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AN ANTI-SYMMETRIC VERSION OF MALLIAVIN CALCULUS

JIRÔ AKAHORI*, TOMO MATSUSITA, AND YASUFUMI NITTA

Dedicated to the memory of Professor Hiroshi Kunita

Abstract. In the present paper we will introduce an anti-symmetric version of Malliavin calculus which consists of operators with anti-commuting relations, which actually form an infinite-dimensional Clifford algebra.

1. Introduction

About 40 years ago, H. Kunita [7] pointed out that a solution of a stochastic differential equation (SDE) on a Lie group can be expressed as an exponentiation of random series in the associated Lie algebra by using the Baker-Campbell-Hausdorff formula. The observation has been extended in various ways since then. As a whole, one can now say that a solution of an SDE can be regarded as a random action of the exponentiation of the Lie algebra generated by the vector fields associated with the coefficients of the SDE. In the preset paper, aiming to give a new insight to this observation, we introduce an anti-symmetric version of Malliavin calculus. Our new calculus consists of operators with anti-commuting relations, by which we can construct (random) representation of infinite Lie algebras, and its exponentiations as well, though we will not discuss them in detail in the present paper, which concentrates on the construction of the anti-commuting operators.

Let $\mathsf{W}^d = C(\mathbb{T} \to \mathbf{R}^d)$ and μ^d be the Wiener measure on W^d, where d is a positive integer and $\mathbb{T}$ is a closed interval, say, $[0,1]$. We will denote the Lebesgue measure restricted to a subset K of $\mathbf{R}^n$, $n \in \mathbf{N}$, by $\mathrm{Leb}(K)$.

It is well known that $L^2(\mathsf{W}^d \to \mathbf{R}, \mu^d) =: L^2(\mathsf{W}^d)$ is isomorphic to the symmetric (boson) Fock space of $L^2(\mathbb{T} \to \mathbf{R}^d, \mathrm{Leb}(\mathbb{T})) =: L^2(\mathbb{T})$;

$$L^2(\mathsf{W}^d) \simeq \bigoplus_{n=0}^{\infty} \odot^n L^2(\mathbb{T}),$$

where $\odot$ denotes symmetric tensor product. Here, and later on in a similar situation, the space of $n=0$ is $\mathbf{R}$. The isomorphism between the two can be constructed

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from the Wiener-Ito expansion [4], which is actually claiming

$$L^2(\mathsf{W}^d \to \mathbf{R}, \mu^d) \simeq \bigoplus_{n=0}^{\infty} L^2(\Delta_n \to (\mathbf{R}^d)^{n\otimes}, \mathrm{Leb}(\Delta_n)) =: \bigoplus_{n=0}^{\infty} L^2(\Delta_n), \tag{1.1}$$

where

$$\Delta_n := \{(s_1, \cdots, s_n) \in \mathbb{T}^n : s_1 > \cdots > s_n\},$$

by the isomorphism

$$L^2(\mathsf{W}^d) \ni X \mapsto (f_0, f_1, \cdots, f_n, \cdots) \in \bigoplus_{n=0}^{\infty} L^2(\Delta_n)$$

with

$$X = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle f_n(s_1, \cdots, s_n), dW_{s_n} \otimes \cdots \otimes dW_{s_1} \rangle_{(\mathbf{R}^d)^{n\otimes}}. \tag{1.2}$$

In particular, we have

$$\|X\|^2_{L^2(\mathsf{W}^d)} = \sum_{n=0}^{\infty} \|f_n\|^2_{L^2(\Delta_n)}.$$

It is well known that symmetric (fermion) Fock space is isomorphic to antisymmetric one and P. A. Meyer pointed out in his book [8, pp71] that

> *... boson Fock space over $L^2(\mathbf{R}_+)$ is isomorphic with $L^2(\Omega)$, where Ω denotes Wiener space, (...) on the other hand, it is also true for antisymmetric Fock space, a fact whose significance is not generally appreciated.*

Based on the *fact*, in the present paper we will introduce a "fermionic" calculus as a variant of Malliavin calculus, which is "bosonic" in our terminology.

The organization of the paper is as follows. In section 2, we introduce the "anti-gradient" $\mathcal{A}$ in a parallel way as we do for the Malliavin -Shigekawa derivative $\mathcal{D}$. Then in section 3, we introduce directional anti-gradient $\psi(h)$ and its adjoint $\psi^*(h)$ for $h \in L^2(\mathbb{T})$ and show that they satisfy anti-commuting relations (Theorem 3.2). In section 4, we point out that $\psi(h_k), \psi^*(h_k)$, for a given CONS of $L^2(\mathbb{T})$ form a Clifford algebra, which proves the *fact* above (Theorem 4.2). In the final section, we comment on potential applications.

2. Anti-gradient $\mathcal{A}$

Let us recall that the Malliavin-Shigekawa derivative $\mathcal{D} : \mathrm{Dom}(\mathcal{D}) \to L^2(\mathbb{T})$ can be defined through the Wiener-Ito expansion (1.2) by

$$\mathcal{D}_t X \equiv (\mathcal{D}X)_t = \sum_{n=1}^{\infty} \int_{\Delta_{n-1}} \langle \tilde{f}_n(t, s_1 \cdots, s_{n-1}), dW_{s_{n-1}} \otimes \cdots \otimes dW_{s_1} \rangle_{(\mathbf{R}^d)^{(n-1)\otimes}}, \tag{2.1}$$

where, for each n, $\tilde{f}_n \in L^2(\mathbb{T}^n \to (\mathbf{R}^d)^{n\otimes})$ is defined by

$$\begin{aligned} &\tilde{f}_n(s_1, \cdots, s_n) := f_n(s_{\sigma(1)}, \cdots, s_{\sigma(n)}), \\ &\text{with } \sigma \in \mathfrak{S}_n \text{ such that } s_{\sigma(1)} > \cdots > s_{\sigma(n)}, \end{aligned} \tag{2.2}$$

with the convention that the first term in (2.1) $= f_1(t)$. Here, and later on in a similar situation, the bracket means the coupling of the tensor products; for example, $a \in V_1 \otimes V_2$ and $b \in V_2$, $\langle a, b\rangle_{V_2} \in V_1$. In (2.1), the set of all permutations over $\{1, \cdots, n\}$ is denoted by $\mathfrak{S}_n$.

Since

$$\begin{aligned}\|\mathcal{D}X\|_{L^2(\mathbb{T})\otimes L^2(\mathsf{W}^d)} &= \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}}|\tilde{f}_n(t,s_1,\cdots,s_{n-1})|^2 dtds_{n-1}\cdots ds_1\\ &= \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}}\sum_{i=1}^{n}1_{(s_{i-1},s_y)}(t)|\tilde{f}_n(t,s_1,\cdots,s_{n-1})|^2 dtds_{n-1}\cdots ds_1\\ &= \sum_{n=1}^{\infty} n\int_{\Delta_n}|f_n(s_1,\cdots,s_n)|^2 ds_n\cdots ds_1\end{aligned}$$

we have

$$\mathrm{Dom}(\mathcal{D}) := \{X \in L^2(W^d) : \sum_{n=1}^{\infty} n\|I_n X\|_{L^2(\Delta_n\to(\mathbf{R}^d)^{n\otimes})} < \infty\},$$

where I_n is the projection from $L^2(\mathsf{W}^d)$ to $L^2(\Delta_n)$ in the expansion (1.1).

By replacing the symmetrization (2.2) with anti-symmetrization

$$\begin{aligned}&\check{f}_n(s_1,\cdots,s_n) := \mathrm{sgn}(\sigma)f_n(s_{\sigma(1)},\cdots,s_{\sigma(n)}),\\ &\text{with } \sigma\in\mathfrak{S}_n \text{ such that } s_{\sigma(1)} > \cdots > s_{\sigma(n)},\end{aligned}$$

we set

$$(\mathcal{A}X)_t \equiv \mathcal{A}_t X = \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\langle \check{f}_n(t,s_1\cdots,s_{n-1}), dW_{s_{n-1}}\otimes\cdots\otimes dW_{s_1}\rangle_{(\mathbf{R}^d)^{(n-1)\otimes}}, \tag{2.3}$$

with the same convention at $n = 1$ as in (2.1). Since

$$\begin{aligned}\|\mathcal{A}X\|_{L^2(\mathbb{T})\otimes L^2(\mathsf{W}^d)} &= \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}}|\check{f}_n(t,s_1,\cdots,s_{n-1})|^2 dtds_{n-1}\cdots ds_1\\ &= \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}}\sum_{i=1}^{n}1_{(s_{i-1},s_i)}(t)|\check{f}_n(t,s_1,\cdots,s_{n-1})|^2 dtds_{n-1}\cdots ds_1\\ &= \sum_{n=1}^{\infty} n\int_{\Delta_n}|f_n(s_1,\cdots,s_n)|^2 ds_n\cdots ds_1 = \|\mathcal{D}X\|^2_{L^2(\mathbb{T})\otimes L^2(\mathsf{W}^d)},\end{aligned}$$

it is defined on the same domain as $\mathcal{D}$.

3. Fermions in Wiener Space

For $h \in L^2(\mathbb{T})$ and $X \in \mathrm{Dom}(\mathcal{A})$ define

$$\psi(h)X := \langle \mathcal{A}X, h\rangle_{L^2(\mathbb{T})}. \tag{3.1}$$

Proposition 3.1. *For $X \in L^2(\mathsf{W}^d)$ given by (1.2), define*

$$\psi^*(h)X = \sum_{n=1}^{\infty} \int_{\Delta_n} \langle h \wedge f_{n-1}(s_1, \cdots, s_n), dW_{s_n} \otimes \cdots, \otimes dW_{s_1} \rangle_{(\mathbf{R}^d)^{n\otimes}}, \tag{3.2}$$

where the operation $h\wedge$ is defined as a map $L^2(\Delta_n) \to L^2(\Delta_{n+1})$ by

$$h \wedge f_n(s_1, \cdots, s_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} h(s_i) \otimes f_n(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{n+1}).$$

Then, for $Y \in \mathrm{Dom}(\mathcal{A})$ and $X \in L^2(\mathsf{W}^d)$,

$$\mathbf{E}[Y\psi^*(h)X] = \mathbf{E}[X\psi(h)Y].$$

Proof. Suppose that Y is given by

$$Y = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle g_n(s_1, \cdots, s_n), dW_{s_n} \otimes \cdots \otimes dW_{s_1} \rangle.$$

By (2.3) we have

$$\begin{aligned} &\mathbf{E}[X\langle \mathcal{A}Y, h\rangle_{L^2(\mathbb{T})}] = f_0 \langle g_1, h\rangle_{L^2(\mathbb{T})} \\ &+ \sum_{n=1}^{\infty} \int_0^1 \int_{\Delta_n} \langle h(t) \otimes f_n(s_1, \cdots, s_n), \check{g}_{n+1}(t, s_1, \cdots, s_n) \rangle_{(\mathbf{R}^d)^{(n+1)\otimes}} \, ds_n \cdots ds_1 dt. \end{aligned} \tag{3.3}$$

Since we have, for each n,

$$\begin{aligned} &\langle h(t) \otimes f_n(s_1, \cdots, s_n), \check{g}_{n+1}(t, s_1, \cdots, s_n) \rangle_{(\mathbf{R}^d)^{(n+1)\otimes}} \\ &= \sum_{i=1}^{n+1} 1_{(s_i, s_{i-1})}(t)(-1)^{i+1} \langle h(t) \otimes f_n(s_1, \cdots, s_n), \\ &\qquad\qquad g_{n+1}(s_1, \cdots, s_{i-1}, t, s_i, \cdots, s_n) \rangle_{(\mathbf{R}^d)^{(n+1)\otimes}}, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \int_0^1 \int_{\Delta_n} \sum_{i=1}^{n+1} 1_{(s_i, s_{i-1})}(t)(-1)^{i+1} ds_n \cdots ds_1 dt \\ &\quad \times \langle h(t) \otimes f_n(s_1, \cdots, s_n), g_{n+1}(s_1, \cdots, s_{i-1}, t, s_i, \cdots, s_n) \rangle_{(\mathbf{R}^d)^{(n+1)\otimes}} \\ &= \sum_{n=0}^{\infty} \int_{\Delta_{n+1}} \sum_{i=1}^{n} (-1)^{i+1} \langle h(s_i) \otimes f_n(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{n+1}), \\ &\qquad\qquad g_{n+1}(s_1, \cdots, s_{n+1}) \rangle ds_{n+1} \cdots ds_1 = \mathbf{E}[Y\psi^*(h)X], \end{aligned}$$

where we used the convention that $s_0 := 1$ and $s_{n+1} := 0$ for $s \in \Delta_n$. □

The operators $\psi(h), \psi^*(h)$, $h \in L^2(\mathbb{T})$, satisfy the following anti-commuting relations:

Theorem 3.2. *(i) The operators $\psi(h), \psi^*(h)$, $h \in L^2(\mathbb{T})$ are bounded operators with*

$$\|\psi^*(h)X\|^2_{L^2(\mathsf{W}^d)} + \|\psi(h)X\|^2_{L^2(\mathsf{W}^d)} = \|h\|^2_{L^2(\mathbb{T})}\|X\|^2_{L^2(\mathsf{W}^d)}, \quad (X \in L^2(\mathsf{W}^d)), \tag{3.4}$$

and in particular $\|\psi(h)\|_{B(L^2(\mathsf{W}^d))}, \|\psi^*(h)\|_{B(L^2(\mathsf{W}^d))} \leq \|h\|_{L^2(\mathbb{T})}$.

(ii) For g, $h \in L^2(\mathbb{T})$,

$$\psi(g)\psi^*(h) + \psi^*(h)\psi(g) = \langle g, h\rangle_{L^2(\mathbb{T})}1, \tag{3.5}$$

and

$$\psi(g)\psi(h) + \psi(h)\psi(g) = \psi^*(g)\psi^*(h) + \psi^*(h)\psi^*(g) = 0 \tag{3.6}$$

in $B(L^2(\mathsf{W}^d))$.

Proof. Let $X, Y \in L^2(\mathsf{W}^d)$ be given by

$$X = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle f_n(s_1, \cdots, s_n), dW_{s_n} \otimes \cdots \otimes dW_{s_1}\rangle_{(\mathbf{R}^d)^{n\otimes}}$$

and

$$Y = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle e_n(s_1, \cdots, s_n), dW_{s_n} \otimes \cdots \otimes dW_{s_1}\rangle_{(\mathbf{R}^d)^{n\otimes}}$$

with $f_n, g_n \in L^2(\Delta_n)$ for $n = 1.2, \cdots$.

First we shall show

$$\begin{aligned}
&\langle(\psi(g)\psi^*(h) + \psi^*(h)\psi(g))X, Y\rangle_{L^2(W^d)} \\
&= \langle\psi^*(h)X, \psi^*(g)Y\rangle_{L^2(W^d)} + \langle\psi(g)X, \psi(h)Y\rangle_{L^2(W^d)} \\
&= \langle h, g\rangle_{L^2(\mathbb{T})}\langle X, Y\rangle_{L^2(\mathsf{W}^d)}
\end{aligned} \tag{3.7}$$

for $X, Y \in \mathrm{Dom}(\mathcal{A})$.

Observe

$$\begin{aligned}
&\langle\psi^*(h)X, \psi^*(g)Y\rangle_{L^2(W^d)} \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \langle h \wedge f_{n-1}(s_1, \cdots, s_n), g \wedge e_{n-1}(s_1, \cdots, s_n)\rangle_{(\mathbf{R}^d)^{n\otimes}}\, ds_n \cdots ds_1 \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \sum_{i=1}^{n}\sum_{i'=1}^{n} (-1)^{i+i'}\langle h(s_i) \otimes f_{n-1}(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{n-1}), \\
&\qquad g(s_{i'}) \otimes e_{n-1}(s_1, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_{n-1})\rangle_{(\mathbf{R}^d)^{n\otimes}} \\
&= \sum_{n=1}^{\infty} \int_{\Delta_n} \sum_{i=1}^{n}\sum_{i'=1}^{n} (-1)^{i+i'}\langle h(s_i), g(s_{i'})\rangle_{\mathbf{R}^d}\langle f_{n-1}(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{n-1}), \\
&\qquad e_{n-1}(s_1, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_{n-1})\rangle_{(\mathbf{R}^d)^{(n-1)\otimes}}.
\end{aligned} \tag{3.8}$$

We divide the rightmost of (3.8) into the two parts: the diagonal

$$\mathrm{diag} := \sum_{n=1}^{\infty}\int_{\Delta_n}\sum_{i=1}^{n}\langle h(s_i), g(s_i)\rangle_{\mathbf{R}^d}\langle f_{n-1}(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_{n-1}), \\ e_{n-1}(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_{n-1})\rangle_{(\mathbf{R}^d)^{(n-1)\otimes}}.$$

and the off-diagonal part:

$$\text{off-diag} \\ := \sum_{n=1}^{\infty}\int_{\Delta_n}\sum_{i\neq i'}^{n}(-1)^{i+i'}\langle h(s_i), g(s_{i'})\rangle_{\mathbf{R}^d}\langle f_{n-1}(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_{n-1}), \\ e_{n-1}(s_1,\cdots,s_{i'-1},s_{i'+1},\cdots,s_{n-1})\rangle_{(\mathbf{R}^d)^{(n-1)\otimes}}.$$

Observing

$$\left(\int_{\mathbb{T}} a(s)ds\right)\left(\int_{\Delta_{n-1}} b(s_1,\cdots,s_{n-1})ds_{n-1}\cdots ds_1\right) \\ = \int_{\Delta_n}\sum_{i=1}^{n} a(s_i)b(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_n)ds_n\cdots ds_1,$$

we obtain

$$\mathrm{diag} = \langle h, g\rangle_{L^2(\mathbb{T})}\sum_{n=1}^{\infty}\langle f_{n-1}, g_{n-1}\rangle_{L^2(\Delta_n)} = \langle h, g\rangle_{L^2(\mathbb{T})}\mathbf{E}[XY].$$

On the other hand, by the defining formula (3.1) and the expression (2.3), we have

$$\langle \psi(g)X, \psi(h)Y\rangle_{L^2(W^d)} = \mathbf{E}[\langle \mathcal{A}X, g\rangle_{L^2(\mathbb{T})}\langle \mathcal{A}Y, h\rangle_{L^2(\mathbb{T})}] \\ = \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}^2} ds_{n-1}\cdots ds_1 dt dt'\langle g(t), h(t')\rangle_{\mathbf{R}^d} \\ \times \langle \check{f}_n(t, s_1,\cdots,s_{n-1}), \check{e}_n(t', s_1,\cdots,s_{n-1})\rangle_{(\mathbf{R}^d)^{n\otimes}} \\ = \sum_{n=1}^{\infty}\int_{\Delta_{n-1}}\int_{\mathbb{T}^2} ds_{n-1}\cdots ds_1 dt dt'\langle g(t), h(t')\rangle_{\mathbf{R}^d} \\ \times \sum_{i=1}^{n}\sum_{i'=1}^{n} 1_{(s_i,s_{i-1})}(t)1_{(s_{i'},s_{i'-1})}(t')\langle \check{f}_n(t, s_1,\cdots,s_{n-1}), \check{e}_n(t', s_1,\cdots,s_{n-1})\rangle_{(\mathbf{R}^d)^{n\otimes}} \\ = -\sum_{n=1}^{\infty}\int_{\Delta_{n+1}} ds_{n-1}\cdots ds_1\sum_{i\neq i'}(-1)^{i+i'}\langle g(s_i), h(s_{i'})\rangle_{\mathbf{R}^d} \\ \times \langle f_n(s_1,\cdots,s_{i'-1},s_{i'+1},\cdots s_n), e_n(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_n)\rangle_{(\mathbf{R}^d)^{n\otimes}} \\ = -\sum_{n=1}^{\infty}\int_{\Delta_{n+1}} ds_{n+1}\cdots ds_1\sum_{i\neq i'}(-1)^{i+i'}\langle g(s_i)\otimes e_n(s_1,\cdots,s_{i-1},s_{i+1},\cdots,s_{n+1}), \\ h(s_{i'})\otimes f_n(s_1,\cdots,s_{i'-1},s_{i'+1},\cdots s_{n+1})\rangle_{(\mathbf{R}^d)^{(n+1)\otimes}} \\ = -\text{off-diag}.$$

Thus we have shown the relation (3.7). By taking $X = Y$ and $g = h$ in (3.7), we obtain (3.4), which ensures that $\psi(h)$ and $\psi^*(h)$ are in $B(L^2(\mathsf{W}^d))$.

Next, we shall show (3.6). For $g, h \in L^2(\mathbb{T})$ and $X, Y \in L^2(\mathsf{W}^d)$ given as above,

$$\langle \psi^*(h)X, \psi(g)Y\rangle_{L^2(\mathsf{W}^d)} = \sum_{n=1}^{\infty} \int_{\Delta_n} \langle h \wedge f_{n-1}(s), \int_{\mathbb{T}} \langle g(t), \check{e}_{n+1}(t,s)\rangle_{\mathbf{R}^d} dt\rangle_{(\mathbf{R}^d)^{n\otimes}} ds$$

$$= \sum_{n=1}^{\infty} \int_{\Delta_n} ds \int_{\mathbb{T}} dt \sum_{i=1}^{n} \sum_{i'=1}^{n+1} 1_{(s_{i'}, s_{i'-1})}(t)(-1)^{i+i'} \langle g(t),$$
$$\langle h(s_i) \otimes f_{n-1}(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n),$$
$$e_{n+1}(s_1, \cdots, s_{i'-1}, t, s_{i'}, \cdots, s_{n+1})\rangle_{(\mathbf{R}^d)^{n\otimes}}\rangle_{\mathbf{R}^d}$$

$$= \sum_{n=1}^{\infty} \int_{\Delta_{n+1}} ds \sum_{i=1}^{n+1} \Big(\sum_{i<i'} (-1)^{i+i'} \langle h(s_i),$$
$$\langle g(s_{i'}) \otimes f_{n-1}(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_{n+1}),$$
$$e_{n+1}(s_1, \cdots, s_{n+1})\rangle_{(\mathbf{R}^d)^{n\otimes}}\rangle_{\mathbf{R}^d}$$
$$- \sum_{i'<i}^{n+1} (-1)_{i+i'} \langle h(s_i), \langle g(s_{i'}) \otimes f_{n-1}(s_1, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{n+1}),$$
$$e_{n+1}(s_1, \cdots, s_{n+1})\rangle_{(\mathbf{R}^d)^{n\otimes}}\rangle_{\mathbf{R}^d} \Big)$$

$$= \sum_{n=1}^{\infty} \int_{\Delta_n} ds \int_{\mathbb{T}} dt \sum_{i=1}^{n+1} 1_{(s_i, s_{i-1})} \Big(\sum_{i<i'} (-1)^{i+i'-1} \langle h(t),$$
$$\langle g(s_{i'}) \otimes f_{n-1}(s_1, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_n),$$
$$e_{n+1}(s_1, \cdots, s_{i-1}, t, s_i, \cdots, s_n)\rangle_{(\mathbf{R}^d)^{n\otimes}}\rangle_{\mathbf{R}^d}$$
$$- \sum_{i'<i} (-1)^{i+i'} \langle h(t), \langle g(s_{i'}) \otimes f_{n-1}(s_1, \cdots, s_{i'-1}, s_{i'+1}, \cdots, s_n),$$
$$e_{n+1}(s_1, \cdots, s_{i-1}, t, s_i, \cdots, s_n)\rangle_{(\mathbf{R}^d)^{n\otimes}}\rangle_{\mathbf{R}^d}$$
$$= -\langle \psi^*(g)X, \psi(h)Y\rangle_{L^2(\mathsf{W}^d)}.$$

□

4. Clifford Algebras in Wiener Space

Let $\{h_k : k \in \mathbf{N}\}$ be an orthogonal basis of $L^2(\mathbb{T})$, and $\mathfrak{A}$ be the subalgebra of $B(L^2(\mathsf{W}^d))$ generator by $\{\psi(h_k), \psi^*(h_k) : k \in \mathbf{N}\}$. The following facts are easy to verify.

Lemma 4.1. *For $k, l \in \mathbf{N}$,*

$$[\psi(h_k), \psi(h_l)]_+ = [\psi^*(h_k), \psi^*(h_l)]_+ = 0, \ [\psi(h_k), \psi^*(h_l)]_+ = \delta_{k,l},$$

where $[E, F]_+ = EF + FE$ is the anti-commutator.

Proof. They are direct consequences of anti-commutation relations (3.5) and (3.6).
□

The following is, in an abstract sense, a well-established fact in quantum field theory but we state it as a theorem and give a short proof to complete our exposition as a tour in anti-symmetric Malliavin calculus.

Theorem 4.2. *The $\mathfrak{A}$-module $\mathfrak{A}(1)$ is dense in $L^2(\mathsf{W}^d)$. In other words, the L^2 space of Wiener functionals $L^2(\mathsf{W}^d)$ is a representation space of an irreducible representation of the Clifford algebra $\mathfrak{A}$.*

Proof. As is well-known and easily checked that

$$\mathfrak{A} = \mathrm{span}\{\psi^*(h_{k_1})\cdots\psi^*(h_{k_n})\psi(l_1)\cdots\psi(l_m) : k_1,\cdots,k_n,l_1,\cdots,l_m \in \mathbf{N}, n,m \in \mathbf{N}\}.$$

We have

$$\psi^*(h_{k_1})\cdots\psi^*(h_{k_n})\psi(l_1)\cdots\psi(l_m)(1) = 0$$

in the cases where $m \geq 1$, and when $m = 0$, by (3.2),

$$\begin{aligned} &\psi^*(h_{k_1})\cdots\psi^*(h_{k_n})(1) \\ &= \int_{\Delta_n} \langle h_{k_1} \wedge \cdots \wedge h_{k_n}(s_1,\cdots,s_n), dW_{s_n} \otimes \cdots \otimes dW_{s_1}\rangle_{(\mathbf{R}^d)^{n\otimes}}, \end{aligned}$$

which form an orthogonal basis of $L^2(W^d)$. □

Remark 4.3. In quantum field theory, $\psi^*(h_{k_1})\cdots\psi^*(h_{k_n})(1)$ is normally denoted by $\psi^*(h_{k_1})\cdots\psi^*(h_{k_n})|\mathrm{vac}\rangle$. The expectation with respect to Wiener measure plays the role of "vacuum expectation".

5. Comments on Potential Applications

(1) It is well-recognized that quadratic forms of the elements in a Clifford algebra form a Lie algebra. Its infinite-dimensional versions have rich applications in various areas of mathematics. Among them, the celebrated theory by Date-Kashiwara-Jimbo-Miwa states that the orbit of "Lie group" of an infinite-dimensional Lie algebra form Sato's Grassmanian, which consists of "tau-functions" of KP/KdV hierarchy of integrable non-linear partial differential equations (see [9] and references therein). In fact, the stochastic representations of tau-functions ([2], [10], [11], [5], [3], [1], [6], and so on) motivated our introduction of anti-symmetric version of Malliavin calculus.

(2) Though we have not explicitly stated, an anti-symmetric version of Malliavin's integration by parts can be obtained and potentially applicable to the analysis of stochastic equations.

(3) Applications to stochastic numerical analysis based on the observations by K. Yoshikawa [12] might also be interesting.

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