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AN APPLICATION OF YAU'S MAXIMUM PRINCIPLE TO CONFORMALLY FLAT SPACES

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ABSTRACT. Results of M. Tani on compact conformally flat manifolds and of M. Okumura on compact hypersurfaces of Euclidean space are extended to complete spaces by an application of S.-T. Yau's "maximum principle".

1. Introduction. M. Tani [3] proved that a compact and orientable Riemannian manifold admitting a conformally flat metric of positive Ricci curvature and constant scalar curvature is a space form, that is, it is a constant curvature space. It is our purpose to extend this result to complete Riemannian manifolds with Ricci curvature bounded from below. This will be accomplished by employing a "maximum principle" due to S.-T. Yau. In fact, the following statement is obtained.

THEOREM 1. *Let M be a d-dimensional, $d > 3$, complete, conformally flat Riemannian manifold whose Ricci curvature is bounded from below. If its scalar curvature r is a positive constant and* $\operatorname{tr} Q^2 < r^2/(d-1)$, *then M is a space form.*

2. Definitions and notation. Let (M, g) be a Riemannian manifold with metric g. The curvature transformation $R(X, Y)$, $X, Y \in M_m$, where M_m is the tangent space at $m \in M$, and g are related by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

where ∇ is the Riemannian connection. In terms of a basis $X_1, \ldots, X_d$ of M_m, we set

$$R_{ijkh} = g(R(X_i, X_j)X_k, X_h), \qquad R_{ij} = \operatorname{tr}(X_k \to R(X_i, X_k)X_j),$$
$$t_{i_1 \ldots i_p} = t(X_{i_1}, \ldots, X_{i_p}), \qquad \nabla_i t_{i_1 \ldots i_p} = (\nabla_{X_i} t)(X_{i_1}, \ldots, X_{i_p}).$$

We denote the scalar curvature by r, that is, $r = \operatorname{tr} Q$, where $Q = (R^i_j)$ and $R^i_j = g^{ik}R_{jk}$. The manifold (M, g) is *conformally flat* if g is conformally related to a locally flat metric.

3. The Laplacian of $\operatorname{tr} Q^2$. The following formula may be found in [1]:

$$\tfrac{1}{2} \Delta \operatorname{tr} Q^2 = g^{ab} \nabla_a R^{ij} \nabla_b R_{ij} + R^{ij} g^{ab} \nabla_a (\nabla_b R_{ij} - \nabla_i R_{bj}) + \tfrac{1}{2} R^{ij} \nabla_j \nabla_i r + K, \tag{3.1}$$

where $\operatorname{tr} Q^2$ is the square length of the Ricci tensor, and

$$K = R^{ik}(R^j_i R_{jk} + R^{hj} R_{ijhk}).$$

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If r is a constant, the third term on the r. h. s. of (3.1) vanishes. If, moreover, M is conformally flat and $d \geq 3$, the second term on the right also vanishes (see [1]) and (3.1) reduces to

$$\tfrac{1}{2}\, \Delta \operatorname{tr} Q^2 = K + g(\nabla Q, \nabla Q).$$

4. Proof of Theorem 1. Since M is conformally flat it can be shown that

$$(d-1)(d-2)K = d(d-1)\operatorname{tr} Q^3 - r(2d-1)\operatorname{tr} Q^2 + r^3.$$

Put $S = Q - (r/d)I$, I = identity. Then, from $\operatorname{tr} S^2 \geq 0$, we see that $\operatorname{tr} Q^2 \geq r^2/d$ with equality holding if and only if, M is an Einstein space. Since r is a constant, the Laplacian Δf^2 of the function $f^2 = \operatorname{tr} S^2, f \geq 0$, satisfies $\Delta f^2 = \Delta \operatorname{tr} Q^2$. Thus,

$$\tfrac{1}{2}\, \Delta f^2 = K + g(\nabla Q, \nabla Q). \tag{4.1}$$

Moreover,

$$(d-1)(d-2)K = d(d-1)\left(\operatorname{tr} S^3 + \frac{3r}{d} f^2 + \frac{r^3}{d^2}\right) - r(2d-1)\left(f^2 + \frac{r^2}{d}\right) + r^3. \tag{4.2}$$

The following lemma may be found in [2].

LEMMA 1. *Let a_i, $i = 1, \ldots, d$, be real numbers with*

$$\sum_{i=1}^{d} a_i = 0, \quad \sum_{i=1}^{d} a_i^2 = k^2, \qquad k = \text{const} \geq 0.$$

Then,

$$-\frac{d-2}{\sqrt{d(d-1)}}\, k^3 \leq \sum_{i=1}^{d} a_i^3 \leq \frac{d-2}{\sqrt{d(d-1)}}\, k^3.$$

Applying Lemma 1 to the eigenvalues of S, (4.2) yields the inequality

$$(d-1)K \geq f^2\left(r - \sqrt{d(d-1)}\, f\right).$$

We conclude from (4.1) that

$$\frac{d-1}{2}\, \Delta f^2 \geq f^2\left(r - \sqrt{d(d-1)}\, f\right). \tag{4.3}$$

LEMMA 2 (S.-T. YAU [4]). *Let M be a complete Riemannian manifold with Ricci curvature bounded below. Let u be a C^2 function with* $\sup u < \infty$. *Then, there exists a sequence $\{p_\nu\}$ in M such that*

$$\lim_{\nu\to\infty} \|du(p_\nu)\| = 0, \qquad \lim_{\nu\to\infty} (\Delta u)(p_\nu) \leq 0, \qquad \lim_{\nu\to\infty} u(p_\nu) = \sup u.$$

Applying Lemma 2, the inequality (4.3) gives rise to the inequality

$$\lim_{\nu\to\infty} f^2(p_\nu)\left\{r - \sqrt{d(d-1)}\, f(p_\nu)\right\} \leq 0.$$

Hence, either $f^2 \equiv 0$ or $\sup f \geq r/\sqrt{d(d-1)}$, the latter implying $\sup \operatorname{tr} Q^2 \geq r^2/(d-1)$. The former says that $\operatorname{tr} Q^2 = r^2/d$, so g is an Einstein metric. However, since g is conformally flat, it is a constant curvature metric.

The condition tr $Q^2 < r^2/(d-1)$ is essential. For, if $M = M_1 \times N$, where M_1 has constant curvature and N is 1-dimensional, then M is conformally flat, its Ricci curvature is bounded below, r is constant and tr $Q^2 = r^2/(d-1)$.

In a similar manner, we obtain the following extension of a theorem of Okumura [2].

THEOREM 2. *Let M be a d-dimensional complete connected hypersurface of R^{d+1} with Ricci curvature bounded from below. If its mean curvature* tr H *is constant and* tr $H^2 < (\mathrm{tr}\, H)^2/(d-1)$, *then M is a totally umbilical hypersurface.*

The inequality in Theorem 2 is the best possible as one sees by considering $M = S^{d-1} \times R$.

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