AN APPLICATION OF GAME THEORY: COST ALLOCATION

By Jean Lemaire

Université Libre de Bruxelles

Summary

The allocation of operating costs among the lines of an insurance company is one of the toughest problems of accounting; it is first shown that most of the methods used by the accountants fail to satisfy some natural requirements. Next it is proved that a cost allocation problem is identical to the determination of the value of a cooperative game with transferable utilities, and 4 new accounting methods that originate from game theory are proposed. One of those methods, the proportional nucleus, is recommended, due to its properties. Several practical examples are discussed throughout the paper.

Keywords

Game theory, cost allocation.

1. Cost allocation in practice

Cost allocation is one of the toughest problems of accounting. It occurs whenever cooperation between several departments of a company produces economies of scale: the benefits of cooperation have to be allocated to the participating departments. In insurance, such problems are numerous, especially in countries where companies are allowed to operate on a multi-class basis; the accountants of the company are then compelled to divide the operating costs between the different classes. The amount of time spent and the complexity of the methods used in cost allocation are absolutely startling: for instance a large Belgian company that operates in three classes (life, fire and accident) uses no less than 11 different criteria or "keys".

Key No. 1: Direct Imputation

Some operating costs can be directly assigned to a class: the salary of the employees that work exclusively in that class, the brokers' commissions, the surveyors' fees, ... Note that only 57% of the operating costs of the company can be allocated directly.

Key No. 2: In Proportion to Key No. 1

The salaries of the employees who do not work exclusively for one class, the premiums of their insurance policies, the employer's contribution to the Social Security system, ... are allocated in proportion of the total observed under key No 1.

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Key No. 3: *In Proportion to the Number of Files*

The salaries, the telephone bills, the travel expenses of the administrative inspectors of the company are allotted according to the number of files they have to consider monthly.

Key No. 4: *In Proportion to the Number of Policies and Endorsements*

Costs allocated according to this key include the salaries of the producing inspectors, of the premium collectors, the agents' solidarity fund in case of illness, etc. . . . .

Key No. 5: *One Third to Each Class*

The company operates a training center, where its agents now and then come for a full week of lessons. All costs relating to this activity (instructors' salaries, food and beverages, caretaker's wage, heating of the center, . . .) are simply distributed evenly among the classes!

Key No. 6: *Average of Keys Nos.* 3, 4 *and* 5

The premiums of the insurance policies of the inspectors are the only costs allocated by this key.

Key No. 7: *In Proportion to the Surface Occupied*

Heating costs, water, electricity, telephone bills, cleaners' salaries, lift maintenance, . . . are apportioned according to the surface occupied by the three classes in the building.

Key No. 8: *In Proportion to Premium Income*

The list of costs divided according to this key is nearly endless and very diversified: subsidies to various organizations, subscriptions to papers and magazines, gifts for the employees' children at Christmas, prizes for competitions between the agents, advertising, travel costs of the directors, maintenance of the company cars, reception costs of the foreign visitors, printing of the company's newsletter, ...

Key No. 9: *In Proportion to the Average Number of Employees of each Class*

In this section we have the maintenance costs of the printing department, the operating costs of the restaurant, the stationery supplies, ...

Key No. 10: *In Proportion to the Number of Computing Hours + the Average Number of Disks and Tapes*

This key was selected to subdivide the computer costs.

Key No. 11: *In Proportion to the Total of Keys No.* 1 *to* 10

This last key includes the postage costs, the operating costs of the company's local offices, the insurance policies of the company cars, the medical aid for the employees, ...

In addition to this complexity, quite large amounts (millions of Belgian francs!) are arbitrarily transferred from one class to another whenever it is felt that one of the keys acts unfairly.

The accountants unanimously acknowledge that their methods are extremely complex and in some ways completely arbitrary. They admit that the grand total for one class may be wrong by quite a few percent, but pretend that this is not too important: since the total profit of the company is the sum of its three components, they claim that an allocation error simply increases the profit of one class at the expense of another, and does not influence the total result. This is not correct: unfair allocations may lead to actions that decrease the total profit of the company, as shown by the following examples.

EXAMPLE 1. In the case where service department costs are allocated to producing divisions, the overcharged division has an incentive to independently contract out such services, and avoid the use of the service department. While the division reports a cost savings from such a move, overall corporate profits may suffer. For instance, in one company, some of the policies of one class are printed outside the printing department: the manager of this class has noticed that, due to the selected allocation key, it is cheaper to have its policies printed outside than at the company's printing department. This is a nice example of an individually optimal decision that turns out to be a collective error: the class manager has increased his profit, but the company profit has decreased, since the printing department's salaries and maintenance have to be paid anyway.

EXAMPLE 2. Key No. 10 penalizes the computer programs that use a lot of disks and tapes. So there is an incentive for class managers to have those programs run outside the company: this reduces the operating expenses of the class, but increases the company's expenses.

EXAMPLE 3. In many countries the technical results of a class influence the commissions paid to the agents and/or the bonus paid at the end of the year to the employees. Also the profit of a class is one of the criteria for the evaluation of the performance of its manager. All those persons would certainly not be very happy if they were to learn that their class is subsidizing another one, by way of some unfair cost allocation procedures that have distorted the relative profitabilities of their products.

EXAMPLE 4: The worst error that could be induced by an incorrect allocation procedure is under-pricing, selling a type of policy below the "break-even" price, without being aware of it. Typically this may happen if the selected key fails to identify the high operating costs of a line, like travel assistance or familial responsibility, that produces numerous small claims. For example, if an allocation key is: "For all policies of the accident class, the operating costs equal 20% of the commercial premium", that amounts to have the travel assistance line subsidized by motorcar third-party liability.

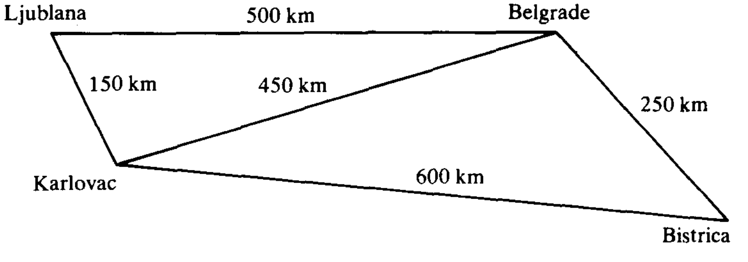
Those examples show that it is of uttermost importance to develop "fair" cost allocation techniques. We shall attempt to show that game theory may be "the" solution to this problem. First (Section 2) an introductory example shows why the classical cost allocation methods, failing to satisfy some important properties, have to be rejected. Then, we show (Section 3) that the cost allocation problem is identical to the problem of computing the value of a n-person cooperative game with transferable utilities. We propose (Section 4) four new methods, adapted from game theory, and compare them (Section 5) by means of three important properties. In Section 6 the case of games without core is briefly considered. In Section 7 we present an extensive list of other applications of game theory that show that cost allocation is an area where game theoretic ideas are effectively implemented. Finally, in Section 8, we completely solve a practical example.

2. AN INTRODUCTORY EXAMPLE

EXAMPLE 1. During the first week of April 1983, three Belgian drivers were involved in an accident in Yugoslavia.

Policy-holder	Company	Place of accident	Amount of the claim (×1.000 dinars)
J_1	1	Ljublana	$s_1 = 300$
J_2	2	Karlovac	$s_2 = 1000$
J_3	3	Bistrica	$s_3 = 200$

The three concerned companies need a damage survey of the cars. They happen to have the same local correpondent in Belgrade, the appraisal bureau Y. Observing the location of the three claims on the map, Y notices that it is much cheaper



(in total mileage) to sens an expert for a round trip, than to come back to Belgrade after each evaluation.

Let S be any subset of $N=\{1,2,3\}$. Denote by $c(S)$ the total mileage driven in order to inspect the vehicle(s) of S.

$$c(1)=1000$$

$$c(2)=900$$

$$c(3)=500$$

$$c(12)=1100$$

$$c(13)=1500$$

$$c(23)=1300$$

$$c(N)=c(123)=1500$$

(for simplicity we denote $c(12)$ for $c(\{1,2\})$, *etc*).

So a round trip produces a total gain of $1000+900+500=900$ km. This however creates a problem to Y: what amount x_i should be charged to each company? Clearly the fixed costs (hotel nights in each city, adjuster's fee for each vehicle, . . .) can be assigned directly to the corresponding claim, so we only need to consider the repartition of the variable costs, the travel expenses. We suppose that the expert's reimbursement indemnity is proportional to the mileage driven. The classical cost allocation methods used in accounting are the following.

Method 1: *Equal Repartition of the Total Gain*

$$x_i=c(i)-\frac{1}{3}\left[\sum_j c(j)-c(N)\right].$$

This leads to the allocation vector $\bar{x}=(x_1, x_2, x_3)$:

$$\bar{x}=(700, 600, 200).$$

Method 2: *Proportional Repartition of the Total Gain* (*or Moriarity's Method*)

$$x_i=c(i)-\frac{c(i)}{\sum_j c(j)}\left[\sum_k c(k)-c(N)\right]=\frac{c(i)}{\sum_j c(j)}c(N).$$

In our example $\bar{x}=(625, 562.5, 312.5)$.

Method 3: *Equal Repartition of the Non-Marginal Costs*

Define the marginal cost for i:

$$CM(i)=c(N)-c(N\backslash\{i\}).$$

$CM(i)$ (sometimes called the separable cost) is the additional mileage to be driven if $\{i\}$ is considered to be the last claim, if it is added to the group $N\backslash\{i\}$, already formed. The method advocates

$$x_i = CM(i) + \frac{1}{3}\left[c(N) - \sum_k CM(k)\right]$$

i.e., an allocation $\bar{x} = (500, 300, 700)$ (the marginal costs are $(200, 0, 400)$).

Method 4: *Proportional Repartition of the Non-Marginal Costs*

$$x_i = CM(i) + \frac{CM(i)}{\sum_j CM(j)}\left[c(N) - \sum_k CM(k)\right] = \frac{CM(i)}{\sum_j CM(j)} c(N)$$

We obtain $\bar{x} = (500, 0, 1{,}000)$.

Method 5: *Repartition Proportional to the Claim Amounts*

$$x_i = \frac{s_i}{\sum_j s_j} c(N)$$

i.e., $\bar{x} = (300, 1{,}000, 200)$.

The five methods recommend wildly different allocations. They can be compared by their properties. In order for a method to be "fair", it certainly has to satisfy the two following natural properties.

Property 1: *Individual Rationality*

$$x_i \leq c(i).$$

A company cannot be charged more than if its policy-holder had been alone to cause an accident. It is inconceivable that a company should suffer from a global saving.

Property 2: *Collective Rationality* (*or Marginality Principle*)

$$x_i \geq c(N) - c(N\backslash\{i\}) = CM(i).$$

No company should be charged less than its marginal cost; if the property is not satisfied for a company, it is effectively subsidized by the other two, who have interest to secede.

The two properties limit the range of the acceptable values for x_i:

$$200 \leq x_1 \leq 1000$$

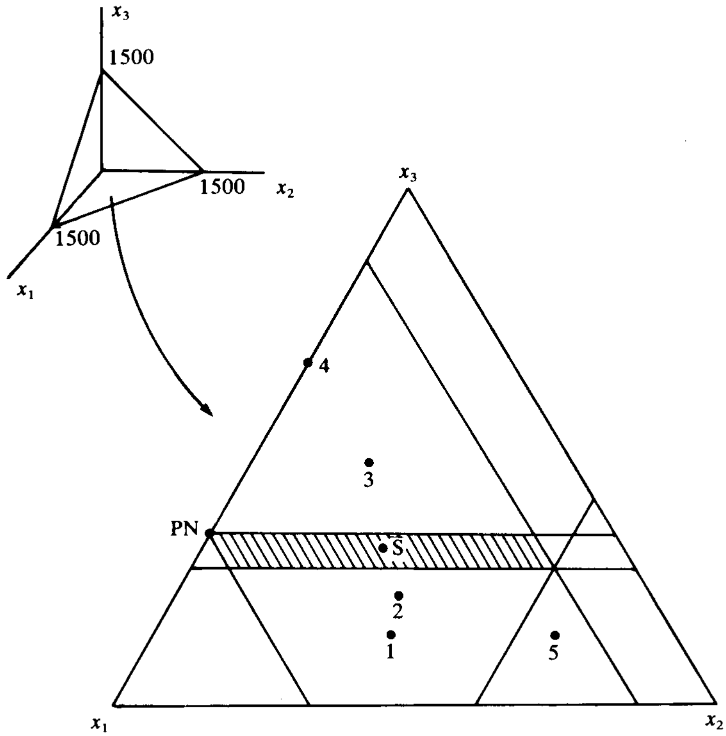
$$0 \leq x_2 \leq 900$$

$$400 \leq x_3 \leq 500.$$

Consequently *all* of the above methods have to be rejected.

The different allocations can be represented in the so-called "fundamental triangle of costs".

$$\left\{(x_1, x_2, x_3) \middle| x_i \geqslant 0, \sum_i x_i = 1500\right\}.$$



The hatched surface is the set of the acceptable allocations, delimited by the two properties. The repartitions are indicated by the number of the method.

3. LINK WITH COOPERATIVE GAME THEORY

We shall show in this section that the cost allocation problem is identical to the determination of the value of a game with transferable utilities.

Cost Allocation

Let N be a set of n departments $\{1, 2, \ldots, n\}$ involved in a given job or project. A cost $c(S)$ is attached to each subset or coalition S of departments. A consequence

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of scale economies is that the set function $c(S)$ has to be sub-additive

$$c(S)+c(T) \geq c(S \cup T) \quad \forall S, T \ni S \cap T = \varnothing:$$

it is cheaper for two departments to collaborate on a job than to act independently.

A cost allocation is a vector $\bar{x}=(x_1, \ldots, x_n)$, such that $x_i \geq 0$, $\forall i$ and $\sum_{i=1}^n x_i = c(N)$.

$\bar{x}$ is said to be individually rational if $x_i \leq c(i) \quad \forall i$.

$\bar{x}$ is said to be collectively rational if, $\forall S$, $\sum_{i \in S} x_i \leq c(S)$.

Imputation of a Game

A n-person cooperative game with transferable utilities is a pair $[N, v(S)]$, where $N=\{1, 2, \ldots, n\}$ is the set of the players, and $v(S)$, the characteristic function of the game, is a super-additive set function that associates a real number $v(S)$ to each coalition S of players.

$$v(S)+v(T) \leq v(S \cup T) \quad \forall S, T \ni S \cap T = \varnothing$$

(it is not limitative to assume that $v(i)=0 \, \forall i$).

An imputation is a vector $\bar{y}=(y_1, \ldots, y_n)$ such that $y_i \geq v(i)$, $\forall i$ and $\sum_{i=1}^n y_i = v(N)$.

The core is the set of imputations such that $\sum_{i \in S} y_i \geq v(S)$, $\forall S$.

Clearly the two problems are identical if we define

$$v(S)=\sum_{i \in S} c(i)-c(S):$$

the characteristic function associated to each coalition is the saving it can achieve. An imputation of this game defines a cost allocation by

$$x_i = c(i) - y_i.$$

So it is equivalent to define a cost allocation game by $[N, v(S)]$ or $[N, c(S)]$. In the sequel, all formulas will be expressed in terms of $c(S)$.

Note that properties 1 and 2 define the core of the game (in the 3-player case). Obviously none of the preceding five methods will provide a point that always belongs to the core, since none explicitly considers all the $c(S)$.

NOTE. The core of a game may be void (a necessary and sufficient condition for a non void core in a 3-person game is $c(12)+c(13)+c(23) \geq 2c(123)$). In that case, there exists no acceptable cost allocation: there is always at least a set of players who have right to complain and who have interest to separate from the rest of the group. Fortunately, in most of the applications, economies of scale are so large that the game is convex.

DEFINITION. A game is convex if, $\forall S$, T (not necessarily disjoint)

$$c(S)+c(T) \geq c(S \cup T)+c(S \cap T).$$

In the three-player case, convexity reduces to 3 conditions

$$c(12)+c(13)\geqslant c(123)+c(1)$$

$$c(12)+c(23)\geqslant c(123)+c(2)$$

$$c(13)+c(23)\geqslant c(123)+c(3).$$

In the four-player case, there are already 30 conditions!

An equivalent definition of convexity is

DEFINITION. A game is convex if, $\forall i$, $\forall S\subseteq T\subseteq N$

$$c(T\cup\{i\})-c(T)\leqslant c(S\cup\{i\})-c(S).$$

So in a convex game there is a "snow-balling" effect: it becomes more and more interesting to enter a coalition as its number of members increases, since the "admission cost" $c(S\cup\{i\})-c(S)$ decreases. Particularly, it is always preferable to be the last to enter the grand coalition N (this justifies our definition of the marginal cost in Section 2: it is only in the case of a convex game that one can assert that the sum of the marginal costs is less than or equal to the total cost $c(N)$).

In a convex game, the study of the different value concepts is considerably easier, since one can show that the core of such a game is always non void and that it satisfies interesting regularity properties: it is a compact convex polyhedron, of dimension at most $n-1$ (Shapley, 1971). Moreover, it coincides with the bargaining set and the Von Neumann and Morgenstern solution (Maschler, Peleg and Shapley, 1972).

4. FOUR NEW COST ALLOCATION METHODS

4.1. *The Shapley Value*

Shapley (1953) has proved that there exists one and only one allocation $\bar{x}$ that satisfies the following 3 axioms.

AXIOM 1. *Symmetry. For all permutations* Π *of players such that* $c[\Pi(S)]=c(S)$, $\forall S$, $x_{\Pi(i)}=x_i$.

A symmetric problem has a symmetric solution. If there are two players that cannot be distinguished by the cost function, if their contribution to each coalition is the same, it is normal to award them the same amount (this axiom is sometimes called "anonymity").

AXIOM 2. *Inessential players. If, for a player* i, $c(S)=c(S\backslash\{i\})+c(i)$ *for each coalition* S *to which he can belong, then* $x_i=c(i)$.

Such a player does not contribute any scale economy to any coalition; he is called an inessential player, and cannot claim to receive a share of the total gain.

AXIOM 3. *Additivity. Let* $[N, c(S)]$ *and* $[N, c'(S)]$ *be two games, and* $x_i(c)$ *and* $x_i(c')$ *the associated allocations. Then*

$$x_i(c+c') = x_i(c) + x_i(c') \quad \forall i.$$

This axiom has been subject to a lot of criticisms, since it excludes the interactions between both games. In the present case, however, those critiques do not appear to have much ground; it is indeed quite natural, in accounting, to add profits that originate from different sources.

Denote by s the number of members of a coalition S. The only imputation that satisfies the axioms is

$$x_i = \frac{1}{n!} \sum_S (s-1)!(n-s)![c(S) - c(S\backslash\{i\})].$$

INTERPRETATION. The Shapley value is the mathematical expectation of the admission cost when all orders of formation of the grand coalition are equiprobable. Everything happens as if the players enter the coalition one by one, each of them receiving the entire saving he offers to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight $1/n!$ in the computation. The Shapley value can also be written

$$x_i = c(i) - \frac{1}{n!} \sum_S (s-1)!(n-s)![c(S\backslash\{i\}) + c(i) - c(S)].$$

The term between square brackets is the saving achieved by incorporating i to coalition S. The cost charged to i is consequently his individual cost less a weighted sum of savings.

The allocation, proposed by Shapley, for example 1, is

$$\bar{x} = (600, 450, 450).$$

It is represented by an S in the fundamental triangle of costs.

4.2. *The Nucleolus* (*Schmeidler*, 1969)

The nucleolus measures the attitude of a coalition towards a proposed allocation by the difference between the cost it can secure and the proposed cost. Define the excess

$$e(\bar{x}, S) = c(S) - \sum_{i \in S} x_i$$

that measures the "happiness degree" of each coalition S. If the excess is negative, the proposed allocation is outside the core; if it is positive, the allocation is acceptable, but the coalition nevertheless has an interest in obtaining the highest possible $e(\bar{x}, S)$. The nucleolus is the imputation that maximizes (lexicographically) the minimal excess.

Let $z(\bar{x})$ be the vector (with 2^{n-1} components) of the excesses of all coalitions $S \subset N$ $(S \neq \varnothing, S \neq N)$, ordered by increasing magnitude. A lexicographic ordering of the vectors $z(\bar{x})$ [i.e., $z(\bar{x}) \geqslant_L z(\bar{x}')$ if $\bar{x}=\bar{x}'$ or if $z_k(\bar{x}) > z_k(\bar{x}')$ for the first component k for which $\bar{x}$ differs from $\bar{x}'$] defines a semi order L. The nucleolus is the first element (=the maximal element) of this semi-order: $z(\bar{x}) \geqslant_L z(\bar{x}')\ \forall \bar{x}'$. To compute the nucleolus amounts to award a subsidy δ, as large as possible, to each proper sub-coalition of N. So one has to solve the linear program

$$\max \delta$$

$$\sum_{i\in S} x_i + \delta \leqslant c(S) \quad \forall S \subset N, S \neq \emptyset, S \neq N,$$

$$\sum_{i=1}^{n} x_i = c(N) \qquad x_i \geqslant 0 \quad \forall i.$$

In the case of example 1, the maximal value of δ is 50; this leads to the same allocation

$$\bar{x} = (600, 450, 450)$$

as the one proposed by Shapley.

4.3. *The Proportional Nucleolus* (*Young et al.*, 1980).

The proportional nucleolus is obtained when the excess is defined by the formula

$$e(\bar{x}, S) = \frac{c(S) - \sum_{i\in S} x_i}{c(S)};$$

instead of granting the same amount to each proper coalition of N, a subsidy proportional to $c(S)$ is awarded. One has to solve the linear program

$$\max s$$

$$\sum_{i\in S} x_i \leqslant c(S)(1-s), \quad \forall S \subset N, S \neq \emptyset, S \neq N,$$

$$\sum_{i\in N} x_i = c(N) \qquad x_i \geqslant 0 \quad \forall i.$$

In the case of example 1, we obtain the allocation (denoted PN on the fundamental triangle)

$$\bar{x} = (1000, 0, 500):$$

all the profit of cooperation goes to the second player, who makes the most out of his veto right; without him, indeed, players 1 and 3 cannot achieve any saving.

4.4. *The Disruptive Nucleolus* (*Littlechild and Vaidya*, 1976) (*Michener, Yuen and Sakurai*, 1981)

For each allocation $\bar{x}$ define the propensity to disrupt for coalition S as the ratio between what $N\backslash S$ and S would lose if $\bar{x}$ were to be abandoned.

$$d(\bar{x}, S) = \frac{c(N\backslash S) - \sum_{i \in N\backslash S} x_i}{c(S) - \sum_{i \in S} x_i}.$$

The disruptive nucleolus is computed like the nucleolus, replacing $e(\bar{x}, S)$ by $d(\bar{x}, S)$: let $z(\bar{x})$ be the vector whose components are the $d(\bar{x}, S)$, $\forall S \neq \varnothing$, N, ranged in increasing order. By lexicographically ordering the $z(\bar{x})$, we obtain a semi-order; its first element is the disruptive nucleolus.

In the case of a 3-person game, we obtain the allocation

$$x_i = CM(i) + \frac{c(i) - CM(i)}{\sum_{j=1}^{3} [c(j) - CM(j)]} \cdot \left[c(N) - \sum_{k=1}^{3} CM(k) \right].$$

This leads, for example 1, to

$$\bar{x} = (600, 450, 450),$$

the same allocation as the Shapley value.

5. PROPERTIES

In Section 4, we have proposed 4 new cost allocation methods, that originate from game theory. Which of them should be selected? The study of the following theoretical properties will help us in this choice.

PROPERTY 1. *Collective rationality. The method should provide an imputation within the core (when it is non void).*

Examples 1 and 2 of Section 1 show that this is a very desirable property. An allocation outside the core effectively means that some departments are unwillingly subsidizing some others; therefore the department managers are enticed to quit the grand coalition and to have the work done outside the company. Allocations within the core are necessary to remove the incentive for sub-coalitions to act independently of the grand coalition.

By construction, the three lexicographic concepts always belong to the core. On the other hand, the Shapley value may fall outside. For instance, in the 3-person game defined by $c(1) = c(2) = c(3) = c(12) = 12$, $c(13) = c(23) = 20$, $c(123) = 23$, the Shapley allocation is $\bar{x} = (6\frac{1}{3}, 6\frac{1}{3}, 10\frac{1}{3})$, while the core is defined by the inequalities $3 \leqslant x_1$, $x_2 \leqslant 12$, $11 \leqslant x_3 \leqslant 12$. In the case of a convex game, however, the Shapley value always belongs to the core (Shapley, 1971); it even lies in its center, since it is the center of gravity of the core's extremal points.

PROPERTY 2. *Monotonicity in costs. All the players contribute to an increase in the project's global cost $c(N)$.*

More often than not, negotiations related to the allocation of the cost of a project take place before it is even started: an electric power company will accept to contribute to the cost of erecting a dam only if it knows in advance how much it will cost (or at least if a good estimation of the total cost is known). But it is rather infrequent that the final cost of a project is known as early as the first discussions: the general rule is rather that it exceeds the forecasts. The monotonicity property demands that each player participates to a rise in the total cost: it would be unfair to have a player benefit from an increase of $c(N)$ (it is assumed that $c(S)$, $\forall S \subset N$ is not modified).

The Shapley value is monotonic. Suppose $c(N)$ increases by a. In the expression

$$x_i = \frac{1}{n!} \sum_S (s-1)!(n-s)![c(S) - c(S \backslash \{i\})],$$

$c(N)$ appears only once, when i enters coalition $N\backslash\{i\}$ to form N. This term (and thus x_i)

$$\frac{1}{n!}(n-1)!1![c(N) - c(N\backslash\{i\})]$$

increases by $[(n-1)!/n!]a = a/n$. Consequently, any budget overstepping is spread evenly among the participants. This is open to criticism: it does not seem fair that all players must contribute equally to unforeseen costs, while their shares in the project may be very different; a "small" department, that only has to pay a small share of the initial allocation, gets the same increase as a "large" participant.

The proportional nucleolus is also monotonic: each increase of the global cost is shared among the players in proportion of their profit $c(i) - x_i$: this is intuitively far more satisfying (see Young, Okada and Hashimoto (1980) for the proof).

On the other hand the nucleolus and the disruptive nucleolus are not monotonic. In the case of the nucleolus, a counter-example was presented by Megiddo (1974). As for the disruptive nucleolus, consider the following example

$$c(1) = 4, \quad c(2) = c(3) = 6, \quad c(12) = c(13) = 7.5, \quad c(23) = 12, \quad c(123) = 13.$$

One verifies that the disruptive nucleolus proposes the allocation

$$\bar{x} = (1.75, 5.625, 5.625).$$

If we now put $c(123) = 13.1$, we obtain

$$\bar{x} = (1.727, 5.6865, 5.6865):$$

while the total cost of the project has increased, the contribution of player 1 has decreased.

PROPERTY 3: *Additivity. A subdivision of a player into two should not affect the allocation.*

Let $[N, c(S)]$ be an allocation game and $\bar{x} = (x_1, \ldots, x_n)$ the proposed allocation. Let $[N^*, c^*(S)]$ be the game that results from the splitting of the cost center j into two centers j_1 and j_2. The cost functions has to be such that, for all $S \subset N\backslash\{j\}$,

$$c^*(S) = c(S) \quad \text{and} \quad c^*(S \cup \{j_1\}) = c^*(S \cup \{j_2\}) = c^*(S \cup \{j_1, j_2\}) = c(S \cup \{j\})$$

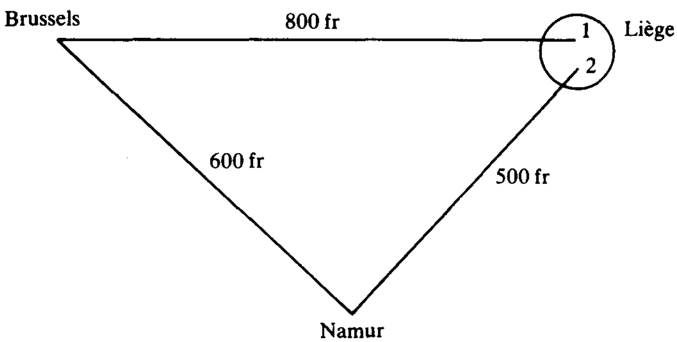
(in words: either fragment, in the absence of the other, incurs the same costs that the two together would incur). Then additivity demands that the allocation $\bar{x}^* = (x_1^*, \ldots, x_{j_1}^*, x_{j_2}^*, \ldots, x_n^*)$ satisfies

$$x_{j_1}^* + x_{j_2}^* = x_j,$$

while for the remaining players i,

$$x_i^* = x_i.$$

Example 2. An insurance company whose head office lies in Brussels wants to install two computer terminals in its local office in Liège, and one in Namur. The renting costs of the telephone lines are indicated in the following figure.



What amount should be charged to each local office? If we reason in terms of terminals, we face a 3-person game, with the cost function

$$c^*(1) = c^*(2) = c^*(12) = 800$$
$$c^*(3) = 600$$
$$c^*(13) = c^*(23) = c^*(123) = 1100.$$

If we think in terms of offices, we have a 2-person (L and N) game

$$c(L) = 800$$
$$c(N) = 600$$
$$c(LN) = 1100.$$

A solution concept is additive iff it amounts to the same to reason in terms of terminals or of offices. The values of the four concepts proposed in Section 4 are

	3-person game			2-person game	
	1	2	3	Liège	Namur
Shapley	350	350	400	650	450
Nucleolus	325	325	450	650	450
Prop. Nucl.	$314\frac{2}{7}$	$314\frac{2}{7}$	$471\frac{3}{7}$	$628\frac{4}{7}$	$471\frac{3}{7}$
Disr. Nucl.	$336\frac{16}{19}$	$336\frac{16}{19}$	$426\frac{6}{19}$	650	450

So the Shapley value and the disruptive nucleolus do not satisfy the property. The nucleolus and the proportional nucleolus are additive. Let us check this for the nucleolus in the case of a 3-person game (the proof is similar for the proportional nucleolus and can be easily generalized to any number of players).

Consider the 3-person game $[\{1, 2, 3\}, c(S)]$ and assume player 3 is split into 3_1 and 3_2 to form the 4-person game $[\{1, 2, 3_1, 3_2\}, c^*(S)]$, where

$$c^*(1) = c(1)$$
$$c^*(2) = c(2)$$
$$c^*(3_1, 3_2) = c^*(3_1) = c^*(3_2) = c(3)$$
$$c^*(1, 2) = c(1, 2)$$
$$c^*(1, 3_1, 3_2) = c^*(1, 3_1) = c^*(1, 3_2) = c(1, 3)$$
$$c^*(2, 3_1, 3_2) = c^*(2, 3_1) = c^*(2, 3_2) = c(2, 3)$$
$$c^*(1, 2, 3_1) = c^*(1, 2, 3_2) = c^*(1, 2, 3_1, 3_2) = c(123).$$

The linear program to compute the nucleolus of the 4-person game is
max δ

(1) $x_1^* + \delta \leq c^*(1)$

(2) $x_2^* + \delta \leq c^*(2)$

(3) $x_{3_1}^* + \delta \leq c^*(3_1)$

(4) $x_{3_2}^* + \delta \leq c^*(3_2)$

(5) $x_1^* + x_2^* + \delta \leq c^*(1, 2)$

(6) $x_1^* + x_{3_1}^* + \delta \leq c^*(1, 3_1)$

(7) $x_1^* + x_{3_2}^* + \delta \leq c^*(1, 3_2)$

(8) $x_2^* + x_{3_1}^* + \delta \leq c^*(2, 3_1)$

(9) $x_2^* + x_{3_2}^* + \delta \leq c^*(2, 3_2)$

(10) $x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(3_1, 3_2)$

(11) $x_1^* + x_2^* + x_{3_1}^* + \delta \leq c^*(1, 2, 3_1)$

(12) $x_1^* + x_2^* + x_{3_2}^* + \delta \leq c^*(1, 2, 3_2)$

(13) $x_1^* + x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(1, 3_1, 3_2)$

(14) $x_2^* + x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(2, 3_1, 3_2)$

(15) $x_1^* + x_2^* + x_{3_1}^* + x_{3_2}^* = c^*(1, 2, 3_1, 3_2)$.

Given the symmetry of $c^*(S)$, $x_{3_1}^*$ will be equal to $x_{3_2}^*$. So conditions (4), (7), (9) and (12) are unnecessary. (10), that can be written $2x_{3_1}^* + \delta \leq c^*(3_1)$, is stronger than (3), so the latter can be deleted. Also (6) and (8) are superfluous, due to (13) and (14). Finally (11) is automatically satisfied, due to (15). Consequently only 7 constraints remain, namely

$$x_1^* + \delta \leq c^*(1) = c(1)$$
$$x_2^* + \delta \leq c^*(2) = c(2)$$
$$x_1^* + x_2^* + \delta \leq c^*(1, 2) = c(1, 2)$$
$$x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(3_1, 3_2) = c(3)$$
$$x_1^* + x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(1, 3_1, 3_2) = c(1, 3)$$
$$x_2^* + x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(2, 3_1, 3_2) = c(2, 3)$$
$$x_1^* + x_2^* + x_{3_1}^* + x_{3_2}^* = c^*(1, 2, 3_1, 3_2) = c(1, 2, 3).$$

Setting $x_{3_1}^* + x_{3_2}^* = x_3$, these are the constraints of the linear program that computes the nucleolus of the 3-person game $[\{1, 2, 3\}, c(S)]$.

In summary, the only method that satisfies the three properties is the proportional nucleolus; we propose it as the best cost allocation method.

6. GAMES WITH EMPTY CORE

If the core of the game is empty, any cost allocation proposal is unstable, since at least one coalition has an incentive to back out of the group. Cooperation between the players is not spontaneous any more, it has to be enforced by an external authority. If one wishes to single out one point, it is necessary to relax some of the collective rationality conditions until a core appears. One can for instance impose a uniform tax ε to each proper subcoalition of N. The least core is obtained by computing the smallest acceptable tax by means of the linear program

$$\min \varepsilon$$
$$\sum_{i \in S} x_i \leq c(S) + \varepsilon \quad \forall S \subset N$$
$$\sum_{i=1}^{n} x_i = c(N).$$

If one feels that the tax has to be proportional to $c(S)$, one obtains the proportional least core by introducing a tax rate t and solving the program

$$\min t$$
$$\sum_{i \in S} x_i \leq c(S)(1 + t) \quad \forall S \subset N$$
$$\sum_{i=1}^{n} x_i = c(N).$$

Notice the similarity with the nucleolus and the proportional nucleolus: in one case coalitions are taxed in order to make the core exist, in the other case coalitions are subsidized in order to reduce the core to a single imputation.

Contrary to the nucleolus and the proportional nucleolus, the Shapley value and the disruptive nucleolus always exist, whether the core is empty or not.

7. COMMENTS

Cooperative game theory presently faces an interesting turning-point of its history. It was born out of practical problems of considerable importance; for instance engineers of the Tennessee Valley Authority (Ramsmeier, 1943), as early as 1930, have considered several cost allocation methods to share among the beneficiaries of the project the costs of improving the existing water communications and constructing dams. The concepts of core, nucleolus and disruptive nucleolus were formulated in an embryonic form, a quarter of a century before those notions were presented in game theory, several years before the publication of the celebrated book of Von Neumann and Morgenstern (1944).

As the problem of the repartition of scale economies occurs in so many commercial activities, it was by no means a surprise to witness the independent development, in numerous areas, of notions very close to game theory. So the disruptive nucleolus is called (in its 3-player version) the "separable costs remaining benefits method", the Gately method, the Louderback method, the Glaeser method, or furthermore the "alternate cost avoided method", depending on the kind of literature one consults.

This enormous duplication of scientific work fortunately seems to come to an end; the contacts between researchers of different areas are improving, the authors more and more explicitly refer to game theory (Hamlen, Hamlen and Tschirhart (1977, 1980), Jensen (1977)) to propose cost allocations. We may now have come full circle, since game theory begins to be applied to the kind of problems that created it.

Many practitioners (and actuaries) still consider game theory as a mathematical toy without any possibility of practical implementation. Let us undeceive them by mentioning several effective applications of solution concepts of game theory:

- —tax allocation among the divisions of McDonnell–Douglas Corporation (Verrechia, 1982)
- —repartition of the renting costs of WATS telephone lines at Cornell University (Billera, Heath and Raanan, 1978)
- —allocation of tree logs after transportation between the Finnish pulp and paper companies (Sääksjärvi, 1976, 1982)
- —maintenance costs of the Houston medical library (new books, periodicals, furniture) shared between the participating hospitals (Bres *et al.*, 1979)
- —financing of large water resource development projects in Tennessee (Straffin and Heaney, 1981)
- —construction costs of multipurpose reservoirs in the U.S. (Inter-Agency Committee on Water Resources (1958))

as well as several domains where a concept of game theory has been proposed
—depreciation problems in financial analysis (Callen, 1978)
—construction of an 80-kilometer water supply tunnel in Sweden (Young *et al.*, 1980)
—building of a power plant in India (Gately, 1974)
—subsidization of public transportation in Bogota (Diaz and Owen, 1979)
—landing fees at Birmingham airport (Littlechild and Thompson, 1977)
—allotment of water between agricultural communities in Japan (Suzuki and Nakayama, 1976)
—construction of a waste treatment center in the U.S. (Heaney, 1979)
—building of a water-filtering plant, financed by three "polluting" factories (Loehmann *et al.*, 1979, Bogardi and Sziderovski, 1976)

. . .

Also in insurance, the possibilities of application are numerous:

—allocation between companies of the costs
- of a professional union (like U.P.E.A. in Belgium)
- of a statistical bureau (like A.G.S.A.A. in France or Försakringstekniska Forskningsnämnden in Sweden)
- of risks supervision and claims appraisal in case of coinsurance;

—allocation between the different classes of a company of most operating costs (see Section 1).

Allocations based on game theoretical considerations have the only disadvantage of requiring more information, since it is necessary to obtain $2^n - 1$ costs $c(S)$, one for each non-void coalition of N.

8. A PROBLEM OF INTEREST ALLOCATION

EXAMPLE 3. The treasurer of ASTIN (player 1) wishes to invest the amount of 1 800 000 Belgian Francs on a short term (3 months) basis. In Belgium, the yield of such an investment is a function of the sum deposited.

Deposit	Annual interest rate
0–1 000 000	7.75%
1 000 000–3 000 000	10.25%
3 000 000–5 000 000	12%

Player 1 contacts the I.A.A. (player 2) and A.A.Br.* (Player 3) treasurers in order to make a group investment. I.A.A. deposits 900 000 fr in the commun fund, A.A.Br. 300 000 fr. How should the interests be split among the 3 players?

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The solution always adopted in practice amounts to award the same yield (12%) to everyone. This allotment is acceptable, since it belongs to the core; it however implies perfect solidarity between the players, who all accept not to use their various threat possibilities. As this allocation is not the only acceptable one, it is interesting to compare the different methods. It is easy to check that

$$v(1) = 46\,125$$
$$v(2) = 17\,437.5$$
$$v(3) = 5812.5$$
$$v(12) = 69\,187.5$$
$$v(13) = 53\,812.5$$
$$v(23) = 30\,750$$
$$v(123) = 90\,000.$$

Core:

$$46\,125 \leqslant y_1 \leqslant 59\,250$$
$$17\,437.5 \leqslant y_2 \leqslant 36\,187.5$$
$$5812.5 \leqslant y_3 \leqslant 20\,812.5.$$

Proportional repartition: 54 000 (12%), 27 000 (12%), 9000 (12%)

Shapley value: 51 750 (11.5%), 25 875 (11.5%), 12 375 (16.5%).

According to the Shapley value, the third player takes a great advantage from the fact that he is essential to reach the 3-million mark; his admission value is very high when he comes in last.

Nucleolus: 52 687.5 (11.71%), 24 937.5 (11.08%), 12 375 (16.5%).

The nucleolus, as generous towards A.A.Br. as the Shapley value, also takes into account the fact that ASTIN is in a better situation than I.A.A., since it can achieve a yield of 10.25% by playing alone, while I.A.A. would only make 7.75% in that case. Note that ASTIN and I.A.A. receive the same amount, in francs, over what they would have earned by playing alone:

$$y_1 - v(1) = y_2 - v(2) = \delta = 6562.5$$

Proportional nucleolus: 54 000 (12%), 27 000 (12%), 9000 (12%).

We obtain in this case the "intuitive" proportional repartition. We shall see later on that this is not always the case.

Disruptive nucleolus: 51 900 (11.53%), 25 687.5 (11.42%), 12 412.5 (16.55%)

The strategic possibilities of the players depend on the amounts they provide. Let us consider two variations of example 3.

Example 3′

ASTIN:	1 700 000 fr
I.A.A.:	1 100 000 fr
A.A.Br.:	300 000 fr

Proportional repartition: 51 000 (12%), 33 000 (12%), 9000 (12%).

Shapley value: 48 395.83 (11.39%), 33 020.83 (12.01%), 11 583.33 (15.44%).

Nucleolus: 48 708.33 (11.46%), 33 333.33 (12.12%), 10 958.33 (14.61%).

Proportional nucleolus: 51 000 (12%), 33 000 (12%), 9000 (12%).

Disruptive nucleolus: 48 481.65 (11.41%), 33 106.65 (12.04%), 11 411.7 (15.22%).

Notice the effects of the more favourable situation of I.A.A., who owns more than a million and can achieve alone a yield of 10.25%: this improves its bargaining power.

Example 3″

ASTIN:	1 700 000 fr
I.A.A.:	1 400 000 fr
A.A.Br.:	300 000 fr.

Proportional repartition: 51 000 (12%), 42 000 (12%), 9000 (12%).

Shapley value: 51 093.75 (12.02%), 43 406.25 (12.4%), 7500 (10%).

Nucleolus: 51 140.625 (12.03%), 43 453.125 (12.41%), 7406.25 (9.875%).

Proportional nucleolus: 52 378.37 (12.32%), 43 621.63 (12.46%), 6000 (8%).

Disruptive nucleolus: 51 127.01 (12.03%), 43 439.52 (12.41%), 7 433.47 (9.91%).

Notice the deep change: the share of A.A.Br., which is not necessary any more to reach 3 millions, is considerably reduced, even in the case of the proportional nucleolus.

The Shapley value and the nucleolus do not seem to be good solution concepts to this problem; in both cases the reasoning is performed in an additive way while the spirit of the problem is multiplicative. When two players form a coalition, the Shapley value simply shares the benefits of cooperation in two equal parts, and equal amounts do not lead to equal percentages. In addition to its theoretical properties, the proportional nucleolus proceeds in a multiplicative way, and seems more adapted to this specific problem.

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