United States
Department of
Agriculture

Forest Service

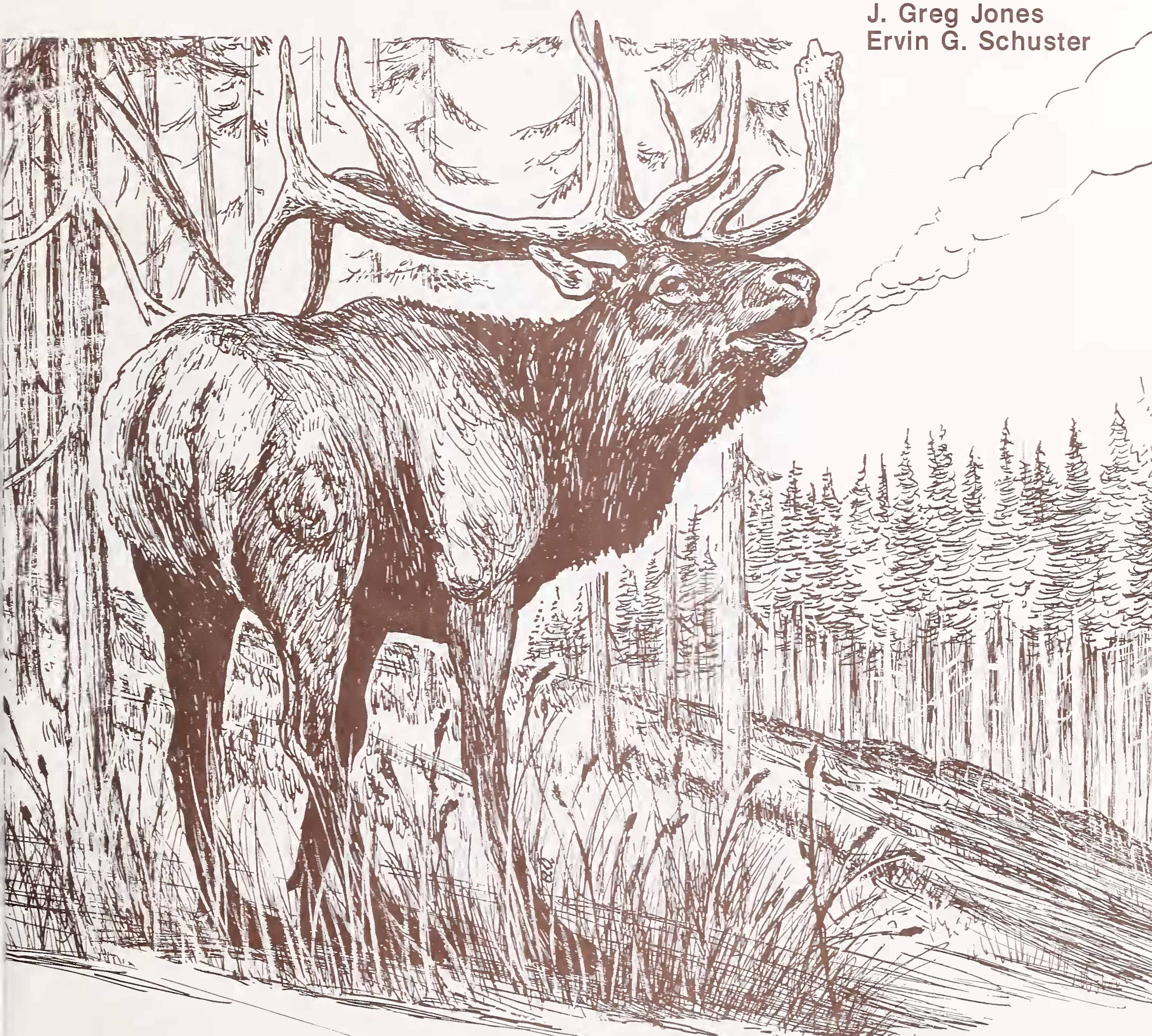
Intermountain
Forest and Range
Experiment Station
Ogden, UT 84401

General Technical
Report INT-178

March 1985

An Application of Discrete Optimization for Developing Economically Efficient Multiple-Use Projects

J. Greg Jones
Ervin G. Schuster



THE AUTHORS

J. GREG JONES is a research forester with the Economics Research Work Unit, Forestry Sciences Laboratory, Missoula, MT. His research includes economic efficiency modeling for use in multiple-use management, measurement of multiple-use costs, and analysis of economic impact.

ERVIN G. SCHUSTER is a research forester and project leader of the Economics Research Work Unit, Forestry Sciences Laboratory, Missoula, MT. His research includes modeling of timber harvesting, measurement of nontimber outputs, and analysis of economic impact.

RESEARCH SUMMARY

This paper presents a model formulation useful (1) for planning multiple-use projects and (2) for identifying efficient management prescriptions and/or aggregate emphasis projects to build into future forest planning models. The formulation is a discrete version of the continuous joint production model in economic theory. Economic efficiency can be analyzed both in terms of type of project and scale of project.

The model can be formulated and solved graphically or as a mixed-integer programming (MIP) problem. The graphic approach rather clearly depicts the nature of economic efficiency in multiple-use production and requires little in the way of equipment. It is, however, limited to problems that can be depicted in two-dimensional space. The MIP approach has the following advantages over the graphic approach: (1) it can accommodate more than two outputs, (2) intertemporal analysis is easier to conduct, (3) capability to conduct sensitivity analysis is enhanced, and (4) it lends itself well to automation.

The MIP formulation contains decision variables that are formulated as whole decision alternatives, which assume values of 0 (do not do project) or 1 (do project). This differs from mathematical programming formulations common in forestry (for example, FORPLAN, MUSYC, and Timber RAM) in which decision variables are formulated on a per-acre basis. The advantages of the MIP formulation are that diminishing marginal productivity can be modeled and the level of site specificity is enhanced. The main disadvantage of this MIP approach is that only a limited number of management alternatives can be handled effectively, making it best suited to problems of a relatively small geographic scope, for example, a project planning area.

The MIP formulation is easy to solve and sufficiently small to be processed on a small computer. Combined with front-end data processing software, it could be useful for conducting multiple-use efficiency analysis. The potential lies not as a substitute for current forest planning methods, but rather as a tool to aid in identifying efficient management prescriptions to place in forest planning models and as a means of analyzing projects for implementation.

An Application of Discrete Optimization for Developing Economically Efficient Multiple-Use Projects

J. Greg Jones
Ervin G. Schuster

INTRODUCTION

In recent history, the focus of land management economic analysis on National Forests has been in forest planning. Large-scale planning models, such as FORPLAN (Gilbert and others 1982), are being used to conduct economic analysis of multiple-use management in this planning process. For a variety of reasons, however, forest planning analysis has to be conducted at a relatively low level of resolution. As a result, there may be many spatial configurations and timing sequences for implementing the general management direction identified in forest planning.

There remains a need for economic analysis in project design to aid in identifying projects that efficiently implement forest plans. Clearly, if projects are not efficient, overall management will not be efficient, because projects are the means by which management is implemented on the ground. Unfortunately, economics of project planning has largely been ignored by economists and analysts. As a result analytical techniques or models for this purpose are lacking. This may be particularly critical for projects with considerable multiple-use components, where efficient designs are particularly difficult to identify.

In this paper we present a model formulation we believe may be useful in planning multiple-use projects. In addition, it could have application in identifying efficient management prescriptions and/or aggregate emphasis projects to build into FORPLAN models in future forest planning efforts.

First, the model is presented in graphical terms for a hypothetical but realistic project planning situation. Next, a mixed-integer mathematical programming formulation of the model is presented and solved. Then, sensitivity analysis techniques applicable to the mixed-integer programming formulation are discussed. Finally, several topics are discussed regarding the operational feasibility of this formulation.

THE CONCEPT

Gregory (1955) presented the case that an appropriate economic formulation for multiple use is the joint production model in microeconomic theory. Joint production occurs when two or more outputs are produced simultaneously (jointly) by a single production process, meat and hides, for example. The joint production model is comprised of a "production surface," which identifies the combinations of outputs that can be produced on a tract of land (or by some fixed production plant), given efficient use of variable inputs. For the two-output case, this production surface is often depicted by a series of "iso-cost" (or constant cost) lines. Each corresponds to a unique level for variable cost, and identifies the combinations of outputs that can be produced with that cost. Unit values for outputs are then introduced to find: (a) the combination of outputs on each iso-cost curve that provides the greatest total value and (b) which of these best points (the expansion path) maximizes net benefit.

The joint production model appears to fit multiple-use management, where the intention is to produce multiple outputs from a tract of land. The problem with applying this theoretical model is that it is not yet operationally feasible in a real-world planning situation. A major impediment is the lack of adequate continuous mathematical functions relating variable cost to the quantities of outputs that can be jointly produced (the production surface).

The formulation we present is a discrete version of Gregory's joint production model that builds on an approach suggested by Muhlenberg (1964). It is comprised of a finite number of points that approximate the continuous production surface of the theoretical model. These points are believed to be more operationally feasible to estimate than continuous mathematical production relationships. Yet, this discrete formulation provides the same type of analysis as the continuous model.

MODEL FORMULATION

We shall illustrate this discrete formulation of the theoretical joint production model by employing a simple but realistic example. The example pertains to a hypothetical 4,000-acre (1 619-ha) tract of forest land. This area is part of an important elk summer range and is currently overstocked with a homogeneous stand of low-quality but merchantable timber. The tree canopy is so dense that forage production is severely restricted and there is an excess of cover. The forest planning process has identified this area for a potential timber sale, the purpose of which is twofold: (1) to open up parts of the area to promote a better balance between cover and forage production and (2) to harvest timber to help meet the established cut goals for the forest.

The purpose of the model we present is to aid in identifying the type and scale of the timber sale project that most efficiently meets the two stated objectives. The scope of the problem is limited to project design. The planning horizon is 30 years—the length of time the cover/forage combination resulting from this management activity would be sustained. No additional harvests are scheduled for this area over the next 30 years. Finally, it is assumed that no other outputs from this area would be sufficiently affected as to warrant their inclusion in the model.

Before proceeding, we should make clear that the example we develop on the following pages is purely for illustrating the analytical approach. It would be inappropriate to generalize the management responses or subsequent results to other areas for several reasons. First, the results would be expected to be sensitive to existing conditions of an area, which could vary greatly. Second, appropriate output responses, costs, and unit values likely vary greatly as well.

The Alternatives

The five series of timber sale alternatives (A to E) presented in table 1 approximate the production surface for this problem. Each series reflects a specific theme, differing in the amount of emphasis given to promoting effective wildlife habitat on each acre harvested. Within a series, the alternatives employ common management practices and cutting unit design. Alternatives within a series differ only by the amount of harvesting that would be conducted, which is directly related to costs. Note that the first alternative in each series has a budget of $200,000, the second a budget of $400,000, and so on. A "no action" alternative (0) is also considered. It is used as a reference point against which output quantities and costs for the other alternatives are measured.

Series A.—These alternatives are designed to harvest timber at the lowest possible cost, thereby yielding the greatest net dollar return to the Federal treasury. These alternatives have relatively large cutting units (35 to 40 acres [12 to 16 ha]) located primarily on the basis of cost efficiency in logging and road building. All basic environmental constraints are satisfied, but no additional activities are undertaken for habitat improvement. Roads are left open and public use of the area is not restricted.

Series B.—These alternatives are the same as series A, except that the roads will be closed to motorized use by the public following harvest.

Series C.—The cutting units in these alternatives are distributed essentially the same as in the previously described alternatives. As in series B, the roads will be closed to public traffic. These alternatives differ mainly in that the logging slash will be broadcast burned to promote forage and browse production.

Series D.—These alternatives are characterized by smaller cutting units (average about 20 acres [8 ha]) with wildlife considerations being the primary basis for location. Roads will be closed to public access, and road slash will be cleaned up to eliminate its effect as a barrier to wildlife movement. Logging slash will be broadcast burned.

Series E.—These alternatives are designed to maximize wildlife benefits while still harvesting timber. The cutting units are either small or shaped to provide a good "edge effect." As in series D, roads will be closed, road slash will be cleaned up, and logging slash will be broadcast burned.

Outputs

Two outputs are included in the model: timber and summer range effectiveness. Both are measured in terms of marginal change from the "no action" alternative.

The quantity of timber is simply the volume that would be harvested under the alternatives (sixth column in table 1). Volume was assumed to be 8.5 M bd ft per acre across the 4,000-acre (1 619-ha) area. Although a constant volume per acre is not a requirement for this model, it is convenient for this example.

Summer range habitat effectiveness is measured in terms of change in the number of animals the 4,000-acre (1 619-ha) area can be expected to support annually (last column in table 1). In order to maintain as much simplicity as possible, carrying capacity response is expressed as an annual average over the planning horizon. Later, we shall discuss an approach for handling changing output response over time in the graphical formulation. Changing output quantities over time does not present any particular difficulty in the mixed-integer programming approach.

Figure 1 provides a good basis for describing the process of estimating change in carrying capacity due to harvesting activities. Under the existing conditions, 20 percent of the area is assumed to be in forage production, and the remaining 80 percent is classified as cover. Current carrying capacity is estimated at 116 animals, and is projected to stay constant if no harvesting is accomplished. This corresponds to the beginning point on each response curve in figure 1. The response curves then show average annual carrying capacity as a function of acres harvested for each series of harvest alternatives. The change in average annual carrying capacity reported for the alternatives in table 1 is the difference between these responses (for the appropriate level and type of harvest) and the annual carrying capacity of 116 animals for the no-action alternative.

Table 1.—Alternatives for hypothetical timber sale

Alternatives	Discounted total cost	Discounted agency cost	Discounted purchaser cost	Size of harvest	Timber harvest	Change in elk-carrying capacity
	Thousands of dollars			*Acres*	*M bd ft*	*Number of animals*
0	0	0.0	0.0	0.0	0.0	0.0
A2	200	35.7	164.3	166.7	1,417.0	– 9.8
A4	400	71.3	328.7	333.3	2,853.1	–17.0
A6	600	107.0	443.0	500.0	4,250.0	–24.3
A8	800	142.7	657.3	666.7	5,667.0	–31.8
A10	1,000	178.3	821.7	833.3	7,083.1	–37.6
A12	1,200	214.0	986.0	1,000.0	8,500.0	–41.2
A14	1,400	249.7	1,150.3	1,166.7	9,917.0	–45.6
A16	1,600	285.3	1,314.7	1,333.3	11,333.1	–49.1
A18	1,800	321.0	1,474.0	1,500.0	12,750.0	–52.8
B2	200	36.9	163.1	165.4	1,406.0	4.7
B4	400	72.6	327.4	332.1	2,822.7	9.2
B6	600	108.2	491.8	498.8	4,239.4	13.1
B8	800	143.9	656.1	665.4	5,656.0	15.5
B10	1,000	179.6	820.4	832.1	7,072.7	17.0
B12	1,200	215.2	984.8	998.8	8,489.4	17.6
B14	1,400	250.9	1,149.1	1,165.4	9,906.0	16.9
B16	1,600	286.6	1,313.4	1,332.1	11,322.7	15.2
B18	1,800	322.2	1,477.8	1,498.8	12,739.4	13.0
C2	200	33.9	166.1	151.5	1,288.0	8.8
C4	400	66.6	333.4	304.2	2,585.7	17.0
C6	600	99.3	500.7	456.9	3,883.4	24.2
C8	800	131.9	668.1	609.5	5,181.1	29.7
C10	1,000	164.6	835.4	762.2	6,478.8	32.9
C12	1,200	197.3	1,002.7	914.9	7,776.5	34.7
C14	1,400	230.0	1,170.0	1,067.0	9,074.2	34.8
C16	1,600	262.0	1,337.4	1,220.2	10,371.9	32.9
C18	1,800	295.3	1,504.7	1,372.9	11,669.7	29.2
D2	200	31.4	168.6	139.8	1,188.2	9.4
D4	400	61.5	338.5	280.6	2,385.4	18.0
D6	600	91.7	508.3	421.5	3,582.6	25.6
D8	800	121.8	678.2	562.3	4,779.8	32.0
D10	1,000	152.0	848.0	703.2	5,976.9	35.9
D12	1,200	182.1	1,017.4	844.0	7,174.1	38.3
D14	1,400	212.3	1,157.7	984.9	8,371.3	39.5
D16	1,600	242.4	1,357.6	1,125.7	9,568.5	38.6
D18	1,800	272.5	1,527.5	1,266.5	10,765.7	35.9
E2	200	28.1	171.9	124.1	1,054.5	9.3
E4	400	54.8	345.2	249.1	2,117.0	17.9
E6	600	81.6	518.4	374.1	3,179.5	25.6
E8	800	108.3	691.7	499.1	4,242.0	32.5
E10	1,000	135.1	864.9	624.1	5,304.5	37.6
E12	1,200	161.8	1,038.2	749.1	6,367.0	41.0
E14	1,400	188.6	1,211.4	874.1	7,429.5	42.9
E16	1,600	215.3	1,384.7	999.1	8,492.0	44.0
E18	1,800	242.1	1,557.9	1,124.1	9,554.5	42.9

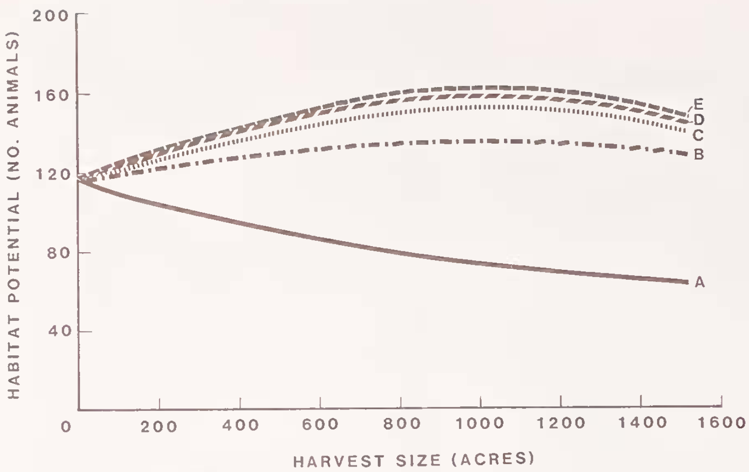


Figure 1.—Average annual elk habitat potential for the example area as a function of size of harvest, by series of alternatives A-E.

The responses in carrying capacity presented in figure 1 were based on the relationships presented in figure 2 (habitat effectiveness as a function of the percent of land in forage production), figure 3 (habitat effectiveness as a function of miles of road per section), and other information presented in a recent annual report on the Montana Cooperative Elk-Logging Study (Lyon and others 1982). These relationships were selected from many alternatives being evaluated in the study mentioned. A different selection of curves would produce somewhat different results.

In applying these relationships, the potential carrying capacity under ideal conditions (40 percent of area in forage production, 60 percent in cover, and no road effects) is estimated at 160 animals per year, which is fairly high but not unrealistic. The road effects shown in figure 3 were assumed to hold only when roads are left open to motorized use by the public. Roads closed to public vehicular traffic are thought to have no effect on habitat quality once harvesting activities are completed.

One final point should be made regarding the predicted output responses. The responses in carrying capacity illustrated in figure 1 exhibit decreasing marginal physical product. Along any given series of alternatives (with the exception of series A), as the size of harvest increases, carrying capacity increases but at a decreasing rate (that is, the slope is decreasing as scale of harvest gets larger). Slope stays positive out to a point (the maximum carrying capacity possible within each series), after which the carrying capacity decreases as size of harvest is further increased. The presence of decreasing marginal physical product is critical, for without it an optimal size of cut would not exist—more would always appear better.

Values

Timber is valued as mill-delivered logs at $140 per M bd ft. An explanation of the rationale for this basis (as opposed to valuing timber as standing trees) may be useful. Land managers can (and do) accomplish management objectives by the way roads and timber sales are

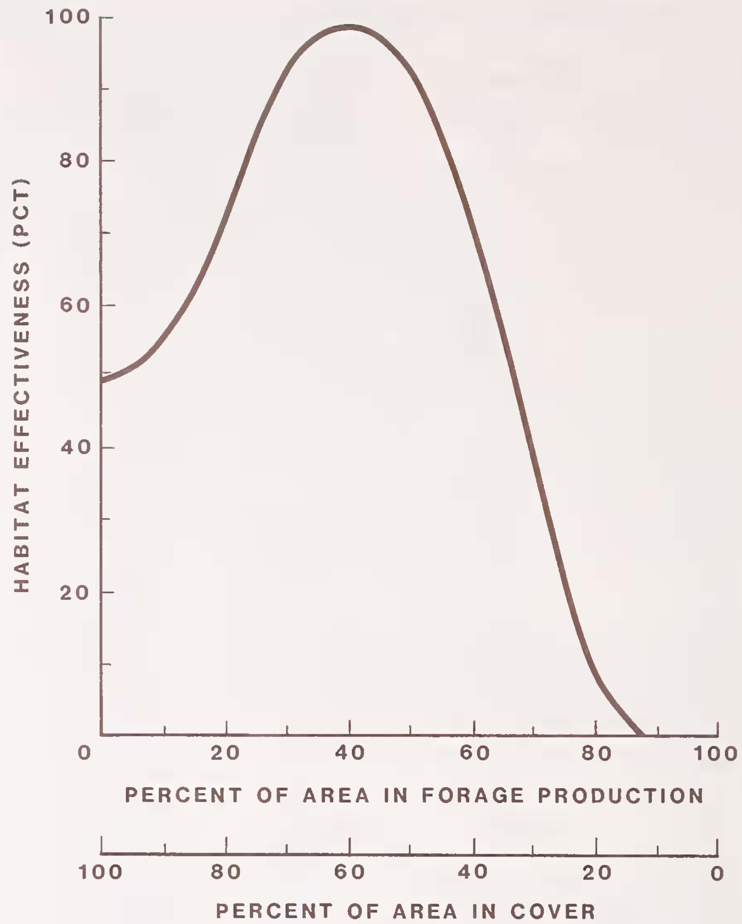


Figure 2.—Effectiveness of elk habitat as a function of percentage of area in cover and forage production (source: Lyon and others 1982).

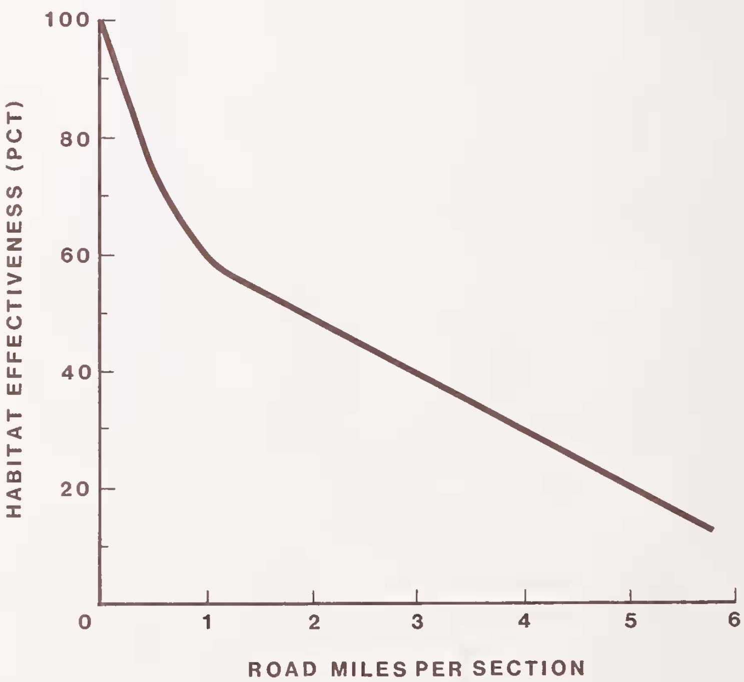


Figure 3.—The effect of road density on elk habitat (source: Lyon and others 1982).

designed and by specifications included in timber sale contracts. These things can affect stump-to-truck costs, haul costs, purchaser slash disposal costs, and other costs that must be paid by the purchaser of the timber, or a purchaser's subcontractor. Assuming competitive markets, any $1 cost imposed on a purchaser (or a purchaser's subcontractor) can on the average be expected to result in $1 less the land manager receives for the timber sold. Thus, purchaser costs can be expected to have the same effect on the seller of timber as a cost incurred directly by the seller. Valuing timber as delivered logs allows purchaser costs to be identified explicitly as part of the "budget" available to the timber seller for conducting land management activities.

The value of the change in elk-carrying capacity was based on the value of the recreational experience of elk hunting. This implicitly assumes that the change in carrying capacity presented in table 1 (last column) correctly measures the change in the number of animals that would be carried by the area. First, the value of an elk living 1 year, V, was estimated as follows:

$$V = [\$/RVD] \times [RVD/Elk] \quad (1)$$

where:

$/RVD = $31.78, the RPA willingness to pay for a recreation-visitor day (RVD) of elk hunting expressed in 1982 dollars

RVD/Elk =the average number of elk hunting RVD's supported by an elk each year, estimated to be seven. Given these numbers, V rounded to the nearest $10 equals $220.

The present value of the change in elk-carrying capacity over the next 30 years for the j[th] alternative, V_{oj}^{ELK}, can be expressed in general terms as:

$$V_{oj}^{ELK} = \sum_{t=1}^{30} V_t\, Q_{jt} \left[\frac{1}{(1+i)^t}\right] \quad (2)$$

where:

V_t = the value of an elk in year t, expressed in constant dollars

Q_{jt} = the change in carrying capacity in year t for the j[th] alternative (last column in table 1)

i = the discount rate in real dollars.

This generalized form can be handled in the mathematical programming formulation, but must be simplified for the more restrictive graphic formulation. Let us assume no real price increase for V. Since Q_{jt} is constant over time in table 1 (change in carrying capacity is constant over 30 years within each alternative), V_{oj}^{ELK} can be written as:

$$V_{oj}^{ELK} = Q_j\, V_1 \sum_{t=1}^{N} \left[\frac{1}{(1+i)^t}\right]$$

or

$$V_{oj}^{ELK} = Q_j\, V_1 \left[\frac{(1+i)^t - 1}{i\,(1+i)^t}\right]$$

Because V is constant across the j alternatives, it is convenient for the graphic formulation to set:

$$P = V \left[\frac{(1+i)^t - 1}{i\,(1+i)^t}\right]$$

Using a discount rate of 4 percent (in real dollar terms) and the previously calculated value of $220 for V, P equals $3,800 when rounded to the nearest hundred dollars. The present value of the change in carrying capacity, V_{oj}^{ELK}, can then be expressed in the familiar terms of price times quantity:

$$V_{oj}^{ELK} = 3{,}800 \bullet Q_j$$

Costs

Total cost for the alternatives in the second column of table 1 is in terms of change relative to no action. It has two major components. The first, Forest Service cost (third column), includes the sale-related costs that are paid with appropriated funds: sale preparation, sale administration, agency overhead, and road closure costs. The second cost component, purchaser-related costs (fourth column), include stump-to-truck, hauling, broadcast burning, and road construction and reconstruction. They represent the costs that must be covered by the value of the timber (when valued as delivered logs) for the sale to be financially viable. Given the objective of increased forage production for improved elk habitat, activities for regenerating the timber will not be undertaken. Thus regeneration costs were not included.

GRAPHIC APPROACH

The graphic formulation presented in figure 4 follows the logic of the continuous theoretical model. The first step in developing this formulation is to construct the iso-cost curves, which identify combinations of outputs that can be produced for given levels of cost. This is simply a matter of plotting the combinations of outputs predicted for each alternative presented in table 1. The iso-cost curve labeled 200 includes the alternatives with a total cost of $200,000, the curve labeled 400, the $400,000 alternatives, and so on. The order of the series (A-E) is illustrated on the curve labeled 600, and is the same on each iso-cost line. Technically, each iso-cost curve consists only of the points representing the alternatives, because linear combinations of projects have no logical interpretation. The points are connected here merely for convenience in identifying alternatives with common costs.

Next, benefits are entered in the form of iso-benefit lines, which arise from the simple price times quantity relationship. An iso-benefit line identifies combinations of outputs that have common total present value of benefits. To illustrate, an increase in carrying capacity of 35 animals (point W) would have a present value benefit of $133,000 (35 times the $3,800 discounted unit price identified earlier). Given the price of $140 per M bd ft, the same amount of benefit would be created by harvesting 950 M bd ft of timber (point T). Each combination of outputs lying on the line connecting points W and T has a total present value benefit of $133,000. An infinite number of iso-benefit curves could be drawn, each corresponding to a different level of total benefit. Nevertheless, location of one iso-benefit line establishes the entire family, because each has the same slope (slope equals the negative ratio of the output prices, with the price of the output on the ordinate as the denominator).

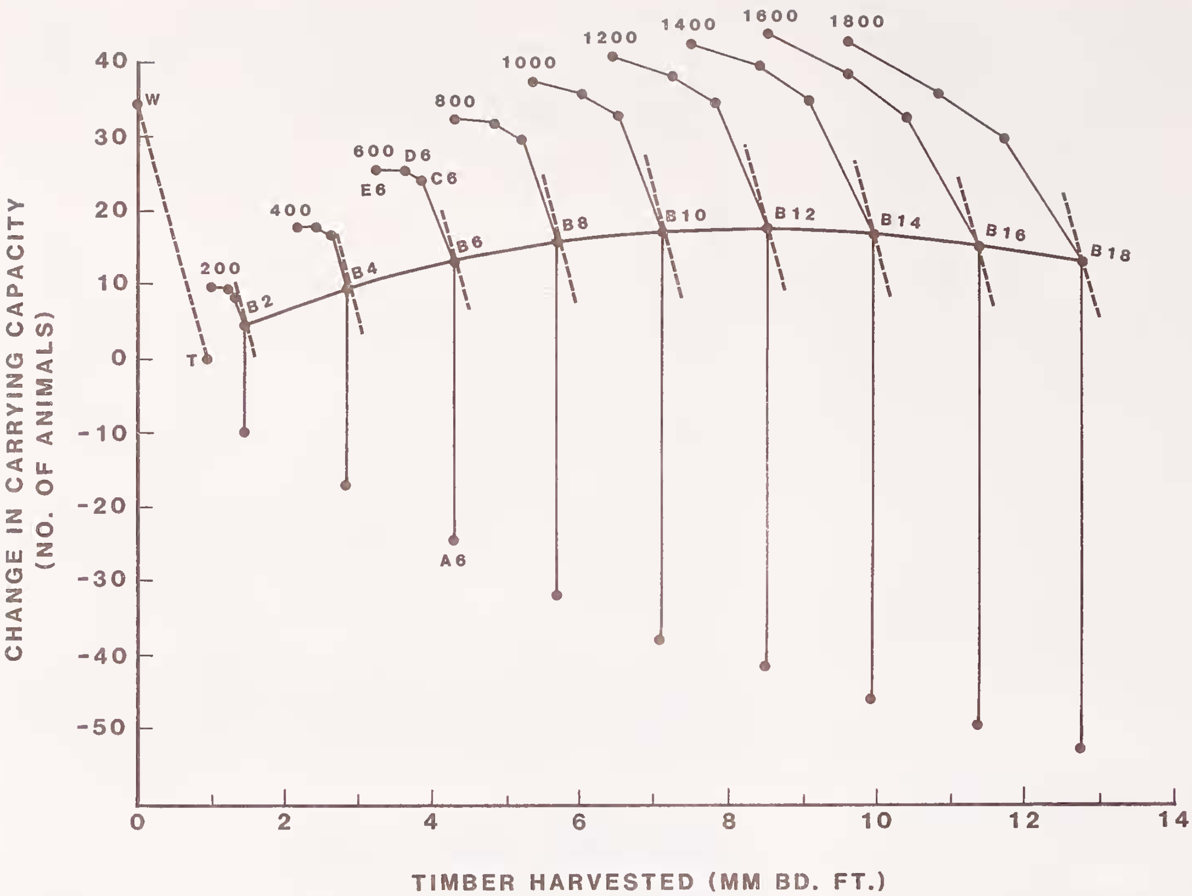


Figure 4.—Iso-cost curves, iso-benefit lines, and expansion path for timber sale example.

Solution

The graphic formulation is solved in two steps. First, the alternative with the highest present value is found for each iso-cost curve. For a given iso-cost curve this is the alternative that lies on the highest iso-benefit line. For iso-cost curve 600, this is alternative B6. There exists a comparable best point for each iso-cost curve. The locus of these points, the expansion path, identifies the best alternative for each budget level. In this example, the expansion path follows the alternatives in series B.

The next step is to identify which of the points along the expansion path maximizes present net value (PNV). This is most easily done by calculating PNV for each alternative on the expansion path, as illustrated in table 2. Alternative B12 is indicated as the best of the alternatives, having a PNV of $55,400. It would harvest about a thousand acres of timberland by means of 30- to 40-acre (12- to 16-ha) cutting units. About 8.5 million board feet of timber would be harvested, and habitat carrying capacity would be increased by an average annual amount of 17.6 elk over the 30 years following harvest.

Table 2.—Calculation of net benefit for alternatives lying on the expansion path

Alternatives	Timber harvest	Change in elk-carrying capacity	Discounted benefits[1]	Discounted cost	Present value
	M bd ft	*Number of animals*	*-----------Thousands*	*of*	*dollars-----------*
B2	1,406.0	4.7	214.7	200	14.7
B4	2,822.7	9.2	430.2	400	30.2
B6	4,239.4	13.1	643.3	600	43.3
B8	5,656.0	15.5	850.7	800	50.7
B10	7,072.7	17.0	1,054.8	1,000	54.8
B12	8,489.4	17.6	1,255.4	1,200	55.4*
B14	9,906.0	16.9	1,451.0	1,400	51.0
B16	11,322.7	15.2	1,643.0	1,600	43.0
B18	12,739.4	13.0	1,832.9	1,800	32.9

[1]Calculated using per unit values of $140 per M bd ft for timber and $3,800 per animal-carrying capacity over 30 years.
*Identifies maximum net benefit.

Intertemporal Analysis

The timber sale example contained only one intertemporal output—the carrying capacity. It was handled by assuming output quantity is constant over time, and by expressing unit value as the present value of the constant annual quantity over 30 years. In reality, multiple-use projects can be comprised of many intertemporal costs and outputs, all of which could vary in magnitude over time. Expressing output as an annual average (as in the timber sale example) may not always be acceptable. Here we discuss several approaches for handling such intertemporal problems graphically. It is suggested that readers who lack a specific interest in techniques for integrating intertemporal analysis into the graphic approach skip directly to the next subtopic, Discussion of Graphic Approach.

Formulating a graphic model in intertemporal terms requires expressing iso-cost and iso-benefit relationships so that the benefits and costs of the alternatives are compared at a common point in time. Following custom, we shall express these relationships in present-value terms.

Expressing iso-cost curves in present-value terms is straightforward. Simply discount the costs of all the resources used in a project to the present. Handling intertemporal output is somewhat more difficult. Both output quantities and unit values can be changing over time. Including these changes in graphic analyses is difficult for two reasons. First, the graphic approach requires that each output for an alternative be expressed as a single number. This number represents one dimension on the base graph (example, in figure 4, carrying capacity was expressed on an average annual basis). Second, unit values must be expressed such that when multiplied by the single output response number, the product is in terms of discounted dollars.

There are several ways outputs and unit values can be expressed to handle this problem, if either output or unit value is constant over time. To explain, let us first rewrite equation 2 (the present value of elk-carrying capacity) in more general terms:

$$V_o = \sum_{t=1}^{n} P_t \, Q_t \left[\frac{1}{(1+i)^t}\right] \tag{3}$$

where:

V_o = present value of the flow of output Q over n years
P_t = unit value of output in year t
Q_t = quantity of output in year t
i = discount rate.

The first approach requires that unit value be constant over time. If P represents a constant unit value, it can be factored out of the summation:

$$V_o = P \sum_{t=1}^{n} Q_k \left[\frac{1}{(1+i)^t}\right] \tag{4}$$

In this formulation, output is expressed as a single number by the term:

$$Q_o = \sum_{t=1}^{n} Q_t \left[\frac{1}{(1+i)^t}\right] \tag{5}$$

Iso-cost curves would then be expressed in terms of Q_o per discounted cost. Unit value used in computing iso-benefit is simply P, the stated value of a unit of Q.

A potential disadvantage of formulating output in this manner is that people may have difficulty relating to quantity expressed as Q_o. It may be easier for some to relate to quantity if it were expressed in terms of an annual equivalent output, Q^A. This can be accomplished as follows:

$$Q^A = Q_o \left[\frac{i(1+i)^n}{(1+i)^n-1}\right] \tag{6}$$

To maintain the correct calculation for V_o, unit value must be multiplied by the inverse of the factor multipled by Q_o:

$$P_o = P \left[\frac{(1+i)^n-1}{i(1+i)^n}\right] \tag{7}$$

Present value of the flow of output can then be written as:

$$V_o = P_o \bullet Q^A \tag{8}$$

Here, unit value (P_o) is the present value of a series of annual outputs. The single value for output, Q^A, is an annual flow equivalent of the actual output flow.

Q^A differs from an "ordinary annual average." The product of Q^A times P_o is equivalent to the present value that would be calculated by discounting each year's benefit (quantity times price in each year) separately and summing. This equality does not hold if annual output is computed as a simple arithmetic average unless, of course, annual output quantity actually is constant.

Both approaches discussed thus far require a constant unit value over time. It is possible to allow unit value to vary over time if the annual quantity of output is constant over time. If Q represents a constant annual flow, it can be factored out of the summation in equation 3:

$$V_o = Q \sum_{t=1}^{n} P_t \left[\frac{1}{(1+i)^t}\right] \tag{9}$$

In this formulation, unit value is expressed as:

$$P_o = \sum_{t=1}^{n} P_t \left[\frac{1}{(1+i)^t}\right] \tag{10}$$

This differs from equation 7 in that P_t is allowed to vary here. Output is expressed in the iso-cost curves as a constant annual quantity occurring over n years.

The reader should note that none of these approaches allow both unit value and output to vary over time. In fact, it does not appear possible to allow for this occurrence using the graphical approach. The order of multiplication and summation indicated in equation 3 must be maintained if both P_k and Q_k vary over time ($\Sigma[P_k \bullet Q_k] \neq \Sigma P_k \bullet \Sigma Q_k$). Only when one of these variables was held constant was it possible to factor them out of the summation to develop the approaches presented.

Discussion of Graphic Approach

The graphic approach rather clearly depicts the nature of economic efficiency in multiple-use production. Consider figure 4. Each iso-cost curve shows the opportunity cost of producing increased amounts of one output at the expense of the other. The specific production points (output combinations) comprising each iso-cost curve are readily available for inspection and verification. The expansion path shows optimal solutions associated with various scales of activity. Finally, sensitivity analyses can be performed graphically to determine the change in relative prices needed to change the preferred alternative on an iso-cost curve. This is done by rotating the iso-benefit line and observing the slope required to identify a new preferred alternative (recall slope of the iso-benefit line equals the inverse ratio of the unit values). The need to more accurately estimate unit values can thereby be assessed.

The graphic approach, however, is inherently limited. Perhaps the most significant limitation is that the number of outputs that can be handled effectively is limited to two. Second, intertemporal analysis imposes restrictions as discussed in the previous section. Third, sensitivity analyses regarding the effect of changes in costs or output quantities can only be conducted by recalculating the iso-cost relationships.

MIP APPROACH

The discrete joint production model presented graphically can also be formulated as a mixed-integer programming (MIP) problem. This approach alleviates the limitations of the graphic formulation discussed in the previous section. It can handle more than two joint outputs. Second, multiple time periods can be handled more easily than in the graphic approach. Third, the MIP formulation provides useful capability for identifying how sensitive the choice of the preferred alternative is to underlying assumptions and projections. Finally, it lends itself to automation. Software could be written such that all the user has to do is enter the data. The computer would take the data, generate the appropriate matrix, and calculate the solution.

The General Model

MIP is a special case of linear programming. Like linear programming, it has decision variables (columns in the matrix), linear constraint rows, and a linear objective function. The major difference is that some of the decision variables are restricted to integer values of either 0 or 1 in the MIP formulation. This provides the ability to express decision variables as whole projects. If in a solution a 0,1 integer variable equals 1, the project represented by that variable was chosen to be accomplished. A value of 0 for project variables indicates those projects were not selected. (Readers interested in a more thorough discussion of MIP are referred to Hillier and Lieberman [1974] or Plane and McMillan [1971].)

The MIP formulation proposed is:

$$\text{Maximize PNV} = \sum_{i=1}^{L} -TC_i X_i + \sum_{j=1}^{M}\sum_{t=1}^{N} DP_{jt} V_{jt} + \sum_{j=1}^{M}\sum_{t=1}^{N} - DP_{jt} W_{jt} \qquad (11)$$

subject to:

$$\sum_{i=1}^{L} X_i \leq 1 \qquad (12)$$

$$\sum_{i=1}^{L} Y^{+}_{ijt} X_i - V_{jt} = 0 \text{ (for all } V_{jt}) \qquad (13)$$

$$\sum_{i=1}^{L} Y^{-}_{ijt} X_i + W_{it} = 0 \text{ (for all } W_{jt}) \qquad (14)$$

$$\text{all } X_i = 0 \text{ or } 1 \qquad (15)$$

THE VARIABLES

There are three sets of variables in this formulation—X_i's, V_{jt}'s, and W_{jt}'s. The X_i's are the project alternatives. Each X_i represents a whole project, and is restricted to the values of either 0 or 1 as indicated by equation 15. The coefficients for the X_i variables are expressed on a project basis (example, TC_i represents total cost for project X_i).

The variables labeled V_{jt} store the positive quantity of the j^{th} output in time period t expected from the alternatives. Unlike the X_i's these are continuous variables that can assume any nonnegative value.

The final set of variables, W_{jt}, measure negative quantity of the j^{th} output in time period t expected from the alternatives. This situation can arise when output is defined as change in volume relative to the no-action alternative (as in the example in table 1). These variables are necessary to avoid infeasibilities that would occur if a V_{jt} variable were to be set equal to a negative output volume (algorithms generally require all variables be nonnegative). Instead, W_{jt} measures the absolute value of the negative volume, and the negative sign is attached to its objective function coefficient ($-DP_{jt}$). A W_{jt} variable is needed only when there is a negative volume predicted for one or more projects for the j^{th} output in time period t. Thus, there should be only a few W_{jt} variables in most applications.

THE ROWS

Equations 11-14 represent the rows in the MIP model. Equation 15 is a restriction placed on the model, but does not appear as a row in the matrix. The objective function to be maximized is PNV (equation 11). The coefficients for the X_i variables, $-TC_j$, are the discounted total costs for the X_i projects. These costs are preceded by a negative sign, because this row measures PNV. The output variable coefficients, DP_{jt} and $-DP_{jt}$, are the unit values for output j in time period t, discounted to present value terms. As explained earlier, W_{jt} variables measure decreases in outputs and therefore have negative unit value coefficients.

The first constraint (equation 12) specifies that not more than one project can be chosen. (Because the X_i's are restricted to values of 0 or 1, combinations of parts of projects that sum to 1.0 are not permitted.) The less-than-or-equal-to form of this constraint does, however, permit a solution in which none of the project alternatives are chosen—the no-action alternative. This would occur if the PNV for each alternative is negative. The model can be forced to choose a project alternative other than the no-action alternative by reformulating this row to equal 1.0.

Equation 13 actually represents a set of rows whose function is to "transfer" positive output quantities from the resource project in solution (X_i) to the variables measuring output volume (V_{jt}). There is one of these rows for each combination of output and time period (i.e., for each V_{jt}). The Y^t_{ijt} coefficients in these rows measure the positive quantity of the j^{th} output produced by project X_i in time period t.

Equation 14 represents the set of rows that "transfer" negative output quantities from the project in solution to the variables measuring negative volume (W_{jt}). The Y^-_{ijt} coefficients in these rows measure the negative quantity of the j^{th} output produced by project X_i in time period t. There is one such row needed for each W_{jt} present, which (as explained earlier) should only be a few in most applications.

WHY THIS FORMULATION?

Thoughtful readers may be wondering at this point why output values are not simply included in the objective function coefficients for the project variables. This would alleviate the need for the output variables V_{jt} and W_{jt} and for equations 13 and 14. The reason is that handling output as separate variables provides advantages for conducting sensitivity analyses on unit values and output quantities.

To illustrate, assume output value has been included in the objective function coefficients for the project variables in a model. The analyst now wants to determine what effect a unit value change would have on a previously obtained optimal solution. The shadow prices from this previous solution are not useful for this purpose. Shadow price measures how much the objective function coefficient for a project variable would have to increase for that variable to become part of the optimal solution, assuming all other coefficients remain unchanged. Other objective function coefficients, however, would change as a result of a unit value change as long as those projects are also producing the same product.

The most straightforward way to determine the effect of a unit value change is to implement that change in the original model and resolve. This, however, would require recalculating and changing every one of the objective function coefficients. In contrast, the equations 11-15 formulation would require changing only the objective function coefficient(s) for that output (one coefficient for each time period t that V_{jt} is quantified), prior to resolving the model. Similar advantages exist in applying some of the other postoptimization techniques for conducting sensitivity analyses that will be discussed later.

Solving The MIP Formulation

There are several options for solving the formulation presented by equations 11-15. One option would be to use algorithms specifically designed for solving MIP problems, such as the branch and bound technique. These algorithms have several disadvantages. First, the capabilities for conducting sensitivity analyses are limited. They do not, for example, offer the majority of the postoptimality techniques available in continuous linear programming software. Second, they are rather restrictive in terms of the size of model (number of rows and columns) that can be handled efficiently. This, however, does not appear to be a significant problem for the class of programming problem created by the equations 11-15 formulation. Third, computer software for solving MIP problems is not as readily available as, say, software for solving continuous linear programming problems, particularly for small computers.

Another option for solving this MIP formulation is to use a conventional continuous linear programming algorithm. This involves simply treating equations 11-14 in the general model as a continuous linear programming problem. If no additional constraint types are added to this equation 11-14 formulation (several will be discussed later), the optimal continuous solution will be the optimal MIP solution.

An explanation might be helpful at this point. Equations 13 and 14 merely ensure that the output variables (V_{jt} and W_{jt}) equal the correct quantity. The key constraint is equation 12. Linear programming algorithms will maximize the PNV objective function by entering as much of the most profitable project as possible. When the upper limit of the equation 12 constraint is reached, the most profitable project variable will equal 1.0. All other project variables (the X_i's) will equal zero at this point. This is an integer solution. Furthermore, it is the optimal solution, because adding any other project to the solution would require the amount of the most profitable project to be reduced to continue to satisfy equation 12. Any such change would reduce the value of the objective function.

Use of continuous linear programming algorithms to solve this MIP formulation provides several advantages. Most importantly, it makes the standard linear programming postoptimization techniques available for conducting sensitivity analyses. Secondly, it makes using a small computer for solving this type of problem more viable, because software for solving continuous linear programming problems is more readily available than MIP software.

The disadvantage of the continuous linear programming approach is that it may not yield integer solutions if additional constraints are added to the equations 11-15 model, an option that will be discussed later. In instances when continuous algorithms do not yield integer solutions, optimal integer solutions would be most easily found using an MIP algorithm.

The Timber Sale Example

The timber sale example presented earlier was formulated as an MIP problem to illustrate how the generalized model can be applied in practice. The following discussion covers the formulation and solution of this model.

FORMULATION

The MIP formulation for the timber sale example is presented in table 3. The project alternatives (the X_i's in equations 11-15) are the alternatives A2 through E18 listed in table 1. Two positive output variables (V_{jt} in equations 11-15) are present. They are TIMB and WILD, which respectively measure positive quantities of timber

Table 3.—Formulation of the timber sale example as an MIP problem[1]

Row name	A2	A4 ...	E18	TIMB	WILD	NWILD	RHS
PNV	−200.0	−400.0 ...	−1,800.0	0.140	3.8	−3.8	
EQN 12	1.0	1.0 ...	1.0				≤1.0
TVOL	1,417.0	2,833.1 ...	9,554.5	−1.0			=0.0
WVOL			42.9		−1.0		=0.0
NWVOL	−9.8	−17.0				−1.0	=0.0

[1]Variables A2 through E18 are treated as 0, 1 integer variables.

and change in elk-carrying capacity. Negative change in elk-carrying capacity (corresponding to W_{jt} in equations 11-15) is measured by NWILD.

The objective function to be maximized is the row labeled PNV, which measures present net value in thousands of dollars. The coefficients for the project alternatives are the discounted total costs from the second column in table 1. The objective function coefficients for TIMB and WILD are the unit values for these outputs developed earlier. Finally, the coefficient for NWILD is the negative unit value for elk-carrying capacity, since NWILD measures decrease in carrying capacity.

The first constraint shown is row EQN 12, which corresponds to equation 12 in the general formulation. The coefficient for each of the project variables is 1.0, and the row is set less-than-or-equal-to 1.0. This specifies that no more than one project can be chosen, but allows for the possibility of not choosing any of the project alternatives—the no-action alternative. (Recall, outputs and costs for the projects are expressed in terms of change from the no-action alternative.)

The next row is TVOL, which corresponds to equation 13 in the general model. It sets the variable TIMB equal to the positive quantity of timber expected from the project alternative selected. The coefficients for the project alternatives predict total timber yield for each alternative and come from the sixth column in table 1.

Row WVOL sets the variable WILD equal to the positive change in elk-carrying capacity in the same manner as TVOL "transfers" timber quantity to TIMB. The project alternative coefficients measure the positive change in carrying capacity and come from the last column in table 1. No coefficients exist in this row for project alternatives A2 through A18 (note, this is equivalent to a coefficient of zero) because the change in carrying capacity is negative for these alternatives.

Row NWVOL corresponds to equation 14 of the general model, and sets NWILD equal to the project coefficients measuring decrease in elk-carrying capacity. These coefficients also come from the last column in table 1. No coefficients are present in this row for alternatives B2 through E18 because these projects are expected to result in an increase in elk-carrying capacity.

THE SOLUTION

The timber sale example in table 3 was solved using the continuous linear programming option in the Functional Mathematical Programming System (FMPS) available at the USDA Fort Collins Computer Center. The solution is presented in figure 5. Although the format used in this figure is specific to FMPS, the information presented is standard among mathematical programming packages.

The first item of interest is the value of the objective function, row PNV. It is found in the portion labeled SECTION 1 - ROWS under the column headed ACTIVITY. The value identified here (55.396) deviates slightly from the value of the selected alternative identified in table 2, due to rounding.

Next, examine the second portion of the solution output labeled SECTION 2 - COLUMNS. The values for the decision variables in the optimal solution are presented in the column headed ACTIVITY. Glancing down this column, one sees that project B12 equals 1.0. This means B12 was the alternative selected—the same project selected earlier in table 2. The other project variables equal zero (represented by a decimal) identifying that they were not chosen in the solution process.

The outputs predicted for the selected alternative B12 are the entries in the activity column for the output variables. TIMB equals 8,489.4 M bd ft, WILD equals an increase of 17.6 animals in carrying capacity, and NWILD equals zero, because change in carrying capacity is predicted to increase rather than decrease.

Sensitivity Analyses

Output responses, costs, and unit values included in such a model are predicted future outcomes, and thus are not known with certainty. Sensitivity analysis can aid the analyst in dealing with uncertainty. It can help determine the range of predicted outcomes over which an alternative identified as optimal remains optimal. Secondly, it can be used to identify what other alternatives are preferred when predicted outcomes are outside the limits for which a given alternative is optimal.

Unfortunately, most of the postoptimization techniques used in linear programming for sensitivity analyses are not available in the branch and bound MIP algorithm commonly used in MIP computer packages. If branch and bound algorithms are used, sensitivity analysis is limited to changing the parameter(s) of interest and resolving. If integer solutions can be obtained with standard linear programming algorithms, however, then some of the more sophisticated postoptimal techniques for conducting sensitivity analyses could be useful. Here we discuss changing parameters and resolving, and several postoptimization techniques available in linear programming that appear particularly useful in the formulation presented by equations 11-15.

SECTION 1 - ROWS PRIMAL-DUAL OUTPUT

NUMBER	..NAME..	AT	..ACTIVITY..	SLACK ACTIVITY	.LOWER LIMIT	.UPPER LIMIT	DUAL ACTIVITY	.INPUT COST.	REDUCED COST
1	PNV	FR	55.396000	-55.396000	NONE	NONE	-1.000000	.	-1.000000
2	EQN 12	EQ	1.000000	.	1.000000	1.000000	55.396000	.	55.396000
3	TVOL	EQ	.	.	.	.	-.140000	.	-.140000
4	WVOL	EQ	.	.	.	.	-3.800000	.	-3.800000
5	NWVOL	EQ	.	.	.	.	-3.800000	.	-3.800000
6	APV	FR	203.716000	-203.716000	NONE	NONE	.	.	.
7	FSCOST	FR	215.199999	-215.199999	NONE	NONE	.	.	.

SECTION 2 - COLUMNS PRIMAL-DUAL OUTPUT

NUMBER	..NAME..	AT	..ACTIVITY..	.INPUT COST.	.LOWER LIMIT	.UPPER LIMIT	REDUCED COST
8	A2	LL	.	-200.000000	.	NONE	94.256000
9	A4	LL	.	-400.000000	.	NONE	123.362000
10	A6	LL	.	-600.000000	.	NONE	152.735998
11	A8	LL	.	-800.000000	.	NONE	182.855999
12	A10	LL	.	-1000.000000	.	NONE	206.641998
13	A12	LL	.	-1200.000000	.	NONE	221.955999
14	A14	LL	.	-1400.000000	.	NONE	240.296000
15	A16	LL	.	-1600.000000	.	NONE	255.341999
16	A18	LL	.	-1800.000000	.	NONE	271.035999
17	B2	LL	.	-200.000000	.	NONE	40.696000
18	B4	LL	.	-400.000000	.	NONE	25.258000
19	B6	LL	.	-600.000000	.	NONE	12.100000
20	B8	LL	.	-800.000000	.	NONE	4.656000
21	B10	LL	.	-1000.000000	.	NONE	.618000
22	B12	BS	1.000000	-1200.000000	.	NONE	.
23	B14	LL	.	-1400.000000	.	NONE	4.336000
24	B16	LL	.	-1600.000000	.	NONE	12.458000
25	B18	LL	.	-1800.000000	.	NONE	22.480000
26	C2	LL	.	-200.000000	.	NONE	41.636000
27	C4	LL	.	-400.000000	.	NONE	28.798000
28	C6	LL	.	-600.000000	.	NONE	19.760000
29	C8	LL	.	-800.000000	.	NONE	17.182000
30	C10	LL	.	-1000.000000	.	NONE	23.344000
31	C12	LL	.	-1200.000000	.	NONE	34.826000
32	C14	LL	.	-1400.000000	.	NONE	52.768000
33	C16	LL	.	-1600.000000	.	NONE	78.309999
34	C18	LL	.	-1800.000000	.	NONE	110.677999
35	D2	LL	.	-200.000000	.	NONE	53.328000
36	D4	LL	.	-400.000000	.	NONE	53.040000
37	D6	LL	.	-600.000000	.	NONE	56.552000
38	D8	LL	.	-800.000000	.	NONE	64.624000
39	D10	LL	.	-1000.000000	.	NONE	82.209999
40	D12	LL	.	-1200.000000	.	NONE	105.481999
41	D14	LL	.	-1400.000000	.	NONE	133.313999
42	D16	LL	.	-1600.000000	.	NONE	169.125999
43	D18	LL	.	-1800.000000	.	NONE	211.778000
44	E2	LL	.	-200.000000	.	NONE	72.426000
45	E4	LL	.	-400.000000	.	NONE	90.995999
46	E6	LL	.	-600.000000	.	NONE	112.985999
47	E8	LL	.	-800.000000	.	NONE	138.015999
48	E10	LL	.	-1000.000000	.	NONE	169.886000
49	E12	LL	.	-1200.000000	.	NONE	208.216000
50	E14	LL	.	-1400.000000	.	NONE	252.245998
51	E16	LL	.	-1600.000000	.	NONE	299.315998
52	E18	LL	.	-1800.000000	.	NONE	354.745998
53	TIMB	BS	8489.399902	.140000	.	NONE	.
54	WILD	BS	17.600000	3.800000	.	NONE	.
55	NWILD	BS	.	-3.800000	.	NONE	.

Figure 5.—Solution to MIP formulation of timber sale example.

Table 4.—The unit values over which project B12 remains optimal

Outputs	Lowest value	Highest value	Project selected if unit value is below the identified lowest value	Project selected if unit value is above the identified highest value
	Dollars			
WILD	2,770.00	5,220.00	B10	C8
TIMB	139.56	143.06	B10	B14

UNIT VALUES

Several types of sensitivity analyses for unit values are potentially useful. The choice depends on the question being asked. The effect of some specific change in unit value on a previously optimal solution is best determined by making that change in the formulation and resolving. This is accomplished by changing the objective function coefficient for the output variables associated with the change in unit value. This can be done easily with a text editor because only a few numbers would change. The model is then resolved using standard procedures. No knowledge of the more sophisticated postoptimization procedures is needed.

Analysts may also be interested in determining the range in unit values over which a particular solution remains optimal. This could be calculated by systematically changing unit values and resolving, but this process would likely require a large number of solutions. An easier approach would be to use a postoptimization technique available in most linear programming packages which calculates this directly. To illustrate, the EXRANGE procedure in FMPS was used to calculate the range in unit values over which the figure 5 solution remains optimal. The results are summarized in table 4. The lowest and highest unit values for WILD are, respectively, $2,770 and $5,220. As long as the unit value for WILD is within this range, project B12 is preferred, assuming other parameters constant.

In addition to the range in unit values, linear programming ranging procedures can be expected to identify what project would be preferred if the unit value falls below or rises above the indicated range (see the last two columns of table 4). For example, if the unit value for WILD were to fall below $2,770, then project B10 would be preferred. This does not imply that B10 is preferred for all unit values less than $2,770, but rather for some range, whose lower unit is unspecified and whose upper limit is $2,770.

If the question to be asked is how does the preferred project change over a wide range in unit values, then parametric programming can be used to good advantage. Parametric programming involves reformulating the objective function from:

$$Z = \sum_{j=1}^{N} C_j X_j \qquad (16)$$

a general expression for equation 11, to:

$$Z(\theta) = \sum_{j=1}^{N} (C_j + \alpha_j \theta) X_j \qquad (17)$$

Here, α_j represents constant changes to be applied to the objective function coefficients (C_j). The symbol θ represents a scalar that, when multiplied times the α_j values, results in proportional change in the objective function coefficients. In the parametric programming procedure, θ is incremented upward, starting at zero (where equations 16 and 17 are equivalent) to some user-specified upper limit. In this process, the values for θ, where the optimal solution changes, are identified.

To illustrate the use of parametric programming, assume we desire to investigate how the preferred alternatives change over the range of timber prices from $120 per M bd ft to $200 per M bd ft, all else remaining equal. The changes that would be made to the matrix presented in table 3 are as follows: First, change the objective function coefficient for TIMB from 0.140 to 0.120 ($120 expressed in thousands). Next, a row corresponding to α_j in equation 17 must be added to the matrix. Because the objective function coefficient for TIMB is the only coefficient to be changed in this analysis, the only nonzero coefficient in this new α_j row would be the coefficient for TIMB. Set this coefficient equal to 0.120. The scalar θ then measures the percentage of change (decimal form) from the starting price of $120 per M bd ft.

The results from this parametric programming analysis are summarized in table 5. Project C2 is optimal over the range in timber prices from $120 to $129.33 per M bd ft. As timber price was increased from $129.33 per M bd ft, the optimal solution moves out series B of project alternatives. The selection of the scale of project within series B is shown to be sensitive to timber price. However, the type of harvesting in series B is clearly preferred over the approach in the other series of alternatives over the range in timber prices.

Table 5.—Preferred alternatives and the range in timber prices over which they are optimal[1]

Project alternative	Range in timber price over which project is optimal
	Dollars per M bd ft
C2	120.00 – 129.33
B4	129.33 – 130.71
B6	130.71 – 134.74
B8	134.74 – 137.15
B10	137.15 – 139.56
B12	139.56 – 143.06
B14	143.06 – 145.73
B16	145.73 – 147.07
B18	147.07 – 200.00

[1]All other parameters held constant at the levels in table 1.

OUTPUTS

In the model formulation depicted by equations 11-15, it is typical for an output to be produced (at least at some level) by most, if not all, projects. It would seem that the question most frequently asked regarding outputs would be how much effect would systematically underestimating or overestimating outputs across the projects have on the preferred alternative. If such a systematic change can be expressed as a percentage of change from the previously predicted outputs, investigating this effect is relatively easy. The suggested approach would be to modify the coefficient(s) for the output variable(s) in the output rows (equations 13 and 14) and resolve the model.

This process is best explained via an example. Assume we desire to determine if a 10 percent increase in elk-carrying capacity over that already predicted would affect which project is chosen. This 10 percent increase would be approximated by changing the coefficient for WILD in row WVOL (table 3) from −1.0 to −0.9. This 10 percent decrease in the coefficient requires a 10 percent larger quantity allocated to WILD to maintain the equality of row WVOL. The model would then be resolved to determine the effect of the change.

In this instance, the 10 percent increase in change in elk-carrying capacity had no effect on the project chosen (B12). The only effect was the value of the objective function increased to $62,800.

COST

Change in virtually any underlying cost (examples, labor costs or equipment costs) would change the objective function coefficient for each project alternative. Therefore, for reasons discussed earlier, shadow prices provide little information regarding how cost changes might affect an optimal solution. The effect of potential changes in costs is best analyzed using parametric programming procedures.

The general formulation for parametric programming described by equation 17 also applies here. The only difference is that here the α_j row to be added to the model should be comprised of the cost changes to be applied to the objective function. We suggest that the α_j row be comprised of the costs included in the objective function coefficients for the resource(s) for which the effect(s) of cost changes is (are) to be investigated. To investigate cost increases, these α_j coefficients should be negative. For example, if the effect of increasing fuel cost is to be measured, α_j would be comprised of the previously calculated total fuel cost for each project. Given this definition for α_j, θ measures the percent change (decimal form) in these costs. The effect of increases in costs is then analyzed when the parametric programming procedure increments θ upwards, starting at zero. The results identify values for θ where the optimal solutions change.

The effect of decreases in cost can be investigated by changing the signs on the coefficients in the α_j row from negative to positive. When formulated in this manner, as θ is incremented upward from zero, the product of θ and α_j is added (rather than subtracted) giving the effect of decreasing costs.

To illustrate this approach, parametric programming was used to analyze the effects of changes in purchaser-related costs. The coefficients for the α_j row (which were added to the model presented in table 3) were the purchaser costs presented in the fourth column of table 1. The signs of these coefficients were negative for the portion of the analysis dealing with cost increases and positive for the portion dealing with cost decreases. Changes from a 30 percent decrease to a 30 percent increase were investigated.

The results are summarized in table 6. Project B12 remains optimal as long as purchaser cost does not decrease more than 2.6 percent or increase more than 0.3 percent. As purchaser cost increases from the original amount, smaller scale series B alternatives are preferred. Decreases in purchaser cost result in larger scale series B alternatives being preferred. The preferred scale within series B is shown to be quite sensitive to changes in purchaser cost. But this analysis indicates the series B method of harvesting is preferred over the other approaches over quite a large range in purchaser cost.

Table 6.—Preferred alternatives and the range in changes in purchaser costs over which they are optimal[1]

Project alternative	Range in purchaser costs over which project is optimal		
	Percent change[2]		
B18	30.0 (decrease)	–	6.1 (decrease)
B16	6.1 (decrease)	–	4.9 (decrease)
B14	4.9 (decrease)	–	2.6 (decrease)
B12	2.6 (decrease)	–	0.3 (increase)
B10	0.3 (increase)	–	2.4 (increase)
B8	2.4 (increase)	–	4.5 (increase)
B6	4.5 (increase)	–	8.0 (increase)
B4	8.0 (increase)	–	9.4 (increase)
B2	9.4 (increase)	–	30.0 (increase)

[1]All other parameters held constant at the levels in table 1.
[2]Percentage of change from the purchaser costs in table 1.

Other Constraints

In actual planning situations there may be management desires that are best handled as constraints. For example, it may be useful to constrain the model to choose an alternative that has a positive appraised sale value or a sediment impact less than some maximum acceptable level. Such constraints could easily be added to the equations 11-15 formulation. The general form for such constraints is as follows:

$$\sum_{i=1}^{L} a_{kit}\, X_i \gtreqless B_{kt} \qquad (18)$$

Here, X_i represents the project alternatives (as before). The coefficients a_{kit} measure the quantity of k (any cost or physical quantity; for example, sediment, water) associated with project X_i in time t. B_{kt} represents the upper and/or lower limits placed on k in time t.

Equation 18 would be modified to the following form for establishing a minimum appraised sale value:

$$\sum_{i=1}^{L} (-PC_{it}\, X_i) + \sum_{j=1}^{N} P_{jt}\, V_{jt} \geq B_t \qquad (19)$$

where V_{jt} measures output quantity of timber in category j in time t. The coefficients for X_i, $-PC_{it}$, are the costs (undiscounted) that must be covered by the value of timber in time t. P_{jt} is the undiscounted unit price for timber in category j in time t. B_t represents the lower limit for sale value specified by the user. There could be a row of this type for each time period there is a potential sale.

Equations 18 and 19 could also be included as "free" or unconstraining rows, which are allowed by most linear programming packages. Such rows do not influence the solution, but the total value of the row is calculated in the solution process. Free rows are useful for monitoring appraised values, costs, and so on.

DISCUSSION

Comparing to Other Linear Programming Formulations

Linear programming formulations common in forestry (FORPLAN, Gilbert and others 1982; Timber RAM, Navon 1971; Resource Allocation Analysis, USDA Forest Service 1975) involve delineating the area being modeled into units, within which the acres are homogenous with regard to one or more characteristics (for example, timber productivity). The decision variables are management prescription alternatives, which are developed for each unit. These prescription alternatives are expressed on a per-acre basis, that is, $X_{ij} = 1$ means 1 acre of prescription j on unit i. All output and input coefficients are therefore on a per-acre basis.

In contrast, the decision variables in the equations 11-15 MIP formulation represent whole alternatives that apply to the entire area. These alternatives are restricted to values of 1 (do project) or 0 (not do project). Differences in the scale of some particular type of activity (scale of a particular type of harvest in the example) are represented by additional decision alternatives.

These differences in structure result in differences in the nature of analyses provided. One difference is that diminishing marginal productivity cannot be modeled in the ordinary linear programming formulation in the same sense as it can in the MIP formulation and the theoretical continuous joint production model. To illustrate the difference, consider modeling the alternatives in the previous example using ordinary linear programming. For simplicity, assume the 4,000-acre (1 619-ha) area is homogeneous, alleviating the need for delineating units. We shall define five prescription alternatives, one for each harvest series. One unit of each variable represents 1 acre (0.4 ha) of harvest activity. We must next formulate a constraint that places an upper limit on the number of acres that can be harvested. Set this upper limit at 1,600 acres (647 ha).

Under this formulation the elk-carrying capacity response to acres harvested is linear—if 1 acre of harvest generates an increase in carrying capacity of Y, 2 acres generate 2Y, etc. No diminishing marginal product is present as was the case in figure 1. The result is that each solution (maximizing PNV) will allocate 1,600 acres (647 ha) to harvest, as long as at least one of the alternatives has a positive PNV per acre. That is, acres are allocated until the upper limit of 1,600 acres (647 ha) of harvest is reached. The only way to obtain a solution with a different level of harvest is to change this upper limit. This formulation is unable to analyze economic efficiency related to scale of harvest as was done by the MIP formulation (recall the various levels of harvest identified as best in tables 4, 5, and 6).

The second difference is in the level of site specificity. The spatial arrangement of activities that comprise a project can be identified in the MIP formulation. In contrast, spatial location is not part of the definition of decision variables in ordinary linear programming formulations. A management prescription simply must be applied somewhere within the homogeneous unit for which it was developed.

This difference is important because spatial arrangement can affect input and output relationships. For example, road cost is usually entered as an average per-acre cost in ordinary linear programming formulations. In reality, however, the road cost associated with implementing an acre of some specific prescription is highly variable, depending on where it occurs. These relationships are handled more precisely in the MIP formulation.

The third difference is that the ordinary linear programming formulations allow one to analyze a greater number of possible outcomes than does the MIP formulation. This can be illustrated most easily by comparing the example MIP formulation (table 3) with the comparable ordinary linear programming formulation described earlier in this section. The MIP formulation contained 45 project variables, which equaled the number of decision alternatives. The linear programming formulation contains one decision variable for each series of harvest alternatives in the MIP—five in all. These five decision variables can represent essentially an infinite number of harvesting alternatives for the area because each variable can assume fractional values.

Specifying Alternatives

Specifying alternatives is a critical step in the integer approximation to the theoretical joint production for the graphical and MIP approaches. The model is limited to choosing among only those alternatives provided. If only inefficient alternatives are specified, then the alternative identified as best will necessarily be inefficient.

Graphs a-d (fig. 6) illustrate this point. In graph a, the decision set (represented by the data) is too narrow with regard to tradeoffs between outputs X and Y. The actual optimum could lie on either side of this rather narrow band of alternatives. Graph b illustrates the opposite, alternatives span the range between outputs, but have little range with regard to scale. The optimal scale could be larger or smaller. In graph c, projects are single-product oriented. The actual optimum may be a joint production alternative lying somewhere in the middle of this decision space. Finally, graph d illustrates a set of alternatives that span the decision space. We believe this to be the best strategy for specifying alternatives because it is the most likely to bound the actual optimal.

A drawback of the MIP formulation is that the distribution of decision alternatives is not visually apparent

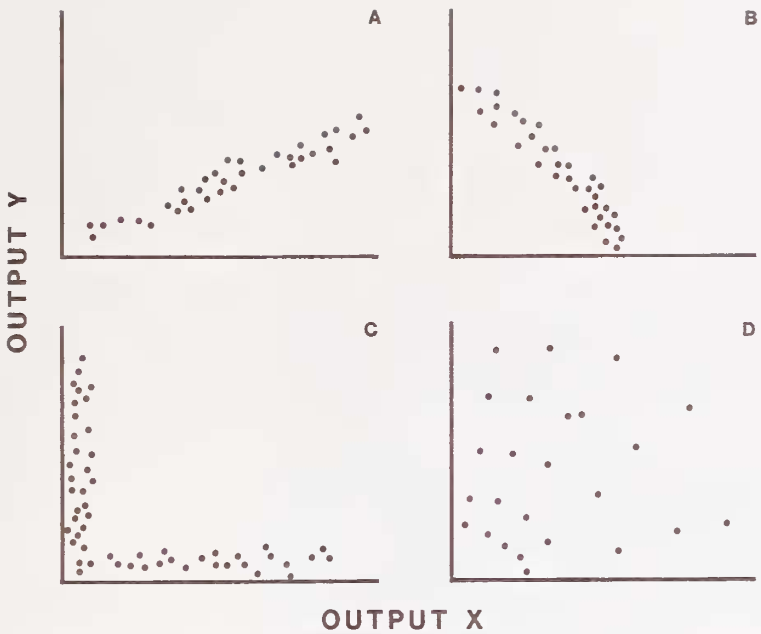


Figure 6.—Project alternatives that span the decision space (graph d) versus several examples of alternative sets that do not (graphs a-c).

as it is in the graphical approach. Perhaps it would be useful to plot project alternatives, even if the MIP approach is used. For problems containing more than two outputs, each combination of two outputs could be plotted for the alternatives. While not perfect, it would at least give a fair idea of the distribution of projects.

An apparent problem with the integer approach presented is that the number of alternatives that can be included in a model is limited by the amount of time available for model construction. If alternatives are held to a modest number, say 40 to 50, there is a chance that none of the alternatives provide a good approximation of the true optimal—even if decision space is spanned as illustrated in graph d. If this is a concern, we suggest constructing a second model that is comprised totally of alternatives in the portion of decision space identified as best with the first model. This would provide the ability to achieve a reasonably good approximation of the true optimal without specifying the large number of alternatives that would be required to achieve the same outcome with one model.

Operationally Viable?

One of the more attractive features of the MIP approach is that it lends itself to automation. Front-end data processing software could be written for data entry and matrix generation. Data entry could be made interactive, leading the user through the process and providing error checking capability. There are numerous ways such a program could be structured. At most, users would be required to enter unit values and costs and output quantities for each project. However, it would likely be possible to structure the process so only codes identifying categories for unit costs and output quantities need be entered. Costs and outputs would then be calculated from information stored internally, either in the form of tables or prediction equations.

A second attractive feature of this MIP formulation is the small size and simplicity—at least when compared to other mathematical programming formulations used in forestry. It is easy to solve and sufficiently small to be processed on a small computer.

Given the front-end software described above, we believe there is little question that the MIP approach for solving discrete joint production models would be operationally viable. It should be no more difficult to use than simulation programs, which are commonly used by resource managers with little or no training in operations research.

Summing It Up

As we have discussed, the discrete joint production model provides the same type of analysis as the continuous joint production model of economic theory. It provides the capability to analyze the economic efficiency of multiple-use management, both in terms of type of project and scale of project (for example, in the timber sale example both the type of cutting alternatives and amount of harvesting were included in the analysis).

The graphic approach to solving these discrete models has the advantage of requiring little in the way of equipment—only paper, pencil, and a straightedge. Little or no start-up time is involved—no need to write computer software or to learn how to use existing software. In addition, it rather clearly depicts the nature of economic efficiency in multiple-use production. The graphic approach, however, has some real limitations enumerated earlier (limited to two outputs and difficulty in conducting intertemporal analysis). Because of these, the graphic approach will likely be limited to special applications.

The MIP approach provides some important advantages over the graphic approach. It lends itself well to automation. With the appropriate software, users relatively inexperienced in computer modeling could conceivably build and solve such a model very efficiently. Next, the mathematical programming formulation provides some very useful sensitivity analysis capability. Finally, the MIP approach is not limited to two outputs and can handle intertemporal analysis more easily.

The discrete joint production model provides a somewhat different type of analysis than what resource allocation mathematical programming formulations common in forestry generally provide. In "ordinary" linear programming formulations, output is a linear function of acres treated, for each decision variable. Questions regarding scale of activities can be addressed only rather crudely by varying the level at which constraints are imposed. The discrete joint production model, on the other hand, can handle nonlinear output and cost relationships, making it a more effective approach by analyzing questions of scale. This can be important, particularly when wildlife and recreation outputs are among the joint products.

The second difference is that the spatial arrangement of activities can be identified more precisely in the discrete joint production model. This is advantageous when location of an activity affects cost or outputs.

Third, the discrete joint production model requires that the user consider fewer alternatives than what can be considered in "ordinary" linear programming formulations. In some respects, the model we have presented has characteristics of both simulation and optimization. Like simulation, it requires the user to formulate whole alternatives. But it does provide some of the optimization and sensitivity analysis capabilities of mathematical programming. Because of the limited number of alternatives that can be handled effectively, the joint production model is best suited to problems of a relatively small geographic scope.

In conclusion, we believe the modeling approach presented in this paper is a practical and useful tool for conducting multiple-use efficiency analysis. The potential lies not as a substitute for current forest planning methods, but rather as a tool to aid in identifying efficient management prescriptions to place in forest planning models, and as a means of analyzing projects for implementation. It would be most effective when spatial arrangement of activities is important, and when outputs or costs are nonlinear with respect to acres treated.

REFERENCES

Gilbert, B.; Johnson, K. N.; Crim, S. FORPLAN Version 2 (DE FORPLAN) user's guide. Unpublished draft on file at USDA Fort Collins Computer Center under file name V2FORPLAN*USERGUIDE.RCNAM. Fort Collins, CO: U.S. Department of Agriculture, Forest Service, Fort Collins Computer Center; 1982.

Gregory, G. R. An economic approach to multiple use. Forest Science. 1: 6-13; 1955.

Hillier, F. S.; Lieberman, G. J. Introduction to operations research. 2d ed. San Francisco: Holden-Day, Inc.; 1974. 800 p.

Lyon, L. J.; Lonner, T. N.; Jones, J.; Marcum, C. L.; Weigand, J.; Sall, D. Montana cooperative elk-logging study. Progress report for period January 1 through December 31, 1981. Missoula, MT: U.S. Department of Agriculture, Forest Service; 1982. 89 p.

Muhlenberg, N. A method for approximating forest multiple-use optima. Forest Science. 10: 209-212; 1964.

Navon, D. I. Timber RAM: a long-range planning method for commercial timber lands under multiple-use management. Research Paper PSW-70. Berkeley, CA: U.S. Department of Agriculture, Forest Service, Pacific Southwest Forest and Range Experiment Station; 1971. 22 p.

Plane, D. R.; McMillan, C., Jr. Discrete optimization: integer programming and network analysis for management decisions. Englewood Cliffs, NJ: Prentice-Hall, Inc.; 1971. 251 p.

U.S. Department of Agriculture, Forest Service. Resource allocation analysis (a user's manual). Fort Collins, CO: U.S. Department of Agriculture, Forest Service, Watershed Systems Development Unit; 1975. n.p.

*U.S. GOVERNMENT PRINTING OFFICE: 1985-0-576-040/10026

Jones, J. Greg; Schuster, Ervin G. An application of discrete optimization for developing economically efficient multiple-use projects. General Technical Report INT-178. Ogden, UT: U.S. Department of Agriculture, Forest Service, Intermountain Forest and Range Experiment Station; 1985. 16 p.

A discrete version of the continuous joint production model in economic theory is presented for use in designing multiple-use projects and identifying efficient management prescriptions for forest planning. Data requirements are less demanding than the continuous theoretical model, yet some of the more important features are maintained. Models can be formulated graphically or as mixed-integer programming problems that are easily solved via computerized routines.

KEYWORDS: economic analysis, multiple-use, decision making, mixed-integer programming

The Intermountain Station, headquartered in Ogden, Utah, is one of eight regional experiment stations charged with providing scientific knowledge to help resource managers meet human needs and protect forest and range ecosystems.

The Intermountain Station includes the States of Montana, Idaho, Utah, Nevada, and western Wyoming. About 231 million acres, or 85 percent, of the land area in the Station territory are classified as forest and rangeland. These lands include grasslands, deserts, shrublands, alpine areas, and well-stocked forests. They supply fiber for forest industries; minerals for energy and industrial development; and water for domestic and industrial consumption. They also provide recreation opportunities for millions of visitors each year.

Field programs and research work units of the Station are maintained in:

Boise, Idaho

Bozeman, Montana (in cooperation with Montana State University)

Logan, Utah (in cooperation with Utah State University)

Missoula, Montana (in cooperation with the University of Montana)

Moscow, Idaho (in cooperation with the University of Idaho)

Provo, Utah (in cooperation with Brigham Young University)

Reno, Nevada (in cooperation with the University of Nevada)