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An anticipating Itô formula for Lévy processes

Elisa Alós, Jorge A. León and Josep Vives

Dpt. d'Economia i Empresa Universitat Pompeu Fabra c/ Ramon Trias Fargas, 25-27 08005 Barcelona, Spain
E-mail address: elisa.alos@upf.edu

Depto. de Control Automático Cinvestav-IPN Apartado Postal 14-740 07000 México, D.F., México
E-mail address: jleon@ctrl.cinvestav.mx

Dpt. de Probabilitat, Lógica i Estadística Universitat de Barcelona Gran Via, 585 08007 Barcelona, Spain
E-mail address: josep.vives@ub.edu

Abstract. In this paper, we use the Malliavin calculus techniques to obtain an anticipative version of the change of variable formula for Lévy processes. Here the coefficients are in the domain of the anihilation (gradient) operator in the "future sense", which includes the family of all adapted and square-integrable processes. This domain was introduced on the Wiener space by Alòs and Nualart (1998). Therefore, our Itô formula is not only an extension of the usual adapted formula for Lévy processes, but also an extension of the anticipative version on Wiener space obtained in Alòs and Nualart (1998).

1. Introduction

It is well–known that the Itô formula, or change of variable formula, is one of the most powerful tools of the stochastic analysis due to its vast range of applications. So, in the last few years, various researchers have studied extensions of the classical Itô formula for different interpretations of stochastic integral (see, for instance, Alòs and Nualart, 1998, Di Nunno et al., 2005, Moret and Nualart, 2001, Nualart and Taqqu, 2006, and Tudor and Viens, 2006). In particular, several authors have been interested in finding extensions of this important formula to the case where the coefficients are not adapted to the underlying filtration (see Di Nunno et al., 2006, León et al., 2003, Nualart and Pardoux, 1988, or Russo and Vallois, 1995).

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The Malliavin calculus or calculus of variations is another important tool of the stochastic analysis that allows us to deal with stochastic integrals whose domains include processes that are not necessarily adapted to the underlying filtration. Recently, the interest of this calculus has increased considerably because of its applications in finance (see, for example, Alòs, 2006, Alòs et al., 2007, Bally et al., 2005, Fournié et al., 2001, 1999, Imkeller, 2003, Nualart, 2006 or Øksendal, 1996), or other theoretical applications (see Alòs and Nualart, 1998, León and Nualart, 1998, Nualart, 1998, 2006 or Sanz-Solé, 2005). This important theory is basically based on the divergence and gradient operators.

The divergence operator has been interpreted as a stochastic integral because it has properties similar to those of the Itô stochastic integral. For instance, the isometry and local properties, the fact that it can be approximated by Riemann sums, the integration by parts formula, etc. (see Nualart, 2006). Hence, it is important to count on a change of variable formula for the divergence operator in order to improve the applications of the Malliavin calculus to different areas of the human knowledge.

On the Wiener space, the divergence operator was defined by Skorohod (1975) and it is an extension of the classical Itô integral. In order to analyze the properties of the Skorohod integral, the adaptability of the integrands (necessary in the Itô's calculus) is changed by some analytic properties that are used to define some spaces, called Sobolev spaces, where a fundamental ingredient is the derivative (gradient) operator (see Sections 2.3 and 2.4 below). For instance, Alòs and Nualart (1998) have considered processes with derivatives "in the future sense".

In this paper the stochastic integral with respect to the continuous part of the underlying Lévy process is in the Skorohod sense. The Skorohod integral can be introduced using different approaches. Namely, the first method is via the Wiener chaos decomposition, and the second one considers the Skorohod integral as the adjoint of the gradient (derivative) operator.

On the Poisson space, the above two methods produce different definitions of stochastic integral (see, for example Carlen and Pardoux, 1990, León and Tudor, 1998, Nualart and Vives, 1995, or Picard, 1996a). Moreover, in this space, we can take advantage of the pathwise characterizations of some stochastic integrals, as we do in this paper, to deal with applications of the stochastic analysis (see León et al., 2001, Picard, 1996b or Privault, 1993). In particular, the gradient operator is a difference one.

Recently, several approaches to develope a calculus of variations for Lévy processes have been introduced in some articles (see, for instance, Di Nunno et al., 2005, Løkka, 2004 and Solé et al., 2007, among others). The gradient and divergence operators are the fundamental tools in this theory again. In this paper, we restrict ourselves to the canonical Lévy space defined in Solé et al. (2007) because, in this space, the gradient operator defined utilizing the chaotic decomposition of a square–integrable random variable is not a "derivative operator" (see Section 2.3 below), but it is the sum of a derivative and an increment quotient operators. This fact is important because we can obtain and use the relation between the stochastic integral introduced via the chaos decomposition and the pathwise stochastic integral, both with respect to the jump part of the involved Lévy process (see Lemma 2.7 below).

The purpose of this paper is to use the Malliavin calculus on the canonical Lévy space given in Solé et al. (2007) to prove an anticipating Itô formula for Lévy processes. Here, the stochastic integrals with respect to the continuous and jump parts of the underlying Lévy process are in the Skorohod and pathwise sense, respectively. The coefficients in this formula have two “derivatives in the future sense”. It means, they are in a class of square-integrable processes u such that u_t is in the domain of the gradient operator D at time r for $r > t$, and $D_r u_t$ is also in the domain of D (see Section 2.4). An example of processes satisfying this property is the square-integrable and adapted processes, whose “derivative” is equal to zero.

The paper is organized as follows. In Section 2 we present the framework that we use in this paper. Namely, we introduce some basic facts of the canonical Lévy space and of the Malliavin calculus on this space. Finally, the anticipating Itô formula is studied in Section 3.

2. Preliminaries

In this section we give the framework that will be used in this article. That is, we introduce briefly the Itô multiple integrals with respect to a Lévy process, and the canonical Lévy process considered by Solé et al. (2007). Then we present some basic facts on the Malliavin calculus for this process. We need to study the anihilation and creation operators corresponding to the Fock space associated with the chaos decomposition on Lévy space, and analyze the Sobolev spaces associated with these operators. Althoug some of these facts are known, we give them for the convenience of the reader.

Throughout, we set $\mathbb{R}_0 = \mathbb{R} - \{0\}$ and $T > 0$. Let ν be a Lévy measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$ (see Sato, 1999). The Borel σ–algebra of a set $A \subset \mathbb{R}$ is denoted by $\mathcal{B}(A)$. The jump of a cádlág process Z at time $t \in [0, T]$ is represented by ΔZ_t (i.e., $\Delta Z_t := Z_t - Z_{t-}$).

2.1. *Itô multiple integrals.* The construction of multiple integrals with respect to Lévy processes is quite similar to that of multiple integrals with respect to the Brownian motion. The reader can consult Itô (1956) for a complete survey on this topic.

Let $X = \{X_t : t \in [0, T]\}$ be a Lévy process with triplet (γ, σ^2, ν). It is well–known that X has the Lévy–Itô representation (see Sato, 1999)

$$X_t = \gamma t + \sigma W_t + \int_{(0,t]\times\{|x|>1\}} x dJ(s,x) + \lim_{\varepsilon\downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\le 1\}} x d\widetilde{J}(s,x). \quad (2.1)$$

Here the convergence is with probability 1, uniformly on $t \in [0, T]$, $W = \{W_t : t \in [0, T]\}$ is a standard Brownian motion,

$$J(B) = \#\{t : (t, \Delta X_t) \in B\}, \quad B \in \mathcal{B}([0, T] \times \mathbb{R}_0),$$

is a Poisson measure with parameter $dt \otimes d\nu$ and $d\widetilde{J}(t, x) = dJ(t, x) - dtd\nu(x)$.

For $E_1, \ldots, E_n \in \mathcal{B}([0, T] \times \mathbb{R})$ such that $E_i \cap E_j = \emptyset$, $i \neq j$, and

$$\mu(E_i) := \sigma^2 \int_{\{t\in[0,T]:(t,0)\in E_i\}} dt + \int_{E_i-(E_i\cap([0,T]\times\{0\}))} x^2 dtd\nu(x) < \infty,$$

we define the multiple integral $I_n(1_{E_1\times\cdots\times E_n})$ of order n with respect to M by

$$I_n(1_{E_1\times\cdots\times E_n}) = M(E_1)\cdots M(E_n), \tag{2.2}$$

with

$$M(E_i) = \sigma\int_{\{t\in[0,T]:(t,0)\in E_i\}} dW_t + \lim_{m\to\infty}\int_{\{(t,x)\in E_i:\frac{1}{m}<|x|<m\}} xd\widetilde{J}(t,x),$$

where the limit is in the $L^2(\Omega)$ sense. By linearity, we can define the multiple integral of order n of an elementary function f of the form

$$f(\cdot) = \sum_{i_1,\ldots,i_n=1}^{N} a_{i_1,\ldots,i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}}(\cdot),$$

where $A_1,\ldots,A_N$ are pairwise disjoint sets of $\mathcal{B}([0,T]\times\mathbb{R})$ and $a_{i_1,\ldots,i_n}=0$ if two of the indices $i_1,\ldots,i_n$ are equal.

The multiple integral I_n is extended to $L^2_n := L^2(([0,T]\times\mathbb{R})^n;\ \mathcal{B}(([0,T]\times\mathbb{R})^n);\mu^{\otimes n})$ due to the fact that the space of all the elementary functions is dense in L^2_n and the property

$$\begin{aligned} &E[I_n(1_{E_1\times\cdots\times E_n})I_m(1_{F_1\times\cdots\times F_m})] \\ &= \delta_n(m)n!\int_{([0,T]\times\mathbb{R})^n} \widetilde{1}_{E_1\times\cdots\times E_n}\widetilde{1}_{F_1\times\cdots\times F_m}d\mu^{\otimes n}, \end{aligned} \tag{2.3}$$

where $\widetilde{f}$ is the symmetrization of the function f and δ_n is the Dirac measure concentrated at n.

It is well–known that if F is a square–integrable random variable, measurable with respect to the filtration generated by X, then F has the unique representation

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.4}$$

where $I_0(f_0)=f_0=E(F)$ and f_n is a symmetric function in L^2_n. This is the so called chaotic representation property for Lévy processes.

2.2. *Canonical Lévy space.* The purpose of this subsection is to present some basic elements of the structure of the canonical Lévy space on the interval $[0,T]$. For a more detailed account of this subject, we refer to Solé et al. (2007).

The construction of the canonical Lévy space is divided in three steps, as follows:

Step 1. Here we introduce the canonical space for a compound Poisson process. Toward this end, let Q be a probability measure on $\mathbb{R}$, supported on $S\in\mathcal{B}(\mathbb{R}_0)$, and $\lambda>0$. Set

$$\Omega_T = \bigcup_{n\geq 0}([0,T]\times S)^n,$$

with $([0,T]\times S)^0=\{\alpha\}$, where α is an arbitrary point. The set Ω_T is equipped with the σ–algebra

$$\mathcal{F}_T = \{B\subset\Omega_T : B\cap([0,T]\times S)^n \in \mathcal{B}(([0,T]\times S)^n), \quad \text{for all} \quad n\geq 1\}.$$

The probability P_T on $(\Omega_T,\mathcal{F}_T)$ is given by

$$P_T(B\cap([0,T]\times S)^n) = e^{-\lambda T}\frac{\lambda^n(dt\otimes Q)^{\otimes n}(B\cap([0,T]\times S)^n)}{n!},$$

with $(dt \otimes Q)^0 = \delta_\alpha$. Here δ_α is the Dirac measure concentrated at α.

The space $(\Omega_T, \mathcal{F}_T, P_T)$ is called the canonical space for the compound Poisson process with Lévy measure λQ. A similar definition for the Poisson process was given in Neveu (1977), and in Nualart and Vives (1995). In $(\Omega_T, \mathcal{F}_T, P_T)$ the process

$$X_t(\omega) = \begin{cases} \sum_{j=1}^{n} x_j 1_{[0,t]}(t_j), & \text{if} \quad \omega = ((t_1, x_1), \cdots, (t_n, x_n)), \\ 0, & \text{if} \quad \omega = \alpha, \end{cases}$$

is a compound Poisson process with intensity λ and jump law given by the probability measure Q.

Step 2. Now we consider the canonical space for a pure jump Lévy process with Lévy measure ν.

Let $S_1 = \{x \in \mathbb{R} : \varepsilon_1 < |x|\}$ and $S_k = \{x \in \mathbb{R} : \varepsilon_k < |x| \leq \varepsilon_{k-1}\}$ for $k > 1$. Here $\{\varepsilon_k : k \geq 1\}$ is a strictly decreasing sequence of positive numbers such that $\varepsilon_1 = 1$, $\lim_{k\to\infty} \varepsilon_k = 0$ and $\nu(S_k) \neq 0$. Note that the fact that ν is a Lévy measure implies that $\nu(S_k) < \infty$ for every $k \geq 1$. Now, the canonical Lévy space with measure ν is defined as

$$(\Omega_J, \mathcal{F}_J, \mathcal{P}_J) = \bigotimes_{k\geq 1} (\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)}),$$

where $(\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)})$ is the canonical space for the canonical compound Poisson process $\{X_t^{(k)} : t \in [0,T]\}$ with intensity $\lambda_k = \nu(S_k)$ and probability measure $Q_k = \frac{\nu(\cdot \cap S_k)}{\nu(S_k)}$. In this case, for $\omega = (\omega^k)_{k\geq 1} \in \Omega_J$ and $t \in [0,T]$, the limit

$$J_t(\omega) = \lim_{n\to\infty} \sum_{k=2}^{n} (X_t^{(k)}(\omega^k) - t \int_{S_k} x d\nu(x)) + X_t^{(1)}(\omega^1)$$

exists with probability 1 and it is a pure jump Lévy process with Lévy measure ν.

Step 3. The canonical Lévy space on $[0,T]$ with Lévy measure ν is

$$(\Omega, \mathcal{F}, P) = (\Omega_W \otimes \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, P_W \otimes P_J),$$

where $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space. Here, for $\omega = (\omega', \omega'') \in \Omega_W \otimes \Omega_J$, the process

$$X_t(\omega) = \gamma t + \sigma \omega'(t) + J_t(\omega'') \tag{2.5}$$

is a Lévy process with triplet (γ, σ^2, ν). For this fact we refer to Sato (1999).

2.3. *The anihilation and creation operators.* Henceforth we suppose that the underlying probability space $(\Omega, \mathcal{F}, P)$ is the canonical Lévy space with Lévy measure ν and that X is the Lévy process defined in (2.5).

We say that the square-integrable random variable F given by (2.4) belongs to the domain of the anihilation operator D ($F \in \mathbb{D}^{1,2}$ for short) if and only if

$$\sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2_n} < \infty. \tag{2.6}$$

In this case we define the random field $DF = \{D_z F : z \in [0,T] \times \mathbb{R}\}$ as

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(z, \cdot)).$$

Note that (2.6) yields that the last series converges in $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$ by (2.3). Thus, in this case, we have that $\sum_{n=0}^{m} I_n(f_n)$ and $\sum_{n=1}^{m} nI_{n-1}(f_n(z,\cdot))$ converge to F and DF in $L^2(\Omega)$ and in $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$ as $m \to \infty$, respectively. D is a closed operator from $L^2(\Omega)$ into $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$, with dense domain. Similarly we can define the iterated derivative $D^n_{z_1,\ldots,z_n} = D_{z_1} \cdots D_{z_n}$ and its domain $\mathbb{D}^{n,2}$.

The following result is due to Solé et al. (2007) and it establishes how we can figure out the random field DF without using the chaos decomposition (2.4). In order to state it, we need the following:
Henceforth $W = \{W_t : t \in [0,T]\}$ is the canonical Wiener process and $\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ denotes the family of $L^2(\Omega_J, \mathcal{F}_J, P_J)$–valued random variables that are in the domain of the derivative operator D^W with respect to W. The reader can consult Nualart (2006) for the basic definitions and properties of this operator. The space $\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ is constructed as follows. We say that a random variable F is an $L^2(\Omega_J)$-valued smooth random variable if it has the form

$$F = f(W_{t_1}, \ldots, W_{t_n})Z,$$

with $t_i \in [0,T]$, $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded), and $Z \in L^2(\Omega_J, \mathcal{F}_J, P_J)$. The derivative of F with respect to W, in the Malliavin calculus sense, is defined as

$$D^W F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_n}) Z 1_{[0,t_i]}.$$

It is easy to see that D^W is a closeable operator from $L^2(\Omega_W; L^2(\Omega_J))$ into $L^2(\Omega_W \times [0,T]; L^2(\Omega_J))$. Thus we can introduce the space $\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ as the completion of the $L^2(\Omega_J)$-valued smooth random variables with respect to the seminorm

$$||F||^2_{1,2,W} = E\left[|F|^2 + |DF|^2_{L^2([0,T])}\right].$$

For $\omega = (\omega', (\omega^k)_{k\geq 1}) \in \Omega$, with $\omega^k = ((t_1^k, x_1^k), \ldots, (t_{n_k}^k, x_{n_k}^k))$, $F \in L^2(\Omega)$ and $z = (t,x) \in (0,T] \times S_{k_0}$, for some positive integer k_0, we define

$$(\Psi_{t,x}F)(\omega) = \frac{F(\omega_z) - F(\omega)}{x},$$

with $\omega_z = (\omega', (\omega_z^k)_{k\geq 1})$ and

$$\omega_z^k = \begin{cases} ((t,x), (t_1^{k_0}, x_1^{k_0}), \ldots, (t_{n_{k_0}}^{k_0}, x_{n_{k_0}}^{k_0})), & \text{if } k = k_0, \\ \omega^k, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let $F \in L^2(\Omega)$ be a random variable such that:*

i) $F \in \mathbb{D}^{1,2}_W(L^2(\Omega_J))$.
ii) $\Psi F \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$.

Then $F \in \mathbb{D}^{1,2}$ and

$$D_{t,x}F = 1_{\{0\}}(x)\sigma^{-1} D_t^W F + 1_{\mathbb{R}_0}(x)\Psi_{t,x}F.$$

Proof. The proof of this result is an immediate consequence of Solé et al. (2007) (Propositions 3.5 and 5.5). ■

Now we establish an auxiliary tool needed for our results.

Lemma 2.2. *Let $F \in \mathbb{D}^{1,2}$. Then there exists a sequence $\{F_n : n \geq 1\}$ of the form*

$$F_n = \sum_{i=1}^{N} H_{i,n} Z_{i,n} \tag{2.7}$$

such that:

i) *$H_{i,n}$ is a smooth functional in $L^2(\Omega_W)$ and $Z_{i,n} \in \mathbb{D}^{2,2} \cap L^\infty(\Omega_J)$.*
ii) *F_n (resp. DF_n) converges to F (resp. DF) in $L^2(\Omega)$ (resp. $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$) as $n \to \infty$.*

Remarks

i) Observe that N in equality (2.7) is a positive integer depending only on n.
ii) By Solé et al. (2007) (Proposition 5.4), $\Psi Z_{i,n} \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes \mu)$.

Proof. Note that it is enough to show the result holds for a multiple integral of the form (2.2). That is

$$F = M(E_1) \cdots M(E_k),$$

where $E_1, \cdots, E_k$ are pairwise disjoint borel subsets of $[0,T] \times \mathbb{R}$. Indeed, in this case, the result is also true for a random variable G with a finite chaos decomposition because, by the definition of the multiple integrals, there exists a sequence $\{G_m : m \geq 1\}$ of linear combinations of multiple integrals of the form (2.2) such that $G_m \to G$ in $L^2(\Omega)$ and $DG_m \to DG$ in $L^2(\Omega \times [0,T] \times \mathbb{R}; P \otimes \mu)$, as $m \to \infty$. Therefore, (2.6) implies that the result is satisfied.

Let $\varphi \in C^\infty(\mathbb{R})$ be a function such that

$$\varphi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Set $\rho_n(x) = x\varphi(\frac{x}{n})$ and

$$F_n = \prod_{i=1}^{k} \left(\rho_n \left(\int_{\{s:(s,0)\in E_i\}} \sigma dW_s \right) + \rho_n \left(\lim_{m\to\infty} \int_{\{(s,y)\in E_i : \frac{1}{m} < |y| < m\}} y d\widetilde{J}(s,y) \right) \right).$$

Then,

$$\begin{aligned} &\Psi_{t,x} \left(\rho_n \left(\lim_{m\to\infty} \int_{\{(s,y)\in E_i : \frac{1}{m} < |y| < m\}} y d\widetilde{J}(s,y) \right) \right) \\ &= \frac{1}{x} \left(\rho_n \left(x 1_{E_i}(t,x) + \lim_{m\to\infty} \int_{\{(s,y)\in E_i : \frac{1}{m} < |y| < m\}} y d\widetilde{J}(s,y) \right) \right. \\ &\qquad \left. - \rho_n \left(\lim_{m\to\infty} \int_{\{(s,y)\in E_i : \frac{1}{m} < |y| < m\}} y d\widetilde{J}(s,y) \right) \right), \end{aligned}$$

and

$$\begin{aligned}
&\Psi_{r,z}\Psi_{t,x}\Bigg(\rho_n\Bigg(\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Bigg)\Bigg)\\
&= \frac{1}{xz}\Bigg(\rho_n\Bigg(x1_{E_i}(t,x)+z1_{E_i}(r,z)+\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Bigg)\\
&\quad -\rho_n\Bigg(x1_{E_i}(t,x)+\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Bigg)\\
&\quad -\rho_n\Bigg(z1_{E_i}(r,z)+\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Bigg)\\
&\quad +\rho_n\Bigg(\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Bigg)\Bigg).
\end{aligned}$$

Hence, $\rho_n\Big(\lim_{m\to\infty}\int_{\{(s,y)\in E_i:\frac{1}{m}<|y|<m\}} yd\widetilde{J}(s,y)\Big)$ is in $\mathbb{D}^{2,2}$ due to Solé et al. (2007) (Lemma 5.2) or Lemma 2.1.

Now the result follows from the facts that $F_n \to F$ in $L^2(\Omega)$ as $n\to\infty$, $|\rho_n(x)|\le |x|$ and that there is a constant C independent of n such that $|\rho_n'(x)|+|\rho_n''(x)|\le C$. ■

An immediate consequence of the last two lemmas is the following:

Corollary 2.3. *Let F be a random variable in $L^2(\Omega)$. Then $F\in\mathbb{D}^{1,2}$ if and only if $F\in\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ and $\Psi F\in L^2(\Omega\times[0,T]\times\mathbb{R}_0;P\otimes\mu)$.*

Proof. The proof follows from Lemmas 2.1 and 2.2, and from Solé et al. (2007) (Proposition 4.8). ■

We will also need the following result.

Lemma 2.4. *Let $F\in\mathbb{D}^{1,2}$ be a bounded random variable. Then $(FG)\in\mathbb{D}^{1,2}$ for every G of the form (2.7).*

Proof. We first observe that $FG\in\mathbb{D}^{1,2}_W(L^2(\Omega_J))$ due to Corollary 2.3.

Finally, we have

$$\Psi_{t,x}(FG)=(\Psi_{t,x}F)G+F\Psi_{t,x}G+(F(\omega_{(t,x)})-F)\Psi_{t,x}G.$$

Therefore $\Psi(FG)\in L^2(\Omega\times[0,T]\times\mathbb{R}_0;P\otimes\mu)$. Consequently the proof is complete by Lemma 2.1. ■

The creation operator δ is the adjoint of $D:\mathbb{D}^{1,2}\subset L^2(\Omega)\to L^2(\Omega\times[0,T]\times\mathbb{R};P\otimes\mu)$. It means, u belongs to Dom δ if and only if $u\in L^2(\Omega\times[0,T]\times\mathbb{R};P\otimes\mu)$ is such that there exists a square–integrable random variable $\delta(u)$ satisfying the duality relation

$$E\left[\int_{[0,T]\times\mathbb{R}} u(z)(D_zF)d\mu(z)\right]=E[\delta(u)F],\quad \text{for every}\quad F\in\mathbb{D}^{1,2}. \tag{2.8}$$

It is not difficult to show that this duality relation gives that if u has the chaos decomposition

$$u(z)=\sum_{n=0}^{\infty} I_n(u_n(z,\cdot)),\quad z\in[0,T]\times\mathbb{R},$$

where $u_n \in L^2_{n+1}$ is a symmetric function in the last n variables, then $\delta(u)$ has the chaos decomposition (see Nualart, 2006)

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n).$$

The creation operator of a process multiplied by a random variable can be calculated via the following two results, which have been considered by Di Nunno et al. (2005) for pure jump Lévy processes.

Proposition 2.5. *Let F be a random variable as in Lemma 2.4 and $u \in \mathrm{Dom}\,\delta$ such that*

$$E\left[\int_{[0,T]\times\mathbb{R}} (u(t,x)(F + xD_{t,x}F))^2 d\mu(t,x)\right] < \infty.$$

Then $(t,x) \mapsto u(t,x)(F + xD_{t,x}F)$ belongs to Dom δ if and only if

$$\left(F\delta(u) - \int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}F d\mu(t,x)\right) \in L^2(\Omega).$$

In this case

$$\delta(u(t,x)F + xu(t,x)D_{t,x}F) = F\delta(u) - \int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}F d\mu(t,x).$$

Proof. Let G be a random variable as in the right-hand side of (2.7). Then Lemma 2.4 and its proof give

$$\begin{aligned}
& E[GF\delta(u)] \\
= \;& E\left[\int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}(FG)d\mu(t,x)\right] \\
= \;& E\left[\sigma^2 \int_0^T u(t,0)D_{t,0}(FG)dt + \int_{[0,T]\times\mathbb{R}_0} u(t,x)D_{t,x}(FG)d\mu(t,x)\right] \\
= \;& E\left[\sigma^2 \int_0^T u(t,0)(D_{t,0}F)Gdt + \sigma^2 \int_0^T u(t,0)FD_{t,0}Gdt\right] \\
& +E\left[\int_{[0,T]\times\mathbb{R}_0} u(t,x)((D_{t,x}F)G + FD_{t,x}G + x(D_{t,x}F)D_{t,x}G)d\mu(t,x)\right] \\
= \;& E\left[G\int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}Fd\mu(t,x)\right] \\
& +E\left[\int_{[0,T]\times\mathbb{R}} (u(t,x)F + u(t,x)xD_{t,x}F)D_{t,x}Gd\mu(t,x)\right].
\end{aligned}$$

Therefore the proof is complete by Lemma 2.2 and by the duality relation (2.8). ■

The following result is an immediate consequence of the proof of Proposition 2.5.

Corollary 2.6. *Let u and F be as in Proposition 2.5. Moreover assume that $(t,x) \mapsto u(t,x)xD_{t,x}F$ belongs to Dom δ. Then $Fu \in \mathrm{Dom}\,\delta$ if and only if*

$$F\delta(u) - \delta(u(t,x)xD_{t,x}F) - \int_{[0,T]\times\mathbb{R}} u(t,x)D_{t,x}Fd\mu(t,x) \tag{2.9}$$

is a square–integrable random variable. In this case $\delta(Fu)$ is equal to (2.9).

2.4. *Sobolev spaces.* In this subsection we proceed as in Alòs and Nualart (1998) in order to define the spaces that contain the integrands in our Itô formula.

Let $\mathcal{S}_T$ be the family of processes of the form $u(\cdot)=\sum_{j=1}^n F_j h_j(\cdot)$, where F_j is a random variable of the form (2.7) and $h:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a bounded measurable function. Note that the fact that $\int_{\mathbb{R}} x^2 d\nu(x)<\infty$ implies that $h\in L^2([0,T]\times\mathbb{R};\mu)$. Denote by $\mathbb{L}^{1,2,f}$ the closure of $\mathcal{S}_T$ with respect to the seminorm

$$||u||^2_{1,2,f}=E\int_{[0,T]\times\mathbb{R}} u(z)^2 d\mu(z)+E\int_{\Delta_1^T}(D_{s,y}u(t,x))^2 d\mu(s,y)d\mu(t,x),$$

where

$$\Delta_1^T=\Big\{((s,y),(t,x))\in([0,T]\times\mathbb{R})^2 : s\geq t\Big\}.$$

A random field $u=\{u(s,y):(s,y)\in[0,T]\times\mathbb{R}\}$ in $\mathbb{L}^{1,2,f}$ belongs to the space $\mathbb{L}^{1,2,f}_-$ if there is $D^-u\in L^2(\Omega\times[0,T]\times\mathbb{R};P\otimes\mu)$ such that

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}}\sup_{(s-\frac{1}{n})\vee 0\leq r<s, y\leq x\leq y+\frac{1}{n}} E[|D_{s,y}u(r,x)-D^-u(s,y)|^2]d\mu(s,y)=0.$$

The random field D^-u has been introduced in Nualart (2006) for the Wiener case, and in Di Nunno et al. (2005) for the pure jump case.

The next result will be a useful tool to state the Itô formula for the operator δ. Remember that we are using the notation $\Delta X_s=X_s-X_{s-}$.

Lemma 2.7. *Let $u=\{u(s,x):(s,x)\in[0,T]\times\mathbb{R}\}$ be a measurable random field and $\varepsilon_1>\varepsilon>0$ such that:*

i) *There exists a constant $c>0$ such that $|u(s,y)|<c$, for all $(s,y)\in[0,T]\times\{\varepsilon<|x|\leq\varepsilon_1\}$.*

ii) *For any sequences $\{s_n\in[0,s):n\in\mathbb{N}\}$ and $\{y_n\in\{\varepsilon<|x|\leq\varepsilon_1\}:n\in\mathbb{N}\}$ that converge to $s\in[0,T]$ and $y\in\{\varepsilon<|x|\leq\varepsilon_1\}$, respectively, we have that the limit*

$$u(s-,y)=\lim_{n,m\to\infty}u(s_n,y_m)$$

is well–defined.

iii) $u(\cdot-,\cdot)\in\mathbb{L}^{1,2,f}_-$.

Then

$$\begin{aligned}\sum_{0<s\leq t} & u(s-,\Delta X_s)\Delta X_s 1_{\{\varepsilon<|\Delta X_s|\leq\varepsilon_1\}}\\ &= \delta((u(s-,y)+yD^-u(s-,y))1_{\{\varepsilon<|y|\leq\varepsilon_1\}}1_{[0,t]}(s))\\ &\quad+\int_0^t\int_{\{\varepsilon<|y|\leq\varepsilon_1\}}u(s-,y)yd\nu(y)ds\\ &\quad+\int_0^t\int_{\{\varepsilon<|y|\leq\varepsilon_1\}}D^-u(s-,y)d\mu(s,y),\quad t\in[0,T].\end{aligned}$$

Proof. The definition of the space $\mathbb{L}^{1,2,f}$ implies that there exists a sequence $\{u^{(m)} \in \mathcal{S}_T : m \in \mathbb{N}\}$ such that

$$E\Big[(u(t-,y) - u^{(m)}(t,y))^2 + \int_t^T \int_{\mathbb{R}} (D_{s,x}(u(t-,y) - u^{(m)}(t,y)))^2 d\mu(s,x)\Big] \to 0, \tag{2.10}$$

as $m \to \infty$, for μ-a.a. $(t,y) \in [0,T] \times \mathbb{R}$. Hence we can choose a sequence $\mathcal{A}_n = \{(s_i^{(n)}, y_j^{(n)}) : i,j \in \{1,\dots,N\}\}$ such that:

- N is a positive integer that depends on n and goes to ∞ as $n \to \infty$.
- $0 \le s_1^{(n)} < \dots < s_N^{(n)} \le T$, $-\varepsilon_1 \le y_1^{(n)} < y_2^{(n)} < \dots < y_N^{(n)} \le \varepsilon_1$.
- $0 = \lim_{n\to\infty} s_1^{(n)}, T = \lim s_N^{(n)}, -\varepsilon_1 = \lim_{n\to\infty} y_1^{(n)}$ and $\varepsilon_1 = \lim_{n\to\infty} y_N^{(n)}$.
- $\max_i (s_{i+1}^{(n)} - s_i^{(n)}) \to 0$ and $\max_i (y_{i+1}^{(n)} - y_i^{(n)}) \to 0$ as $n \to \infty$.
- Property (2.10) holds when we write $(s_i^{(n)}, y_{j+1}^{(n)})$ instead of (t,y).

Thus, from the duality relation (2.8), Proposition 2.5, (2.10) and Solé et al. (2007) (Theorem 6.1), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{N-1} u(s_i^{(n)}-, y_{j+1}^{(n)}) \int_{]s_i^{(n)}, s_{i+1}^{(n)}]} \int_{y_j^{(n)}}^{y_{j+1}^{(n)}} y 1_{\{\varepsilon<|y|\le\varepsilon_1\}} 1_{[0,t]}(s) d\widetilde{J}(s,y) \\
&\qquad = \sum_{i,j=1}^{N-1} u(s_i^{(n)}-, y_{j+1}^{(n)}) \delta\left(1_{\{\varepsilon<|y|\le\varepsilon_1\}} 1_{]s_i^{(n)}, s_{i+1}^{(n)}]}(s) 1_{]y_j^{(n)}, y_{j+1}^{(n)}]}(y) 1_{[0,t]}(s)\right) \\
&\qquad = \sum_{i,j=1}^{N-1} \Big\{ \delta\Big(1_{[0,t]}(s) 1_{\{\varepsilon<|y|\le\varepsilon_1\}} 1_{]s_i^{(n)}, s_{i+1}^{(n)}]}(s) 1_{]y_j^{(n)}, y_{j+1}^{(n)}]}(y) \\
&\qquad\qquad \times (u(s_i^{(n)}-, y_{j+1}^{(n)}) + y D_{s,y} u(s_i^{(n)}-, y_{j+1}^{(n)}))\Big) \\
&\qquad\quad + \int_{s_i^{(n)}}^{s_{i+1}^{(n)}} \int_{y_j^{(n)}}^{y_{j+1}^{(n)}} 1_{[0,t]}(s) 1_{\{\varepsilon<|y|\le\varepsilon_1\}} D_{s,y} u(s_i^{(n)}-, y_{j+1}^{(n)}) d\mu(s,y) \Big\}.
\end{aligned}$$

Indeed, by Proposition 2.5 we have that the last equality holds when we change $u(s_i^{(n)}-, y_{j+1}^{(n)})$ by $u^{(m)}(s_i^{(n)}, y_{j+1}^{(n)})$. Consequently, we prove that our claim is true using (2.8) with a random variable as in the right-hand side of (2.7) and letting m go to ∞. So, we can conclude the proof because of the dominated convergence theorem, the hypotheses of this lemma and the fact that δ is a closed operator. ■

The space $\mathbb{L}_F$ is the closure of $\mathcal{S}_T$ with respect to the norm

$$||u||_F^2 = ||u||_{1,2,f}^2 + E\int_{\Delta_2^T} (D_{r,x} D_{s,y} u(t,z))^2 d\mu(r,x) d\mu(s,y) d\mu(t,z),$$

with $\Delta_2^T = \{((r,x),(s,y),(t,z)) \in ([0,T]\times\mathbb{R})^3 : r \vee s \ge t\}$.

The following result was stated on the Wiener space by Alòs and Nualart (1998).

Lemma 2.8. *Let $u \in \mathbb{L}_F$. Then $u \in \mathrm{Dom}\,\delta$ and*

$$E[\delta(u)^2] \le 2||u||_F^2. \tag{2.11}$$

Proof. We first observe that it is enough to show that (2.11) is true for $u \in \mathcal{S}_T$ because δ is a closed operator. In this case, we have by Solé et al. (2007) (Section 6) or by Alòs and Nualart (1998),

$$
\begin{aligned}
E[\delta(u)^2] &= E\Bigg[\int_0^T \int_{\mathbb{R}} u(t,x)^2 d\mu(t,x) \\
&\quad + \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} D_{s,y}u(t,x) D_{t,x}u(s,y) d\mu(t,x) d\mu(s,y)\Bigg]. \qquad (2.12)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& E\Big[\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} D_{s,y}u(t,x) D_{t,x}u(s,y) d\mu(t,x) d\mu(s,y)\Big] \\
&= 2E\Big[\int_0^T \int_{\mathbb{R}} u(s,y)\delta(1_{[0,s]} D_{s,y}u) d\mu(s,y)\Big] \\
&\le E\Big[\int_0^T \int_{\mathbb{R}} u(s,y)^2 d\mu(s,y)\Big] + E\Big[\int_0^T \int_{\mathbb{R}} [\delta(1_{[0,s]} D_{s,y}u)]^2 d\mu(s,y)\Big] \\
&\le E\Big[\int_0^T \int_{\mathbb{R}} u(s,y)^2 d\mu(s,y)\Big] \\
&\quad + E\Big[\int_0^T \int_{\mathbb{R}} \int_0^s \int_{\mathbb{R}} (D_{s,y}u(t,x))^2 d\mu(t,x) d\mu(s,y)\Big] \\
&\quad + E\Big[\int_0^T \int_{\mathbb{R}} \int_{([0,s]\times\mathbb{R})^2} D_{t,x}D_{s,y}u(r,z) d\mu(r,z) d\mu(t,x) d\mu(s,y)\Big].
\end{aligned}
$$

Thus (2.12) yields that (2.11) holds. ■

Inequality (2.11) allows us to consider Lemma 2.7 with $\varepsilon = 0$ or $\varepsilon_1 = \infty$ to obtain the relation between the pathwise integral and the operator δ.

Corollary 2.9. *Let u satisfy the hypotheses of Lemma 2.7 for each $\varepsilon, \varepsilon_1 \in (a,b)$, $0 \le a < b \le \infty$. Moreover assume that the random fields $(s,y) \mapsto u(s-,y)$, $yD^- u(s-,y)$ belong to $\mathbb{L}_F$ and $(s,y) \mapsto u(s-,y)y$ is pathwise integrable with respect to $\widetilde{J}$ on $[0,T] \times \{a < |y| < b\}$. Then*

$$
\begin{aligned}
& \int_{]0,t]} \int_{\{a<|y|<b\}} u(s-,y) y d\widetilde{J}(s,y) \\
&= \delta\left((u(s-,y) + yD^- u(s-,y)) 1_{[0,t]}(s) 1_{\{a<|y|<b\}}(y)\right) \\
&\quad + \int_0^t \int_{\{a<|y|<b\}} D^- u(s-,y) d\mu(s,y), \quad t \in [0,T].
\end{aligned}
$$

Proof. The result is an immediate consequence of Lemmas 2.7 and 2.8. ■

3. The Itô formula

Here we assume that, for $i \in \{1, \ldots, n\}$,

$$\begin{aligned} Y_t^{(i)} &= Y_0^{(i)} + \int_0^t u_i(s)dW_s + \int_0^t \sigma_s^{(i)} ds + \int_{]0,t]} \int_{\{|x|>1\}} v_{i1}(s-,x)xdJ(s,x) \\ &\quad + \int_{]0,t]} \int_{\{0<|x|\leq 1\}} v_{i2}(s-,x)xd\tilde{J}(s,x), \quad t \in [0,T]. \end{aligned}$$

The stochastic integrals with respect to W and J are in the Skorohod and pathwise sense, respectively, and

- (H1) $Y_0^{(i)} \in \mathbb{D}^{1,2}$.
- (H2) $u_i \in \mathbb{L}_F$ is such that $\{\int_0^t u_i(s)dW_s : t \in [0,T]\}$ has continuous paths and there is a constant $M > 0$ such that $\int_0^T u_i(s)^2 ds \leq M$ with probability 1.
- (H3) $\sigma^{(i)} \in \mathbb{L}^{1,2,f}$ and $\int_0^T (\sigma_s^{(i)})^2 ds \leq M$ with probability 1, for some positive constant M.
- (H4) v_{i1} satisfies the assumptions of Corollary 2.9 for $a = 1$ and $b = \infty$. Moreover assume that there is a positive constant M such that $|v_{i1}| < M$ for $(s, y) \in [0,T] \times \{1 < |x| < \infty\}$.
- (H5) The hypotheses of Corollary 2.9 hold for v_{i2} with $a = 0$ and $b = 1$, and there is a positive constant M such that $|v_{i2}(s-, y)| \leq M$, for $(s, y) \in [0,T] \times \{0 \leq |x| \leq 1\}$. Moreover assume that $D^- v_{i2} \in \mathbb{L}^{1,2,f}$.

Observe that by Lemma 2.8 and Corollary 2.9, we have that

$$\int_{]0,t]} \int_{\{0<|x|\leq 1\}} v_{i2}(s-,x)xd\widetilde{J}(s,x)$$

belongs to $L^2(\Omega)$, for all $t \in [0,T]$. Also observe that in Alòs and Nualart (1998) (Theorem 1) we can find sufficient conditions that guarantee the continuity of the stochastic integral $\{\int_0^t u_i(s)dW_s : t \in [0,T]\}$.

To show our Itô formula, we first need to assume that our Lévy process defined in (2.5) has no small side jumps. So, for $\varepsilon > 0$, we need to use the notation

$$\begin{aligned} Y_t^{(i),\varepsilon} &= Y_0^{(i)} + \int_0^t u_i(s)dW_s + \int_0^t \sigma_s^{(i)} ds + \int_{]0,t]} \int_{\{|x|>1\}} v_{i1}(s-,x)xdJ(s,x) \\ &\quad + \int_{]0,t]} \int_{\{\varepsilon<|x|\leq 1\}} v_{i2}(s-,x)xd\widetilde{J}(s,x), \quad t \in [0,T]. \end{aligned} \tag{3.1}$$

The i–th jump time of the compound Poisson process $\{\int_{]0,t]} \int_{\{\varepsilon<|x|\}} xdJ(s,x) : t \in [0,T]\}$ is denoted by T_i^ε. We also use the notation $T_0^\varepsilon = 0$.

Theorem 3.1. *Assume that (H1)–(H5) hold, for $i \in \{1, \ldots, n\}$, and that $F \in C_b^2(\mathbb{R}^n)$. Then, the processes*

$$\begin{aligned} &\big(\partial_i F(Y_{s-})(u_i(s)1_{\{y=0\}} + v_{i2}(s-,y)1_{\{0<|y|\leq 1\}}) \\ &\qquad + y1_{\{0<|y|<1\}} D^-(v_{i2}\partial_i F(Y_{\cdot-}))(s,y)\big)\, 1_{[0,t]}(s) \end{aligned}$$

belong to Dom δ *and*

$$
\begin{aligned}
&F(Y_t) - F(Y_0) \\
&= \delta\left(\left[\partial_i F(Y_{s-})(u_i(s)1_{\{y=0\}} + v_{i2}(s-,y)1_{\{0<|y|\leq 1\}})\right.\right. \\
&\quad \left.\left. + y1_{\{0<|y|\leq 1\}} D^-(v_{i2}\partial_i F(Y_{\cdot -}))(s,y)\right] 1_{[0,t]}(s)\right) \\
&\quad + \frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s)u_i(s)u_j(s)ds + \int_0^t \partial_i F(Y_s)\sigma_s^{(i)} ds \\
&\quad + \int_0^t \partial_i\partial_j F(Y_s)(D^- Y^{(j)})(s,0)u_i(s)ds \\
&\quad + \int_0^t \int_{\{0<|y|\leq 1\}} D^-(\partial_i F(Y_{\cdot -})v_{i2})(s,y)d\mu(s,y) \\
&\quad + \sum_{0\leq s\leq t} \{F(Y_{s-} + \Delta Y_s) - F(Y_{s-}) - \partial_i F(Y_{s-})v_{i2}(s-,\Delta X_s)\Delta X_s\}1_{\{0<|\Delta X_s|\leq 1\}} \\
&\quad + \sum_{0\leq s\leq t} (F(Y_{s-} + \Delta Y_s) - F(Y_{s-}))1_{\{1<|\Delta X_s|\}}, \quad t\in[0,T].
\end{aligned}
$$

Here we use the convention of summation over repeated indexes.

Remark By (2.1), we have $\Delta Y_s 1_{\{0<|\Delta X_s|\leq 1\}} = v_{i2}(s-,\Delta X_s)\Delta X_s 1_{\{0<|\Delta X_s|\leq 1\}}$.

Proof. We first observe that the process $Y^{(i),\varepsilon}$ given by (3.1) evolves as

$$
Y_t^{(i),\varepsilon} = Y_{T_j^\varepsilon}^{(i),\varepsilon} + \int_{T_j^\varepsilon}^t u_i(s)dW_s + \int_{T_j^\varepsilon}^t \sigma_s^{(i)} ds - \int_{]T_j^\varepsilon,t]}\int_{\{\varepsilon<|x|\leq 1\}} v_{i2}(s-,x)x\nu(dx)ds
$$

on the stochastic interval $]T_j^\varepsilon, T_{j+1}^\varepsilon[$. Consequently, proceeding as in Alòs and Nualart (1998) and using that W and J are independent, and Corollary 2.9, we have that $1_{[0,t]}\partial_i F(Y)u_i$ belongs to Dom δ^W, for $i\in\{1,\ldots,n\}$ and

$$
\begin{aligned}
F(Y_t^\varepsilon) - F(Y_0) &= \sum_{i=1}^\infty \left(F(Y^\varepsilon_{t\wedge T_i^\varepsilon -}) - F(Y^\varepsilon_{t\wedge T_{i-1}^\varepsilon})\right) \\
&\quad + \sum_{i=1}^\infty \left(F(Y^\varepsilon_{t\wedge T_i^\varepsilon}) - F(Y^\varepsilon_{t\wedge T_i^\varepsilon -})\right) \\
&= \int_0^t \partial_i F(Y_s^\varepsilon)u_i(s)dW_s + \int_0^t \partial_i F(Y_s^\varepsilon)\sigma_s^{(i)} ds \\
&\quad - \int_0^t \partial_i F(Y_s^\varepsilon)\int_{\{\varepsilon<|x|\leq 1\}} v_{i2}(s-,x)xd\nu(x)ds \\
&\quad + \frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s^\varepsilon)u_i(s)u_j(s)ds \\
&\quad + \int_0^t \partial_i\partial_j F(Y_s^\varepsilon)(D^- Y^{(j),\varepsilon})(s,0)u_i(s)ds \\
&\quad + \sum_{0\leq s\leq t} (F(Y_{s-}^\varepsilon + \Delta Y_s^\varepsilon) - F(Y_{s-}^\varepsilon)), \quad t\in[0,T], \quad (3.2)
\end{aligned}
$$

with

$$\begin{aligned} D^{-}Y^{(j),\varepsilon}(s,0) &= D_{s,0}Y_0^{(j)} + \int_0^s D_{s,0}u_j(r)dW_r + \int_0^s D_{s,0}\sigma_r^{(j)}dr \\ &\quad + \delta(D_{s,0}(v_{j2}(r-,y) + yD^{-}v_{j2}(r-,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,s]}(r)) \\ &\quad + \delta(D_{s,0}(v_{j1}(r-,y) + yD^{-}v_{j1}(r-,y))1_{\{1<|y|\}}1_{[0,s]}(r)) \\ &\quad + \int_0^s \int_{\{\varepsilon<|y|\le 1\}} D_{s,0}(D^{-}v_{j2}(r-,y))d\mu(r,y) \\ &\quad + \int_0^s \int_{\{1<|y|\}} D_{s,0}(D^{-}v_{j1}(r-,y))d\mu(r,y) \\ &\quad + \int_0^s \int_{\{1<|y|\}} yD_{s,0}v_{j1}(r-,y)d\nu(y)dr. \qquad (3.3) \end{aligned}$$

Now we divide the proof in several steps.

Step 1. Here we see that $Y_t^{(i),\varepsilon} \to Y_t^{(i)}$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$, for every $t \in [0,T]$.
It follows, from (3.1) and Lemma 2.7,

$$\begin{aligned} &Y_t^{(i),\varepsilon} \\ &= Y_0^{(i)} + \int_0^t u_i(s)dW_s + \int_0^t \sigma_s^{(i)}ds \\ &\quad + \delta\Big((v_{i1}(s-,y) + yD^{-}v_{i1}(s-,y))1_{\{1<|y|\}}1_{[0,t]}(s)\Big) \\ &\quad + \delta\Big((v_{i2}(s-,y) + yD^{-}v_{i2}(s-,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,t]}(s)\Big) \\ &\quad + \int_0^t \int_{\{\varepsilon<|y|\le 1\}} D^{-}v_{i2}(s-,y)d\mu(s,y) \\ &\quad + \int_0^t \int_{\{1<|y|\}} v_{i1}(s-,y)yd\nu(y)ds \\ &\quad + \int_0^t \int_{\{1<|y|\}} D^{-}v_{i1}(s-,y)d\mu(s,y). \qquad (3.4) \end{aligned}$$

Thus our claim follows by Corollary 2.9. Indeed, by Lemma 2.8, we have that

$$\delta\Big((v_{i2}(s-,y) + yD^{-}v_{i2}(s-,y))1_{\{\varepsilon<|y|\le 1\}}1_{[0,t]}(s)\Big) + \int_0^t \int_{\{\varepsilon<|y|\le 1\}} D^{-}v_{i2}(s-,y)d\mu(s,y)$$

converges in $L^2(\Omega)$ to the pathwise integral

$$\int_0^t \int_{\{0<|y|\le 1\}} v_{i2}(s-,y)d\tilde{J}(s,y).$$

Step 2. Now we show that $\partial_i F(Y_{\cdot-}^{\varepsilon})v_{i2}(\cdot-,\cdot)$ is in $\mathbb{L}_{-}^{1,2,f}$.
We first observe that (2.11), (3.4) and Solé et al. (2007) (Section 6) yield $Y^{(i),\varepsilon} \in \mathbb{L}_{-}^{1,2,f}$, $i \in \{1,\dots,n\}$. Hence, $Y_{\cdot-}^{(i),\varepsilon} \in \mathbb{L}_{-}^{1,2,f}$ due to $E[|Y_t^{(i),\varepsilon} - Y_{t-}^{(i),\varepsilon}|] = 0$, for

$t \in [0,T]$, which follows from (3.1). Thus, $D^- Y^{(i),\varepsilon} = D^- Y^{(i),\varepsilon}_{\cdot-}$. Therefore, it is clear the fact that $F(Y^\varepsilon)$ and v_{i2} are bounded implies that

$$\begin{aligned} D^-(\partial_i F(Y^\varepsilon_{\cdot-}) v_{i2}(\cdot-,\cdot))(s,0) &= \partial_i\partial_j F(Y^\varepsilon_{s-}) v_{i2}(s-,0) D^- Y^{(j),\varepsilon}(s,0) \\ &\quad + \partial_i F(Y^\varepsilon_{s-}) D^- v_{i2}(s-,0). \end{aligned} \tag{3.5}$$

On the other hand, the definition of the operator Ψ leads to write, for $r > t$,

$$\begin{aligned} &\Psi_{r,x}(\partial_i F(Y^\varepsilon_{t-}) v_{i2}(t-,y)) \\ &\quad = (\Psi_{r,x}\partial_i F(Y^\varepsilon_{t-})) v_{i2}(t-,y) + \partial_i F(Y^\varepsilon_{t-}) \Psi_{r,x} v_{i2}(t-,y) \\ &\qquad + x(\Psi_{r,x} v_{i,2}(t-,y)) \Psi_{r,x}\partial_i F(Y^\varepsilon_{t-}) \\ &\quad = v_{i2}(t-,y) \frac{\partial_i F(Y^\varepsilon_{t-} + x D_{r,x} Y^\varepsilon_t) - \partial_i F(Y^\varepsilon_{t-})}{x} + \partial_i F(Y^\varepsilon_{t-}) D_{r,x} v_{i2}(t-,y) \\ &\qquad + (\partial_i F(Y^\varepsilon_{t-} + x D_{r,x} Y^\varepsilon_t) - \partial_i F(Y^\varepsilon_{t-})) D_{r,x} v_{i2}(t-,y), \end{aligned}$$

which, together with (3.5) and Corollary 2.3, gives that $\partial_i F(Y^\varepsilon) v_{i2} \in \mathbb{L}^{1,2,f}_-$, with

$$\begin{aligned} &D^-(\partial_i F(Y^\varepsilon_{\cdot-}) v_{i2}(\cdot-,\cdot))(s,y) \\ &\quad = \Big(\partial_i\partial_j F(Y^\varepsilon_{s-}) v_{i2}(s-,0) D^- Y^{(j),\varepsilon}(s,0) + \partial_i F(Y^\varepsilon_{s-}) D^- v_{i2}(s-,0)\Big) 1_{\{y=0\}} \\ &\qquad + \Bigg(v_{i2}(s-,y) \frac{\partial_i F(Y^\varepsilon_{s-} + y D^- Y^\varepsilon(s,y)) - \partial_i F(Y^\varepsilon_{s-})}{y} + \partial_i F(Y^\varepsilon_{s-}) D^- v_{i2}(s,y) \\ &\qquad + (\partial_i F(Y^\varepsilon_{s-} + y D^- Y^\varepsilon(s,y)) - \partial_i F(Y^\varepsilon_{s-})) D^- v_{i2}(s,y) \Bigg) 1_{\mathbb{R}_0}(y). \end{aligned}$$

Step 3. From Step 2, Lemma 2.7 and (3.2), we get

$$\begin{aligned} F(Y^\varepsilon_t) =& F(Y_0) + \int_0^t \partial_i F(Y^\varepsilon_s) u_i(s) dW_s + \int_0^t \partial_i F(Y^\varepsilon_s) \sigma^{(i)}_s ds \\ &+ \delta\left((\partial_i F(Y^\varepsilon_{s-}) v_{i2}(s-,y) + y(D^-\partial_i F(Y^\varepsilon_{\cdot-}) v_{i2})(s,y)) 1_{\{\varepsilon<|y|\le 1\}} 1_{[0,t]}(s) \right) \\ &+ \int_0^t \int_{\{\varepsilon<|y|\le 1\}} D^-(\partial_i F(Y^\varepsilon_{\cdot-}) v_{i2})(s,y) d\mu(s,y) \\ &+ \frac{1}{2} \int_0^t \partial_i\partial_j F(Y^\varepsilon_s) u_i(s) u_j(s) ds \\ &+ \int_0^t \partial_i\partial_j F(Y^\varepsilon_s)(D^- Y^{(j),\varepsilon})(s,0) u_i(s) ds \\ &+ \sum_{0\le s\le t} \Big(F(Y^\varepsilon_{s-} + \Delta Y^\varepsilon_s) - F(Y^\varepsilon_{s-}) - \partial_i F(Y^\varepsilon_{s-}) v_{i2}(s-,\Delta X_s) \Delta X_s \Big) \\ &\qquad\qquad \times 1_{\{\varepsilon<|\Delta X_s|\le 1\}} \\ &+ \sum_{0\le s\le t} (F(Y^\varepsilon_{s-} + \Delta Y^\varepsilon_s) - F(Y^\varepsilon_{s-})) 1_{\{1<|\Delta X_s|\}}. \end{aligned} \tag{3.6}$$

Step 4. Now we analyze the convergence in $L^2(\Omega)$ of the terms in (3.6).

$$\begin{aligned}
&E\Big[|\sum_{0\le s\le t}(F(Y^\varepsilon_{s-}+\Delta Y^\varepsilon_s)-F(Y^\varepsilon_{s-}))1_{\{1<|\Delta X_s|\}}|^2\Big]\\
&\quad= E\left[|\sum_{0\le s\le t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-}))1_{\{1<|\Delta X_s|\}}|^2\right]\\
&\quad\le CE\left[\left(\sum_{i=1}^n\sum_{0\le s\le t}|v_{i1}(s-,\Delta X_s)\Delta X_s|1_{\{1<|\Delta X_s|\}}\right)^2\right]\\
&\quad\le n^2CE\left[\left(\sum_{0\le s\le t}|\Delta X_s|1_{\{1<|\Delta X_s|\}}\right)^2\right]\\
&\quad\le CE\left[\left(\int_{]0,t]}\int_{\{|x|>1\}}|x|d\widetilde{J}(s,x)+\int_{]0,t]}\int_{\{|x|>1\}}|x|d\nu(x)ds\right)^2\right]\\
&\quad\le C\int_{]0,t]}\int_{\{|x|>1\}}x^2d\nu(x)ds+\left(\int_{]0,t]}\int_{\{|x|>1\}}xd\nu(x)ds\right)^2\\
&\quad\le C\int_{]0,t]}\int_{\mathbb{R}_0}x^2d\nu(x)ds<\infty.
\end{aligned}$$

Also

$$\begin{aligned}
&E\Big[\Big(\sum_{0\le s\le t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-})-\partial_iF(Y_{s-})v_{i2}(s,\Delta X_s)\Delta X_s)1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big]\\
&\le E\Big[\Big(\sum_{i=1}^n\sum_{0\le s\le t}|\Delta Y^{(i)}_s|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big]\\
&= E\Big[\Big(\sum_{i=1}^n\sum_{0\le s\le t}|v_{i2}(s-,\Delta X_s)\Delta X_s|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big]\\
&\le CE\Big[\Big(\sum_{0\le s\le t}|\Delta X_s|^2 1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big]\\
&\le CE\Big[\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}x^2d\widetilde{J}(s,x)\Big)^2\Big]+C\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}x^2d\nu(x)ds\Big)^2\\
&\le C\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}x^2d\nu(x)ds\to 0\quad\text{as}\quad\varepsilon\to 0.
\end{aligned}$$

It is not difficult to deduce, from Step 1,

$$E\left[\int_0^t|\partial_iF(Y_s)u_i(s)-\partial_iF(Y^\varepsilon_s)u_i(s)|^2ds\right]\to 0$$

and, from Step 2, (2.11) and the dominated convergence theorem,

$$E\Big[\int_{]0,t]}\int_{\{0<|y|\leq 1\}}\Big|\Big(\partial_i F(Y_s^\varepsilon)v_{i2}(s-,y)+yD^-(\partial_i F(Y^\varepsilon)v_{i2})(s,y)\Big)$$

$$\times 1_{\{\varepsilon<|y|\leq 1\}}-\partial_i F(Y_s)v_{i2}(s-,y)+yD^-(\partial_i F(Y)v_{i2})(s,y)\Big|^2 d\mu(s,y)\Big]$$

$$\to 0 \quad \text{as} \quad \varepsilon\downarrow 0.$$

The missing terms can be analyzed similarly.

Step 5. Finally the result follows from the fact that δ is a closed operator and from Steps 1-4. ■

Theorem 3.2. *Assume that*

$$\int_{\mathbb{R}_0}|x|d\nu(x)<\infty.$$

Then the hypotheses of Theorem 3.1 imply that

$$\begin{aligned} F(Y_t) \quad = \quad & F(Y_0)+\int_0^t \partial_i F(Y_s)u_i(s)dW_s+\int_0^t \partial_i F(Y_s)\sigma_s^{(i)}ds \\ & -\int_0^t \partial_i F(Y_s)\int_{\{0<|x|\leq 1\}} v_{i2}(s-,x)xd\nu(x)ds \\ & +\frac{1}{2}\int_0^t \partial_i\partial_j F(Y_s)u_i(s)u_j(s)ds \\ & +\int_0^t \partial_i\partial_j F(Y_s)(D^-Y^{(j)})(s,0)u_i(s)ds \\ & +\sum_{0\leq s\leq t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-})), \quad t\in[0,T]. \end{aligned}$$

Proof. The fact that $\int_{\mathbb{R}_0}|x|d\nu(x)<\infty$ yields

$$E\Big[\Big(\int_0^t\int_{\{0<|x|\leq 1\}}|v_{i2}(s-,x)x|d\nu(x)ds\Big)^2\Big]\leq C\Big(\int_0^t\int_{\{0<|x|\leq 1\}}|x|d\nu(x)ds\Big),$$

which implies

$$E\Big[\Big(\int_0^t \partial_i F(Y_s^\varepsilon)\int_{\{\varepsilon<|x|\leq 1\}} v_{i2}(s-,x)xd\nu(x)ds$$

$$-\int_0^t \partial_i F(Y_s)\int_{\{0<|x|\leq 1\}} v_{i2}(s-,x)xd\nu(x)ds\Big)^2\Big]\to 0.$$

Also we have

$$
\begin{aligned}
& E\Big[\Big(\sum_{0\le s\le t}(F(Y_{s-}+\Delta Y_s)-F(Y_{s-}))1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big] \\
&\le\ CE\Big[\Big(\sum_{i=1}^{n}\sum_{0\le s\le t}|v_{i2}(s-,\Delta X_s)\Delta X_s|1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big] \\
&\le\ CE\Big[\Big(\sum_{0\le s\le t}|\Delta X_s|1_{\{0<|\Delta X_s|\le\varepsilon\}}\Big)^2\Big] \\
&\le\ CE\Big[\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}|x|d\widetilde{J}(s,x)+\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}|x|d\nu(x)ds\Big)^2\Big] \\
&\le\ C\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}x^2d\nu(x)ds+C\Big(\int_{]0,t]}\int_{\{0<|x|\le\varepsilon\}}|x|d\nu(x)ds\Big)^2.
\end{aligned}
$$

Thus the result is a consequence of the proof of Theorem 3.1. ∎

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