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AN APPLICATION OF THE RUSCHEWEYH DERIVATIVES

SHIGEYOSHI OWA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577, Japan

SEIICHI FUKUI

Department of Mathematics
Wakayama University
Wakayama 640, Japan

KOICHI SAKAGUCHI

Department of Mathematics
Nara University of Education
Nara 630, Japan

and

SHOTARO OGAWA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577, Japan

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ABSTRACT. Let $D^{\alpha}f(z)$ be the Ruscheweyh derivative defined by using the Hadamard product of $f(z)$ and $z/(1-z)^{1+\alpha}$. Certain new classes S_{α}^{*} and K_{α} are introduced by virtue of the Ruscheweyh derivative. The object of the present paper is to establish several interesting properties of S_{α}^{*} and K_{α}. Further, some results for integral operator $J_c(f)$ of $f(z)$ are shown.

KEY WORDS AND PHRASES. Ruscheweyh derivative, Hadamard product, starlike function, convex function, integral operator.
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1. INTRODUCTION. Let A denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Let S denote the subclass of of A consisting of univalent functions in the unit disk U. A function $f(z)$ belonging to A is said to be starlike with respect to the origin in the unit disk U if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.2}$$

for all $z \in U$. We denote by S^{*} the class of all starlike functions with respect to the origin in the unit disk U. A function $f(z)$ belonging to A is said to be convex in the unit disk $\mathcal{U}$ if it satisfies

$$\mathrm{Re}\ \{1 + \frac{zf''(z)}{f'(z)}\} > 0 \qquad (1.3)$$

for all $z \in \mathcal{U}$. We denote by $\mathcal{K}$ the class of all convex functions in the unit disk $\mathcal{U}$. We note that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$ and that

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}.$$

Let $f_j(z)$ $(j = 1,2)$ in $\mathcal{A}$ be given by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1} \qquad (a_{1,j} = 1).$$

Then the Hadamard product (or convolution product) $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}. \qquad (1.5)$$

By the Hadamard product, we define

$$D^{\alpha} f(z) = \frac{z}{(1 - z)^{1+\alpha}} * f(z) \qquad (\alpha \geqq -1) \qquad (1.6)$$

for $f(z) \in \mathcal{A}$. The symbol $D^{\alpha} f(z)$ was introduced by Ruscheweyh [1], and is called the Ruscheweyh derivative of $f(z)$.

To derive our results, we have to recall here the following lemmas.

LEMMA 1 ([2]). Let $\phi(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$ and satisfy $\phi(0) = g(0) = 0$, $\phi'(0) \neq 0$, $g'(0) \neq 0$. Suppose that for each σ $(|\sigma| = 1)$ and δ $(|\delta| = 1)$, we have

$$\phi(z) * \frac{1 + \delta\sigma z}{1 - \sigma z} g(z) \neq 0 \qquad (0 < |z| < 1) \qquad (1.7)$$

Then for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying $\mathrm{Re}\ \{F(z)\} > 0$ $(z \in \mathcal{U})$, we have

$$\mathrm{Re}\ \{\frac{\phi * G(z)}{\phi * g(z)}\} > 0 \qquad (z \in \mathcal{U}). \qquad (1.8)$$

where $G(z) = F \cdot g(z)$.

LEMMA 2 ([3]). Let $w(z)$ be regular in the unit disk $\mathcal{U}$, with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ $(0 \leqq r < 1)$ at a point z_0, we can write

$$z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geqq 1$.

LEMMA 3 ([4]). For a real number α $(\alpha > -1)$, we have

$$z(D^{\alpha} f(z))' = (\alpha + 1) D^{\alpha+1} f(z) - \alpha D^{\alpha} f(z). \qquad (1.9)$$

REMARK. Note that (1.9) holds true for $\alpha = -1$.

LEMMA 4 ([5]). Let $\phi(u,v)$ be a complex function, $\phi: D \to \mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that ϕ satisfies the following conditions

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0)\in D$ and $\mathrm{Re}\{\phi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\phi(iu_2, v_1)\} \leqq 0$ for all $(iu_2, v_1)\in D$ and such that $v_1 \leqq -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be regular in the unit disk U, such that $(p(z), zp'(z))\in D$ for all $z\in U$. If $\mathrm{Re}\{\phi(p(z), zp'(z))\} > 0$ $(z\in U)$, then $\mathrm{Re}\, p(z) > 0$ for $z\in U$.

2. PROPERTIES OF $D^\alpha f(z)$. Applying Lemma 1, we prove

THEOREM 1. Let $f(z)$ be in the class S^* and satisfy the condition $D^\alpha f(z) \neq 0$ $(0 < |z| < 1)$ for $\alpha \geqq -1$. Then $D^\alpha f(z)$ is also in the class S^*.

PROOF. We note that

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{D^\alpha(zf'(z))}{D^\alpha f(z)} = \frac{\dfrac{z}{(1-z)^{1+\alpha}} * (zf'(z))}{\dfrac{z}{(1-z)^{1+\alpha}} * f(z)}. \tag{2.1}$$

Setting $\delta = -1$, $\phi(z) = z/(1-z)^{1+\alpha}$, $g(z) = f(z)$, and $F(z) = zf'(z)/f(z)$ in Lemma 1, we have

$$\mathrm{Re}\left\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)}\right\} > 0 \qquad (z\in U), \tag{2.2}$$

which implies $D^\alpha f(z)\in S^*$.

THEOREM 2. Let $f(z)$ be in the class K and satisfy the condition $D^\alpha(zf'(z)) \neq 0$ $(0 < |z| < 1)$ for $\alpha \geqq -1$. Then $D^\alpha f(z)$ is also in the class K.

PROOF. Since $f(z)\in K$ if and only if $zf'(z)\in S^*$, Theorem 1 derives $z(D^\alpha f(z))' = D^\alpha(zf'(z))\in S^*$. Hence we have $D^\alpha f(z)\in K$.

3. THE CLASSES S^*_α AND K_α. In view of Theorems 1 and 2, we can introduce the following classes;

$$S^*_\alpha = \{f(z)\in A:\ D^\alpha f(z)\in S^*,\ \alpha \geqq -1\}$$

and

$$K_\alpha = \{f(z)\in A:\ D^\alpha f(z)\in K,\ \alpha \geqq -1\}.$$

Now, we derive:

THEOREM 3. For $\alpha \geqq 0$, we have $S^*_{\alpha+1} \subset S^*_\alpha$.

PROOF. For $f(z)\in S^*_{\alpha+1}$, we define the function $w(z)$ by

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{1+w(z)}{1-w(z)} \qquad (w(z) \neq 1). \tag{3.1}$$

Then, with Lemma 3, we have

$$\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} = \frac{1}{\alpha+1}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} + \alpha\right\}$$

$$= \frac{(1+\alpha) + (1-\alpha)w(z)}{(1+\alpha)(1-w(z))}. \tag{3.2}$$

Differentiating both sides of (3.2) logarithmically, it follows that

$$\frac{zD^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} = \frac{1+w(z)}{1-w(z)} + \frac{2zw'(z)}{(1-w(z))\{(1+\alpha)+(1-\alpha)w(z)\}}. \tag{3.3}$$

Suppose that for $z_0 \in U$

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq \pm 1). \tag{3.4}$$

Then it follows from Lemma 2 that

$$z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geqq 1$. Setting $w(z_0) = e^{i\theta_0}$, we obtain

$$\mathrm{Re}\left\{\frac{z_0(D^{\alpha+1}f(z_0))'}{D^{\alpha+1}f(z_0)}\right\}$$

$$= \mathrm{Re}\left\{\frac{1+w(z_0)}{1-w(z_0)}\right\} + \mathrm{Re}\left\{\frac{2mw(z_0)}{(1-w(z_0))\{(1+\alpha)+(1-\alpha)w(z_0)\}}\right\}$$

$$= -\frac{m\alpha(1-\cos\theta_0)}{M} \leqq 0, \tag{3.5}$$

where $M = \{\alpha(1-\cos\theta_0) + (1-\alpha)\sin^2\theta_0\}^2 + \{\alpha + (1-\alpha)\cos\theta_0\}^2\sin^2\theta_0$.
This contradicts the hypothesis that $f(z) \in S^*_{\alpha+1}$. Therefore, $w(z)$ has to satisfy that $|w(z)| < 1$ for all $z \in U$. Thus we have

$$\mathrm{Re}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right\} = \mathrm{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \tag{3.6}$$

which implies $f(z) \in S^*_{\alpha}$.

THEOREM 4. For $\alpha \geqq 0$, we have

$$\bigcap_{\alpha} S^*_{\alpha} = \{id\},$$

where id is the identity function $f(z) = z$.

PROOF. Note that $D^{\alpha}z = z$ for all α, and that

$$\text{Re}\left\{\frac{z(D^{\alpha}z)'}{D^{\alpha}z}\right\} = 1 > 0 \qquad (z \in U)$$

for all α. Consequently, we conclude that $\text{id} \in S_{\alpha}^{*}$ for all α.

For the converse, we assume that the function $f(z)$ belonging to A is in the class $\bigcap_{\alpha} S_{\alpha}^{*}$. Then we have

$$D^{\alpha}f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} a_{n+1}z^{n+1} \in S^{*}$$

for all $\alpha \geqq 0$. It is well known that

$$|a_{n+1}| \leqq n + 1 \qquad (n \geqq 1)$$

for $f(z) \in S^{*}$. This implies

$$\frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} |a_{n+1}| \leqq n + 1 \qquad (n \geqq 1), \tag{3.7}$$

or

$$|a_{n+1}| \leqq \frac{(n + 1)!\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} \qquad (n \geqq 1) \tag{3.8}$$

for all $\alpha \geqq 0$. Therefore, we have $f(z) = z$.

By virtue of Theorem 3, we prove:

THEOREM 5. For $\alpha \geqq 0$, we have $K_{\alpha+1} \subset K_{\alpha}$.

PROOF. By Theorem 3, it follows that

$$\begin{aligned} f(z) \in K_{\alpha+1} &\iff D^{\alpha+1}f(z) \in K \\ &\iff z(D^{\alpha+1}f(z))' \in S^{*} \\ &\iff D^{\alpha+1}(zf'(z)) \in S^{*} \\ &\iff zf'(z) \in S_{\alpha+1}^{*} \\ &\implies zf'(z) \in S_{\alpha}^{*} \\ &\iff D^{\alpha}(zf'(z)) \in S^{*} \\ &\iff z(D^{\alpha}f(z))' \in S^{*} \end{aligned}$$

This asserts the result of the theorem.

THEOREM 6. FOR $\alpha \geqq 0$, we have

$$\bigcap_{\alpha} K_{\alpha} = \{id\},$$

where id is the identity function $f(z) = z$.

The proof of Theorem 6 is similar to that of Theorem 4.

Furthermore, an application of Lemma 4 to the classes S_{α}^{*} and K_{α} gives:

THEOREM 7. Let $f(z)$ be in the class S_{α}^{*} with $\alpha \geqq -1$. Then

$$\operatorname{Re}\left\{\left(\frac{D^{\alpha}f(z)}{z}\right)^{\beta-1}\right\} > \frac{1}{2\beta - 1} \qquad (z \in U), \tag{3.9}$$

where $1 < \beta \leqq 3/2$.

PROOF. We define the function $p(z)$ by

$$A\left(\frac{D^{\alpha}f(z)}{z}\right)^{\beta-1} = p(z) + (A - 1), \tag{3.10}$$

where $A = 1 + 1/2(\beta-1)$. Differentiating (3.10) logarithmically, we have

$$\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} = \frac{1}{\beta - 1}\,\frac{zp'(z)}{p(z) + (A - 1)} + 1. \tag{3.11}$$

Since $f(z) \in S_{\alpha}^{*}$, it follows that

$$\operatorname{Re}\left\{\frac{1}{\beta - 1}\cdot\frac{zp'(z)}{p(z) + (A - 1)} + 1\right\} > 0 \qquad (z \in U). \tag{3.12}$$

Let $p(z) = u = u_1 + iu_2$ and $zp'(z) = v = v_1 + iv_2$, and define the function $\phi(u,v)$ by

$$\phi(u,v) = \frac{1}{\beta - 1}\cdot\frac{v}{u + (A - 1)} + 1. \tag{3.13}$$

Then $\phi(u,v)$ is continuous in $D = (C-\{1-A\}) \times C$, and together with $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} = 1 > 0$. Moreover, for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1 + u_2^2)/2$, we can show that

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2,v_1)\} &= \frac{1}{\beta - 1}\operatorname{Re}\left\{\frac{v_1}{iu_2 + (A - 1)}\right\} + 1\\ &\leqq \frac{-1}{\beta - 1}\cdot\frac{(A - 1)(1 + u_2^2)}{2\{u_2^2 + (A - 1)^2\}} + 1 \leqq 0,\end{aligned} \tag{3.14}$$

for $1 < \beta \leqq 3/2$. Hence the function $\phi(u,v)$ satisfies the conditions in Lemma 4. It follows from this fact that $\operatorname{Re} p(z) > 0$ for $z \in U$, that is,

$$\operatorname{Re}\left\{A\left(\frac{D^{\alpha}f(z)}{z}\right)^{\beta-1} - (A - 1)\right\} > 0 \qquad (z \in U). \tag{3.15}$$

This completes the assertion of Theorem 7.

Taking $\beta = 3/2$ in Theorem 7, we have:

COROLLARY 1. Let $f(z)$ be in the class S_{α}^{*} with $\alpha \geqq -1$. Then

$$\operatorname{Re}\left\{\left(\frac{D^{\alpha}f(z)}{z}\right)^{1/2}\right\} > \frac{1}{2} \qquad (z \in U), \tag{3.16}$$

COROLLARY 2. Let $f(z)$ be in the class K_{α} with $\alpha \geqq -1$. Then

$$\operatorname{Re} \quad \{(D^{\alpha}f(z))'\}^{\beta-1} \quad > \frac{1}{2\beta - 1} \qquad (z \in U), \tag{3.17}$$

where $1 < \beta \leqq 3/2$.

PROOF. Note that

$$\begin{aligned} f(z) \in K_{\alpha} &\iff D^{\alpha}f(z) \in K \\ &\iff z(D^{\alpha}f(z))' \in S^{*} \\ &\iff D^{\alpha}(zf'(z)) \in S^{*} \\ &\iff zf'(z) \in S_{\alpha}^{*}, \end{aligned}$$

which implies

$$\frac{D^{\alpha}(zf'(z))}{z} = (D^{\alpha}f(z))'.$$

Therefore, we have the corollary with the aid of Theorem 7.

4. INTEGRAL OPERATOR $J_c(f)$. We define the integral operator $J_c(f)$ by

$$J_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c > -1) \tag{4.1}$$

for $f(z) \in A$. The operator $J_c(f)$ when $c \in N = \{1, 2, 3, \ldots\}$ was studied by Bernardi [6]. In particular, the operator $J_1(f)$ was studied by Libera [7] and Livingston [8].

THEOREM 8. Let $f(z)$ be in the class S_{α}^{*} with $\alpha \geqq 0$. Then $J_{\alpha}(f)$ is also in the class S_{α}^{*}.

PROOF. Define the function $w(z)$ by

$$\frac{z(D^{\alpha}J_{\alpha}(f))'}{D^{\alpha}J_{\alpha}(f)} = \frac{1 + w(z)}{1 - w(z)} \qquad (w(z) \neq 1). \tag{4.2}$$

Then, by taking the differentiation of both sides logarithmically, we have

$$\frac{z^2(D^{\alpha}J_{\alpha}(f))'' + z(D^{\alpha}J_{\alpha}(f))'}{z(D^{\alpha}J_{\alpha}(f))'} - \frac{z(D^{\alpha}J_{\alpha}(f))'}{D^{\alpha}J_{\alpha}(f)} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))} \tag{4.3}$$

Since

$$z(z(D^{\alpha}f(z))')' = z^2(D^{\alpha}f(z))'' + z(D^{\alpha}f(z))', \tag{4.4}$$

we can see that

$$z^2(D^{\alpha}f(z))'' = (\alpha + 1)z(D^{\alpha+1}f(z))' - (\alpha + 1)z(D^{\alpha}f(z))' \tag{4.5}$$

by Lemma 3. Furthermore, it follows from the definition of $J_\alpha(f)$ that

$$D^\alpha f(z) = D^{\alpha+1}J_\alpha(f). \tag{4.6}$$

By using (4.5) and (4.6), we have

$$\begin{aligned} z^2(D^\alpha J_\alpha(f))'' &= (\alpha + 1)z(D^{\alpha+1}J_\alpha(f))' - (\alpha + 1)z(D^\alpha J_\alpha(f))' \\ &= (\alpha + 1)z(D^\alpha f(z))' - (\alpha + 1)z(D^\alpha J_\alpha(f))'. \end{aligned} \tag{4.7}$$

With the aid of Lemma 3, we have

$$z(D^\alpha J_\alpha(f))' = (\alpha + 1)D^\alpha f(z) - \alpha D^\alpha J_\alpha(f). \tag{4.8}$$

Consequently, from (4.3), we obtain

$$\frac{(\alpha + 1)z(D^\alpha f(z))'}{z(D^\alpha J_\alpha(f))'} - \frac{(\alpha + 1)D^\alpha f(z)}{D^\alpha J_\alpha(f)} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))}, \tag{4.9}$$

or

$$\frac{(\alpha + 1)D^\alpha f(z)}{z(D^\alpha J_\alpha(f))'}\left\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} - \frac{z(D^\alpha J_\alpha(f))'}{D^\alpha J_\alpha(f)}\right\} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))}. \tag{4.10}$$

Since (4.8) implies

$$\frac{(\alpha + 1)D^\alpha f(z)}{z(D^\alpha J_\alpha(f))'} = 1 + \frac{\alpha D^\alpha J_\alpha(f)}{z(D^\alpha J_\alpha(f))'} = \frac{(1 + \alpha) + (1 - \alpha)w(z)}{1 + w(z)}, \tag{4.11}$$

it follows from (4.10) that

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{(1 - w(z))\{(1 + \alpha) + (1 - \alpha)w(z)\}}. \tag{4.12}$$

By assuming

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq \pm 1)$$

for $z_0 \in \mathcal{U}$ and using the same technique as in the proof of Theorem 3, we can show that

$$\mathrm{Re}\left\{\frac{z(D^\alpha J_\alpha(f))'}{D^\alpha J_\alpha(f)}\right\} = \mathrm{Re}\left\{\frac{1 + w(z)}{1 - w(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{4.13}$$

Thus we conclude that $J_\alpha(f)$ is in the class $\mathcal{S}_\alpha^*$.

COROLLARY 3. Let f(z) be in the class $\mathcal{S}_\alpha^*$ with $\alpha \geq 0$. Then, for $p \in \mathbb{N}$,

$$(z\,{}_{p+1}F_p(\alpha+1,\dots, \alpha+1,1;\ \alpha+2,\dots, \alpha+2;z))*f(z) \in \mathcal{S}_\alpha^*,$$

where ${}_{p+1}F_p(\alpha_1,\dots, \alpha_{p+1};\ \beta_1,\dots, \beta_p;z)$ denotes the generalized hypergeometric function.

PROOF. It is easy to see that

$$J_\alpha(f) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1}\left(\sum_{n=0}^{\infty} a_{n+1}t^{n+1}\right)dt$$

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha+1}{n+\alpha+1}\right) a_{n+1}z^{n+1}$$

$$= (z_2F_1(\alpha+1,1;\alpha+2;z))*f(z) \tag{4.14}$$

for $f(z) \in A$. Therefore, by Theorem 8, we have

$$(z_2F_1(\alpha+1,1;\alpha+2;z))*f(z) \in S_\alpha^*.$$

Repeating the same manner, we conclude that

$$\begin{aligned} f(z) \in S_\alpha^* &\Longrightarrow (z_2F_1(\alpha+1,1;\alpha+2;z))*f(z) \in S_\alpha^* \\ &\Longrightarrow (z_3F_2(\alpha+1,\alpha+1,1;\alpha+2,\alpha+2;z))*f(z) \in S_\alpha^* \\ &\Longrightarrow (z_{p+1}F_p(\alpha+1,\cdots,\alpha+1,1;\alpha+2,\cdots,\alpha+2;z))*f(z) \in S_\alpha^*. \end{aligned}$$

Finally, we prove

THEOREM 9. Let $f(z)$ be in the class K_α with $\alpha \geq 0$, Then $J_\alpha(f)$ is also in the class K_α.

PROOF. In view of Theorem 5, we can see that

$$\begin{aligned} f(z) \in K_\alpha &\Longleftrightarrow z(D^\alpha f(z))' \in S^* \\ &\Longleftrightarrow D^\alpha(zf'(z)) \in S^* \\ &\Longleftrightarrow zf'(z) \in S_\alpha^* \\ &\Longrightarrow J_\alpha(zf'(z)) \in S_\alpha^* \\ &\Longleftrightarrow D^\alpha(J_\alpha(zf')) \in S^* \\ &\Longleftrightarrow z(D^\alpha J_\alpha(f))' \in S^* \\ &\Longleftrightarrow D^\alpha J_\alpha(f) \in K \\ &\Longleftrightarrow J_\alpha(f) \in K_\alpha, \end{aligned}$$

which completes the proof of Theorem 9.

COROLLARY 4. Let $f(z)$ be in the class K_α with $\alpha \geq 0$. Then, for $p \in N$, $(z_{p+1}F_p(\alpha+1,\ldots,\alpha+1,1;\alpha+2,\ldots,\alpha+2;z))*f(z) \in K_\alpha$, where ${}_{p+1}F_p(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;z)$ denotes the generalized hypergeometric function.

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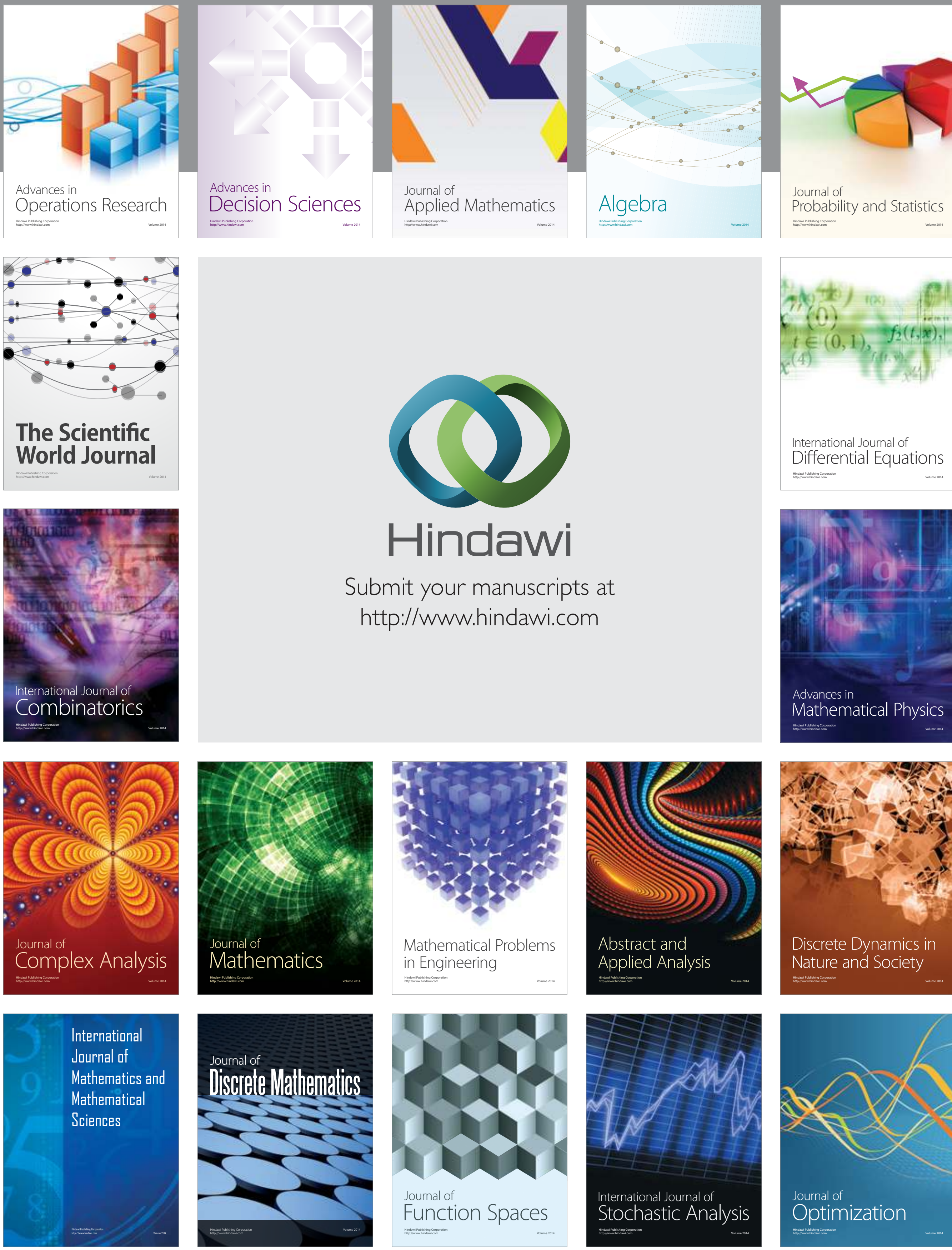
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