An Analytically Tractable Approximation for the Gaussian Q-Function

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***Abstract*— In this letter we propose an approximation for the Gaussian Q-function that enables simpler evaluation of important communication system performance metrics. The approximation enables derivation of closed-form expressions for metrics such as average symbol, bit and block error probabilities which are known to be analytically involved as they require computation of the expectation of Q-function and its integer powers, for any m of Nakagami-m fading. The tightness of the approximation is verified by simulations. The usefulness of the approximation is demonstrated by obtaining a simple closed-form expression for the average symbol error probability of differentially encoded QPSK in Nakagami-m fading.**

Index Terms: Gaussian Q-function, Nakagami-m fading, Gaussian approximation

I. Introduction

The Gaussian Q-function plays an important role in the performance analysis of many communication problems [1]. Obtaining closed form expressions for a number of wireless communication performance metrics, particularly average symbol, bit and block error probabilities of various digital communication schemes typically involve taking the expectation of the Gaussian Q-function and its integer powers w.r.t a random variable that captures the fading environment and is quite involved [2]. The analytical problems associated with evaluating expectation of the Q-function spurred the interest in finding alternate representations, as well as approximations that are both tight and analytically simple in form [2]-[7] (and the references therein).

The appeal of the alternate representation of the Q-function, presented in detail in [2], is limited to the first two powers of the Q-function and it requires a double integration to get the expectation of the Q-function w.r.t a standard Nakagami-m distribution, m being the Nakagami fading parameter ($m = 1$ represents the popular Rayleigh fading environment). Among the approximations, to the best of our knowledge, the one presented in [7] performs well in terms of how accurately it resembles the actual Q-function combined with its relatively simple form. As pointed out in [8], there are better approximations for Q-function compared to that in [7]. However, the approximation in [7] is sufficient for the purposes of this work. Our interest in approximating the Q-function is two fold, one is the accuracy and the second is the simplicity of the form that lets further performance analysis of fading communication systems possible in an easy manner. The form of approximation given in [7] is still not easy to integrate and is limited to a restricted m of Nakagami-m distributions. In this letter, building upon the approximation in [7], we suggest a modified approximation which is simple and is easily integrable w.r.t any m of a Nakagami-m fading distribution in closed-form while preserving the tightness of the approximation.

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II. Approximation for the Gaussian Q-Function

The approximation for Q-function given in [7] is

$$Q(x) \approx \frac{\left(1 - e^{\frac{-Ax}{\sqrt{2}}}\right) e^{\frac{-x^2}{2}}}{B\sqrt{2\pi}x}\,, \tag{1}$$

where $A = 1.98$ and $B = 1.135$[1]. The accuracy with which (1) represents the actual Q-function is quite remarkable. However, the presence of x in the denominator of (1) makes it difficult to evaluate $E\left[Q^N(x)\right]$ in many scenarios. Higher integer powers of the Q-function appear in the evaluation of average block error probabilities. It is straightforward to show that if the maximum integer power of the Gaussian Q-function in the performance metric[2] is N, then with (1) replacing the actual Q-function, closed-form expression for $E[Q^N(x)]$ in Nakagami-m fading is possible only for $m > \frac{N}{2}$. For the performance metric of average SEP (ASEP) for a differentially encoded QPSK in Nakagami-m studied in [7], the maximum integer power of Q-function is 4 and so the analytical results are limited to $m > 2$. Unfortunately this limitation implies that important cases such as the popular Rayleigh fading ($m = 1$) are not covered.

Building on the approximation given in [7], we develop a slightly modified version of (1) that will avoid the presence of x in the denominator and is easily integrable for any m. We begin with Taylor series expansion of $e^{\frac{-Ax}{\sqrt{2}}}$

$$e^{\frac{-Ax}{\sqrt{2}}} = \sum_{n=0}^{\infty} \frac{(-Ax)^n}{\sqrt{2}^n\, n!}\,. \tag{2}$$

[1]Please refer to [7] for a discussion on the selection of A and B

[2]with x appearing inside the square root of the Q-function's argument ($Q(\sqrt{w\,x})$) as is the case in [7].

Substituting (2) in (1) and truncating the series we arrive at

$$Q(x) \approx \frac{\left(1-e^{\frac{-Ax}{\sqrt{2}}}\right) e^{\frac{-x^2}{2}}}{B\sqrt{2\pi}x} \approx e^{\frac{-x^2}{2}} \sum_{n=1}^{n_a} c_n\, x^{n-1}\,, \quad (3)$$

$$c_n = \frac{(-1)^{n+1}(A)^n}{B\sqrt{\pi}\,(\sqrt{2})^{n+1}\, n!}\,.$$

In (3) we truncated the infinite series by taking the first n_a terms. The presence of $(\sqrt{2}^{n+1}\, n!)$ in the denominator of (3) ensures that as n increases c_n approaches zero quickly. So we can approximate the complete infinite series with a relatively small n_a. Obtaining (3) requires only a minor modification to (1) but is very important from a performance metric evaluation point of view. The analytical simplicity of (3) is significant, (3) doesn't have x in the denominator, and the expression is a simple finite weighted summation of the terms of the form $x^{\alpha}\, e^{-\frac{x^2}{2}}$, $\alpha \geq 0$, which are easily integrable for any m of a Nakagami-m distribution. This simplicity also extends to powers of the Q-function. Let N be the maximum integer power of Q-function in the performance metric, then using multinomial expansion, we can write $Q^N(x)$ as

$$\begin{aligned} Q^N(x) &\approx e^{\frac{-Nx^2}{2}} \left(c_1 + c_2\, x + \cdots + c_{n_a}\, x^{n_a-1}\right)^N, \\ &= \sum_{k_1,k_2,\ldots,k_{n_a}} K_M\, C_M\, x^{f_m}\, e^{\frac{-Nx^2}{2}}, \end{aligned} \quad (4)$$

the summation is over all sequences of nonnegative integers $k_1, \ldots, k_{n_a}$ such that $k_1 + \cdots + k_{n_a} = N$. In (4)

$$K_M = \frac{N!}{(k_1)!(k_2)!\ldots(k_{n_a})!}\,, \quad (5)$$

$$C_M = (c_1)^{k_1}(c_2)^{k_2}\ldots(c_{n_a})^{k_{n_a}}\,, \quad (6)$$

$$f_m = k_2 + 2k_3 + \cdots + (n_a-1)k_{n_a}\,. \quad (7)$$

Notice that the final form for the approximation of $Q^N(x)$, given in (4), is still a simple finite linear combination of terms of the form $x^{\alpha} e^{-\frac{N}{2}x^2}$, $\alpha \geq 0$. So it is relatively easy to evaluate $E[Q^N(x)]$ for any m of a Nakagami-m distribution. Also we believe that due to the nature of the term $x^{\alpha} e^{-\frac{N}{2}x^2}$ and the availability of a vast number of integration tables [9], $E[Q^N(x)]$ can be evaluated for a wide range of other distributions as well. Note that with the help of generalized binomial expansion, it is also possible to express the tighter Q-approximations given in [3] (eq. 9 and 13) in a form that is suitable for integration w.r.t a Nakagami-m distribution. However from the perspective of understanding the performance of communication systems we find (4) to be both simple and close enough to the results obtained by the actual Q-function.

We now examine how accurately (4) represents the actual Q-function. For $N \in \{3, 4, 5\}$, where N is the exponent of the Gaussian Q-function, Fig. 1 shows that the approximation to the Gaussian Q-function given in (4) can be seen to be quite tight. The approximation in (4) obviously depends on n_a. With $N = 4$, Fig. 2 plots the additional loss incurred as we further approximated (1) to arrive at (3). From Fig. 2 we observe that $n_a = 8$ is a reasonable choice as the additional loss is almost zero.

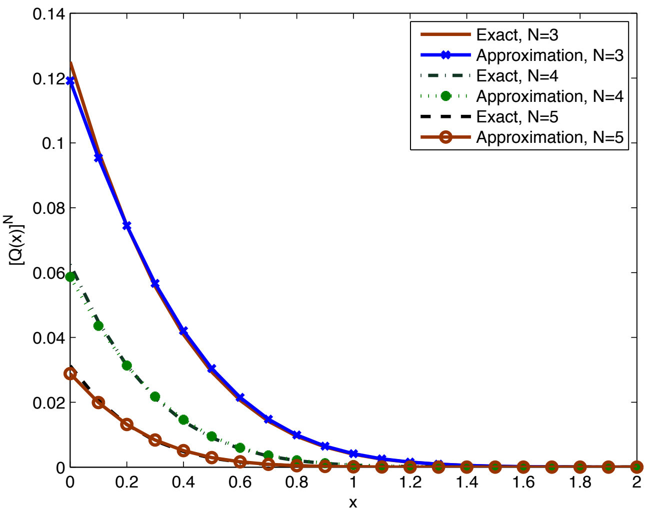

Fig. 1. Verification of the accuracy of the Gaussian Q-function approximation given in (4).

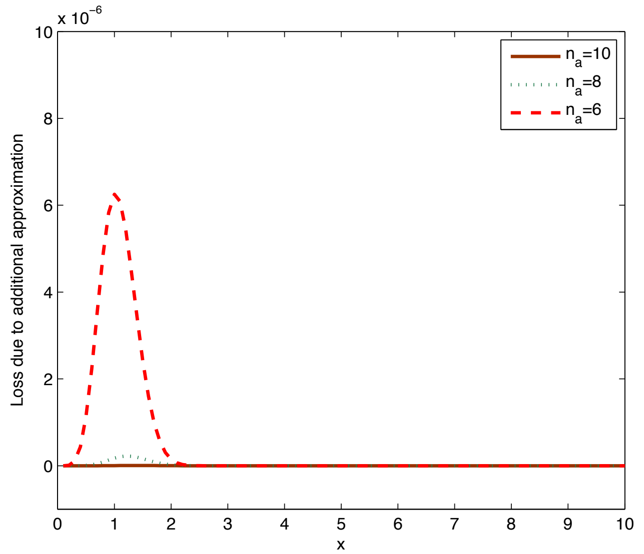

Fig. 2. Additional loss due to further approximation of (1) to arrive at (3), N=4.

III. Average SEP of Differentially Encoded QPSK

To demonstrate the usefulness of the approximation developed, we determine the expression for the Average SEP of Differentially Encoded QPSK. Conditioned on the fading related parameter γ, the SEP of differentially encoded QPSK (DE-QPSK) is given by [2]

$$P_s(\gamma_s, m, \gamma) = 4Q\left(\sqrt{\gamma}\right) - 8Q^2\left(\sqrt{\gamma}\right) + 8Q^3\left(\sqrt{\gamma}\right) - 4Q^4\left(\sqrt{\gamma}\right). \quad (8)$$

The pdf of γ is given by

$$p_\gamma(\gamma) = \frac{m^m\,\gamma^{m-1}}{\gamma_s^m\,\Gamma(m)}\,e^{-\frac{m\gamma}{\gamma_s}}\,, \tag{9}$$

where γ_s is the SNR per symbol. Averaging over fading, with the help of (4) and (9), the average SEP is given by

$$\begin{aligned} &P_s(\gamma_s, m) = E_\gamma\left[P_s(\gamma_s, m, \gamma)\right] \\ &= E_\gamma\Big[4\,Q\left(\sqrt{\gamma}\right) - 8\,Q^2\left(\sqrt{\gamma}\right) + 8\,Q^3\left(\sqrt{\gamma}\right) - 4\,Q^4\left(\sqrt{\gamma}\right)\Big] \\ &= 4\,\mathcal{F}\left(\gamma_s, m, 1\right) - 8\,\mathcal{F}\left(\gamma_s, m, 2\right) + 8\,\mathcal{F}\left(\gamma_s, m, 3\right) \\ &-4\,\mathcal{F}\left(\gamma_s, m, 4\right) \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\mathcal{F}\left(\gamma_s, m, N\right) = E_\gamma\left[Q^N\left(\sqrt{\gamma}\right)\right] \\ &= \sum_{k_1,k_2,\ldots,k_{n_a}} \frac{m^m K_M C_M}{\gamma_s^m \Gamma(m)} \int_0^\infty \gamma^{\frac{f_m+2m-2}{2}}\, e^{-\left(\frac{N}{2}+\frac{m}{\gamma_s}\right)\gamma} d\gamma, \\ &= \sum_{k_1,k_2,\ldots,k_{n_a}} \frac{m^m K_M C_M \Gamma\left(\frac{f_m}{2}+m\right) 2^{\frac{f_m}{2}+m} \gamma_s^{\frac{f_m}{2}}}{\Gamma(m)\,\left(N\,\gamma_s + 2m\right)^{\frac{f_m}{2}+m}} \end{aligned} \tag{11}$$

where K_M, C_M and f_m are defined in (5), (6), and (7) respectively and $\Gamma(.)$ is the standard Gamma function. To simplify the above expression we used the identity [9]

$$\int_0^\infty x^p e^{-\omega x}\,dx = \Gamma(p+1)\,\omega^{-(p+1)}\,, \quad \omega > 0\,, \quad p > -1\,. \tag{12}$$

Without approximating the Q-function, the single integral based expression given for ASEP of DE-QPSK in [2] is only valid for integer values of m. With the Q-approximation of [7], as pointed out earlier, the complicated ASEP expression for DE-QPSK given in [7] is valid for $m > 2$, and it involves the confluent hypergeometric function ${}_1F_1$. The final ASEP expression derived in this paper (10) is valid for any m of Nakagami-m fading and is a simple finite series. In Fig. 3 it can be seen that the analytical form of average SEP given in (10) matches well with the simulated average SEP for different values of m.

IV. Conclusion

A principal reason for approximating the Gaussian Q-function is to have a simple form for the Q-function that facilitates further mathematical analysis of communication system's performance. Through a modification to the approximation given in [7], the new approximation (4) proposed in this paper preserves the tightness of the approximation in [7] yet allowing the closed-form analytical expression for $E[Q^N(x)]$ for any m of a Nakagami-m fading distribution. Average SEP of DE-QPSK in Nakagami-m fading is evaluated using the proposed approximation and its accuracy is validated by simulations.

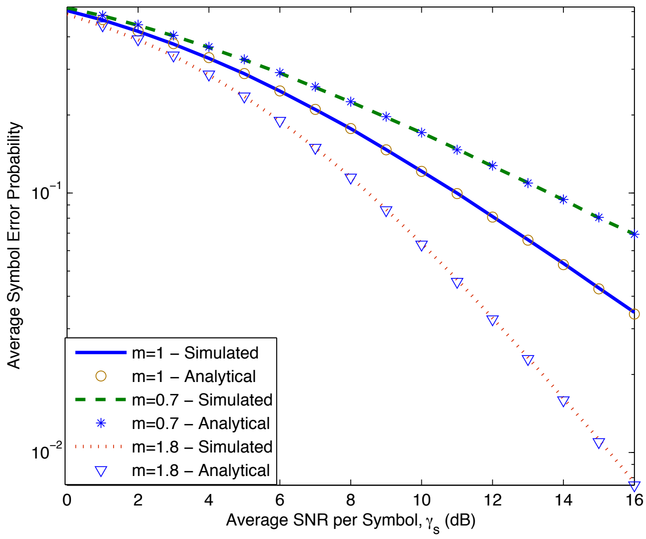

Fig. 3. Comparison of simulated and analytical (10) average SEP of DE-QPSK, m is the Nakagami-m fading parameter.

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