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AN APPLICATION OF THE CORRELATION STRUCTURE OF A MARKOV CHAIN FOR THE ESTIMATION OF SHIFT PARAMETERS IN QUEUEING SYSTEMS

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Abstract. The problem on the estimation of shift parameters of a queueing system $M/M/1/0$ from distorted data observed during a time interval between two sequential states of the system is considered in this paper. The information about the states of the system is not available. Asymptotic properties of the estimators are studied.

Consider a queueing system $M/M/1/0$ containing a single server whose service intensity is ν. The input of the server is characterized by the intensity λ. Let $s_1, s_2, \dots, s_k, \dots$ be the sequence of moments when customers either arrive at the system or leave it. Assume also that we do not know whether a certain moment s_k, $k = 0, 1, \dots$, is a time when the service of a customer is terminated or it is a moment when a customer leaves the system because it is overloaded. Denote by $\mu(t)$ the number of customers in the queueing system at the moment t and note that $\mu(t)$, $t \geq 0$, is a stochastic process. Let the stochastic process $x(k)$, $k = 0, 1, \dots$, be defined as follows:

$$x(k) = \mu(s_k), \qquad k = 0, 1, \dots . \tag{1}$$

Assume that the system is under observation during a time interval $[0, T]$ and let the observations

$$\bar{\xi}_T = \left\{\xi_1, \xi_2, \dots, \xi_{K(T)}, \xi'_{K(T)}\right\}$$

be made with some delay during the intervals

$$\tau_k = s_k - s_{k-1}, \qquad k = 1, 2, \dots, \qquad s_0 = 0.$$

The delay θ_μ depends on the state μ of the queueing system at the moment s_k. In other words, if

$$K(T) = \max\{k, s_k \leq T\},$$

then

$$\xi_k = \tau_k + \theta_{x(k-1)}, \qquad k = 1, 2, \dots, K(T), \qquad \xi'_{K(T)} = T - s_{K(T)} + \theta_{x(K(T))}.$$

Note that the information about the states $\mu(t)$ of the system is not available; thus the vector $\{x(0), x(1), \dots, x(K(T))\}$ is unknown. The problem therefore is to estimate the shift parameters $\theta = \{\theta_0, \theta_1\}$ by using partially distorted observations $\bar{\xi}_T$ and known parameters (λ, ν). It is worthwhile mentioning that the classical statistical methods of estimation of parameters (say, the maximum likelihood method) are not helpful for the queueing system described above because of lack of statistical information.

A method of estimation based on the correlation structure of a Markov chain is proposed in [3]. Some applications of this method for the estimation of the vector of shift

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parameters $\theta^0 = \{\theta_0^0, \theta_1^0\}$ is announced in [1, 2]. We state and prove the corresponding results in this paper.

For the sake of simplicity we assume that the estimators are constructed from a vector of observations $\bar{\xi}_N = \{\xi_1, \xi_2, \dots, \xi_N\}$ of a given size N. We study the properties of these estimators as N is increasing. The results can easily be reformulated for the case where the system is observed on an increasing time interval $[0, T]$ and the size of the vector of observations $\bar{\xi}_T = \left\{\xi_1, \xi_2, \dots, \xi_{K(T)}, \xi'_{K(T)}\right\}$ is random and equals $K(T)+1$.

The process $\mu(t)$, $t \geq 0$ (thus the process $x(t)$, $t \geq 0$, too) has only two states for systems of the $M/M/1/0$ type and therefore the vector θ of the unknown shift of parameters is two-dimensional:

$$\mu(t) \in \{0, 1\}, \qquad \theta = \{\theta_0, \theta_1\}.$$

Let $\bar{\xi}_N = \{\xi_1, \xi_2, \dots, \xi_N\}$ be the statistical data used to construct estimates for unknown coordinates of the vector θ. Note that

$$\xi_k = s_k - s_{k-1} + \theta_{x(k-1)}, \qquad k = 1, 2, \dots, N.$$

Consider the following statistics:

$$q_N^{(1)} = \frac{1}{N-1} \sum_{l=1}^{N-1} \xi_l \xi_{l+1}, \tag{2}$$

$$q_N^{(2)} = \frac{1}{N-2} \sum_{l=1}^{N-2} \xi_l \xi_{l+2}. \tag{3}$$

Lemma 1. *The statistics $q_N^{(1)}$ converges in probability as $N \to \infty$ to the random variable Q_1 defined as follows:*

$$Q_1 = \frac{1}{\lambda + 2\nu} \left(\theta_1 + \frac{1}{\lambda + \nu}\right) \left(2\nu \left(\theta_0 + \frac{1}{\lambda}\right) + \lambda \left(\theta_1 + \frac{1}{\lambda + \nu}\right)\right). \tag{4}$$

This means that

$$\lim_{N \to \infty} \mathsf{P} \left\{ \left| q_N^{(1)} - Q_1 \right| > \varepsilon \right\} = 0 \tag{5}$$

for an arbitrary positive number $\varepsilon > 0$.

The statistics $q_N^{(2)}$ converges in probability as $N \to \infty$ to the random variable Q_2 defined as follows:

$$\begin{aligned} Q_2 = {} & \frac{\nu^2}{(\lambda+\nu)(\lambda+2\nu)} \left(\theta_0 + \frac{1}{\lambda}\right)^2 + \frac{2\lambda\nu}{(\lambda+\nu)(\lambda+2\nu)} + \left(\theta_0 + \frac{1}{\lambda}\right)\left(\theta_1 + \frac{1}{\lambda+\nu}\right) \\ & + \frac{\nu^2 + \lambda\nu + \lambda^2}{(\lambda+\nu)(\lambda+2\nu)} \left(\theta_1 + \frac{1}{\lambda+\nu}\right)^2. \end{aligned} \tag{6}$$

This means that

$$\lim_{N \to \infty} \mathsf{P} \left\{ \left| q_N^{(2)} - Q_2 \right| > \varepsilon \right\} = 0 \tag{7}$$

for an arbitrary number $\varepsilon > 0$.

Proof. It follows from the definition of the queueing system $M/M/1/0$ and by using properties of general right continuous Markov chains with continuous time ([4, Chapter VI]) that the stochastic process $x(k)$, $k = 0, 1, \dots$, defined by relation (1) is a homogeneous ergodic Markov chain with discrete time whose matrix of one-step transition probabilities is given by

$$P = \begin{Vmatrix} 0 & 1 \\ \frac{\nu}{\lambda+\nu} & \frac{\lambda}{\lambda+\nu} \end{Vmatrix}. \tag{8}$$

The stationary distribution $\pi = \{\pi_0, \pi_1\}$ of the chain $x(k)$, $k = 0, 1, \dots,$ is defined by the following equalities:

$$\pi_0 = \frac{\nu}{\lambda + 2\nu}, \qquad \pi_1 = \frac{\lambda + \nu}{\lambda + 2\nu}. \tag{9}$$

Without loss of generality we assume in what follows that the initial distribution of the chain $x(k)$, $k = 0, 1, \dots,$ coincides with its stationary distribution (see [4, Chapter V]), and therefore $x(k)$, $k = 0, 1, \dots,$ is a second-order stationary φ-mixing stochastic process (see [5]). Moreover there exist two numbers $C > 0$ and $0 < \rho < 1$ such that the mixing coefficient can be represented in the following form:

$$\varphi(n) = C\rho^n, \qquad n = 1, 2, \dots .$$

Applying the general result on the structure of purely discontinuous Markov chains with continuous time ([4, Chapter VI]) we prove that the vector of observations

$$\bar{\xi}_N = \{\xi_1, \xi_2, \dots, \xi_N\}$$

is such that

$$\xi_t \sim \xi^{(t)}(x(t)), \qquad t = 1, 2, \dots, N, \tag{10}$$

where the symbol "$\sim$" stands for the stochastic equivalence of random variables and

$$\left\{\xi^{(l)}(x), x \in \{0, 1\}, l = 0, 1, 2, \dots\right\}$$

is a family of random variables that are independent of the stochastic process $x(k)$, $k = 0, 1, \dots,$ are mutually independent for different l, and whose distribution does not depend on l. Moreover

$$\mathsf{P}\left\{\xi^{(0)}(0) > t\right\} = e^{-\lambda(t-\theta_0)}, \qquad t > 0, \tag{11}$$

$$\mathsf{P}\left\{\xi^{(0)}(1) > t\right\} = e^{-(\lambda+\nu)(t-\theta_1)}, \qquad t > 0. \tag{12}$$

It is easy to see that the process defined by relation (10) also is a second-order stationary φ-mixing stochastic process. Its mixing coefficient $\bar{\varphi}(n)$ is of the following form:

$$\bar{\varphi}(n) = 2C\rho^n, \qquad n = 1, 2, \dots . \tag{13}$$

Consider two stochastic processes $\eta_i(t)$, $i = 1, 2$, $t = 1, 2, \dots,$ defined as follows:

$$\eta_i(t) = \xi_t \xi_{t+i}, \qquad i = 1, 2, \ t = 1, 2, \dots . \tag{14}$$

Obviously $\eta_i(t)$, $i = 1, 2$, $t = 1, 2, \dots,$ are second-order stationary φ-mixing stochastic processes and their mixing coefficient is given by (13). The ergodic theorem for second-order φ-mixing stochastic processes [4, 5] implies that

$$\frac{1}{n} \sum_{t=1}^{n} \eta_i(t) \to \mathsf{E}(\eta_i(t)), \qquad i = 1, 2,$$

as $N \to \infty$ with probability one. A straightforward calculation completes the proof of Lemma 1. $\square$

Now let the vector π of the stationary distribution of the chain $x(k)$, $k = 0, 1, \dots,$ be defined by relation (9) and let $P^{(2)} = \left\|p_{ij}^{(2)}\right\|$ be its matrix of two-step probabilities, that is,

$$P^{(2)} = P^2.$$

Consider the following statistics:

$$Y_n(0) = \sqrt{q_n^{(1)} + \frac{\nu(\lambda+\nu)\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}}, \qquad Y_n(1) = \sqrt{\frac{(\lambda+\nu)^2\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}},$$

where $q_N^{(1)}$ and $q_N^{(2)}$ are defined by relations (2) and (3), respectively.

Theorem 1. *The probability that the system of equations*

$$\begin{cases} \sum_{i=0}^{1}\sum_{j=0}^{1} \pi_i p_{ij} x_i x_j = q_N^{(1)}, \\ \sum_{i=0}^{1}\sum_{j=0}^{1} \pi_i p_{ij}^{(2)} x_i x_j = q_N^{(2)} \end{cases} \tag{15}$$

has solutions approaches one as $N \to \infty$.

If the parameters of an $M/M/1/0$ system are such that

$$(\theta_1 - \theta_0) - \frac{\nu}{\lambda(\lambda+\nu)} > 0, \tag{16}$$

then

$$\theta_0^*(n) = Y_n(0) - \frac{\sqrt{\lambda+\nu}}{\sqrt{\nu}} Y_n(1) - \frac{1}{\lambda}, \tag{17}$$

$$\theta_1^*(n) = Y_n(0) + \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}} Y_n(1) - \frac{1}{\lambda+\nu}. \tag{18}$$

Furthermore, if

$$(\theta_1 - \theta_0) - \frac{\nu}{\lambda(\lambda+\nu)} < 0, \tag{19}$$

then the statistics

$$\theta_0^*(n) = Y_n(0) + \frac{\sqrt{\lambda+\nu}}{\sqrt{\nu}} Y_n(1) - \frac{1}{\lambda}, \tag{20}$$

$$\theta_1^*(n) = Y_n(0) - \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}} Y_n(1) - \frac{1}{\lambda+\nu} \tag{21}$$

are consistent estimators of the parameters $\{\theta_0, \theta_1\}$. This means that

$$\lim_{n\to\infty} \mathsf{P}\{|\theta_i^*(n) - \theta_i| > \varepsilon\} = 0, \qquad i = 0, 1, \tag{22}$$

for any positive number $\varepsilon > 0$.

Proof. First we consider the system of equations

$$\begin{cases} x^T \pi P x = Q_1, \\ x^T \pi P^2 x = Q_2, \end{cases} \tag{23}$$

where $x = (x_0, x_1)$ is the vector of unknowns, x^T means the transposition of x, and Q_1 and Q_2 are defined by equalities (4) and (6), respectively. Put

$$\Phi_0(\theta_0, \theta_1) = \mathsf{E}\left(\xi^{(0)}(0)\right) = \theta_0 + \frac{1}{\lambda}, \tag{24}$$

$$\Phi_1(\theta_0, \theta_1) = \mathsf{E}\left(\xi^{(0)}(1)\right) = \theta_1 + \frac{1}{\lambda+\nu}. \tag{25}$$

It follows from the proof of Lemma 1 that the vector $x^* = (x_0^*, x_1^*)$ is one of the solutions of the system of equations (23) where

$$x_0^* = \Phi_0(\theta_0, \theta_1), \qquad x_1^* = \Phi_1(\theta_0, \theta_1).$$

Consider the matrix

$$\hat{\Pi} = \begin{Vmatrix} \frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}} & 0 \\ 0 & \frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}} \end{Vmatrix}$$

and let the matrix A be defined by

$$A = \hat{\Pi} P \hat{\Pi}^{-1}$$

or, in the explicit form, by

$$A = \begin{Vmatrix} 0 & \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}} \\ \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}} & \frac{\lambda}{\lambda+\nu} \end{Vmatrix}.$$

It is easy to see that

$$AA^T = A^T A.$$

Now we change the variables $x = (x_0, x_1)$ for $z = (z_0, z_1)$ as follows:

$$z = \hat{\Pi} x^T. \tag{26}$$

Then the system of equations (23) becomes of the form

$$\begin{cases} zAz^T = Q_1, \\ zA^2 z^T = Q_2. \end{cases}$$

The eigenvalues of the matrix A are

$$a_1 = 1, \qquad a_2 = -\frac{\nu}{\lambda+\nu},$$

while

$$u^{(1)} = \left(\frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}}, \frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}} \right), \qquad u^{(2)} = \left(-\frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}}, \frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}} \right)$$

are the corresponding eigenvectors. Note that the matrix

$$U = \begin{Vmatrix} u^{(1)} \\ u^{(2)} \end{Vmatrix}$$

is orthogonal, that is $U^{-1} = U^T$, or in other words,

$$UU^T = U^T U = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Denote by Λ the diagonal matrix of the eigenvalues of the matrix A:

$$\Lambda = \begin{Vmatrix} a_1 & 0 \\ 0 & a_2 \end{Vmatrix}.$$

Then

$$UAU^T = \Lambda.$$

Moreover

$$UA^2U^T = \Lambda^2.$$

Now we use the variables $z = (z_0, z_1)$ such that

$$y^T = Uz^T \quad \text{or} \quad z^T = U^T y^T.$$

The transformation of variables can be written explicitly as

$$z_0 = \frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}} y_0 - \frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}} y_1, \tag{27}$$

$$z_1 = \frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}} y_0 + \frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}} y_1. \tag{28}$$

Then the system of equations (23) becomes of the form

$$\begin{cases} y\Lambda y^T = Q_1, \\ y\Lambda^2 y^T = Q_2. \end{cases}$$

One can easily find four solutions of the latter system by considering all possible combinations of

$$y_0 = \pm Y(0), \qquad y_1 = \pm Y(1),$$

where

$$Y(0) = \sqrt{Q_1 + \frac{\nu(\lambda+\nu)(Q_2 - Q_1)}{\nu^2 + (\lambda+\nu)^2}}, \qquad Y(1) = \sqrt{\frac{(\lambda+\nu)^2(Q_2 - Q_1)}{\nu^2 + (\lambda+\nu)^2}}.$$

Taking into account the above changes of variables (27), (28), and (26) we conclude that all the solutions of the system of equations (23) are given by

$$x_0 = (\pm Y(0)) - \frac{\sqrt{\lambda+\nu}}{\nu}(\pm Y(1)), \tag{29}$$

$$x_1 = (\pm Y(0)) + \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}}(\pm Y(1)). \tag{30}$$

It follows from (24) and (25) that both coordinates x_0^* and x_1^* are positive for the solution x^* of the system of equation (23). Since x^* is one of the above four solutions, it is of the form

$$x_0^* = Y^*(0) - \frac{\sqrt{\lambda+\nu}}{\nu} Y^*(1), \qquad x_1^* = Y^*(0) + \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}} Y^*(1).$$

Thus $Y^*(0) = Y(0)$.

Now we determine the sign of $Y(1)$ in relations (29) and (30) for the solution x^*. Condition (16) is equivalent to the condition that the difference $\Phi_1(\theta_0, \theta_1) - \Phi_0(\theta_0, \theta_1)$ is positive. Similarly, condition (19) is equivalent to the condition that the same difference is negative. On the other hand, it follows from (27), (28), and (26) that

$$Y^*(1) = -\frac{\sqrt{\lambda+\nu}}{\sqrt{\lambda+2\nu}} z_0^* + \frac{\sqrt{\nu}}{\sqrt{\lambda+2\nu}} z_1^* = \frac{\sqrt{(\nu)(\lambda+\nu)}}{\lambda+2\nu}(\Phi_1(\theta_0,\theta_1) - \Phi_0(\theta_0,\theta_1)). \tag{31}$$

Therefore Theorem 1 follows from (31) and Lemma 1. □

Now we study the rate of convergence of the statistics $\theta^*(n) = (\theta_0^*(n), \theta_1^*(n))$ defined in Theorem 1 to the vector of unknown parameters $\theta = (\theta_0, \theta_1)$. Consider the vector

$$\Delta\theta^*(n) = \theta^*(n) - \theta$$

and let us study the behavior of the distribution of $\Delta\theta^*(n)$ as $n \to \infty$. This allows one to obtain the rate of convergence of the vector of estimators $\theta^*(n)$ to the vector of unknown parameters θ as well as to construct interval estimators for unknown parameters and statistical tests for testing hypotheses about the parameters.

Consider a vector stochastic process $\eta(t) = (\eta_1(t), \eta_2(t))$, $t = 1, 2, \dots$, where $\eta_i(t)$, $i = 1, 2$, $t = 1, 2, \dots$, are defined by relations (14). We also consider the random vectors $q_n = \left(q_n^{(1)}, q_n^{(2)}\right)$ and

$$\Delta q_n = \left(\sqrt{n}\left(q_n^{(1)} - \mathsf{E}\left(q_n^{(1)}\right)\right), \sqrt{n}\left(q_n^{(2)} - \mathsf{E}\left(q_n^{(2)}\right)\right)\right),$$

where the statistics $q_n^{(1)}$ and $q_n^{(2)}$ are defined by relations (2) and (3), respectively. Using the relationship between the statistics $q_n^{(i)}$, $i = 1, 2$, and stochastic processes

$$\eta_i(t), \qquad i = 1, 2, \ t = 1, 2, \dots$$

(cf. (2), (3), and (14)) and the central limit theorem for second-order stationary φ-mixing stochastic processes whose mixing coefficient satisfies condition (13) [4, 5] one can obtain the following result.

Lemma 2. *The distribution of the random vector Δq_n weakly converges as $n \to \infty$ to the multivariate normal distribution with zero vector of expectations and the covariation matrix*

$$B = \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

Moreover,

$$b_{ii} = \mathsf{E}(\eta_i(1) - Q_i)^2 + 2\sum_{t=2}^{\infty} \mathsf{E}(\eta_i(1) - Q_i)(\eta_i(t) - Q_i), \qquad i = 1, 2, \tag{32}$$

$$\begin{aligned} b_{12} = b_{21} = {} & \mathsf{E}(\eta_1(1) - Q_1)(\eta_2(1) - Q_2) \\ & + \sum_{t=2}^{\infty} \mathsf{E}\big[(\eta_1(1) - Q_1)(\eta_2(t) - Q_2) + (\eta_1(t) - Q_1)(\eta_2(1) - Q_2)\big]. \end{aligned} \tag{33}$$

Let the vector $x^*(n) = (x_0^*(n), x_1^*(n))$ be defined by the equalities

$$x_0^*(n) = \sqrt{q_n^{(1)} + \frac{\nu(\lambda+\nu)\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}} - \frac{\sqrt{\lambda+\nu}}{\sqrt{\nu}}\sqrt{\frac{(\lambda+\nu)^2\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}},$$

$$x_1^*(n) = \sqrt{q_n^{(1)} + \frac{\nu(\lambda+\nu)\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}} + \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}}\sqrt{\frac{(\lambda+\nu)^2\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}}$$

if condition (16) holds for the parameters of a system $M/M/1/0$, or by the equalities

$$x_0^*(n) = \sqrt{q_n^{(1)} + \frac{\nu(\lambda+\nu)\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}} + \frac{\sqrt{\lambda+\nu}}{\sqrt{\nu}}\sqrt{\frac{(\lambda+\nu)^2\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}},$$

$$x_1^*(n) = \sqrt{q_n^{(1)} + \frac{\nu(\lambda+\nu)\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}} - \frac{\sqrt{\nu}}{\sqrt{\lambda+\nu}}\sqrt{\frac{(\lambda+\nu)^2\left(q_n^{(2)} - q_n^{(1)}\right)}{\nu^2 + (\lambda+\nu)^2}}$$

if condition (19) holds.

Consider the random vector

$$\Delta x^*(n) = \left(\sqrt{n}(x_0^*(n) - x_0^*), \sqrt{n}(x_1^*(n) - x_1^*)\right),$$

where the random variables $x^* = (x_0^*, x_1^*)$ are defined by relations (24) and (25). It follows from equalities (17), (18), (20), (21), (24), and (25) that

$$\sqrt{n}\Delta\theta^*(n) = \Delta x^*(n).$$

It is seen from the proof of Theorem 1 that the vector $x^*(n) = (x_0^*(n), x_1^*(n))$ and the vector $y^*(n) = (y_0^*(n), y_1^*(n))$ of solutions of the system

$$\begin{cases} y\Lambda y^T = q_n^{(1)}, \\ y\Lambda^2 y^T = q_n^{(1)} \end{cases}$$

are such that

$$(y^*(n))^T = U\hat{\Pi}(x^*(n))^T$$

(irrespective of either condition (16) or condition (19) holding).

We introduce the vector $y^* = (y_0^*, y_1^*)$ by putting

$$(y^*)^T = U\hat{\Pi}(x^*)^T$$

and consider the vector

$$\Delta y^*(n) = \left(\sqrt{n}(y_0^*(n) - y_0^*), \sqrt{n}(y_1^*(n) - y_1^*)\right).$$

Then

$$(\Delta y^*(n))^T = U\hat{\Pi}(\Delta x^*(n))^T \quad \text{or} \quad (\Delta x^*(n))^T = \hat{\Pi}^{-1}U^T(\Delta y^*(n))^T.$$

On the other hand, the following system of equations

$$\begin{cases} a_1\left[(y_0^*(n))^2 - (y_0^*)^2\right] + a_2\left[(y_1^*(n))^2 - (y_1^*)^2\right] = q_n^{(1)} - \mathsf{E}\left(q_n^{(1)}\right), \\ a_1^2\left[(y_0^*(n))^2 - (y_0^*)^2\right] + a_2^2\left[(y_1^*(n))^2 - (y_1^*)^2\right] = q_n^{(2)} - \mathsf{E}\left(q_n^{(2)}\right) \end{cases} \tag{34}$$

holds, since $\mathsf{E}(q_n^{(i)}) = Q_i$, $i = 1, 2$, where a_1 and a_2 are eigenvalues of the matrix A defined in the proof of Theorem 1. Using the matrices

$$W = \begin{Vmatrix} a_1 & a_2 \\ a_1^2 & a_2^2 \end{Vmatrix}, \qquad H_n = \begin{Vmatrix} y_0^*(n) + y_0^* & 0 \\ 0 & y_1^*(n) + y_1^* \end{Vmatrix}$$

the system of equations (34) can be rewritten in the following form:

$$W H_n \Delta y^*(n) = \Delta q_n. \tag{35}$$

Now we apply Theorem 1 and use properties of multivariate normal distributions, asymptotic behavior of the random vector Δq_n obtained in Lemma 1, and the relationship between the vectors $\Delta\theta^*(n)$, $\Delta x^*(n)$, $\Delta y^*(n)$, and Δq_n to prove the following result.

Theorem 2. *The distribution of the random vector $\Delta\theta^*(n)$ of errors of the estimation of the vector of unknown parameters $\theta = (\theta_0, \theta_1)$ weakly converges as $n \to \infty$ to the multivariate normal distribution with zero vector of expectations and the covariation matrix*

$$K = \left(\hat{\Pi}^{-1}U^T H^{-1} W^{-1}\right) \cdot B \cdot \left(\hat{\Pi}^{-1}U^T H^{-1} W^{-1}\right)^T,$$

where the matrix B is defined in Lemma 2, while the matrices W and H are given by

$$W = \begin{Vmatrix} 1 & -\frac{\nu}{\lambda+\nu} \\ 1 & \frac{\nu^2}{(\lambda+\nu)^2} \end{Vmatrix}, \qquad H = \begin{Vmatrix} 2y_0^* & 0 \\ 0 & 2y_1^* \end{Vmatrix}.$$

Bibliography

1. O. A. Voǐna and O. O. Zhigaǐlo, *The estimation of parameters for Markov systems with partially deformed observations*, Dopovidi Akad. Nauk Ukrainy Ser. A (2001), no. 2, 70–74. (Ukrainian) MR1837074 (2002c:62128)
2. O. A. Voǐna and E. Czapla, *The estimation of shift parameters for queueing systems from observations of mixed input and output flows*, Dopovidi Akad. Nauk Ukrainy Ser. A (2001), no. 8, 54–57. (Ukrainian) MR1887243
3. A. A. Voǐna, *Statistical estimation in a scheme of random variables on Markov chains, with incomplete observations*, Teor. Veroyatnost. Matem. Statist. **37** (1987), 16–26; English transl. in Theory Probab. Math. Statist. **37** (1988), 19–28. MR0913905 (88k:62164)

4. J. Doob, *Stochastic Processes*, John Wiley–Chapman and Hall, New York, 1953. MR0058896 (15:445b)
5. P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York–London–Sydney–Toronto, 1968. MR0233396 (38:1718)

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