An angular displacement interferometer based on total internal reflection

Weidong Zhou and Lilong Cai

Department of Mechanical Engineering, Hong Kong University of Science and Technology, Hong Kong, People's Republic of China

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Abstract. This paper describes a new angular displacement interferometer based on the internal-reflection effect. In order to make use of the internal-reflection effect, a novel prism assembly is designed, which yields a very compact optical configuration. As a result, the linearity of the angular displacement interferometer is greatly improved. Both the linearization of the measurement equation and the experimental verification of the analysis have been conducted and the results are presented here in detail. The main factors affecting the angle measurement are also addressed. The results of the experimental verification are in good agreement with the theoretical analysis.

Keywords: angular displacement measurement, internal reflection, right-angle prism, interferometer

1. Introduction

Huang and coworkers recently reported an angle-measurement method based on internal reflection [1–3]. Since angle-measurement methods using the internal-reflection effect are simple in structure, low in cost and easy to conduct, there may be potential wide applications in the field of alignment, assembly and calibration of machine tools and in other fields of small-angle measurement. In Huang's method, a light beam reflected from a target mirror is divided into two beams. The two beams are incident on two separate prisms and are reflected in the vicinity of the critical angle. Any rotation of the target will increase the incidence angle within a prism and decrease the incidence angle within another prism. The small rotation angle of the target is thus estimated by measuring the difference in reflectance between the beams reflected respectively by the two prisms. Use of elongated prisms with multiple reflection can increase the angle sensitivity. Owing to the differential optical configuration, the method has good linearity and high resolution (0.02 arcsec) and the apparatus is compact in size and low in cost, but the method is also very sensitive to variations in intensity of the light source and stray light. This is so because the angle measurement is directly proportional to the intensity of reflected light. The measurement range is rather small (about 3 arcmin). To overcome this problem, Chiu and Su proposed a heterodyne interferometry method [4, 5]. The rotation angle in this case is measured from the phase difference between parallel and perpendicular polarization states of beams reflected from prisms by total internal reflection. The method is hence independent of light-intensity variations and has higher anti-turbulence ability. However, there are still problems waiting to be solved before this interferometric method can become more useful in practice. First of all, since the direction of the beams reflected by the right-angle prism doubly varies with the rotation of the prism, it is difficult to receive the reflected beams directly. Hence we require a retro-reflector which can return an incident beam in parallel and displace the beam at a distance, while maintaining the relationship of the phase difference between the two polarization states and the rotation angle unchanged. In addition, the phase difference between the two polarization states is a nonlinear function of the rotation angle. To use this relationship to measure rotation angles, linearization of the measurement equation is imperative.

In order to solve these problems, a novel prism assembly is designed, which can always return the incoming light beams in parallel so that the reception of reflected beams is easy. Meanwhile the optical configuration becomes a differential common path and the linearity of the angular displacement interferometer is improved. By utilizing the proposed prism assembly, we have constructed a novel angular displacement heterodyne interferometer. Both theoretical analysis and the linearization of the measurement equation are conducted for this interferometer. The main factors affecting angle measurement are addressed. The results of the experimental verification are presented here.

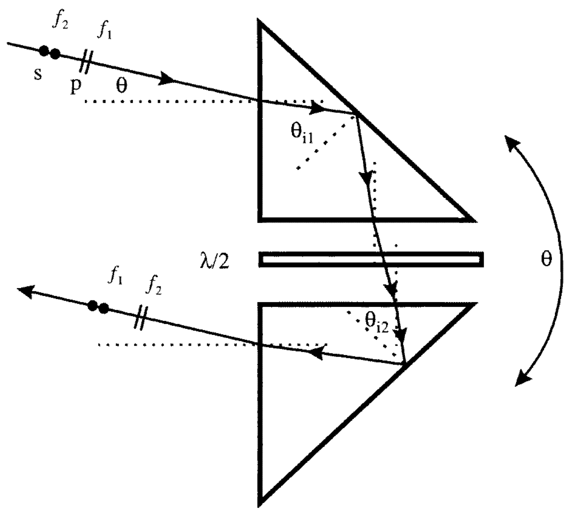

Figure 1. The principle of the prism assembly.

2. Principles of operation

It is known that, when a total reflection takes place, the phase difference of the total reflection light between parallel and perpendicular polarization states is a function of the incidence angle according to Fresnel's law [6]. The total reflection effect may be used to measure rotation angles. However, the direction of the reflected light is not constant during measurement. This makes the reception of light difficult, especially if the measurement angle is large. To make use of the total reflection effect, a novel prism assembly which can always retro-reflect the incoming light beams in parallel has been designed. The principle of the prism assembly is schematically shown in figure 1. The prism assembly comprises a $\lambda/2$ waveplate and two right-angle prisms which are parallel with each other and the $\lambda/2$ waveplate is placed between them. The fast or slow axis direction of the $\lambda/2$ waveplate is adjusted to be 45° with respect to the polarization direction of incidence light.

Suppose that two orthogonally linearly polarized laser beams from a stabilized laser are incident normally on one side surface of the first right-angle prism. Frequency f_1 is the parallel polarization component p; frequency f_2 is the perpendicular polarization component s. The beams will be incident at $\theta_{i1} = \pi/4$ upon the hypotenuse surface and totally reflected by the hypotenuse surface and emitted out of the first prism from another side. The polarization state of the beams is changed after the beams pass through the $\lambda/2$ waveplate and the frequency f_1 becomes the perpendicular polarization component s and the frequency f_2 becomes the parallel polarization component p. They are incident normally on one side surface of the second right-angle prism. The beams are incident at $\theta_{i2} = \pi/4$ on the hypotenuse surface and emitted out of the second prism from another side. The beams exiting from the prism assembly are now parallel with the incoming ones. Therefore, the optical configuration can be very compact and its use for measurement is possible. The prism assembly is placed on a rotary table for testing. For simplicity, suppose that the refractive index of air is 1. Thus, from the refraction law, when the table rotates an angle θ, the angles at which the beams are incident on the hypotenuse surfaces of the two right-angle prisms are expressed as follows:

$$\theta_{i1} = \pi/4 + \sin^{-1}\left(\frac{\sin\theta}{n}\right) \tag{1}$$

$$\theta_{i2} = \pi/4 - \sin^{-1}\left(\frac{\sin\theta}{n}\right) \tag{2}$$

where n is the refractive index of the right-angle prisms and θ_{i1} and θ_{i2} are the incidence angles on the hypotenuses of the two right-angle prisms respectively.

According to Fresnel's equation [6–7], after the laser beams have passed through the prism assembly the phase difference between the frequency components f_1 and f_2 can be expressed as

$$\begin{aligned}
\varphi = \varphi_0 &+ 2\tan^{-1}\left(\frac{n(n\sin^2\theta_{i1} - 1)^{1/2}}{\cos\theta_{i1}}\right) \\
&-2\tan^{-1}\left(\frac{(n\sin^2\theta_{i1} - 1)^{1/2}}{n\cos\theta_{i1}}\right) \\
&-2\tan^{-1}\left(\frac{n(n\sin^2\theta_{i2} - 1)^{1/2}}{\cos\theta_{i2}}\right) \\
&+2\tan^{-1}\left(\frac{(n\sin^2\theta_{i2} - 1)^{1/2}}{n\cos\theta_{i2}}\right) + \Delta\varphi \\
= \varphi_0 &+ 2\tan^{-1}\left(\frac{\cos\theta_{i1}(n^2\sin^2\theta_{i1} - 1)^{1/2}}{n\sin^2\theta_{i1}}\right) \\
&-2\tan^{-1}\left(\frac{\cos\theta_{i2}(n\sin^2\theta_{i2} - 1)^{1/2}}{n\sin^2\theta_{i2}}\right) + \Delta\varphi
\end{aligned} \tag{3}$$

where φ_0 is the initial phase difference between the frequency components f_1 and f_2. The magnitude of φ_0 depends on the distance travelled by the beams before they arrive at the receiver and on the difference in frequency between f_1 and f_2. $\Delta\varphi$ is the phase change caused by the rotation of the prism assembly. $\Delta\varphi$ can be estimated by considering the rotation of the equivalent glass plate of the prism assembly as shown in figure 2. According to the refraction law and using geometrical analysis, the optical path difference (OPD) introduced by the rotation of a glass plate is

$$OPD = n\overline{AC} - n\overline{AB} - \overline{BE} \tag{4}$$

where $\overline{AB} = t$

$$\overline{AC} = nt/(n - \sin^2\theta)^{1/2}$$

$$\overline{BE} = \overline{AC}\cos(\theta - \theta') - t$$

$$\sin\theta = n\sin\theta'.$$

On substituting the above equations into equation (4), we have

$$OPD(\theta) = \frac{n^2 t - t\cos\theta(n^2 - \sin^2\theta)^{1/2} - t\sin^2\theta}{(n^2 - \sin^2\theta)^{1/2}} - (n-1)t \tag{5}$$

where t is the thickness of the equivalent glass plate of the prism assembly and θ is the incidence angle on the equivalent glass plate, which is equal to the rotation angle of the prism assembly. Therefore the phase change introduced by the rotation of the prism assembly can be given by

$$\Delta\varphi = \frac{2\pi}{\lambda}\frac{\Delta f}{f} OPD(\theta)$$
$$= \left(\frac{n^2 - \cos\theta(n^2 - \sin^2\theta)^{1/2} - \sin^2\theta}{(n^2 - \sin^2\theta)^{1/2}} - (n-1)\right)t \times \frac{\Delta f}{f}\frac{2\pi}{\lambda} \tag{6}$$

where f is the mean frequency of the laser, $f = (f_1 + f_2)/2$, Δf is the difference in frequency between f_1 and f_2, λ is the mean wavelength of the laser, $\lambda = c/f$, and c is the speed of light. On substituting equation (6) into equation (3), we get

$$\varphi = \varphi_0 + 2\tan^{-1}\left(\frac{\cos\theta_{i1}(n^2\sin^2\theta_{i1} - 1)^{1/2}}{n\sin^2\theta_{i1}}\right) - 2\tan^{-1}\left(\frac{\cos\theta_{i2}(n^2\sin^2\theta_{i2} - 1)^{1/2}}{n\sin^2\theta_{i2}}\right)$$
$$+ \left(\frac{n^2 - \cos\theta(n^2 - \sin^2\theta)^{1/2} - \sin^2\theta}{(n^2 - \sin^2\theta)^{1/2}} - (n-1)\right)t \times \frac{\Delta f}{f}\frac{2\pi}{\lambda}. \tag{7}$$

From equation (7), it seems that an angular displacement interferometer can be set up using the proposed prism assembly. The rotation angle of the prism assembly is estimated by sensing the phase difference between the frequency components f_1 and f_2. However, in practice, equation (7) cannot yet be directly used as a measurement equation for the rotation angle because the initial phase difference, the first term φ_0 in equation (7), cannot be exactly determined unless the difference in frequency between f_1 and f_2 is zero. Therefore, depending on whether the difference in frequency between f_1 and f_2 is zero, angle measurements based on equation (7) can be divided into two categories: absolute and relative measurements. If the difference in frequency between f_1 and f_2 is zero, the first and last terms are zero; thus the rotation angle can be directly determined by measuring the phase difference. In this case, the absolute measurement of an angle can be performed directly without any error in principle. That means that a laser with a single frequency must be adopted. With present techniques, it is not as easy to measure the phase difference of single-frequency interferometers dynamically as it is to do so for two-frequency interferometers with the same resolution. However, the absolute measurement does promise to mark the zero angular position and measure the rotation angle more accurately and effectively. If the difference in frequency between f_1 and f_2 is not zero, the measurement of an angle can be conducted only relatively. Hence a linear measurement equation is imperative.

When θ is small, the Taylor series expansion of equation (6) is

$$\Delta\varphi(\theta) = \Delta\varphi(0) + \Delta\varphi'(0)\theta + \frac{1}{2!}\Delta\varphi''(0)\theta^2 + \frac{1}{3!}\Delta\varphi'''(0)\theta^3 + \cdots \tag{8}$$

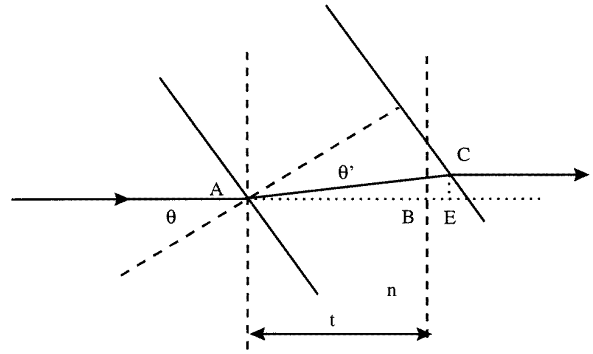

Figure 2. The optical path difference introduced by the rotation of a glass plate.

where

$$\Delta\varphi(0) = 0$$
$$\Delta\varphi'(0) = 0$$
$$\Delta\varphi''(0) = \frac{\Delta f}{f}\frac{2\pi}{\lambda}t\frac{n-1}{n}$$
$$\Delta\varphi'''(0) = 0.$$

Neglecting terms of order higher than 3, equation (6) can be simplified to

$$\Delta\varphi = \frac{\Delta f}{f}\frac{2\pi}{\lambda}t\frac{n-1}{n}\frac{\theta^2}{2}. \tag{9}$$

It is clear that $\Delta\varphi$ is a quadratic function of θ. If the refractive index of the prisms $n = 1.515\,09$ and $t = 50$ mm, $\lambda = 0.6328$ μm, $f = 4.74 \times 10^{14}$ Hz and $\Delta f = 2$ MHz, then $\Delta\varphi$ is approximately

$$\Delta\varphi \approx 0.000\,356\theta^2. \tag{10}$$

Thus, $\Delta\varphi \approx 0.7$ arcsec when $\theta = 5.6°$ and $\Delta\varphi \approx 0.09$ arcsec when $\theta = 2°$. Therefore $\Delta\varphi$ can be neglected when Δf and t are selected to be comparatively small. As a result, the phase difference between the frequency components f_1 and f_2 at the output of the system can be rewritten as

$$\varphi = \varphi_0 + 2\tan^{-1}\left(\frac{\cos\theta_{i1}(n^2\sin^2\theta_{i1} - 1)^{1/2}}{n\sin^2\theta_{i1}}\right) - 2\tan^{-1}\left(\frac{\cos\theta_{i2}(n^2\sin^2\theta_{i2} - 1)^{1/2}}{n\sin^2\theta_{i2}}\right). \tag{11}$$

If we suppose now that θ is small enough, the Taylor series expansion of $\varphi(\theta)$ is

$$\varphi(\theta) = \varphi_0 + \varphi(0) + \varphi'(0)\theta + \frac{1}{2!}\varphi''(0)\theta^2 + \frac{1}{3!}\varphi'''(0)\theta^3 + \cdots \tag{12}$$

where

$$\varphi(0) = 0$$
$$\varphi'(0) = \frac{4(3 - n^2)}{(n^2 - 1)(n^2 - 2)^{1/2}}$$
$$\varphi''(0) = 0$$

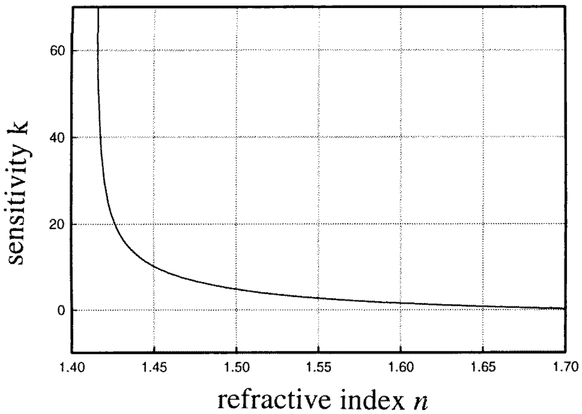

Figure 3. The curve of the sensitivity k versus the refractive index n.

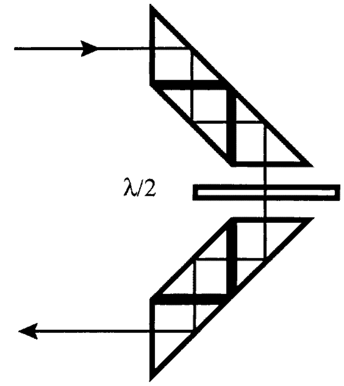

Figure 4. The multi-reflection prism assembly.

$$\Delta\varphi'''(0) = 4\frac{n^{12}-10n^{10}+48n^8-54n^6-65n^4+48n^2+96}{n^2(n^2-2)^{5/2}(n^2-1)^3}.$$

Here, the derivatives of $\varphi(\theta)$ were calculated by a computer with Maple V Release 4. Thus, neglecting higher-order terms, the relation between φ and θ is approximately expressed as

$$\varphi(\theta) \approx \varphi_0 + \frac{4(3-n^2)}{(n^2-1)(n^2-2)^{1/2}}\theta = \varphi_0 + k\theta \qquad (13)$$

where k is the sensitivity,

$$k = \frac{4(3-n^2)}{(n^2-1)(n^2-2)^{1/2}}. \qquad (14)$$

Clearly, the sensitivity, k, depends only on the refractive index n. The relationship between k and n is represented in figure 3. We can see from figure 3 that, when n decreases, k grows greatly. k approaches infinity when n is close to $\sqrt{2}$. For example, when the right-angle prism is made of BK7 glass with refractive index 1.515 09, the sensitivity is 4.0016. If a phase meter with the resolution of 0.01° is used, an angle resolution of about 9 arcsec will be achieved. If $n = 1.415$, the resolution is better than 0.4 arcsec. However, it should be noted that, as the refractive index of the prisms approaches $\sqrt{2}$, the scale factor becomes unstable. This is so, because small differences in temperature alter the refractive indices of the glass and air (the refractive index of air is not unity but very close to it) and hence, as the refractive index of the prisms approaches $\sqrt{2}$, the denominator of $\varphi'''(0)$ can rapidly change value.

In order to increase the resolution of the system, a multi-reflection prism assembly can be used, as shown in figure 4. In this case, the resolution can be up to 0.1 arcsec.

3. Experimental details

To verify the above theoretical analysis, an angular displacement interferometer using the proposed prism assembly has been set up. Figure 5 is a schematic diagram of our experimental arrangement. A HP 5517C laser head was adopted as the light source. It emits two orthogonally linearly polarized light beams, their frequencies being f_1 and f_2. The frequency difference, $\Delta f = f_1 - f_2$, is about 2 MHz. A glass plate is used to sample the beams from the laser head and the sample is sensed by a HP 10870C receiver. The receiver outputs a reference signal with the frequency of Δf. The transmitted part is incident on the prism assembly and retroreflected parallelly. The reflected beams are received by another HP 10870F receiver, whose output is the measurement signal. The reference and measurement signals are transmitted to a phase meter to measure their phase difference. In our experiment, a HP 3575 phase meter with a resolution of 0.1° was used. The right-angle prism is made of BK7 glass with a refractive index of 1.515 09. Therefore the resolution of the experimental system is about 0.025°. In order to verify the measurement equation, our measuring system is compared with an angular displacement interferometer made by Renishaw Transducer Inc. The prism assembly and an angular reflector are placed on a precision rotary table. Renishaw's angular displacement interferometer and M10 laser are set on another side. A photo of the experimental arrangement is shown in figure 6. Figure 7 shows the experimental curve of the phase difference φ versus the rotation angle θ. The regression equation of the curve over the measurement range of ±2° is $\varphi = -0.04 + 4.1045\theta$. There is a minor difference between the experimental sensitivity and the sensitivity calculated by equation (14). There are many factors causing this discrepancy. The first one is the tolerances on alignment of the prism assembly and on the prismatic angles. In addition, the waveplate is not perfect and the angle of incidence varies during the measurement. The final reason is the nonlinearity error of the measurement equation. As the measurement range decreases the nonlinearity error decreases; hence, the difference will be reduced. The drift of the phase difference was also tested and the result is shown in figure 8.

4. The main factors affecting the measurement

The above experiment demonstrated only the feasibility of the proposed method and theoretical analysis. In order

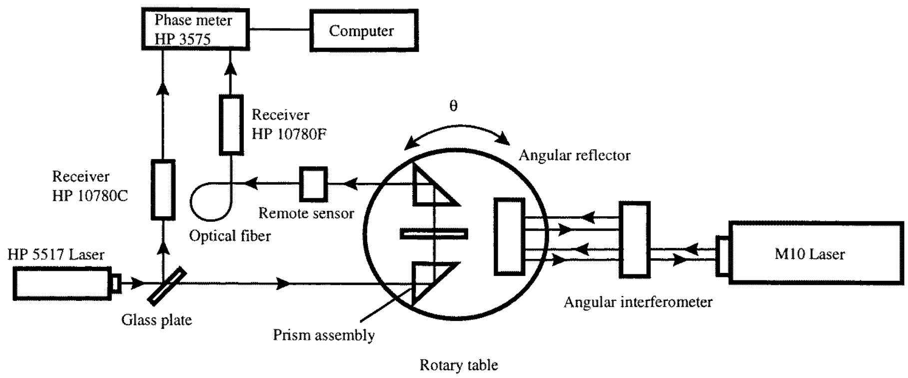

Figure 5. A schematic diagram of the experimental arrangement.

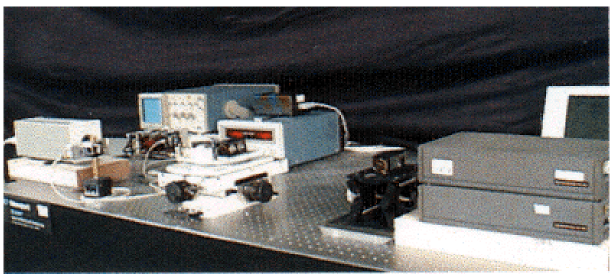

Figure 6. The experimental arrangement for angle measurement.

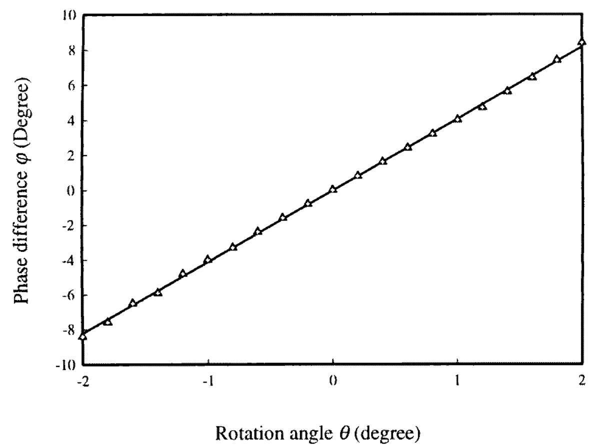

Figure 7. The experimental curve of φ versus θ.

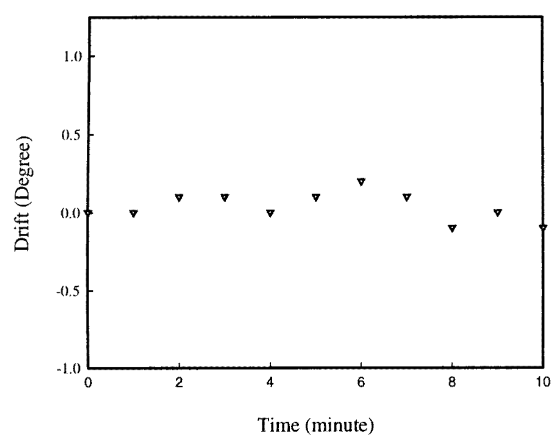

Figure 8. The results of a drift test.

to achieve a more satisfactory performance, the following factors affecting the measurement should be taken into consideration in the design of an angular displacement interferometer using the proposed prism assembly.

4.1. The nonlinearity error

In deriving equation (13), the term $\varphi'''(0)\,\theta^3/3!$ and other higher-order terms were neglected in order to achieve a linear measurement equation. For simplicity, we consider only the third-order term. Thus, the nonlinearity error can be given by

$$\delta = \varphi'''(0)\frac{\theta^3}{3!} = 2\frac{n^{12} - 10n^{10} + 48n^8 - 54n^6 - 65n^4 + 48n^2 + 96}{3n^2(n^2-2)^{5/2}(n^2-1)^3}\,\theta^3. \tag{15}$$

For $n = 1.51509$, the nonlinearity error is

$$\delta \approx 147\theta^3. \tag{16}$$

The nonlinearity error as a function of the rotation angle for $n = 1.515\,09$ is shown in figure 9. It can be seen that the error increases as the rotation angle increases due to nonlinearity. When the rotation angle exceeds $\pm 1^\circ$, the linearity error grows rapidly.

4.2. The error caused by the rotation of the prism assembly

According to equation (9), the phase change introduced by the rotation of the prism assembly is a quadratic function of the rotation angle θ and a linear function of the frequency difference Δf and the thickness of the equivalent glass plate of the prism assembly t. A laser with a low frequency difference should be selected. The lower the frequency difference, the smaller the additional phase change caused by the rotation of the equivalent glass plate

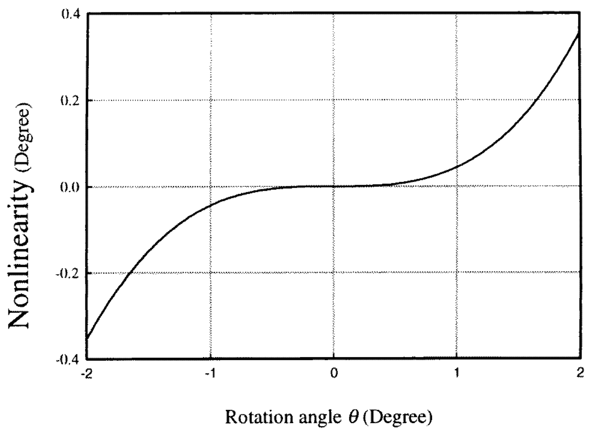

Figure 9. The nonlinearity error versus the rotation angle θ.

of the prism assembly $\Delta\varphi$. When Δf and t are selected to be comparatively small, $\Delta\varphi$ can be neglected. Note that the error caused by the rotation of the prism assembly can be compensated, because, if the rotation angle is relatively small, the additional phase change is related to the rotation angle only and the starting angle is irrelevant.

4.3. The error introduced by environmental variations

The differential common-path optical configuration is naturally designed into the system. Therefore, the angle measurement is theoretically insensitive to variations in the environment. However, in practice, optical components are not perfect; thus both the mechanical tolerances between optical components and temperature gradients and variations make the system environmentally sensitive. Therefore, it is necessary to improve the dimensional accuracy of the prism assembly.

4.4. The phase measurement error

The accuracy and resolution of angle measurement depend directly on those of the phase meter, so a phase meter of high accuracy and resolution should be adopted. In general, the lower the frequency of signals the higher the resolution and accuracy of the phase meter. Therefore, the selection of a laser with a low frequency difference will reduce the frequency of signals and thus can help to improve the resolution and the accuracy of the angle measurement.

4.5. The error caused by rotation of the $\lambda/2$ waveplate

The incidence angle on the $\lambda/2$ waveplate has an influence on the retardation of passed light. This may introduce a nonlinearity error. Compared with multiple-order waveplates, zero-order waveplates are less sensitive to changes in retardation with rotation about the fast or slow axis. This means that light not parallel to the optical axis will experience a smaller change in retardation with zero-order waveplates than it will with multiple-order waveplates. Therefore a zero-order half waveplate should be used to reduce the influence of rotation of the waveplate during measurement.

5. Conclusion

In order for the internal-reflection effect to be used, we designed a novel prism assembly which makes the optical configuration of the system very compact and the linearity of the system much better. A new angular displacement interferometer utilizing the proposed prism assembly based on the internal-reflection effect has been proposed. Therefore, the linearity of the system is greatly improved and angle measurement is insensitive to variations in the environment. The theoretical analysis of the measurement principle indicates that the use of angular displacement interferometer can be divided into absolute and relative measurements depending on whether the frequency difference is zero. To perform the relative angle measurement, the measurement equation must be linearized. The resolution of the angular displacement interferometer is determined by the refractive index of the prisms and the resolution of the phase meter. If a phase meter with a resolution of 0.01° and right-angle prisms made from BK7 glass are selected, the resolution of the interferometer can be better than 3 arcsecond. A multi-reflection prism assembly can be used to increase the resolution of the system. The experimental verification has been conducted. The experimental results are in good agreement with our theoretical analysis. On the basis of results of the analysis and experimental verification, it is concluded that an angular displacement interferometer using the proposed prism assembly is feasible. Such an angular displacement interferometer has the following advantages: compactness, simple structure, high resolution, high anti-turbulence ability, low cost and ease of use.

Acknowledgments

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