interval (1,2]:

$$\left.\frac{d\Lambda_k}{dt}\right|_{t=0} = \frac{1}{N_k}\sum_{i=1}^{n}\frac{\lambda_i}{a_i}\int_{r_{i-1}}^{r_i} g_i^*(r)\Phi_{i,k}(r)dr. \tag{30}$$

Introducing the function

$$Q_k(r,\tau) = \sum_{i=1}^{n}\int_{r_{i-1}}^{r_i}\Phi_{i,k}(r)\, q_i^*(r,\tau)\,dr\,d\tau,$$

the solution of (28–30) can be presented as:

$$\begin{aligned}\Lambda_k(t) = {}&\frac{1}{N_k}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\beta_k^2(t-\tau)^\alpha\right)Q_k(r,\tau) + \\ &+\frac{1}{N_k}E_\alpha\left(-\beta_k^2 t^\alpha\right)\sum_{i=1}^{n}\frac{\lambda_i}{a_i}\int_{r_{i-1}}^{r_i}\left(f_i^*(r) + t\,g_i^*(r)\right)\Phi_{i,k}(r)dr,\end{aligned} \tag{31}$$

wherein the term $tg_i^*(r)$ under the second integral occurs only for $\alpha \in (1,2]$. In (31), $E_{\alpha,\beta}$ is a two-parameter Mittag-Leffler function, which is defined by [29]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{32}$$

where $\Gamma$ is the gamma function. Function $E_\alpha$ is defined by $E_\alpha(z) = E_{\alpha,1}(z)$.

Finally, function $V_i(r, t)$ is given by (17), where the functions $\Phi_{i,k}(r)$ and $\Lambda_k(t)$ are given by (23) and (31), respectively. Taking into account the relationship of (10) and (17), (23) and (31), the temperature $T_i(r, t)$ can be expressed by:

$$\begin{aligned}T_i(r,t) = {}&T_\infty(t) + \frac{1}{r}\sum_{j=1}^{n}\int_0^t\int_{r_{j-1}}^{r_j}(t-\tau)^{\alpha-1}q_j^*(r',\tau)\cdot \\ &\cdot\sum_{k=1}^{\infty}\frac{1}{N_k}E_{\alpha,\alpha}\left(-\beta_k^2(t-\tau)^\alpha\right)\Phi_{i,k}(r)\Phi_{j,k}(r')\,dr'd\tau + \\ &\frac{1}{r}\sum_{j=1}^{n}\frac{\lambda_j}{a_j}\int_{r_{j-1}}^{r_j}\sum_{k=1}^{\infty}\frac{1}{N_k}E_\alpha\left(-\beta_k^2 t^\alpha\right)\Phi_{i,k}(r)\Phi_{j,k}(r')F_j(r',t)dr'\end{aligned} \tag{33}$$

where $F_j(r, t) = f_j^*(r) + tg_i^*(r)$.

In a particular case, for $\alpha = \alpha' = 1$, formula (33) presents the solution of the classical heat conduction problem. Introducing the matrix:

$$\mathbf{G}^{\alpha,\alpha'}(r,t;r',\tau) = \left[G_{i,j}^{\alpha,\alpha'}(r,t;r',\tau)\right]_{1\le i,j\le N},$$

where

$$\begin{aligned}G_{i,j}^{\alpha,\alpha'}(r,t;r',\tau) = {}&(t-\tau)^{\alpha'-1} \\ &\sum_{k=1}^{\infty}\frac{1}{N_k}\tilde{\Phi}_{i,k}(r)\tilde{\Phi}_{j,k}(r')\,E_{\alpha,\alpha'}\left(-\beta_k^2(t-\tau)^\alpha\right),\end{aligned} \tag{34}$$

we can write (33) in a matrix form:

$$\begin{aligned}\mathbf{T}(r,t) = {}&\frac{1}{r}\int_0^t\int_0^b r'\mathbf{G}^{\alpha,\alpha'}(r,t;r',\tau)\mathbf{U}(r',\tau)dr'd\tau + \\ &+\frac{1}{r}\int_0^b r'\mathbf{G}^{\alpha,1}(r,t;r',0)\mathbf{F}(r',t)dr',\end{aligned} \tag{35}$$



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where the column-matrices $\mathbf{T}(r, t)$, $\mathbf{U}(r', t)$ and $\mathbf{U}(r', \tau)$ are:

$$\mathbf{T}(r,t)=\left[T_1(r,t)-T_\infty(t) \quad \ldots \quad T_n(r,t)-T_\infty(t)\right]^T,$$

$$\mathbf{U}(r',t)=\left[U_1(r',t) \quad \ldots \quad U_n(r',t)\right]^T,$$

$$\mathbf{F}(r',\tau)=\left[F_1(r',\tau) \quad \ldots \quad F_n(r',\tau)\right]^T,$$

where:

$$U_j(r',\tau)=q_j(r',\tau)-\frac{\lambda_j}{a_j}\frac{d^\alpha T_\infty(\tau)}{dt^\alpha},$$

$$F_j(r',t)=\frac{\lambda_j}{a_j}\Big(f_j(r')-T_\infty(0)+t\big(g_j(r')-T_\infty'(0)\big)\Big).$$

The matrix $\mathbf{G}^{\alpha,\alpha'}$ is a Green's function matrix for the considered problem.

## 4. Temperature distribution in a sphere with harmonically varying heat generation

We assume that volumetric heat generation in the sphere is described by a function defined by

$$q_i(r,t)=\begin{cases} Q_1+Q_2\sin\nu t, & 0\le r\le r_1, \quad i=1 \\ 0, & r_{i-1}\le r\le r_1, \quad i=2,\ldots,n. \end{cases} \tag{36}$$

The initial temperature in the sphere and the ambient temperature are assumed as constants: $T_i(r, 0) = f_i(r) = T_0$ for $i = 1, \ldots, n$ and $T_\infty(t) = T_a$ for $t \ge 0$. Using (33) and (36), the function $T_i(r, t)$ can be rewritten as:

$$\begin{aligned} T_i(r,t)=T_a+\frac{T_0-T_a}{r}\sum_{k=1}^{\infty}\frac{\gamma_k}{N_k}\Phi_{i,k}(r)E_\alpha\left(-\beta_k^2t^\alpha\right)+ \\ +\frac{1}{r}\sum_{k=1}^{\infty}\frac{\varphi_{1,k}}{N_k}\Phi_{i,k}(r)\left(Q_1J^1_{\alpha,k}(t)+Q_2J^2_{\alpha,k}(t)\right), \end{aligned} \tag{37}$$

where $\gamma_k=\sum_{j=1}^{n}\frac{\lambda_j\varphi_{j,k}}{a_j}$ and

$$\begin{aligned} \varphi_{j,k}=\int_{r_{j-1}}^{r_j} r'\Phi_{j,k}(r')dr'=\frac{1}{\mu_{j,k}^2}\Big[A_{j,k}\Big(\cos\mu_{j,k}\left(r_j-r_{j-1}\right)+ \\ +\mu_{j,k}r_j\sin\mu_{j,k}\left(r_j-r_{j-1}\right)-1\Big)+B_{j,k}\Big(\mu_{j,k}r_{j-1}- \\ -\mu_{j,k}r_j\cos\mu_{j,k}\left(r_j-r_{j-1}\right)+\sin\mu_{j,k}\left(r_j-r_{j-1}\right)\Big)\Big], \end{aligned} \tag{38}$$

$$J^1_{\alpha,k}(t)=\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\beta_k^2\tau^\alpha\right)d\tau, \tag{39}$$

$$J^2_{\alpha,k}(t)=\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\beta_k^2\tau^\alpha\right)\sin\nu(t-\tau)d\tau. \tag{40}$$

Functions $J^1_{\alpha,k}(t)$ and $J^2_{\alpha,k}(t)$ for $\alpha = 1$ and $\alpha = 2$ can be expressed in a simple form by using the properties of the Mittag-Leffler function [29]: $E_1(z) = e^z$, $E_{2,2}(-z) = \sin\sqrt{z}/\sqrt{z}$. After calculations, the integrals in (39, 40), we obtain:

$$J^1_{1,k}(t)=\frac{1}{\beta_k^2}\left(1-e^{-\beta_k^2t}\right),\quad J^1_{2,k}(t)=\frac{1}{\beta_k^2}\left(1-\cos\beta_kt\right), \tag{41}$$

$$J^2_{1,k}(t)=\frac{1}{\beta_k^4+\nu^2}\left[\nu e^{-\beta_k^2t}-\nu\cos\nu t+\beta_k^2\sin\nu t\right], \tag{42a}$$

$$J^2_{2,k}(t)=\frac{1}{\beta_k^2-\nu^2}\left[\sin\nu t-\frac{\nu}{\beta_k}\sin\beta_kt\right], \tag{42b}$$

The integral occurring in (41) for $0 < \alpha \le 2$, can be expressed as [22]:

$$J^1_{\alpha,k}(t)=t^\alpha E_{\alpha,\alpha+1}\left(-\beta_k^2t^\alpha\right). \tag{43}$$

The convolution integral occurring in equation (40) will be determined for a rationale number of order $\alpha$, by applying the properties of the Laplace transform, namely by using the convolution rule, we obtain the Laplace transform of the function $J^2_{\alpha,k}(t)$ in the form of:

$$L\left\{J^2_{\alpha,k}(t)\right\}=\frac{1}{s^\alpha+\beta_k^2}\frac{\nu}{s^2+\nu^2}, \tag{44}$$

where $L$ denotes the Laplace transform: $L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$ and $s$ is the complex variable. The inverse Laplace transform will be determined for rational numbers $\alpha$, i.e. we assume that $\alpha = p/q$, where $p$, $q$ are positive integer relative prime numbers. Introducing the new variable, $z = s^{1/q}$, the right-hand side of equation (44) can be written in the form of:

$$\begin{aligned} \frac{1}{z^p+\beta_k^2}\frac{1}{z^{2q}+\nu^2}=\frac{1}{z^{2q}+\nu^2}\sum_{j=0}^{2q-1}A_jz^j+ \\ +\frac{1}{z^p+\beta_k^2}\sum_{j=0}^{p-1}B_jz^j \end{aligned} \tag{45}$$

Unknowns $A_j$, $B_j$ are determined by solving an equation system, which is obtained by comparing the coefficients in the polynomials received by multiplication of (45) by the denominators.

Function $J^2_{\alpha,k}(t)$ for $\alpha = p/q$ is determined using the equations (44, 45) and the following formula [28]:

$$L\left\{t^{\beta-1}E_{\alpha,\beta}\left(-\lambda t^\alpha\right)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha+\lambda}. \tag{46}$$

Hence, the function $J^2_{p/q,k}(t)$ is as follows:

$$\begin{aligned} J^2_{p/q,k}(t)=\nu\sum_{j=0}^{2q-1}A_j\,t^{1-\frac{j}{q}}E_{2,2-j/q}\left(-\nu^2t^2\right)+ \\ +\nu\sum_{j=0}^{p-1}B_j\,t^{\frac{p-j}{q}-1}E_{p/q,(p-j)/q}\left(-\beta_k^2t^{p/q}\right). \end{aligned} \tag{47}$$



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Finally, the temperature distribution in the $i$-th layer of the sphere with harmonically varying heat generation (36) is given by equation (37), where function $J^1_{\alpha,k}(t)$ is given by (43), and function $J^2_{\alpha,k}(t)$ for $\alpha = p/q$ is given by (47).

## 5. Temperature distribution in the sphere with harmonically varying ambient temperature

Temperature distribution in the sphere without heat generation is given by equation (33), in which it is assumed that $g_j(r, t) = 0$ for $j = 1, \ldots, n$. Moreover, we assume that the initial temperature is constant and the same in all layers: $f_j(r) = T_0$ for $j = 1, \ldots, n$, and ambient temperature $T_\infty(t)$ changes according to the formula:

$$T_\infty(t) = P_1 + P_2 \sin \nu t. \tag{48}$$

Hence, using (33), we find the temperature distribution in the sphere as:

$$T_i(r,t) = T_\infty(t) - \frac{T_0 - P_1}{r}\sum_{k=1}^{\infty}\frac{\gamma_k}{N_k}\Phi_{i,k}(r)E_\alpha\left(-\beta_k^2 t^\alpha\right) - \frac{P_2}{r}\sum_{k=1}^{\infty}\frac{\gamma_k}{N_k}\Phi_{i,k}(r)J^3_{\alpha,k}(t), \tag{49}$$

where

$$J^3_{\alpha,k}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\beta_k^2 (t-\tau)^\alpha\right)\frac{d^\alpha \sin \nu\tau}{d\tau^\alpha}d\tau \tag{50}$$

We calculate the integral (50) by using the property of the Laplace transform of the Caputo derivative:

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0+), \quad n-1 < \alpha \le n \tag{51}$$

Hence, we receive:

$$L\left\{\frac{d^\alpha \sin \nu t}{dt^\alpha}\right\} = \begin{cases} \dfrac{\nu s^\alpha}{s^2+\nu^2}, & 0<\alpha\le 1 \\ \dfrac{\nu^3 s^{\alpha-2}}{s^2+\nu^2}, & 1<\alpha\le 2 \end{cases}. \tag{52}$$

On the basis of equations (50, 52) we obtain the Laplace transform of function $J^3_{\alpha,k}(t)$, which, after some transformation, can be written in the form of:

$$L\left\{J^3_{\alpha,k}(t)\right\} = \begin{cases} \dfrac{\nu}{s^2+\nu^2} - \beta_k^2 \dfrac{1}{s^\alpha+\beta_k^2}\dfrac{\nu}{s^2+\nu^2} & \text{for } 0<\alpha\le 1 \\ \nu\dfrac{s^{\alpha-2}}{s^2+\nu^2} - \dfrac{\nu}{s^2+\nu^2} + \\ +\beta_k^2\dfrac{1}{s^\alpha+\beta_k^2}\dfrac{\nu}{s^2+\nu^2} & \text{for } 1<\alpha\le 2 \end{cases} \tag{53}$$

Function $J^3_{\alpha,k}(t)$, as the inverse Laplace transform of the function (53), can be presented as:

$$J^3_{\alpha,k}(t) = \begin{cases} \sin \nu t - \beta_k^2 J^2_{\alpha,k}(t), & 0<\alpha\le 1 \\ \nu t E_{\alpha,2}\left(-\beta_k^2 t^\alpha\right) - \sin \nu t + \\ +\beta_k^2 J^2_{\alpha,k}(t), & 1<\alpha\le 2 \end{cases} \tag{54}$$

This completes the formula for temperature distribution in the $i$-th layer of the sphere: the temperature is determined by (49), where function $J^3_{\alpha,k}(t)$ is given by (54) and function $J^2_{\alpha,k}(t)$ for $\alpha = p/q$ is given by (47).

## 6. Numerical example

The analytical solution of the fractional heat conduction problem derived in the previous sections will be used to compute the temperature distributions in a layered sphere. Two illustrative numerical examples are presented. In both examples the considered sphere consists of an inner, small solid sphere of radius $r_1$ and five concentric spherical layers of outer radii $r_i$. Non-dimensional radii $r_i/b$, thermal diffusivity $a_i$, and thermal conductivity $\lambda_i$ of the material of the solid inner sphere and the five layers are given in Table 1. The physical units given in Table 1 were discussed in [30]. The heat transfer coefficient is assumed as $a_\infty = 1200.0$ W/(m$^2 \cdot$K). The computations were performed using the Mathematica package [31].

Table 1
Non-dimensional outer radii, thermal diffusivity and thermal conductivity of the sphere layers applied in the numerical examples

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $r_i/b$ | 0.25 | 0.4 | 0.55 | 0.7 | 0.85 | 1.0 |
| $a_i$[m$^2$/s$^\alpha$] | $2.2\cdot10^{-5}$ | $3.3\cdot10^{-6}$ | $6.0\cdot10^{-6}$ | $1.1\cdot10^{-5}$ | $2.0\cdot10^{-5}$ | $3.6\cdot10^{-5}$ |
| $\lambda_i$[W/(m·K)] | 0.016 | 16.0 | 24.0 | 36.0 | 54.0 | 81.0 |

The first example concerns fractional heat conduction in the sphere with harmonically varying heat generation in the inner solid sphere, which is given by formula (36). The frequency of changes of the volumetric heat source intensity is $\nu = 2\pi/12\,000$ s$^{-1}$, and the coefficients in the formula (36) are $Q_1 = Q_2 = 4.2\cdot10^7$ W/m$^3$. The initial temperature in the sphere $T_0$ and the ambient temperature $T_a$ are assumed as constants: $T_0 = 50$°C, $T_a = 40$°C. The non-dimensional tem-



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perature $\overline{T}(\bar{r}, \tau) = T(r, \tau)/T_0$ at the outer surface of the sphere ($\bar{r} = r/b$), as a function of variable $\tau = \frac{a_6}{b^2}t$ for different values of the fractional order, is presented in Fig. 2. The computations were performed for $\alpha = 0.75; 0.8; 0.85; 0.9; 0.95; 1.0$. It can be seen that amplitudes of the temperature oscillations at the outer surface of the sphere decrease for smaller orders of the fractional derivative in the heat conduction model. This observation leads to a physical interpretation of the parameter $\alpha$ as a thermal damping coefficient in the fractional heat conduction model.

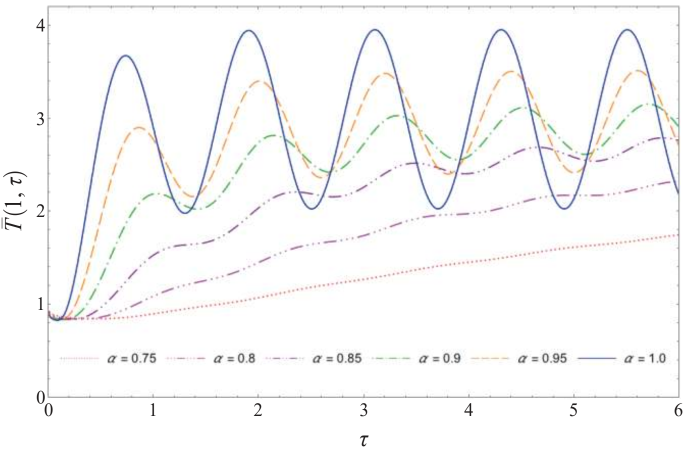


Fig. 2. Non-dimensional temperature $\overline{T}(1, \tau)$ as a function of variable $\tau = \frac{a_6}{b^2}t$ for different values of fractional order $\alpha$

In the second example, the changes of temperature in the sphere follow as a result of oscillation of ambient temperature $T_\infty(t)$, which changes according to formula (48). It is assumed that there is no heat source in the sphere and the initial temperature is $T_0 = 75°C$. The numerical computations were performed for $\alpha = 0.4; 0.6; 0.8; 1.0; 1.2; 1.25; 1.3; 2.0$, and for an oscillation frequency of the ambient temperature $v = 2\pi/12\,000 \text{ s}^{-1}$. The coefficients in the formula (48) are assumed as $P_1 = 75°C$ and $P_2 = 50°C$. The remaining data are the same as in the first example. Non-dimensional temperature $\overline{T}(\bar{r}, \tau)$ as a function of radial coordinate $\bar{r} = r/b$ for different values of variable $\tau$ and different orders of the time-fractional derivative $\alpha$ are presented in Fig. 3.

## 7. Concluding remarks

The solution of the time-fractional, radial heat conduction problem in a multilayered solid sphere in an analytical form has been derived. The temperature distribution in the sphere is obtained by taking into consideration the time-space-dependent volumetric heat source and the time-dependent ambient temperature. A numerical computation was performed to show the temperature time-history at the outer surface of the sphere for different values of the time-derivative fractional orders in the heat conduction equation when the intensity of the inner heat source varies harmonically with time. It is observed that the amplitude of the temperature oscillation at the sphere surface is lower for the heat conduction characterized by a lower order of the fractional derivative. Another numerical example shows the temperature distribution as a function of distance from the center sphere when the ambient temperature varies harmonically with time. The changes of the temperature in the sphere at a fixed time are smaller for lower orders of the fractional derivative. Although the numerical computation was performed for five layers of the solid sphere, the obtained solution can be used for numerical calculation of the temperature in the sphere consisting of an arbitrary number of concentric sphere layers. The approach can also be applied to approximate a solution of the fractional heat conduction problem in the radially, functionally graded sphere.

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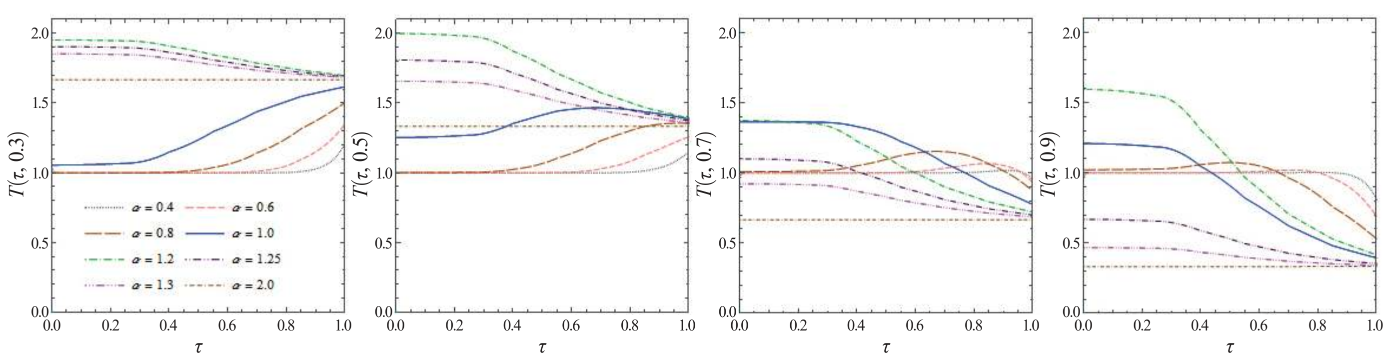


Fig. 3. Non-dimensional temperature $\overline{T}(1, \tau)$, as a function of the radial coordinate $\bar{r} = r/b$, for $\tau = 0.3; 0.5; 0.7; 0.9$ and different values of fractional order $\alpha$



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