# An Appraisal of the Efficiency of Alternative Deterministic Equivalents to the Stochastic Programming Model. 

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BUFFA, Frank Paul, 1942-
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The Louisiana State University and Agricultural and Mechanical College, Ph.D., 1971 Statistics

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An Appraisal of the Efficiency of Alternative Deterministic Equivalents to the Stochastic Programming Model

## A Dissertation

# Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy 

in
The Department of Quantitative Methods

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by
Frank Paul Buffa
B.S., Loyola University, 1964
M.B.A., Louisiana State University in New Orleans, 1967 January, 1971
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## PLEASE NOTE:

Some pages have indistinct print. Filmed as received. UNIVERSITY MICROFILMS.

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#### Abstract

The linear programming model with stochastic elements in the vector of cost coefficients or the vector of resource requirements has been approached in many ways. The foremost attempts at a solution involve the transformation of the model to a deterministic equivalent. There are a number of deterministic equivalents which have been developed for this purpose.

The objective of this study is to develop an experimental model which can be used to evaluate proposed deterministic equivalents to the stochastic programming model. This experimentai model has been designed to determine the responses of a deterministic equivalent to induced changes in the properties and the positions of the stochastic parameters which appear in the Iinear programming model:

Three different linear deterministic equivalents were evaluated in this study. These were the one-stage expected value approach, the two-stage slack approach to programming under uncertainty, and the active approach to linear programming under risk.


The experimental model was used to evaluate, in turn, two different variations of an empirical stochastic
linear programming problem in terms of each deterministic equivalent. Two variations of the empirical problem were analyzed so that conclusions could be stated for either a tightly constrained or a slightly constrained problem. A Monte Carlo simulation of each of these empirical problems was also performed. The results of these simulations were used as standards with which to evaluate the results of each deterministic equivaient.

The experimental procedure was divided into three phases. In the first phase the stochastic parameters were limited to the vector of resource requirements, in the second phase the stochastic parameters appeared only in the vector of cost coefficients, while in the third phase the stochastic parameters appeared in both vectors simultaneously. In all cases the stochastic parameters were assumed to be normally and independently distributed with known means and variances, while the non stochastic parameters in the problem were assumed to be constant and equal to their expected values.

In all three phases of the experiment the deterministic equivalents were analyzed for each experimental problem as the positions of the stochastic parametens changed and as the variances of the stochastic parameters increased. For all initial conditions and for each of the deterministic equivalents, the null hypothesis of no difference between the results of the simulation
approach and the results of the deterministic equivalent was tested at the levels of significance of $\alpha=.01$ and $\alpha=.05$.

In the first phase of the experiment an analysis of the results indicated that the two-stage slack approach yielded better results than either of the other deterministic equivalents evaluated. The results of the two-stage slack approach were generally feasible on the average, were not significantly different from the results of the simulation approach at either level of significance, and were consistent with respect to the two experimental problems considered. The expected value approach was found to yield the best results in phase two of the experiment. This approach yielded results which were generally feasible on the average, were not affected by the increases in the variances of the stochastic parameters, and were very reliable regardless of the type of problem analyzed. In the third phase the two-stage approach again yielded the best results. The results were generally feasible on the average and statistically the same as the results of the simulation approach at both levels of significance.

## CHAPTER I

## INTRODUCTION


#### Abstract

Objective of the Study The linear programming model with stochastic elements in the vector of cost coefficients or the vector of resource requirements has been approached in many ways. The foremost attempts at a solution involve the transm formation of the model to a deterministic equivalent. There are a number of deterministic equivalents which have been developed for this purpose.

When the linear programming model containing stochastic parameters is transformed to a deterministic equivalent, then it is desirable to question the efficiency of the transformation used. In particular, three deterministic equivalents are evaluated. These are the onestage expected value approach, the active approach to programming under risk, and the two-stage slack approach.

In order to determine the effectiveness of each deterministic equivalent, a Monte Cario simulation of the stochastic model is used in this study as a standard for comparison. Fox each set of specifications of the model, the expected value of the optimal solutions derived


from each deterministic equivalent is contrasted with the expected value of the optimal solutions derived from a simulation of the model. These expected values should not be considered optimal solutions in terms of the variables of the model. They are each estimates of the respective expected values of the optimal objective function values which are determined by randomly selecting from the distributions of the stochastic parameters specific values, which are then used to solve for a conditional optimal solution of each of the respective deterministic equivalents and of the simulation model.

The objective of this study is to determine the response of each of the types of deterministic equivalents indicated above to induced changes in the properties and the positions of the stochastic parameters which appear in the linear programming model. Each deterministic equivalent is to be evaluated under these changing conditions by utilizing the simulation solution as a standard. Specifically this study is concerned with the determination of the specific conditions under which any particular deterministic equivalent performs better than the others, and how much efficiency is lost in the application of each of these deterministic equivalents.

Although the primary objective of this study is the evaluation of those deterministic equivalents to the stochastic programming model which were mentioned
above, it should also be pointed out that the development of an experimental model which can be used for this purpose is also an important result of this study. The reader should realize that the experimental model which was developed for this study is flexible in that it can be applied to the analysis of any proposed deterministic equivalent to the stochastic programming model.

## Justification of the Study

The parameters of the linear programming model must be constant in order to use the simplex algorithm to correctly solve the model. The investigator would rarely meet a real world situation fulfilling this requirement for fixed parameters. Due to this fact the use of the simplex algorithm to solve most real world models is not theoretically justified. To overcome the problem created by the presence of the stochastic parameters, a deterministic equivalent to the model can be formulated and then solved. ${ }^{1}$

It should be understood that the use of a deterministic equivalent to solve a stochastic programming model is analogous to the use of the expected value of a
${ }^{I} \operatorname{In}$ this study only Iinear deterministic equivalents are considered. The reader should realize, however, that deterministic equivalents to the stochastic linear programming model can be non-linear. Some of these nonlinear equivalents are also neferred to in the next chapter.
variable to test an hypothesis or make a decision involving that variable. Use of the expected value of a variable in no way implies that the expected value completely describes the properties of the variable or, for that matter, the variable itself. The expected value of a variable is merely an efficient means of taking into account the influence of that variable in a deterministic decision-making procedure.

Similarly a deterministic equivalent can never be exactly the same as the stochastic model that it replaces. The deterministic equivalent represents an attempt to include in the deterministic solution procedure of a stochastic programming model the effects resulting from the presence of the variable parameters in that model.

The optimal solution of the model derived from a deterministic equivalent can only be considered to be an approximation to the true optimal solution of the stochastic model. The closeness of this approximation depends upon the detemministic equivalent which is used, the properties of the stochastic parameters, and the positions of these stochastic parameters in the model. The utilization of any particular deterministic equivalent should be investigated under changing conditions with respect to the properties and the positions of the stochastic parameters in the programming model.

If a relationship can be found between the properties and the positions of the stochastic parameters on the one hand and the closeness of the approximate solution to the true optimal solution on the other, then the investigator can use a particular deterministic equivalent with increased confidence. In effect, this relationship can be used to select the particular deterministic equivalent which minimizes the error incurred in approximating the true optimal solution to a stochastic programming model under a given set of initial conditions.

Scope and Limitations of the Study
The particular linear programming model utilized in this study to achieve the stated objective is an agricultural production model. ${ }^{2}$ The stochastic parameters, when they appear in the model, are assumed to be independently distributed with normal distributions with known means and variances. The location of the stochastic parameters are restricted to the vector of resource requirements and the vector of profitability coefficients.
${ }^{2}$ This model was formulated from data determined from an empirical study presented in M. M. Babbar, "Distributions of Solutions of a Set of Linear Equations (With an Application to Linear Programming)," Journal of the American Statistical Association, L (September, 1955), 854-869. This same problem was used to generate results for a study of linear programming under risk found in J. K. Sengupta and J. H. Portillo-Campbelid, "A Fnactile Approach to Linear Progrianming Under'Risk," Management Science, XVI (January, 197.0), 298-308.

It should be recognized that there are numerous ways in which the stochastic parameters can appear in the two vectors mentioned above. Specifically there are three general cases that can be identified. Stochastic parameters may appear in only the vector of resource requirements, only the vector of profitability coefficients, or in both vectors.

There are numerous ways in which the stochastic elements can appear in either of the vectors. For example, in the first general case all the parameters in the vector of resource requirements may be stochastic, or only some defined subset of these parameters may be stochastic. The same can be said for the vector of profitability coefficients. In the third general case, the matter is only compounded since various combinations of stochastic elements in both vectors must be considered.

In the case where there are $n$ variables and $m$ constraints in a model, then there are n elements in the vector of profitability coefficients and m elements in the vector of resource requirements. If only the vector of resource requirements is assumed to contain stochastic elements, then there are $\mathrm{m}_{\mathrm{i}}$ combinations in which i elements may be stochastic. The total number of ways in which stochastic elements may be combined in the vector of resource requirements is then $\sum_{i=1}^{m} m_{i}{ }^{m}$.

Similarly there are $\cdot \sum_{i=1}^{n} n^{C} C_{i}$ total ways in which stochastic elements can be combined in the vector of profitability coefficients and $\left(\sum_{i=1}^{m} m^{c}\right)\left(\sum_{i=1}^{n} n_{i}\right)$ total ways in which stochastic elements can be combined in both vectors simultaneously.

If the problem under investigation contains a large number of variables, a large number of constraints, or a large number of both variables and constraints; then an investigation of these three general cases would be quite lengthy. Because of this, the agricultural production model utilized in this study is restricted to a small number of both variables and constraints.

The justification for this restriction is reinforced by the fact that each initial formulation of the model can also be expanded. For example, once it has been determined which elements of the model are stochastic elements; then the properties of these stochastic elements can be changed. In this study only the effects induced by a change in the variances of the stochastic parameters are to be evaluated.

## Outline of the Study

The formal presentation of the study is divided into five parts. Chapter one presents the objective; the justification, and the scope and limitation of the
study. The theories of stochastic linear programming models are presented in chapter two. Special emphasis is placed upon the development of deterministic equivalents to these models. Chapter three discusses the requirements of simulation studies, with special emphasis given to the generation and testing of pseudorandom numbers. The particular deterministic equivalents which are tested in this study are highlighted in chapter four. This chapter includes a detailed statement of the experimental procedure used to achieve the stated objective. Chapter five is a summary of the results obtained from the simulation experiments and a statement of the conclusions drawn from this study.

## CHAPTER II

## STOCHASTIC LINEAR PROGRAMMING MODELS

## Introduction

The initial development of the stochastic linear programming model is attributed to George Dantzig. ${ }^{1}$ Since the appearance of this initial formulation, there have been many contributions made to the development of a theory of stochastic linear programming. It is necessary to assimulate this existing knowledge into a suitable format which can serve as a means of relating the results of the present study to the existing reservoir of understanding.

The objective of this chapter is to present the general theories of stochastic linear programming models. This objective can be accomplished through a dual classification system. The various stochastic linear programming models which have appeared in the literature and the various solution techniques which have been developed can both be classified. In attempting to classify the solution techniques special emphasis will be placed
${ }^{\text {George }}$ B. Dantzig, "Linear Programming Under Uncertainty," Management Science, I (April-July, 1955), 197-206.
upon the development of the different deterministic equivalents to the various stochastic linear programming models.

## General form of the stochastic

linear programming model
The generalized primal linear programming model (LP)
can be formulated as follows:
Maximize: $\quad Z=C^{\prime} X$,
subject to: $A X \leq B$, and

$$
\begin{equation*}
x \geq 0 \tag{1}
\end{equation*}
$$

In this model $C$ is a ( $n \times 1$ ) vector of profitability coefficients, A is a ( $\mathrm{m} \times \mathrm{n}$ ) matrix of technological coefficients, and $B$ is a ( $\mathrm{m} \times 1$ ) vector of resource restrictions. The dual associated with this model can be stated as:

$$
\begin{align*}
\text { Minimize: } \quad Z & =B^{\prime} W, \\
\text { subject to: } A^{\prime} W & \geqq C, \quad \text {, and } \\
W & \geq 0, \tag{2}
\end{align*}
$$

The parameters $A, B$, and $C$ in the models stated above are deterministic. In the primal model the set of inequalities form a convex polyhedral set over which the objective function is to be maximized. There are many solution procedures which can be used to find the maximum value of the objective function of the model. The simplex algorithm is one such procedure which is designed to move from one basic feasiable solution to the next while simultaneously increasing the value of
the objective function along its searching path. Once. the basic feasible solution, which maximizes the value of the objective function is found, the algorithm indicates that this maximum has been found. If the basis which maximizes the objective function contains $m$ non-zero elements in the solution vector $X$ then this is a nondegenerate basic feasible solution. If there are less than $m$ non-zero elements in $X$, then the solution is degenerate. A similar presentation can be made for the dual problem. ${ }^{2}$

If some of the parameters of the LP model are considered to be stochastic, then the model fits the general description of a stochastic linear programming model (SLP). The SLP model takes account of the fact that there is a probability associated with each specific set (A, B, C) which forms the structure of the model. For example

$$
\begin{equation*}
P[A, B, C]=(A, B, C)_{k}=P_{k} \tag{3}
\end{equation*}
$$

where $P$ stands for probability. It can be seen that:

$$
\begin{align*}
P_{k} & =\left[P\left(A=A_{h}\right) \cap P\left(B=B_{i}\right) \cap P\left(C=C_{j}\right)\right], \text { and } \\
\sum_{k} P_{k} & =1 \quad, \text { for all possible } k, \tag{4}
\end{align*}
$$

where $A_{h}, B_{i}$, and $C_{j}$ indicate specific values that these parameters can take on.
${ }^{2}$ George Hadley, Linear Programming (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1963), pp. 221-272.

The general formulation of the SLP model provides an appropriate starting point from which the different specific models can be deduced. These specific models can be classified under three broad headings: ${ }^{3}$. chanceconstrained programming, two-stage programming under uncertainty, and linear programming under risk.

The chance-constrained programming model ${ }^{4}$ replaces the set of constraints of the LP model with a new set of conditions which can be stated as

$$
\begin{align*}
\mathrm{P}[\mathrm{AX} \leq \mathrm{B}] & \geq \alpha \quad, \text { and } \\
\mathrm{X} & \geq 0 \tag{5}
\end{align*}
$$

where $P$ stands for probability and $\alpha$ is a ( $m \times 1$ ) column vector such that any particular $\alpha_{i}$ satisfies the condition $0 \leqq \alpha_{i} \leq 1$. This vector contains a prescribed set of constants that are probability measures of the extent to which constraint violations are allowed.

The two-stage programming under uncertainty model ${ }^{5}$ can be briefly stated as
${ }^{3}$ J. K. Sengupta, G. Tintner, and C. Millham, "On Some Theorems of Stochastic Linear Programming with Applications," Management Science, $X$ (October, 1963), 144-145.
${ }^{4}$ A. Charnes and W. W. Cooper, "Chance-Constrained Programming," Management Science, VI (October, 1959), 73-79.
${ }^{5}$ G. B. Dantzig and A. Madansky, "On the Solution of Two-Stage Linear Programs Under Uncertainty," in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, ed. by Jerzy Neyman (Berkeley: University of California Press, 1961); I, 165-176; and A. Madansky, "Dual Variables in Two-Stage Linear Programming Under Uncertainty," Journal of Mathematical Analysis and Applications, VI (February, 1963), 98-108.

$$
\operatorname{minimize}: \quad Z=C^{\prime} X+E_{\min _{y}}\left[F^{\prime} Y\right]
$$

under the conditions

$$
\begin{align*}
& A X+D Y \geq B \\
& X \geq 0, Y \geq 0 \tag{6}
\end{align*}
$$

In this formulation the vector $B$ contains random elements, $E$ is the expected value operator, $F$ and $Y$ are ( $r x$ ) column vectors, and $D$ is a ( $m \mathrm{x} r$ ) matrix. This formulation introduces an additional variable $Y$ into the model. The model in [6] must be minimized with respect to both $X$ and $Y$.

The linear programming under risk classification encompasses all the approaches which are concerned with the statistical distribution of the objective function. 6 Models of this type consider the parameters of the LP model to be random variables with known probability distributions. Given this premise these models attempt either the optimization of the expected value of the objective function or the derivation of the statistical distribution of the objective function values.

## Chance-Constrained Programming Models

Definition
In this section the chance-constrained interpretation
${ }^{6}$ G. Tintner, "iStochastic Linear Programming with Applications to Agricultural Economics," in Proceedings of the Second Symposium on Linear Programming, ed. by H. A. Antosiewicz (Washington: National Bureau of Standards, 1955), I; and J. K. Sengupta, G. Tintner, and B. Morrison,
and its corresponding deterministic equivalents are defined. The different deterministic equivalents, nelated to chanceconstrained programming, result from the different objectives which can be first formalized and then optimized by using these models.
A. Charnes and W. W. Cooper initially interpreted the SLP model as a chance-constrained model. They define the general class of SLP models in this way:

The problem of stochastic (or better, chanceconstrained) programming is here defined as follows: Select certain random variables as function of random variables with known distributions in such a manner as (a) to maximize a functional of both classes of random variables subject to (b) constraints on these variables which must be maintained at prescribed levels of probability. More loosely, the problem is to determine optimal stochastic decision rules under these circumstances. 7

This definition equates all SLP models to chance-constrained programming models. The "optimal stochastic decision rule" mentioned in the definition refers to the transformation of the model to a specific deteministic equivalent which depends upon the form of the decision rule employed.

Chames and Cooper are not alone in their formulation of deteministic equivalents to SLP models based upon a chance-constrained interpretation of the model. 8

[^0]The justification for this interpretation is that whenever stochastic parameters appear in a constraint of a programming model, then the question of that constraint being satisfied, once the optimal solution is found, can only be stated in probabilistic terms. H. Theil, for example, reports that "It is hardly reasonable to require that such an inequality [constraint] holds with certainty; indeed, it is much more reasonable to require that it holds with a sufficiently large probability." ${ }^{9}$

Charnes and Cooper emphasize that "...optimization under risk immediately raises very important questions concerning a choice of rational objectives. ${ }^{10}$ In accordance with this feeling these authors examined three different types of objectives. These objectives include (1) an expected value optimization, (2) a minimum variance objective, and (3) a maximum probability model.
with Joint Constraints," Operations Research, XXIII (Novem-ber-December, 1965), $930-945$; J. K. Sengupta, "Safety First Rules Under Chance-Constrained Linear Programming," Operations Research, XVII, (January-February, 1969), 112132; Gifford H. Symonds, "Chance-Constrained Equivalents of Some Stochastic Programming Problems," Operations Research, XVI (November-December, 1968), 1152-1159; H. Theil, "Some Reflections on Static Programing Under Uncertainty," Weltwirtschaftliches Archiv, LXXXVII No. 1 (1961), 124-138; C. Van de Panne and W. Popp, "MinimumCost Cattle Feed Under Probabilistic Protein Constraints," Managemen't Science, IX (April, 1963), 405-430.
${ }^{9}$ Theil, "Some Reflections on Static Programming," 124-125.
${ }^{10}$ A. Charnes and W. W. Cooper, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," Operations Research, XI (January-February, 1963), 22.

These objectives are referred to respectively as (1) the E-model, (2) the V-model, and (3) the P-model. ${ }^{I l}$

Regardless of the objective sought and the model which results from the formalization of that objective, all chance-constrained models can be transformed to some type of deterministic equivalent. The specific structure of this deterministic equivalent depends upon the choice of an objective and a suitable transformation.

The transformation is referred to as a decision rule. The decision rule is stated in terms of the parameters of the model, that is, the decision rule is some function of $A, B$, and $C$. The unknowns of the chance-constrained model are transformed according to this decision rule which can be stated generally as

$$
\begin{equation*}
X=\phi(A, B, C) . \tag{7}
\end{equation*}
$$

The function, $\phi$, should be chosen and applied in a manner that "...guarantees that the $X$ values, as generated, will satisfy the chance constraints and optimize [the objective function]...."12
$E$-model ${ }^{13}$
The expected value model can be stated as
${ }^{11}$ Ibid., 23.
${ }^{12}$ Ibid., 19-20. Fon a discussion of different classes of the decision rule see this source and also A. Charnes and M. J. L. Kirby, "Some Special P-Models on Chance-Constrained Programming," Management Science, XIV (November, 1967), 183-195.
${ }^{13}$ Charnes and Cooper, "Deterministic Equivalents,".
25-30.

$$
\begin{array}{ll}
\text { Maximize: } \quad E\left[C^{\prime} X\right] \\
\text { subject to: } P[A X \leqq B] & \geq \alpha \\
& X=D B \tag{8}
\end{array}
$$

Where the symbols have the same meanings as before. In the model $E$ is the expected value operator and $P$ is the symbol for probability. Charnes and Cooper consider $A$ to be a matrix of known constants and $B$ and $C$ to be uncorrelated vectors each containing at least one random element. 14

The last expression in the model in [8] indicates that a linear transformation of the variables in $X$ is to be performed before a solution to the model is attempted. This transformation serves the purpose of converting a chance-constrained model, formulated in terms of the variables in $X$, into a deterministic equivalent in terms of the variables in $D$. The matrix $D$ contains ( $n \times m$ ) unknowns which, when determined, are then used to specify the value of the variables in $X$.

Substituting the decision rule into the objective function of the model yields

$$
\begin{equation*}
E\left[C^{\prime} X\right]=E\left[C^{1} D B\right]=E\left[C^{\prime}\right] D E[B] \tag{9}
\end{equation*}
$$

If the $E\left[C^{\prime}\right]$ and the $E[B]$ are defined respectively as $\mu_{C}^{\prime}$ and $\mu_{B}$, then the objective function can be written as

Maximize: $\quad \mu_{\mathrm{C}}^{\prime} D \mu_{\mathrm{B}} \quad$ -
A consideration of the function in [10] reveals that this

$$
14 \text { Ibid., } 26 .
$$

function contains deterministic parameters with the variables being contained in the matrix $D$.

The conversion of the probabilistic constraints to corresponding deterministic constraints begins with substituting the function $\mathrm{X}=\mathrm{DB}$ into the set of constraints. Making this substitution the constraint set can be written

$$
\begin{equation*}
P[A D B \leqq B] \geq \alpha \tag{1.1}
\end{equation*}
$$

This set of constraints contains stochastic elements in the $B$ vector. If $a_{i}$ and $b_{i}$ are the $i$ th row of $A$ and $B$ respectively, then the ith constraint can be written

$$
\begin{equation*}
P\left[a_{i} D B \leqq b_{i}\right] \geq \alpha_{i} \tag{12}
\end{equation*}
$$

If $\mu_{B}$ is defined as the mean vector of the parameters $B$, then the mean of the ith element, $b_{i}$, can be written $\mu_{b_{i}}$. In addition $\hat{B}$ and $\hat{b}_{i}$ can be defined respectively as

$$
\hat{B}=B-\mu_{B}
$$

$$
\begin{equation*}
\text { and } \hat{b}_{i}=b_{i}-\mu_{b_{i}} \tag{13}
\end{equation*}
$$

The variate ( $a_{i} D B-b_{i}$ ), which is obtained from the left-hand side of the expression in [12], is assumed to be symmetrical and to be completely specified by its first two moments. Specifically this deviation is assumed to be normally distributed. ${ }^{15}$ Given the assumption of symmetry the left-hand side of [12] can be written as

$$
\begin{equation*}
P\left[a_{i} D B-b_{i} \leqq 0\right]=P\left[b_{i}-a_{i} D B \geqq 0\right] \tag{14}
\end{equation*}
$$

15 Ibid., 26-27.

Solving the expressions in [13] for $B$ and $b_{i}$ and substituting these results into [14] yields the following results

$$
\begin{align*}
P\left[b_{i}-a_{i} D B \geqq 0\right]= & P\left[\hat{b}_{i}+\tilde{\mu}_{b_{i}}-a_{i} D\left(\hat{B}+\mu_{B}\right)\right. \\
& \geqq 0]  \tag{15}\\
= & P\left[\hat{b}_{i}-a_{i} D \hat{B} \geqq-\mu_{b_{i}}\right. \\
& \left.+a_{i} D \mu_{B}\right] . \tag{16}
\end{align*}
$$

If $E\left[b_{i}-a_{i} D B\right]^{2}$ is assumed to be greater than zero, ${ }^{16}$ then this expression can be divided into the two terms of the expression [16] to yield

$$
\begin{equation*}
P\left(\frac{\hat{b}_{i}-a_{i} D \hat{D B}}{\sqrt{E\left[\hat{b_{i}}-a_{i} D B\right]^{2}}} \geqq \frac{-\mu_{b_{i}}+a_{i} D \mu_{B}}{\sqrt{E\left[\hat{b}_{i}-a_{i} D \hat{B}\right]^{2}}}\right) \tag{17}
\end{equation*}
$$

Upon inspection it can be seen that the left side of this expression is a standard deviate. By replacing the left side of the above expression with $\mathrm{Z}_{\mathrm{i}}$ and then substituting the whole expression into [12] yields

$$
\begin{equation*}
P\left(z_{i} \geqq \frac{-\mu_{b_{i}}+a_{i} D \mu_{B}}{\sqrt{E\left[\hat{b}_{i}-a_{i} \hat{D B}\right]^{2}}}\right\} \geqq \alpha_{i} \tag{18}
\end{equation*}
$$

This last expression can also be presented as
${ }^{16}$ Ibid., 27. This assumption is made by the authors to simplify the derivation of the model.

$$
\begin{equation*}
F_{i}\left(\frac{-\mu_{b_{i}}+a_{i} D \mu_{B}}{\sqrt{E\left[b_{i}-a_{i} D B\right]^{2}}}\right) \geq \alpha_{i} \tag{19}
\end{equation*}
$$

where $F_{i}$ is the cumulative distribution function of $Z_{i}$. If $\alpha_{i}$ is assumed to be greater than $0.5,{ }^{17}$ then the expression on the right within the parentheses in [18] must be negative due to the properties of the standard normal deviate. This can be expressed as

$$
\begin{equation*}
\frac{-u_{b_{i}}+a_{i} D \mu_{B}}{\sqrt{E\left[\hat{b}_{i}-a_{i} \hat{D B}\right]^{2}}} \leq F_{i}^{-1}\left[\alpha_{i}\right]=-K_{\alpha_{i}} \tag{20}
\end{equation*}
$$

where $F_{i}^{-1}$ is the inverse distribution function of the standard normal deviate for the ith constraint. In [20] $K_{\alpha_{i}}$ is a positive constant whose value can be determined given the probability level assigned to the ith constraint being satisfied.

In order to develop a deterministic equivalent which is a convex programming problem, each constraint in the constraint set [20] is first rewritten and then separated into an equivalent pair. Rewriting [20] yields
${ }^{17}$ Ibid. This assumption is a realistic one when one considers practical applications relative to managerial policy problems. See for example: A. Charnes and W. W. Cooper, "Chance Constraints and Normal Deviates," Journal of the American Statistical Association, LVII (March, 1962), 134-148; and A. Charnes, W. W. Cooper, and G. H. Symonds, "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil," Management Science, IV (April, 1958), 235-263.

$$
-\mu_{b_{i}}+a_{i} D \mu_{B} \leq-K_{\alpha_{i}} \sqrt{E\left[\hat{b}_{i}-a_{i} D B\right]^{2}}
$$

or

$$
\begin{equation*}
u_{b_{i}}-a_{i} D \mu_{B} \geq K_{\alpha_{i}} \sqrt{E\left[\hat{b}_{i}-a_{i} \hat{D B}\right]^{2}} \tag{21}
\end{equation*}
$$

Since each term on the right hand side of this inequality is positive and their product is positive, then the expression on the left hand side must also be positive. Separating the set of constraints yields

$$
\begin{align*}
\mu_{b_{i}}-a_{i} D \mu_{B} & \geqq v_{i} \\
v_{i} & \geq K_{\alpha_{i}} \sqrt{E\left[\hat{b}_{i}-a_{i} D B\right]^{2}} \tag{22}
\end{align*}
$$

Upon separating the constraints a new set of variables is introduced into the model. The variable, $v_{i}$, serves the role of a slack variable for the $i$ th constraint. ${ }^{18}$ These slack variables are used to coordinate the two constraint sets in [22]. The first set of constraints in [22] is composed of the fixed parameters $\mu_{b_{i}}, a_{i}$, and $\mu_{B}$ as well as the variables $D$ and $v_{i}$. The second set of constraints contains the stochastic parameters $\hat{b}_{i}$ and $\hat{B}$ and the fixed element $K_{\alpha_{i}}$ in addition to the fixed parameter mentioned above. This second group of constraints incorporates the risk-taking elements of the original chance-constrained model into this determination equivalent; while the first group of constraints

[^1]incorporates the original structural parameters into this deterministic equivalent. ${ }^{19}$

Since each term in the second expression in [22]
is positive, the sense of the inequality is unaltered if each term in the inequality is squared. The resulting pair of constraint sets equivalent to the constraint sets is [21] is

$$
\begin{align*}
\mu_{b_{i}}-a_{i} D \mu_{B} & \geqq v_{i} \\
v_{i}{ }^{2} & \geqq K_{\alpha_{i}}^{2} E\left[\hat{b}_{i}-a_{i} D \hat{B B}\right]^{2} \tag{23}
\end{align*}
$$

Rearranging the constraints in [23] yields

$$
\begin{array}{r}
\mu_{b_{i}}-a_{i} D \mu_{B}-v_{i} \geqq 0 \\
-k_{\alpha_{i}}^{2} E\left[\hat{b}_{i}-a_{i} D \hat{D B}\right]^{2}+v_{i}^{2} \geqq 0 \tag{24}
\end{array}
$$

These constraints can be simplified by use of the following definitions

$$
\begin{align*}
& \mu_{i}[D]=\left(\mu_{b_{i}}-a_{i} D \mu_{B}\right) \quad, \text { and } \\
& \hat{\sigma}_{i}^{2}[D]=E\left[\hat{b}_{i}-a_{i} D \hat{D B}\right]^{2}, \tag{25}
\end{align*}
$$

The constraints of the deterministic equivalents can then be written ${ }^{20}$

$$
\begin{align*}
\mu_{i}[D]-v_{i} & \geqq 0 \\
-k_{\alpha_{i}}^{2} \hat{\sigma}_{i}^{2}[D]+v_{i}^{2} & \geqq 0 \tag{26}
\end{align*}
$$

$$
{ }^{19} \text { Ibj.a. }
$$

${ }^{20}$ The presentation of the constraints in this form differs from the presentation of Chames and Cooper. The

Each set of constraints above corresponds to a convex set; so that their intersection is also convex. ${ }^{21}$

The implication is that the deterministic equiva-
lent to the E-model of a chance-constrained programming model is a convex programming problem. This deterministic equivalent can be written if the objective function [10] is combined with the set of constraints above. This yields

Maximize: $\quad \mu_{C}^{\prime} D \mu_{B}$
subject to: $\mu_{i}[D]-v_{i} \geqq 0$,

$$
\begin{align*}
-K_{\alpha_{i}}^{2} \hat{\sigma}_{i}^{2}[D]+v_{i}^{2} & \geqq 0 \\
v_{i} & \geqq 0 \tag{27}
\end{align*}
$$

This deterministic equivalent is stated in terms of the variables $D$ and $v_{i}$ where the $d_{i j}$ 's are the structural variables and the $v_{i}$ 's are slack variables. The value of the slack variables can be increased so that the inequalities in [27] become equalities. In the special case when the elements of the B-vector are perfectly correlated, then the model above contains only linear constraints. ${ }^{22}$ This would allow the use of the simplex algorithm in solving this model since the objective function of the model is also linear.

[^2]V-model. ${ }^{23}$
The minimum variance model can be stated in the following way

Minimize: $\quad E\left[C^{\prime} X-C^{O} X^{\circ}\right]^{2}$
subject to: $P[A X \leqq B] \geqq \alpha$,

$$
\begin{equation*}
\mathrm{X}=\mathrm{DB} . \tag{28}
\end{equation*}
$$

As in the E-model, A is a matrix of known values and $B$ and $C$ are vectors which contain the stochastic elements. The effects, which result from a change in the objective of optimization, are incorporated only into the objective function of the model in [28]. This objective function states that the model seeks to minimize the squared deviation between the value $\mathrm{C}^{\prime} \mathrm{X}$ and some desired value $c^{\circ} x^{\circ}$, which is predetermined by the decision-maker. ${ }^{24}$

The derivation of the deterministic equivalent to this model is similar to the derivation of the deterministic equivalent for the expected value model, in view of the fact that the constraints of this model are the same as the constraints in the expected value model. By applying the decision rule $\mathrm{X}=\mathrm{DB}$ to the constraints in [28]; it can be seen that this operation should yield the same results that were determined in the expected

value model. The application of the decision rule in the objective function of [28] yields

Minimize $E\left[C^{\prime} D B-C^{\circ} X^{\circ}\right]^{2}$
which is the objective function of the deterministic equivalent to the $V$-model. If the expression in [29] is defined as V[D], then the deterministic equivalent to the $V$-model can be written

Minimize: V[D]
subject to: $\mu_{i}[D]-v_{i} \geqq 0$,
$-K_{\alpha_{i}} \hat{\sigma}_{i}^{2}[D]+v_{i}^{2} \geq 0$,
$v_{i} \geq 0 \quad$.
This deterministic equivalent is a convex programming problem since these constraints are the same as those in [27]. In addition any differences in the $d_{i j}{ }^{\prime} s$ and $v_{i}$ 's which result from the solution of the deterministic equivalents of the expected value and the minimum variance models are entirely due to difference in the functional forms of the objective function of the two models.

P-model. 25
In this model the objective is to maximize the probability of achieving some specified $C^{\circ} X^{\circ}$, which is determined from the aspiration level of the decision-maker. This model can be formalized as

[^3]\[

$$
\begin{array}{ll}
\text { Maximize: } & P\left[C^{\prime} X \geq C^{\circ} X^{\circ}\right] \\
\text { subject to: } & P[A X \leq B] \geqq \alpha \\
& X=D B \tag{32}
\end{array}
$$
\]

By utilizing the decision rule $X=D B$ and transforming this model in the same manner as the $E$ and $V$-models were transformed, the deterministic equivalent for the P -model can be written

$$
\begin{align*}
& \text { Maximize: } \quad v_{o} / w_{o} \\
& \text { subject to: } \mu_{C}^{1} D_{B}-v_{0} \geqq \mu_{C}{ }^{0} \text {, } \\
& -V[D]+w_{0}^{2} \geqq 0 \quad \text {, } \\
& \mu_{i}[D]-v_{i} \geq 0 \quad, \\
& -K_{\alpha_{i}} \hat{\sigma}_{i}^{2}[D]+v_{i}^{2} \geqq 0 \text {, } \\
& v_{i} \geq 0 \quad . \tag{32}
\end{align*}
$$

The last three constraints in this deterministic equivalent are the same constraints that appear in the expected value and the minimum variance models. In conjunction with these, two additional constraints result from the derivation. These additional constraints constitute the objective functions of both of the previous deterministic equivalents. The $v^{\prime} s$ and $w^{\prime} s$ which appear in both the objective function and the constraints are slack variables.

The objective function of this model is stated as a fractional and assumes "...a minimax-like character in the sense that maximization of $v_{o} / w_{o}$ represents a
striving toward cooperatively maximizing $v_{0}$ while minimizing $w_{0} .^{26}$ Since the constraints still form a: convex set and since the objective function is a fractional, the following transformation can be performed to replace the fractional programming problem with a simple convex programming problem. ${ }^{27}$ Assuming $w_{o}>0$, define a variable $t$, which is used to transform the variables of the fractional programing problem, as follows

$$
\begin{equation*}
\bar{w}_{0}=t w_{0}=1 \tag{33}
\end{equation*}
$$

The remaining variables $D$ and $v$ are transformed by the following relationships

$$
\begin{align*}
& \overline{\mathrm{D}}=\mathrm{tD} \quad, \text { and } \\
& \overline{\mathrm{v}}=\mathrm{tv} \quad . \tag{34}
\end{align*}
$$

Substituting in [32] for $w_{o}$, $D$, and $v$ the above transformation yields the following convex programming problem

Maximize: $\bar{v}_{0}$
subject to: $\mu_{C}^{\prime} \bar{D} \mu_{B}-\bar{v}_{o} \geq t \mu_{C}^{\circ}$,

$$
\begin{align*}
&-\bar{v}[\bar{D}]+\bar{w}_{o}^{2} \geq 0 \\
& \bar{u}_{i}[\bar{D}]+\bar{v}_{i} \geq 0, \\
&-K_{\alpha_{i}} \hat{\sigma}_{i}^{2}[\overline{\mathrm{D}}]+\bar{v}_{i}^{2} \geq 0, \\
& \bar{w}_{0}=1, \\
& t, \bar{v}_{i} \geq 0, \tag{35}
\end{align*}
$$

${ }^{26}$ Charnes and Cooper, "Deterministic Equivalent," 32.
${ }^{27}$ A. Charnes and W. W. Cooper, "Programming with Linear Fractional Functionals," Naval Research Logistics
where

$$
\begin{align*}
\bar{V}[\bar{D}] & =E\left[C^{\prime} \bar{D} B-t C^{\circ} X^{\circ}\right]^{2} \\
\hat{\sigma}_{i}^{2}[\bar{D}] & =E\left[t \hat{b}_{i}-a_{i} \overline{D B}\right]^{2} \\
\bar{\mu}_{i}[\bar{D}] & =\left(t \mu_{b_{i}}-a_{i} \bar{D} \mu_{B}\right) \tag{36}
\end{align*}
$$

This last model in [35] is the convex programming problem which is the deterministic equivalent to the $P$-model of a chance-constrained programming problem.

The foregoing should not be interpreted as a complete presentation of deterministic equivalents to chance-constrained models. The reader should consider the fact that the deterministic equivalents which result depend upon the type of decision rule which is used to transform the original model. Only a linear decision rule was considered in this discussion. Various transformations that can be used to derive deterministic equivalents to chancemconstrained models have appeared in the literature. 28 These other transformations are not considered

Quarterly, IX (September-December, 1962), 181-186. In this source theorems can be found which establish the criteria for converting a fractional functional programming problem to a convex programming problem.
${ }^{28}$ In addition to references made in other parts of this chapter, the reader can also consider: A. Charnes, W. W. Cooper, and G. L. Thompson, "Constrained Generalized Medians and Hypermedians As Deterministic Equivalents for Two-Stage Linear Programs Under Uncertainty," Management Science, XXII (September, 1965), 83-112; A. Channes, M. J. L. Kirby, and W. M. Raike, "Solution Theorems In Probabilistic Programming: A Linear Programming Approach," Journal of Mathematical Analysis and Applications, XX
here since the purpose of this discussion has been primarily to classify chance-constrained models according to the objective functions used in the model.

Two-Stage Programming Under Uncertainty
The two-stage programming model refers to all stochastic programming models which allow adjustments to be made once the stochastic elements of the model have been observed to be equal to specific values. These adjustments are made by including in the stochastic programming model a new variable which attempts to compensate for infeasible solutions which result from the previous actions of the decision-maker. The solution of a stochastic programming model by utilizing a twostage approach has been referred to as the slack solution. ${ }^{29}$ Slack solution. ${ }^{30}$

The slack solution can be explained if one considers the linear programming model
(December, 1967), 565-582; Fredrik S. Hillier, "ChanceConstrained Programming With 0-1 or Bounded Continuous Decision Variables," Management Science, XIV (September, 1967), 34-57; and Gifford H. Symonds, TDetemministic Solutions For A Class of Chance Constrained Programming Problems," Operations Research, XV (May-June, 1967), 495-512.
29. Madansky, "Methods of Solution of Linear Programs Under Uncentainty," Operations Research, X (JulyAugust, 1962), 463-471.

$$
30_{\text {Ibid. }} 468-470
$$

$$
\begin{align*}
& \text { Minimize: } \quad C^{\prime} X \\
& \text { subject to: } A X \geq B \\
& X \geq 0 \tag{37}
\end{align*}
$$

In this model, if the $A$ matrix and the $B$ vector were to contain stochastic elements, then the possibility would arise that an optimal solution to the model could violate some of the constraints of the model. This possibility is dependent upon the subsequent observations on the elements of the $A$ matrix and the $B$ vector.

Instead of minimizing the objective function C'X over the convex set defined by $A$ and $B$, the two-stage solution procedure allows an adjustment to be made after calculating $X$ and subsequently observing $A$ and $B$. This adjustment for the possible infeasibility of a selected $X$ is in the form of a new variable, $Y$, with a corresponding penalty cost given by $\mathrm{F}^{\mathrm{t}} \mathrm{Y}$, where F is a vector of penalty cost coefficients. Both $F$ and $Y$ are (m $x$ l) vectors corresponding to the dimensions of $B$. The choice of the vector $Y$ depends not only on the original stochastic parameters $A$ and $B$ but also upon the initial solution vector $X$. In view of the inclusion of this new variable, Y, the objective function of the resultant model must also be adjusted to take into consideration both the cost, $C^{\prime} X$, and the penalty cost, F'Y, which may be incurred.

This two-stage programming model is a special case ${ }^{31}$ of a general class of programming models which can be stated
$\operatorname{Minimize}_{X}: \quad E_{\text {min }}^{y}\left[C^{\prime} X+F^{\prime} Y\right]$
subject to: $A X+D Y=B$,

$$
\begin{align*}
& X \geq 0 \\
& Y \geq 0 \tag{38}
\end{align*}
$$

In this general model $A$ and $D$ are ( $m \times n$ ) matrices, $C, X, F$, and $Y$ are ( $n \times I$ ) vectors, and $B$ is $a(m \times I)$ vector composed of stochastic elements with known distributions. The objective function is composed of two types of cost, the cost associated with each element of $X$ and the penalty cost associated with each element of $Y$.

The general model in [38] can be specialized to the two-stage programming model by considering $D Y$ to be equivalent to $\left(Y^{+}-Y^{-}\right)$. The vector $Y$ that yields the smallest penalty cost for each $A, B$, and $X$ would then be composed of two parts.

If $\quad B \geqslant A X$, then
$Y^{+}=B-A X$ and

$$
\begin{equation*}
Y^{-}=0 \tag{39}
\end{equation*}
$$

${ }^{31}$ Ibid., 468. The reader can also consider the treatment presented in Dantzig and Madansky, "Solution of Two Stage Linear Programs," 165-166.

If

$$
\begin{align*}
B & <A X, \text { then } \\
Y^{+} & =0 \text { and } \\
Y^{-} & =A X-B \tag{40}
\end{align*}
$$

In some cases some of the rows of $A$ and $B$ may not be stochastic. If this is the case then the corresponding constraints contain no $Y$ elements. These constraints are then called fixed constraints on $X$. ${ }^{32}$

The objective function of the general model in [38] can also be specialized to accommodate the two-stage programming model. Since the choice of $Y$ depends upon both $B$ and $X$, the objective function can be formulated to minimize $C^{\prime} X$ plus the expected smallest penalty cost.

The two-stage programming model can then be written:
Minimize: $\quad C^{\prime} X+E_{\min _{Y}}\left[F^{\prime} Y\right]$
subject to: $A X+\left(Y^{+}-Y^{-}\right)=B \quad$,

$$
\begin{equation*}
X \geqq 0, \quad Y \geqq 0 \tag{4,1}
\end{equation*}
$$

where $\mathrm{Y}^{+}$and $\mathrm{Y}^{-}$are defined in [39] and [40]. From the format it can be seen that the $Y^{\prime}$ s act as slack variables either reducing the left side of the equation when infeasibility occurs or increasing the left side of the equation when the initial solution $X$ does not utilize the total resources available in $B$. If the ith row of $A$ and $B$ contain deterministic elements, then the ith constraint is a fixed constraint and can therefore be written without the vaxiable Y .

$$
{ }^{32} \text { Madansky, "Methods of Solutions," } 468 .
$$

The assumption implied in the model above is that for each $X \geq 0$, which satisfies all the constraints, and for each $B$ there exists a $Y$ such that ( $X, Y$ ) is feasible. As an alternative to this assumption define "...K as the convex set of the $X^{\prime}$ s such that each $X \varepsilon K$ is nonnegative and has an associated $Y$ for each $A$ and $B$ such that ( $\mathrm{X}, \mathrm{Y}$ ) is feasible. The problem is, then, to find XeK that minimizes $C^{\prime} X+E_{\min _{Y}}\left[F^{\prime} Y\right] . "^{33}$

Linear Programming Under Risk
Those models that are classified under this general heading can be grouped into two distinct classes. The first class contains one-stage linear programming models under uncertainty. There are two different solution procedures that can be used to solve these models. These procedures are called the expected value solution and the "fat" solution. The second class contains the models that are formulated to specify the statistical distribution of the objective function of a stochastic model. These models can be formulated in terms of either an active or a passive approach to the problem. Whether an active or passive approach is used to specify the statistical distribution of the objective function, these models assume that the distributions of the parameters of the stochastic model are known. In effect the models which
${ }^{33}$ Ibid. , The reader can also consult Dantzig and Madansky, "Solution of Two Stage Linear Programs," 166.
fall into this category are referred to as linear programming models under risk.

One-stage model ${ }^{34}$
The expected value solution
The implication of an expected value solution to a one-stage stochastic linear programming model can best be explained if first a deterministic linear programming model is stated in terms of a matrix. game. Consider the deterministic model

Minimize: $\quad C^{\prime} X$
subject to: $A X \geqq B$,

$$
\begin{equation*}
x \geq 0 \tag{42}
\end{equation*}
$$

This model is feasible and finite "...if and only if the matrix game with payoff matrix

$$
Q=\left|\begin{array}{ccc}
0 & A & -B  \tag{43}\\
-A^{\prime} & 0 & C \\
B^{\prime} & -C^{\prime} & 0
\end{array}\right|
$$

has an optimal mixed strategy ( $X_{0}^{\prime}, Y_{0}^{\prime}, t$ ) such that $t>0 .^{15}$ Under these conditions the solutions to the primal and the dual model are given by

$$
\begin{array}{ll}
X=X_{0} / t & \text { and } \\
Y=Y_{0} / t \tag{44}
\end{array}
$$

${ }^{34}$ Madansky, "Methods of Solutions," 464-468.
${ }^{35}$ Ibid., 464-465. The reader should also consider the reference given in the cited text.

When the parameters of the programming model are stochastic and an expected value solution to that model is attempted, then the corresponding matrix game has a payoff matrix E[Q] with an optimal strategy given by ( $\bar{Y}, \bar{X}^{r}, \overline{\mathrm{t}}$ ) where $\overline{\mathrm{t}}>0$ and E is the expected value operator. In this case the expected value solution of the model is $\mathrm{X}^{*}=\mathrm{X} / \overline{\mathrm{E}}$. This solution minimizes the model

Minimize: E[C']X
subject to: $E[A] X \geqq E[B]$,

$$
\begin{equation*}
x \geq 0 \tag{45}
\end{equation*}
$$

The solution vector $X^{*}$ is nonnegative, but may not guarantee that the constraint set in [42] is satisfied. Let $S$ be defined as the set of permanently feasible X's, that is

$$
\begin{equation*}
\{X \varepsilon S: X \geqq 0, \operatorname{Pr}[A X \geqq B]=1\} \tag{46}
\end{equation*}
$$

Now if ( $E[A], E[B]$ ) is a member of the set of values ( $A, B$ ) and if $X^{*}$ is permanently feasible, then $X^{*}$ is a solution to the stochastic programning model.

The necessary and sufficient conditions for the expected value solution $X^{*}$ to be an optimal solution can be determined from the payoff matrix given in [43] if the following definitions are made:

$$
\begin{align*}
\bar{Z}^{\prime} & =\left(\bar{Y}^{\prime}, \bar{X}^{r}, \overline{\mathrm{~T}}\right) \text { and } \\
\mathrm{M}[\mathrm{Q}] & =\bar{Q}^{\prime} . \tag{4.7}
\end{align*}
$$

The term $\overline{\mathrm{Z}}$ ' is defined as the optimal set of strategies for the matrix game $E[Q]$, and the term $M[Q]$
is the product of the original payoff matrix and the vector comprising the optimal strategies.

The first m-nows of the matrix M[Q] can be
written ( $A \bar{X}-B \bar{t}$ ). These rows correspond to the constraints of the primal model stated in [42]. Since $\overline{\mathrm{t}}>0$, then

$$
\begin{align*}
(A \bar{X}-B \bar{t}) & \geq 0 \quad \text { if and only if } \\
A \bar{X} / \bar{t} & =A X^{*} \geq B \tag{48}
\end{align*}
$$

for all $A$ and $B$. The conclusion is that the expected value solution is optimal, if and only if, for all values of $A$ and $B$, the first m-rows of $M[Q]$ are nonnegative.

If the ith now is one of the first m-rows of $M[Q]$, then this row can be written

$$
\begin{equation*}
M_{i}[Q]=a_{i 1} X_{1}+a_{i 2} X_{2}+\ldots+a_{i n} X_{n}-b_{i} \bar{Z} \tag{49}
\end{equation*}
$$

The necessary and sufficient conditions for the expected value solution to be optimal are satisfied if the minimum. of $M_{i}[Q]$ with respect to the elements ( $a_{i I}, \ldots, a_{i n}, b_{i}$ ) is greater than or equal to zero. ${ }^{36}$

The "fat" solution
The "fat" solution was initially proposed as a means of accounting for uncertainties which may develop in the long-run when a deterministic solution to a programming
${ }^{36}$ Ibid., 466. Madansky assumes that the set of possible values ( $a_{i l}$, ...., $a_{i n}$, $b_{i}$ ) form a bounded convex polyhedron and that the minimum of $M_{i}[Q]$ is taken subject to that condition.
model was the basis upon which decisions were made. ${ }^{37}$ The procedure which would yield a "fat" solution involves ignoring the random variation and providing plenty of "fat" in the deterministic version of the model. Consider the model stated in [42] where now the parameters ( $A, B$ ) are stochastic. Utilizing the "fat" solution one would postulate a pessimistic (A, B) and then solve the nonstochastic program. The choice of the appropriate pessimistic values of the stochastic elements should be such that the optimal solution to the program is from the set of permanently feasible X's. ${ }^{38}$ In the situation where there are a finite number of possible (A, B)'s, the set of permanently feasible X's can be described as those $X^{\prime} s$ that satisfy the $m R$ constraints

$$
\begin{align*}
A^{(r)} & X \geqq B^{(r)} \quad \text { for } r=1, \ldots, R, \text { and } \\
X & \geqq 0 \quad . \tag{50}
\end{align*}
$$

The optimal solution of the stochastic program is the $X$, from the set of permanently feasible $X$ 's, which minimizes C'X subject to the constraints stated in [50] above. In the case where $C$ contains stochastic elements, the function to be minimized can be stated as E[C']X.
${ }^{37}$ George B. Dantzig, "Recent Advances in Linear Programming," Management Science, II (January, 1966), 131; Salah E. Elmaghraby, "An Approach to Linear Programming Under Uncertainty," Operations Research, VII (March-April, 1959), 208-209.
${ }^{38}$ Madansky, "Methods of Solution," 467.

## Distribution models

The approach taken toward stochastic linear programming models which concerns itself with the specification of the distribution of the objective function was introduced by Gerhard Tintner. This approach is based upon the assumption that the parameters of a linear programming model are random variables with known probability distributions. These models have been described in the following way.

If all the parameters in a linear programming problem are random variables, the problem becomes a stochastic programming problem. A passive solution exists if the activities are not chosen in advance. We have an active solution if the proportion of the resounces to be devoted to various activities are chosen. By numerical methods we can determine approximations to the various distributions and choose the optimal one [solution] according to some criteria. 39

It can be seen that the distribution model, specified by Tintner, is approached from two different points of view. It is the objective of this section to specify both the passive and the active approaches to the distribution model. 40

[^4]
## The passive approach

In the passive approach the objective is to determine the expected value and variance of the distribution of the optimal objective function values. This can be formally stated as: Find the $E[Z(X)]$ and $V[Z(X)]$ in the model

$$
\begin{align*}
\text { Maximize: } & Z_{k}
\end{align*}=C_{k} X_{k},
$$

The parameters $A, B$, and $C$ are randomly distributed with known distributions, such that $A_{k}, B_{k}$, and $C_{k}$ are specific values that each respective parameter may take on. The specific value of each parameter is determined a priori from the characteristics of its distribution and substituted into the model in [51] to determine an optimal value of the objective function $z_{k}$. If this process is repeated N times, with N optimal values of the objective function being determined, then the expected value and variance of the optimal objective function values can be defined as

$$
\begin{align*}
& E[Z(X)]=\sum_{\sum_{k}^{N} Z(X)_{k} P\left[Z(X)_{k}\right]}^{N}[Z(X)]=\sum_{i}^{N}\left\{Z(X)_{k}-E\left[Z(X)_{k}\right]\right\}^{2} P\left[Z(X)_{k}\right],
\end{align*}
$$

where

$$
\begin{equation*}
P\left[Z(X)_{k}\right]=P\left[A=A_{k} \cap B=B_{k} \cap C=C_{k}\right] \tag{53}
\end{equation*}
$$

This appraoch assumes "...that in almost all possible situations, i.e., for almost all possible variations of
the parameters, the conditions of the simple nonstochastic linear program are fulfilled and the maximum achieved. ${ }^{41}$

The active approach
The model ${ }^{42}$ utilized in the active approach can be formalized as

$$
\begin{array}{rlrl}
\text { Maximize: } & & Z & =C^{\prime} X \\
& & & \\
\text { subject to: } & A \hat{X} & \leqq \hat{B} U  \tag{54}\\
& X & \geqq 0
\end{array}
$$

In this model $U$ is $a(m \times n$ ) matrix with the elements $u_{i j}$ satisfying the conditions

$$
\begin{align*}
u_{i j} & \geq 0 \quad, \text { and } \\
\sum_{j=1}^{n} u_{i j} & =1 \tag{55}
\end{align*}
$$

Further, $X$ and $B$ are square, diagonal matrices with the elements of the $X$ and $B$ vectors making up the respective diagonals of these matrices. The vector of cost coefficients is C.

The $u_{i j}$ 's are allocation ratios such that $u_{i j}$ indicates the proportion of the resource $i$ which is used for activity j. These allocation ratios are exogenous variables and are determined by the policies of the decision-maker. The only conditions placed upon these ratios are those stated in [55] above which imply that all resources must be completely used up.

[^5]In the active approach the objective is also to determine the probability distribution of the optimal values of the objective function given the alternative values that can be assumed by the parameters $A, B$, and $C$. In this case, however, the distribution of the optimal values of the objective function is also dependent upon the allocation matrix $U$. The implication is that a set of distributions of optimal objective function values results from the set of allocation matrices that are employed. In effect the decision-maker can measure the effects of various policy decisions upon the distributions of the optimal objective function values by simply changing the matrix U.

A specific decision rule must be established in order to distinguish among the different distributions which can result due to the employment of alternative sets of allocation ratios. Three different decision rules or criteria of optimization under risk have been utilized to distinguish among the distributions explained above. These are the expected value criteria, the fractile criteria, and the portfolio or aspirations level criteria. ${ }^{43}$

These three criteria are specified for the case where the $C$ vector is randomly distributed in the following: J. K. Sengupta and J. H. Portillo-Campbell, "A Fractile Approach to Linear Programming Under Risk, " Management Science, XVI (January, 1970), 298-308; and A. M. Geoffrion, "Stochastic Programming with Aspiration or Fractile Criteria," Management Science, XIII (May, 1967), 672-679.

Expected value criteria.--Under the expected value criteria the choice among the distributions of the optimal objective function values is made by selecting the distribution with the maximum expected value. If $K$ different allocation matrices are considered, then the maximum value in the set

$$
\begin{align*}
& \left\{E[Z(X)]_{U=U_{1}}, \quad E[Z(X)]_{U=U_{2}},\right. \\
& \left.\ldots, E[Z(X)]_{U=U_{k}}\right\} \tag{56}
\end{align*}
$$

is selected. This selection indicates the set of allocation ratios which optimizes the stochastic model.

Fractile criteria.--The fractile criteria specifies that the $\alpha$-fractile of the distribution of optimal objective function values is to be optimized. Selecting the maximum value from the set

$$
\begin{align*}
& \left\{\mathrm{F}_{\alpha}[\mathrm{Z}]_{\mathrm{U}=\mathrm{U}_{1}} \quad, \quad \mathrm{~F}_{\alpha}[\mathrm{Z}]_{\mathrm{U}=\mathrm{U}_{2}} \quad,\right. \\
& \left.\ldots, F_{\alpha}^{[Z]}{ }_{U=U_{k}}\right\} \tag{57}
\end{align*}
$$

satisfies this decision rule. In this relation the $\alpha$ is a predetermined constant such that $F_{\alpha}[Z]$ is the $\alpha$-fractile of the distribution of optimal objective function values.

Portfolio criteria.--The application of the portfolio criteria requires that the variance of the distribution of optimal objective function values is minimized under the additional constraint that the expected values of these same distributions are at least equal to some
preassigned level. This criteria involves selecting the minimum from the set

$$
\begin{align*}
& \left\{v[z(x)]_{U=U_{1}}, \quad v[z(X)]_{U=U_{2}}\right. \\
& \left.\ldots, v[z(X)]_{U=U_{R}}\right\} \tag{58}
\end{align*}
$$

subject to the additional constraint that the

$$
E[Z(X)]_{U=U_{I}} \quad, \ldots, E[Z(X)]_{U=U_{R}} \geqq P_{0}
$$

where

$$
\begin{equation*}
0 \leqq R \leqq K \quad, \text { and } \tag{59}
\end{equation*}
$$

$P_{o}$ is a predetermined profit level.
The fractile criteria possesses an important characteristic. ${ }^{44}$ Consider the case when $\alpha=0.5$. In this situation the fractile criteria yields the same results as the expected value criteria. Similarly, an appropriate value of $\alpha$ can be determined such that the results from the application of the fractile criteria and the portfolio criteria are the same. In effect the expected value criteria and the portfolio criteria are special cases of the fractile criteria.

Computationally the fractile criteria is difficult to apply since for the most part the objective function is nonlinear. Consider the case where $\alpha>0.5$ and only the $C$ vector is stochastic, having a multivariate normal

[^6]distribution with mean $M$ and variance $V$, the model can then be written ${ }^{45}$

Maximize: $\quad M^{\prime} X-\mathrm{q}^{\left(\mathrm{X}^{\mathrm{t}} \mathrm{VX}\right)^{1 / 2}}$
subject to: $A X \leqq B$,

$$
\begin{equation*}
x \geq 0, \tag{60}
\end{equation*}
$$

where $q$ is a positive standard normal deviate and $M^{\prime} X$ and (X'VX) are the mean and variance of the objective function $Z=$ C'X. Iterative algorithms have been developed to analytically solve a model of this type. ${ }^{46}$

## Summary

The purpose of this chapter has been to classify into groups the stochastic linear programming models which have appeared in the literature. These groups consist of (1) the chance-constrained programming models, (2) the two-stage programming under uncertainty model, and (3) the linear programming under risk model. Within each classīication various solution procedures or deterninistic equivalents to these models were then indicated. The E-model, the V-model, and P-model were shown to be deterministic equivalents to the chance-constrained programming models. The two-stage programming under uncertainty model was approached through a slack solution.

[^7]And the linear programming under risk models were classified as either one-stage models or distribution models.

The reader should realize that, even though the classification scheme above is all-inclusive, not all possible solution techniques or deterministic equivalents to these models have been presented. The emphasis in this chapter has been upon the classification of the models. The more important solution techniques have been included in order to enhance this objective.

CHAPTER III

## REQUIREMENTS FOR SIMULATION STUDIES

## Introduction

The experimental results of this study are genexated by using a simulation procedure. A discussion of the Monte Cario simulation technique is therefore a necessary prerequisite to the presentation of the experimental model used in this study. This chapter meets this objective by defining simulation and briefly presenting the properties and characteristics of Monte Carlo simulation models. Special emphasis is placed upon the development and the testing of the pseudorandom number generator which is used in this study. Operations research models can utilize different problem-solving procedures. These are (1) the analytical, (2) the numerical, and (3) the simulation procedures. ${ }^{1}$ The analytical procedure is based upon mathematical deduction. The decision-maker works from a set of defined assumptions adhering to all the rules of mathematical logic until the solution is derived. The expression of the
${ }^{1}$ W. W. Thompson, Operations Research Techniques (Columbus, Ohio: Charles E. Merrill Books, 1967), pp. 4-6.
model in a mathematically rigid way is the essential prerequisite for using an analytical procedure. In effect the relationships among the variables in the model must be both identified and rigidly defined.

The numerical procedure is one in which the assumptions and the relationships among the variables of the model are exactly defined, but the solution to the model is obtained through a less formal trial-anderror technique. The use of the numerical procedure is restricted to those models where either no analytical procedure is applicable or where the analytical procedure is too inconvenient to apply.

A simulation procedure is used when the model of the system is too complex for the effective use of either of the two other procedures. In the simulation procedure a set of synthetic variables, representing an analogous set of real world variables, is manipulated with the purpose of arriving at conclusions about the real world system being studied. The first step of this procedure is the representation of the real world situation in the form of an abstracted model. The model must then be manipulated in order to generate a set of synthetic outputs which are characteristic of the real world system.

## Monte Carlo Simulation

Simulation has been described as an experimental technique involving logical and mathematical models of a
real world system. The experimentation is performed under stochastic or dynamic conditions. In addition the experimental results from a simulation experiment which is run on a computer may not necessarily be determinable by analytical methods. ${ }^{2}$

The essence of the above statements is that simulation is a special kind of experimentation. Specifically it is mathematical experimentation. Simulation has been considered a form of mathematical experimentation by many authors. For example, J. M. Hammersly and D. C. Handscomb classify simulation as a tool of experimental or theoretical mathematics which relies upon the deductive process. ${ }^{3}$ James R. Jackson concludes that

Mathematical experimentation [simulation] may be an appropriate research technique when interesting problems appear too difficult for the effective application of the traditional deductive approach. From a mathematical point of view, the conclusions reached can rarely be thought of as more than conjectures; but it seems to me that when a high degree of confidence can be placed in experimentally reached conclusions, they are of virtually the same practical interest as would be proven theorems with the same content... It is important to note that I am proposing simulation experimentation as a supplement to mathematical analysis, not as a substitute therefor. 4
${ }^{2}$ Thomas H. Naylor, et al., Computer Simulation Techniques (New York: John Wiley and Sons, 1966), pp. 2-3.
${ }^{3}$ J. M. Hammersly and D. C. Handscomb, Monte Carlo Methods (London: Methuen, 1964), p. 1.
${ }^{4}$ James R. Jackson, "Simulation as Experimental Mathematics," in Symposium on Simulation Models: Methodology and Applications to the Behavioral Sciences ed. by Austin C. Hoggatt and Frederick E. Balderston (Cincinnati, Ohio: South-Westem Publishing Co., 1963), p. 246.

Monte Carlo methods comprise that segment of simulation techniques which is concerned with experiments having a stochastic or probabilistic structure. The stochastic components of the system are included in the model through the use of a random number generator. Monte Carlo simulation is an efficient means of analyzing and solving stochastic models when these models are considerably complex. In some cases a Monte Carlo simulation may be the only means available for analyzing and solving a stochastic model.

The justification for using a Monte Carlo simulation to derive the results of this study is based upon the size and the objective of the study. With regard to size, the reader should recall the numerous initial formulations of the stochastic linear programming model which can be established. Each of these initial formulations is then considered under varying assumptions concerning the stochastic elements and in terms of the different deterministic equivalents to the stochastic model. A Monte Carlo simulation can efficiently analyze such a large scale model.

The objective of this study is the evaluation of the performance of the different deterministic equivalents under varying conditions. Monte Carlo simulation is an efficient means of analyzing the internal interactions among the components of a model when the assumptions and,
or, the parameters of the model are allowed to change. The most important feature of Monte Carlo simulation is the flexibility which it allows the researcher in manipulating the components of a model.

## Random Number Generators ${ }^{5}$

A Monte Carlo simulation is as valid as the random number generator which is used to produce the values of the stochastic elements which are integral parts of the simulation. For each simulation it is important to select a random number generator which has been properly tested. The verification of the random number generator is an essential part of any simulation experiment.

This section contains a general discussion of random number generators with emphasis upon the congruential methods of generating random numbers. The specific random number generator used in the simulation experiment described in this work is explained at the end of this section. The verification of this generator is the topic of the next section of this chapter.

There are many alternative methods available for generating random numbers. This section is limited to a discussion of digital computer methods for their generation. Digital computer methods' provide for the intemal generation of a sequence of digits by a recurrence relationship. The immediate advantage of this procedure

[^8]is the small amount of computer memory required to perform the operation. An additional advantage is that the process is totally reproducible.

The use of a recurrence relationship may appear to be in conflict with the randomness required of the digits in the sequence. Since each digit in the sequence can be determined from the previous digit or from some set of previous digits through the recurrence relationship, then the process is technically not random. The process is defined as random if and only if the sequence meets certain statistical tests of randomness. If these statistical tests are met, the sequence is called a series of pseudorandom numbers.

The desirable properties of a sequence of pseudorandom numbers generated internally on a digital computer are that the numbers be (I) uniformly distributed, (2) statistically independent, (3) reproducible, and (4) nonrepeating for a sufficient length. This last property refers to the period of the sequence of numbers. In addition the generator should require a minimum amount of computer memory and should generate the pseudorandom numbers at a high rate of speed. ${ }^{6}$

[^9]Congruential Methods for Generating
Pseudorandom Numbers
Most computer codes for generating random numbers use some variation of the congruential method developed by D. H. Lehmer. ${ }^{7}$ The congruential method is speedy, reproducible, and requires only a small amount of computer memory. The period of the sequence of random numbers depends upon the particular congruential method used. Whatever congruential method is used statistical tests should be performed on the sequence of numbers to determine whether they are uniformly distributed and statistically independent. Most congruential generators satisfy the statistical tests of uniformity and independence.

Fundamental congruential relationship

Congruential methods are based on a fundamental congruential relationship. This relationship can be expressed in the following way.

Two integers $a$ and $b$ are congruent modulo $m$ if their difference is an integral multiple of m. The congruence relation is expressed by the notation $a \equiv b(\bmod m)$ which reads $" a$ is congruent to $b$ modulo m." 8

[^10]The implications of this definition are (1) that (a - b) is divisible by $m$ and (2) that $a$ and $b$ leave identical remainders when divided by the absolute value of $m$. This relationship can be expressed as the following recursive formula

$$
\begin{equation*}
n_{i+1} \equiv a n_{i}+c \quad(\bmod m) \tag{1}
\end{equation*}
$$

where $n_{i}, a, c$, and $m$ ane all nonnegative integers. 9 Given a constant multiplier a and an additive constant $c$, this formula establishes the relationship between any number in a sequence and the previous number.

The period, $h$, of the above formula is the length of the sequence before a number repeats itself, that is, before $n_{h}=n_{0}$. Once a number in the sequence is repeated, then the series will duplicate itself; that is, $n_{h+1}=n_{1}$, $n_{h+2}=n_{2}$, etc. Theorems are available to show that the congruential methods have a finite period which depends upon the constants in the recursive formula in [1]. 10

## Basic types of congruential

 generatorsThree congruential methods have been developed for generating pseudorandom numbers. Each of these methods is based upon the relation in [1]. Each method
${ }^{9}$ Ibid., p. 48.
${ }^{10}$ Ibid., pp. 65-66. The reader should also refer to M. D. MacLaren and G. Marsaglia, "Uniform Random Number Generators," Jouxnal of the Association for Computing Machinery, XII, (January, 1965), 86-89.
is designed to generate a sequence of pseudorandom numbers with a maximum period in a minimum amount of tine. These methods are (1) the additive congruential method, (2) the multiplicative congruential method, and (3) the mixed congruential method.

Additive congruential method ${ }^{11}$
The additive congruential method assumes that $k$ random numbers are provided in computer memory. The sequence of pseudorandom numbers is computed by means of the congruence relationship

$$
\begin{equation*}
n_{i+1} \equiv n_{i}+n_{i-k} \quad(\bmod m) \tag{2}
\end{equation*}
$$

The pseudorandom numbers generated in this way have a period which depends upon $k$ and $m$. Statistical tests have indicated that $k=16$ is the smallest value which will yield acceptable random numbers. 12

Multiplicative congruential methodl 3

The multiplicative congruential method is based upon the congruence relationship

$$
\begin{equation*}
n_{i+1} \equiv a n_{i} \quad(\bmod m) \tag{3}
\end{equation*}
$$

56-57.
${ }^{1}{ }_{\text {Naylor }}$, Computer Simulation Techniques, p. 49,
12B. F. Green, J. Smith, and L. Klem, "Empirical Tests of an Additive Random Number Generator," Journal of the Association for Computing Machinery, VI Coctober,1959), 537.

51-54.
$13_{\text {Naylor, Computer Simulation Techniques, }}$ pp. 49,
where a is a positive constant. This relationship yields a sequence of positive integers less than $m$. The period of the sequence depends upon both the constant multiplier $a$ and the initial value in the series $n_{0}$. Conditions can be placed upon these constants to insure a maximum period. Statistical tests have been performed which indicate that the multiplicative congruential method generally yields a sequence of pseudorandom numbers which is uniformly distributed and statistically independent. ${ }^{14}$ Mixed congruential method ${ }^{15}$

The mixed congruential method is based upon the recursive relationship as it is expressed in [1] with both a and $c$ not equal to zero. With this method the maximum period depends upon the constants a and $c$. Little or no effect upon the statistical properties of the sequence is attributed to the initial value $n_{0}{ }^{16}$

Some investigators have expressed dissatisfaction with the congruential methods discussed above. The essence of this criticism is the failure of the congruential methods to always satisfy the requirement for serial

[^11]independence. 17 Two remedies have been proposed to alleviate this problem. R. R. Coveyou and M. Greenberger have determined theoretical conditions on the values of $a, c$, and $m$ in the fundamental congruential relationship in [l] which will guarantee a small serial correlation among the numbers in the sequence. These empirical studies specify that $a$ value of $a=\sqrt{m}$ will yield the smallest value for the correlation coefficient regardless of the value of $c .{ }^{18}$ The second remedy proposed by M. D. MacLaren and G. Marsaglia is called the combination method. With this procedure a mixed congruential generator is used to randomly determine an index which is used to select a random number from a set of stored random numbers. The stored random numbers are generated by the multiplicative congruential method such that the ith number is replaced by a new value when $i$ is the index generated by the mixed congruential method. 19

17MacLaren, "Uniform Random Number Generators," pp. 86-89.
${ }^{18}$ R. R. Coveyou, "Serial Correlation in the Generation of Pseudo-Random Numbers," Journal of the Association for Computing Machinery, VII (January, 1960), 72-74; and M. Greenberger, "An a Priori Determination of Serial Correlation in Computer Generated Random. Numbers," Mathematics of Computations, XV COctober, 1961), 384.-386.
${ }^{19}$ MacLaren, "Uniform Random Number Generators," Pp. 83-86.

The random number generator
used in this study
The random number generator used in the simulation experiment of this study is shown in the appendix to this chapter. The generator is a multiplicative congruential generator with multiple initial values. This generator is recommended for use on the IBM 360 computer. ${ }^{20}$ Random real numbers between zero and one and random integers between zero and $2^{31}$ are computed with this generator. The random integer produced at any stage is used as the input value for determining the random number at the next stage. The period of this generator is $h=2^{29}$. The multiple initial values have been added to the generator by this researcher in order to reduce the serial correlation among the numbers in the sequence.

Recalling the multiplicative congruential
relationship

$$
\begin{equation*}
n_{i+1} \equiv a n_{i} \quad(\bmod m) \tag{4}
\end{equation*}
$$

it is necessary to specify the values of the constants in that relationship which guarantee a maximum period and a minimum value of the serial correlation coefficient for the sequence of numbers generated by the relationship. The condition on the multiplicative constant a is that
${ }^{20}$ This generator can be found in the IBM Application Program, System 360 Scientific Subroutine Package, (360A(m - 03X). Version III, p. 77. The reader is also peferred to IBM Manual C20-8011 on random number generators.
it must be odd and relatively prime to $m{ }^{21}$ Since the IBM 360 computer is a binary computer $m=2^{b}$ where $b$ is the number of binary digits in a word. (The value of $b$ is thirty-one for the IBM 360 computer.) The values of a which satisfy the condition above can be expressed as

$$
\begin{equation*}
a=8 t \pm 3, \tag{5}
\end{equation*}
$$

where $t$ is any positive integer. ${ }^{22}$
According to the conditions set down by Coveyou and Greenberger, values of a should be chosen to minimize the first order serial correlation of the numbers in the sequence. A value of a approximately equal to $\sqrt{\mathrm{m}}$, which is equal to $2^{\mathrm{b} / 2}$ in this case, should be chosen. The value of a recommended with this generator is 65539. It was found that this value does not satisfy the condition for minimizing the serial correlation of the sequence. As an alternative to the above integer the value 46331 is used with the generator since this integer satisfies all the conditions stated above.

The condition on the initial value, $n_{0}$, specifies that it be any positive odd number. To meet the design of this particular random number generator eleven initial values of $n_{0}$ are stored in computer memory. The finst ten
${ }^{21}$ Naylor, Computer Simulation Techniques, pp. 63-64. Two integers are relatively prime if the greatest common divisor of the two integers is one.
${ }^{22}$ Ibid., pp. 51-52.
of these values are the multiple starting values to be used to generate a pseudorandom number. The eleventh value is used to randomly determine an index which indicates which starting value is to be used to determine that pseudorandom number. Each time that a particular starting value is used it is updated, that is the ith pseudorandom number in integer form serves as the starting value for the generation of the next pseudorandom number. The eleven values of $n_{0}$ used with this generator are shown in the appendix.

## Verification of the Random

## Number Generator

Three types of statistical tests are used to verify the random number generator described at the end of the last section. These types of tests are (1) a uniform frequency test, (2) a serial correlation test, and (3) a test of runs. ${ }^{23}$ First, second, and third order serial correlation tests are performed on the sequence of pseudorandom numbers generated. Two types of runs tests are performed. These involve runs above and below the mean and runs up and down. The FORTRAN program written to peifform these tests and the results of the tests are presented in the appendix to this chapter.
${ }^{23}$ Ibid., pp. 57-62. The tests described in this section are based. upon the corresponding tests presented in this reference.

Each test is designed to be run on a predetermined number of groups with each group containing a predetermined number of pseudorandor numbers. The purpose for arranging and testing the random numbers in groups is that the consistency of the generator with respect to its meeting a particular test can be observed. In addition subsequent groups, which were not originally planned for, can be tested along with the initial groups at a later time without having to again generate the initial groups.

## Uniform Frequency Test

The uniform frequency test is a chi-square test used to test whether the sequence of pseudorandom numbers is uniformly distributed. The test is performed on a sequence of $A M$ consecutive sets of $A N$ pseudorandom numbers. 24

The generator produces real numbers between zero and one. This interval is divided into ten subintervals. The expected number of pseudorandom numbers in each group which falls into each subinterval is AN divided by ten. Let $f_{j}$ denote the actual number of pseudorandom numbers in the subinterval

$$
\begin{equation*}
(j-1) / 10 \leq r<j / 10, \text { where } j=1,2, \ldots, 10 \tag{6}
\end{equation*}
$$

${ }^{24}$ In explaining the logic behind these tests the variable names used in the FORTRAN program are incorporated into the text whenever such an inclusion is beneficial to the reader.

The statistic

$$
\begin{equation*}
x_{1}^{2}=\sum_{j=1}^{10} \frac{\left(f_{j}-\frac{A N}{I O}\right)^{2}}{\frac{A N}{10}}=\frac{10}{A N} \cdot \sum_{j=1}^{10}\left(f_{j}-\frac{A N}{10}\right)^{2} \tag{7}
\end{equation*}
$$

is then distributed according to a chi-square distribution with nine degrees of freedom. ${ }^{2}$

This chi-square statistics is computed for each of the AM groups of AN pseudorandom numbers. These AM values of $X_{1}^{2}$ are then grouped into four intervals in the following way. Let $F_{i}$ be the number of the resulting values of $x_{1}^{2}$ which lie between the (i - 1) th and the ith quartile of a chi-square distribution with nine degrees of freedom. Since the expected number of $X_{I}^{2}$ values which fall in each interval is AM divided by four, then the statistic

$$
\begin{equation*}
x_{F}^{2}=\frac{4}{A M} \cdot \sum_{i=1}^{4}\left(F_{i}-\frac{A M}{4}\right)^{2} \tag{8}
\end{equation*}
$$

is chi-square distributed with three degrees of freedom. The hypothesis concerning the randomness of the sequence of pseudorandom numbers is rejected if the statistic $\chi_{F}^{2}$ is greater than a critical value of the chi-square statistic with three degrees of freedom. The critical
${ }^{25}$ In this statistic and in all the others to be explained the values of $A M$ and $A N$ must be chosen so as to guarantee that the expected number of elements falling into each subinterval is greater than five. This is based upon the assumptions under which a chi-square test is performed.
value is determined by assuming a given level of significance at which the hypothesis is tested.

## Serial Correlation Test

Serial correlation tests are used to determine the independence of successive pseudorandom numbers in a sequence. First, second, and third order serial correlation tests are run. These tests respectively determine the independence between the ith and the (i +h )th pseudorandom number in the sequence. The value h equals one for the first order test, two for the second order test, and three for the third order test.

The serial correlation tests used in this study are also based upon a chi-square distribution. For each set of $A N$ pseudorandom numbers let $f_{j k}$ denote the number of pseudorandom numbers which fall in the intervals

$$
\begin{align*}
& (j-1) / 10 \leq r_{i}<j / 10 \quad \text { and } \\
& (k-1) / 10 \leq r_{i+h}<k / 10 \tag{9}
\end{align*}
$$

where the intervals are the same as those used in the uniform frequency test. In the expressions above $j$ and $k$ range from one to ten, i ranges from one to (AN - h), and $h$ equals one, two, or three respectively for the first, second, or third order serial correlation test. ${ }^{26}$
${ }^{26}$ Fon the sake of clarity, the remainder of the discussion is in terms of only one serial correlation test. To perform the three tests the only change in procedure involves using a different value of h .

The statistic

$$
\begin{aligned}
& x_{2}^{2}=\frac{10^{2}}{A N-h} \sum_{j=1}^{10} \sum_{k=1}^{10}\left(f_{j k}-\frac{A N-h}{10^{2}}\right)^{2} \\
& (\text { for } h=1,2,3)
\end{aligned}
$$

is chi-square distributed with ninety-nine degrees of freedom since there are one hundred classes into which a pair of successive pseudorandom numbers can fall. This statistic is determined for each of the AM groups of pseudorandom numbers.

The statistic $\left(x_{2}^{2}-x_{1}^{2}\right)$ is calculated for each of the AM groups of pseudorandom numbers. This statistic is distributed according to a chi-square distribution with ninety ( $100-10$ ) degrees of freedom. ${ }^{27}$ Let $s_{j}$ denote the number of values of $\left(x_{2}^{2}-x_{1}^{2}\right)$ which lie between the ( $j-1$ ) th and the $j$ th quartile of a chi-square distribution with ninety degrees of freedom. Then the statistic

$$
\begin{equation*}
x_{S}^{2}=\frac{4}{A M} \sum_{j=1}^{4}\left(s_{j}-\frac{A M}{4}\right)^{2} \tag{11}
\end{equation*}
$$

is chi-square distributed with three degrees of freedom. The serial independence of the sequence of pseudorandom numbers is established at a given level of significance if the values $\chi_{F}^{2}$ and $\chi_{S}^{2}$ are each less than the critical

27 Ibid., p. 59. The reader is also referred to I. J. Good, "On the Serial Test for Random Sequences," Annals of Mathematical Statistics, XXVIII (March, 1957), 262-264.
value of the chi-square statistic with three degrees of freedom.

## Runs Tests

Runs tests are concerned with the particular arrangement of the pseudorandom numbers within the sequence. Since the pseudorandom numbers should be uniformly distributed, the mean of the numbers should equal the median of the numbers.

The test for runs above and below the mean is designed to result in a rejection of the hypothesis of randomness iff the sequence of numbers is such that any number, which is greater than (or less than) the mean or median of the numbers in the sequence, is repeatedly followed by a number which is also greater than (or less than) the mean or median of the numbers in the sequence. If the generator is producing pseudorandom numbers then the conditional probability that some number $r_{i+1}$, is greater than the mean, given that $r_{i}$ is greater than the mean, is equal to the probability that $r_{i+1}$ is less than the mean given that $r_{i}$ is greater than the mean. A similar equality holds for the conditional probabilities in the case when $r_{i}$ is Less than the mean.

A similan argument can be made concerning the test for runs up and down. In this case the hypothesis of randomness is rejected if the numbers in the sequence are repeatedly larger (or smaller) than the pnevious numbers.

## Runs above and below <br> the mean

Since the generator produces pseudorandom numbers over the interval from 0 to 1 , then the mean of the pseudorandom numbers should be 0.5 . Each of the AN pseudorandom numbers in each group can be classified as either greater than, less than, or equal to 0.5. For each sequence of AN pseudorandom numbers a corresponding sequence can be constructed. If $r_{i}<0.5$, define $s_{i}=0 ;$ and if $r_{i}>0.5$, define $s_{i}=1$. Values of $r_{i}$ exactly equal to 0.5 are improbable and are not counted as a run. The runs in $s_{i}$ are accumulated by size and are then compared with the expected number of runs of each size. The expected number of total runs is (AN +1 )/2 and the expected number of runs of any length $k$ is

$$
\begin{equation*}
\frac{(A N-k+3)}{2^{k+1}} \tag{12}
\end{equation*}
$$

The chi-square statistic for this test can be written

$$
\begin{equation*}
x_{A / B}^{2}=\sum_{i=I}^{n} \frac{\left(0_{i}-E_{i}\right)^{2}}{E} \tag{13}
\end{equation*}
$$

where $O_{i}$ is the observed number of runs of a given size $i$ and $E_{i}$ is the expected number of runs of size $i$.

The larger the size of a run the smaller is the probability of a run of that size. Therefore the expected number of runs of fairly large size is smali. Care must be taken in performing this chi-square test to include in
the test only terms whose expected values exceed five. Whenever for a given run size the expected number of runs of that size is less than five, then this run size and all larger run sizes must be grouped into a single class in order for the chi-square test to be performed correctly. The last term in the test compares the observed and expected numbers of runs of some size $j$ and all sizes larger than $j$. The largest run size which has an expected value greater than or equal to five depends upon $A N$, the number of pseudorandom numbers generated in a group.

Once the number of temms to be included in the chi-square test is known, then the degrees of freedom for the test is one less than the number of terms. All the statistical tests in this study were performed on 30 groups of 400 pseudorandom numbers each. That is $A M=30$ and $A N=400$. In the test of runs above and below the mean, run sizes as large as five had expected values greater than five. The last term of the chi-square test then compared the observed and expected number of runs of sizes six or more. The degrees of freedom then associated with this test is five.

The statistic $\chi_{A / B}^{2}$ must be determined for all AM groups of pseudorandom numbers. Let $s_{j}$ denote the number of the resulting $\chi_{A / B}^{2}$ values that lie between the ( $j-1$ ) th and the $j$ th quartile of a chi-square distribution with five degrees of freedom. The statistic

$$
\begin{equation*}
x^{2}=\frac{4}{A M} \cdot \sum_{j=1}^{4}\left(s_{j}-\frac{A M}{4}\right)^{2} \tag{14}
\end{equation*}
$$

is then chi-square distributed with three degrees of freedom. The test is met at a given level of significance i.f this value is less than a critical value of chi-square with three degrees of freedom.

## Runs up and down

The test for runs up and down is analogous to the test for runs above and below the mean. For each sequence of AN pseudorandom numbers again a corresponding sequence can be constructed. If a particular pseudorandom number $r_{i}$ is less than the next number in the sequence $r_{i+1}$ then $s_{i}=0$. When $r_{i}$ is greater than $r_{i+l}$, then $s_{i}=1$. The runs in $s_{i}$ are then accumulated by size. The expected number of total runs is (2AN - 1)/3; the expected number of runs of length $k$ is

$$
\begin{equation*}
\frac{2\left[\left(k^{2}+3 k+1\right) A N-\left(k^{3}+3 k^{2}-k-4\right)\right]}{(k+3)!} \tag{15}
\end{equation*}
$$

for $k$ less than (AN - 1); and the expected number of runs of length (AN - 1) is (2/AN!).

The statistic $X_{U / D}^{2}$ can be defined in a similar fashion as the statistic $X_{A / B}^{2}$ in [13]. Again care must be taken to incluce in this test only run sizes whose expected numbers are greater than five. With $A M=30$ and AN $=400$, run sizes up to a length of three have expected
values greater than five. The last term in the test therefore compares the observed and expected numbers of run sizes of four or more. The degrees of freedom of the statistic $X_{U / D}^{2}$ is three.

The statistic $X_{U / D}^{2}$ is determined for each of the AM groups of pseudorandom numbers. Again the number of the values of $X_{U / D}^{2}$ which fall into the various quartiles of a chi-square distribution with three degrees of freedom are determined. A statistic analogous to that in [14] is then found and the test is completed in a similar fashion.

## The test results

The sequence of pseudorandom numbers is tested by arranging the sequence into 30 (AM) groups with 400 (AN) pseudorandom numbers in each group. The sequence numbers of the pseudorandom numbers in the first set of the 30 groups run from 1 to 12,000 . The test results shown in the appendix were obtained by running and testing 12 sets of groups. The total number of pseudorandom numbers tested is 144,000.

The hypothesis of randomness is tested at a level of significance of $\alpha=.05$ and $\alpha=.01$. The critical values of the chi-square statistic with three degrees of freedom at these levels of significance are respectively 7.814.7.3 and 11.3449.

The results of these different tests are presented in Table 1 in the appendix. The generator passed all
-tests of randomness at a level of significance of $\alpha=.01$. At the significance levell of $\alpha=.05$ the generator failed the various tests on three occasions. The first onder serial correlation test led to a rejection of the hypothesis of independence for the set of pseudorandom numbers with sequence numbers from 120,001 to 132,000 . Similarly the second order sexial correlation test led to a rejection of the hypothesis for the numbers whose sequence numbers are from 108,001 to 120,000 . On one occasion the test of runs above and below the mean led to a rejection of the hypothesis of randomness. This occurred for the pseudorandom numbers with sequence numbers from 132,001 to 144,000.

The failure to accept the hypothesis on these three occasions at the level of significance of $\alpha=.05$ does not lessen the confidence that this generator is producing a sequence of numbers which can be assumed to be random. These three failures represent a smaller proportion of failures than that considered acceptable at the 0.5 level of significance.

## APPENDIX

FLOW CHART OF THE
RANDOM NUMBER GENERATOR


ELOW CHART OF THE STATISTICAL TEST
RUN ON THE RANDOM NUMBER GENERATOR




FORTRAN PROGFAM GAF THE PANDUM HUMEFP GERERATOR

```
G THE RANDOM NIHMGEF {FNEUATOW USIG IS A THO STAGE
```



```
C DETF&MINIS FANHHMI.Y THF IX(I) VALUL TO IGE USFO IAN THF
C DETERMINATTON UF TILE PANDOH NUHBI:N. TIE SECGNLS STAGE
C DETERMINES THE KAPIDO:A NUMBLER ITSELF. THIS FROCLOURE IS
C USED TO MINIMI/E THE SLOIAL.CORIELATIGN. IXIIM IS THE
C
    USEO TA OETFRMINIMG THE RANDGM NUMBERS THENSFLVES.
    DIMENSION IX(11)
    CDUNT=0.0
    DO12 I= 1,11
    IX(I)=C
    2 CONT INIJE
    DO3I=1,11
    READ(5,9C) 1X(I)
90 FORMAT(I7)
    3 CONTINUE
    4 COUNT=COUNT+1.
        I= 1 1
20 I IX=|X(I)
    I Y = IIX:4 46331
    IF(IY)5,6,6
    5 I Y=IY+2147483647+1
    6 YFL=IY
    IX(I)=IY
    YFL=YFL\div.465661 3E-9
    IF(I.NE.11)GO TO Z!I
    IF(YFL.LT..1) 万O TO 11
    IF(YFL.LT..2)GO TOJ 12
    IF(YFL.LT..3) GO TO 1.3
    IF{YFL.LT..4} GG TO L4
    IF(YFL.LT..5) GO TD 15
    IF(YFL.I_T..6) GO TO 16
    IF(YFL.LT..7) GO TO 17
    IF(YFL.LT** G) GO TD 18
    IF(YFL.LT..9).GO 10 19
    J=10
    GO TO 20
11 I=1
    GO TO 20
12 I =2
    GOTO 20
13 I=3
    GO TO 20
14 I=4
    GO TO 20
15 I=5
    GO TO 20
16 I=6
```



RUN TJN THE PANDON NUMBER GENRRATOR
this program contains a multipl.icative CONGRUENTIAL RANDOM NUMBER GENERATUR AND SIX DIFFERENT test that can be used tu test any p seudorandon number GENERATOR. THE TEST INCLUDE (1) A UNIFORM FREQUENCY TEST, (2), (3), (4) TEST FOR FIRST, SECUNO, AND THIRU ORDER SERIAL CORRELATIDN, (5) A TEST FOK RUNS AGCVE AND DELDW THE MEAN, AND (6) A TEST FDR RUNS UP AND doinn. these test are destgned so that they can be AFFIXED TO ANY PROGRAM WHICH WILL GENERATE PSEUDORANDOM NUMBERS. IN ADDITION THE SERIAL CORRELATION testing procedure can be mouified so as to test for. HIGHER ORDER SERIAL CORRELATION. the random numbers arf generated in am' sets of 'AN' NUMBERS. FOR EACH SET UF 'AN' NUMBERS ALL SIX TEST ARE CONDUCTED. THE RESULTS FOR EACH TEST IN TURN ARE THEN COMBINED INTO A GORRESPONDING TEST ON ALL 'AN' NUMBERS.

A LIST OF THE MAJOR VARIAGLES FOLLOWS.

variables in the random number genfrator
IX(I)......INITIAL VALUES. THESE AKE READ IN.
AN........NUMBER GF RANDOM NUMPERS IN A GROUP (GE. 50)
AM........ .NUMBER OF GROUPS (GE. 20 )
YFL....... RANDOM NUMBER IN FLCATING PCINT mGde
IY........ RANDOM NUABER IN FIXED POINT MODE. THE NEM IX(I).
variables in the freduency test
XINTR.... NUMBER GF CLASSES INTO WHICH A RANDOM NUMBER IS PLACEO.
U..........NUMBER OF SUBDIVISIONS INTO WHICH THE AN. different chi-swuare values are placeo for THE OVER-ALL CHI-SQUARE TEST.
CHISQ..... The Chi-SQuare value for the uniformity TEST FOR EACH SET AM.
FCHISQ.... The chi-sQuare value for the uniformity TEST FOR ALL AM SETS.
varlables in the corpelation test
IRHO.......INDICATES HIGHEST CRDER SERIAL TEST TC BE PERFORMED.
TCHISQ....THE CHI-SQUARE VALUE FDR THE SERIAL TEST. FIRST ORIER TEST IS SUBSCRIPTEO L, SECONO ORDER TEST IS SUBSCRIPTED 2, THIKD CRDER TEST IS SUBSCRIPTED 3.
DICHI..... THE DIFFERENCE BETWEEN FCHISQ AND TCHISQ. THIS VALUE IS CHI-SQUARE DISTRIBUTEC AND IS


```
    85 CONTINUE
        DOB4I=1, 2
        DO84J=1,IIU
        CHIKUT(I,JI=0.
    84 CINTINUE
C INITIALIZING STATEMENTS FOR EACH SFT IN TURM
```



```
    KK=0
    86 KK=KK+1
    D083I=1,2
    CHIRUN(I)=0.
    83 CONTINUE
    DO29I=1,INTR
    FJ(I)=0.
    29 CONTINUE
    DO28IS=1, IRHO
    DO 2&LJ=1,INTR
    DO2BLK=1,INTR
    FJK(IS,LJ,LK)=0.
    28 CONTINUE
    LJ=0
    LJK=0
    LKJ=0
    LK=0
    IGR=0
    ILT=0
    ICOUNT=0
    DO1071=1,20
    IRUN(I)=0
    107 CONTINUE
    J COUNT =0
    IUP=0
    IDW=0
    00140I=1,20
    JRUN([)=0
    140 CONTINUE
    D0501I = 1,N
    THE RANDGM NUMBER GENERATUR.
C A MULTIPLICATIVE CONGRUENTIAL GENERATOR WITH MULTIPLE
```



```
    I=11
20 1IX=IX(I)
    IY=1IX* 46331
    IF(IY)5,6,6
    5 IY=IY+2147483647+1
    6 YFL=IY
    IX{I)=IY
    YFL=YFL*.4656613E-9
    IF(I.NE.11)GO TO 21
    IF(YFL.LT..L) GO TO ll
    IF(YFL.LT..2) GU TO 12
    IF(YFL.LT..3) GO TO 13
    IF(YFL..LT..4) GO TO 14
    IF(YFL.LT..5) GO TO 15
```

```
    IF(YFI..LT..G) GO MO 16
    IF(YFL.LT..7) GU TO 17
    IF(YFL.LT..8) GO TO 18
    IF(YFL.LT..9) GO TO 19
    I=10
    GO TO 20
    11 1=1
    GO TO 20
    12 1=2
    GO TO 20
    13 I=3
    GO TO 20
    14 1=4
    GO TO 20
    15 I=5
    GO TO 20
    16 I=6
    GO TO 20
    17 I=7
    GO TO 20
    18 I=8
    GO TO 20
    19 1=9
    GO TO 20
C DETERMINING RUN SILES FCR RUNS (UP EDOWN,) TEST
```



```
    21 IF(II.EQ.l) GO TO 129
    IF(FYFL.GT.YFL) GO TO 12l
    IF(IUP.F.Q.O) GO TO }12
    IUP=IUP+1
    IF(II.EO.N) JRUN(IUP)=JRUN(IUP)+1
    GO TO l29
    120 JCOUNT=JCOUNT+1
    IF(II.GT. 2)JRUN(IDW)=JRUN(IDW)+1
    IOW=0
    IUP=IUP +1
    IF(II.FQ.N) JRUN(IUP)=JRUN(IUP)+1
    GO TO 12.9
    121 IF(IDW.EQ.O) GO TO 123
    IDW=1DW+1
    IF(II.EQ.N) JRUN(IUW)=JRUN(IDW)+1
    GO TO 129
    123 JGUUNT=JCOUNT+1
    IF(iI.GT.2) JRUN(IUP)=JRUN(IUP)+1
    124 IUP=0
        IOW=IDW+1
        IF(II.EQ.N) JRUN(IUW)=JRUN(IDW)+I
    129 FYFL=YFL
C
```



```
    OETERMINING RUN SIZES FGR RUNS (ABOVEE BELOW) TEST
    IF(YFL.GT...5) GO TO 102
    IF(ILT.EQ.O) GO TO 101
    ILT=ILT+I
    IF(II.EU.N) IRUN(ILT)=IRUN(ILT)+I
```

```
    GO TO 105
    101 ICOUNT = 1COUNT +1
        IF(II.EO.1) GO T0 104
    IRUN(IGR)=IRUN(IGR) +I
    104 IGR=0
    IL.T=ILT+1
    IF(II.EO.N) IRUN(ILT)=IRUN(ILT)+1
    GO TO IOS
    102 IF(IGR.IO.O) GO TO 103
    IGR=1GR+1.
    IF(II.E(J.N) IRUN(IGK)=IRUN(IGR)+I
    GO TO 105
    103 ICOUNT=ICOUNT+1
    IF(II.EO.1) GO TO IDG
    IRUN(ILT)=IRUN(ILT)+1
    106 ILT=0
    IGR=IGR+1
    IF(II.EG.N) IRUN(IGR)=IRUN(IGR)+1
C DETERMINING THE INTERVALS INTO GHICH THE RANOGM
C NUMBERS FALL FOR T!HE UN:FORW FREQUENCY TEST AND THE
    1ST,2ND,3RD ORDER SERIAL CORRELATIGN TEST
```



```
105 IF(YFL .LT. .1) GO TO +1
    IF{YFL .LT. .2) GO TO 42
    IF(YFL .LT. .3) GO TO 43
    IF{YFL .LT. .4) GO TO 44
    IF\YFL .LT. .5) CG TO 45
    IF(YFL .LT. .G) OO TO 46
    IF(YFL .LY..7) GG TO 47
    IF{YFL .LT. .8) GO TO 4A
    IF(YFL .LT. .9) GO T0 49
    L=10
    GO TO 51
41 L=1
    GO TO 51
42 L=2
    GO TO 51
43 L=3
    GO TO 51
44 L=4
    GO TO 51
45 L=5
    GO TO 51
46 L=6
    GO TO 51
4 L=7
    GO TO 51
48 L=8
    GO TO 51
49 L=9
51 FJ(L)=FJ(L)+1.
    IF{II.EQ.1) GO TO 75
    IF(II-3) 70,71,72
70 LJK=L
```

GU TU 50
71 LKJ=L
GO TO 50
72 LK=L
FJK (1,LJ,LJK) $=$ FJK (L,LJ,LJK) + 1 .
FJK(2,LJ,LKJ) $=F J K(2, L J, L K J)+1$.
FJK(3,LJ,LK) $=F$ JK(3,LJ,LK )+1.
LJ=LJK
LJK=LKJ
$L K J=L K$
IFIII.NE.NI GU TO 50
FJK (L,LJ,LJK) =FJK(l,LJ,LJK) +1.
FJK(1,LJK,LKJ)=FJK(1,LJK,LKJ)+1.
FJK(2,LJ,LKJ)=FJK(2,LJ,LKJ)+1.
GO TO 50
$75 \mathrm{LJ}=\mathrm{L}$
50 CONTIMUE
ChI-SQUARE TEST UN TOTAL RUNS AND RUN SIZES FOR RUN UP
UP AND DCI 4 N * **
CNT = JCOUNT
EXCNT=(2.*AN-1.)/3.
CHITER=((CNT-EXCNT)**2)/EXCNT
SUEXRU=0.
DO130I $=1, N$
$\mathrm{A} I=1$.
$F A C=A I+3$.
$\mathrm{R}=\mathrm{FAC}$
DOL31J=1,N
$A J=J$
IF( (R-AJ).EQ.1.) GO TO 132
$F A C=F A C *(R-A J)$
131 CONTINUE
132 EXRU(I) $=2 . *(((A I * * 2)+3 . * A I+1) * A N-.((A I * * 3)+3 . *(A I * * 2)-$
1AI-4.1)/FAC
SUEXRU=SUEXRU+EXRU(I)
IF(EXRU(I).GE.5.) GO TO 130
EXRU(I)=((2.*AN-1.)/3.1-(SUE XRU-EXRU(I))
IF(EXRU(I).GE.5.) GO TO 160
EXRU(I-1)=EXRU(I-1)+EXRUTI)
$I R J=I-1$
GO TOl61
160 IRJ=I
GO TO 161
130 CONTINUE
$161 \mathrm{JDUM}=0$
DU170I=1,IRJ
IFII.EQ.IRJIGO TO 170
JOUM = JDUM + JRUN(I)
170 CONTINUE
JRUN(IRJ)=JCOUNT-JDUM
DO162I=1, IRJ
RUDIFF=((JRUN(I)-EXRU(I))**2)/EXRU(I)
$I \mathrm{I}=1$
CHIRUN(II)=CHIRUN(II)+RUDIFF

```
    162 GONTINUE
    C.HIRUN(II)=CHTRUN(II)+CHITER
C CHI-SQUARE TEST ON TOTAL RUNS ANIJ RUN SIZES FOR RUN
```



```
    CNT = ICOUNT
    EXCNT=(AN+1.)/2.
    CHITER=((CNT-EXCNT)**2)/EXCNT
    SUEXRU=0.
    DOL10I=1,N
    Al=1
    EXRU(I)=(AN-AI+3.1/(2.**(I+1))
    SUEXRU=SUEXRU+EXRU(I)
    IF(EXRU(I).GE.5.) GG ro llo
    EXRU(I)=((AN+1.)/2.)-(SUEXRU-EXRU(I))
    IF(EXRU(I).GE.5.) GO TD 164
    EXRU(1-1)=EXRU(I-1)+EXRU(I)
    IRI=I-1
    GO TO 165
    164 IRI=1
    GO TO165
    110 CONTINUE
    265 JDUM=0
    DO171[=1,IRI
    IF(I.EQ.IRIIGO TO 171
    JDUM=JDUM +IRUN(I)
    171 CUNTINUE
    IRUN(IRI)=ICOUNT-JOUM
    0OLG6I=I,IRI
    RUDIFF=((IRUN(I)-EXRU(I))**2)/EXPU(I)
    I I= ?
    CHIRUN(II)=CHIRUN(II)+RUDIFF
    l66 CONTINUE
    CHIPUN(II)=CHIRUN(III)+CHITER
C POSITIONING CHI-SQUARE VALUES OF RUNS TEST INTO
```



```
    DOL50I=1,2
    IF(I.EQ.2) GO TO 167
    IF{CHIRUN(I) .LT.1.92256 JGO TO 151
    IF(CHIRUN(I) .LT.3.35G6S JGG TO 152
    IF(CHIRUN(I) .LT.5.38527 JGO JO 153
    GO TO 1G&
    167 IF(CHIRUN(I) .LT.3.4546C 1GO TO 151
    IF(CHIKUN(I) .LT.5.34&12 IGO TO }25
    IF(CHIRUN:I) .LT.7.84C80 IGOT0 153
    168 IU=4
    GO TO 154
    151 1U=1
    GO TO 154
    152 1U=2
    GO TO 154
    153 IU=3
    154 CHIRUT(I,IU)=CHIRUT(I,IU\+1.
    150 CONTINUE
C
    CALCULATING CHI-SQUARE(1) VALUES
```



```
    SUDIFF=0.
    DU53 J=L, INTR
    OIFF=(FJ(J)-AN/XINTK) **2
    SUDIFF=SUDIFF+DIFF
    53 CONTINUE
    CHISQ=(XINTR/AN)*SUOIFF
C POSITIONING EACH CHI-SOUARE(L) VALUE INTD INTERVALS
```



```
    IF(CHISQ
    .LT.5.098B3) CC TO 63
    IF(CHISQ .LT.&.34283) GO TO 64
    IFICHISQ &LT.ll.3888) GO TO 65
    IU=4
    GO TO 36
    6 3 ~ 1 U = 1
    GO TO 36
    64 IU=2
        GO TO 36
    65 IU=3
    36 CAPFJ(IU)=CAPFJ(IU)+1.
C CALCULATING CHI-SQUARE(2) AND CHI-SQUARE(2) MINUS
```



```
    DO7715=1, IRHO
    SUM(IS)=0.
    77 CONTINUE
        DO76IS=1, IRHO
        D032LJ=1, INTR
        D032LK=1, INTR
        DIFFJK(IS)=(FJK(IS,LJ,LK)-(AN-IS)/XINTR**2)**2
        SUM(IS)=SUM(IS)+DIFFJK(IS)
    32 CONTINUE
        TCHISQ(IS)=((XINTR**2)/(AN-IS))*SUM(IS)
        DICHI(IS)=TCHISQ(IS)-CHISQ
G POSITIONING EACH CHI-SQUARE(2) MINUS CHI-SQUARE(1)
```



```
    IF(DICHI(IS).LT.80.6247 IGU TO 33
    IFIDICHI(IS) .LT.89.3342 )GO TO 34
    IF(DICHI(IS) LTT.98.6499 IGO TO 35
    IU=4
    GO TD 67
    33 [U=1
    GO TO ò7
    34 IU=2
    G0 10 67
    35 IU=3
    67 S(IS,IU)=S(IS,IU}+l.
    76 CONTINUE
    68 [F(KK.LT.M) GO TO 86
C CalCulating overall chi-square value on run siles for
```



```
    DO157I=1,2
    SD[(I)=0.
    157 CONTINUE
    DO156I=1,2
```

```
            00155J=1, IIU
            DI(I)=(CHIRUT (I,N)-AN/U)**2
            SOI(I)=SOI(I)+DI(1)
    155 CONTINUF
            KUCHI(I)=(U/AM)*SOI(I)
    156 CONTINUF.
```



```
    DO82IS=1,IRHO
    SUOI(IS)=0.
    82 CONTINUE
        DO81IS=1, 1RHO
        D037J=1,IIU
        SUIFF(IS)={S(IS,J)-AM/U)我东2
        SUDI(IS)= SUDI(IS) +SDIFF(IS)
        37 CONTINUE
        SCHI(IS)=(U/AM)*SUOT(IS)
        81 CUNTINUE
C
```



```
    SUCADF=0.0
    DO69J=1,IIU
    CADIF=(CAPFJ(J)-AM/U)**2.
    SUCADF=SUCAOF +CADIF
    69 CONTINUE
    FCHISQ=(U/AM) &SUCADF
    IF(INO.NE.1) GO TO 1111
    WRITE(0,1000)
1000 FORMAT(/47X,'TABLE 1')
    WRITE(6,1001)
1001 FORMATI/3IX, 'TEST RESULTS ON RANDOM NUMGER GENERATOR*)
    WRITE(6,1002)
1002 FORMAT(21X;'
    l *)
    WRITE{6,1002)
    WRITE(6,1003)
1003 FORMAT(/22X;'SEQUENCE',3X,'UNIFOR-',7X,'GORRELATIGN',
    112X,'RUNS')
    WRITE(6,1004)
1004 FORMAT(23X,*NUMBER",5X, "MITY*, 2X,"
    1 ')
    WRITE(6,1005)
1005 FORMATI/42X,'FIRST', 2X,'SECONO*, 3X,'THIRD', 2X,'ABOVE-'
    1.4X,'UP-')
    WRITE(6,1C06)
1006 FORMAT(42X, 'ORDER', 3X,'ORDER', 3X,'ORUER', 2X,'RELDM',4X
    1,"DOWN")
    WRITE(6,1002)
1111 WRITE(6,1007)FCHISU,SCHI(1),SCHI(2),SCHI(3),RUCHI(2),
    IRUCHI(1)
1007 FURMAT(// 33X,6(F6.4,2X))
    IF(IQQ.NE.12) GO TO 2000
    WRITE(6,1002)
    WRITE(6,1008)
```

```
1008 FORMAT(/2BX,'thE R,RITIGAL VALUE OF CHi-SMUARE ARE')
    WRITE(6,1009)
1009 FORMAT(57X,'FOR = .05 IS 7.81473')
    WRITE(6,1010)
1010 FORMAT(57X,'FOR =.01 IS 11.3449')
2000 CONTINUE
    STOP
    END
```

TABLE 1
TEST RFSULTS DN KANOOM NUMHEF GENFRATGK

| SEOUENC, NUMGER | UNIFOKMITY | CORRELATIUN |  |  | RUNS |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | FIHST ORDER | $\begin{gathered} \text { SECGNO } \\ \text { OROER } \end{gathered}$ | TIITRD ORDER | ABOVE- <br> BELETH: | $\begin{aligned} & \text { UP- } \\ & \text { UTWN } \end{aligned}$ |
| $\begin{aligned} & 1 \\ & -12000 \end{aligned}$ | 5.4667 | 2.2667 | 2.5333 | 0.6667 | 0.6667 | 6.8000 |
| $\begin{array}{r} 12001 \\ -24000 \end{array}$ | 2.8000 | 0.6667 | 1.2000 | 3.6000 | 0.4000 | 6.0000 |
| $\begin{array}{r} 24001 \\ -36000 \end{array}$ | 1.7333 | 1.4667 | 2.2667 | 3.3333 | 2.2667 | 1.2000 |
| $\begin{array}{r} 360 \mathrm{Cl} \\ -48000 \end{array}$ | 4.6667 | 2.2667 | 4.9333 | 2.2007 | 4.9333 | 6.5333 |
| $\begin{array}{r} 48001 \\ -60000 \end{array}$ | 6.0000 | 3.6000 | 2.2667 | 5.7333 | 2.5333 | 1.46067 |
| $\begin{array}{r} 60001 \\ -72000 \end{array}$ | 2.2667 | 2.5333 | 7.0667 | 2.8000 | 2.5333 | 4.6667 |
| $\begin{array}{r} 72001 \\ -84000 \end{array}$ | 6.0000 | 0.6667 | 1.2000 | 5.4667 | 0.1333 | 3.6000 |
| $\begin{array}{r} 84001 \\ -96000 \end{array}$ | 0.6667 | 0.1333 | 2.8000 | 3.6000 | 1.4607 | 3.6000 |
| $\begin{aligned} & 96001 \\ & \quad-108000 \end{aligned}$ | 1.2000 | 4.4000 | 1.2000 | 3.8667 | 2.8000 | 2.5333 |
| $\begin{aligned} & 108001 \\ & -120000 \end{aligned}$ | 2.2667 | 6.8000 | 3.6667 | 0.1333 | 5.7333 | 2.2667 |
| $\begin{aligned} & 120001 \\ & -132000 \end{aligned}$ | 0.6667 | 8.1333 | 4.9333 | 2.2667 | 2.5333 | 0.8000 |
| ${ }_{-144000}$ | 0.6667 | 2.8000 | 0.6667 | 1.2000 | 9.2000 | C. 4000 |

the gritical value of chi-square

```
FOR \alpha = .05 IS 7.81473
FOR \alpha = .01 IS 11.3449
```


## CHAPTER IV

## A STATEMENT OF THE EXPERTMENTAL PROCEDURE

## Introduction

The simulation model constructed for this study included three alternative deterministic equivalents to the stochastic programming model. These deterministic equivalents were (1) the two-stage slack approach to programming under uncertainty, (2) the active approach to stochastic linear programming under risk, and (3) the one-stage expected value approach. The experimental model evaluated an empirical linear programming problem with stochastic parameters in terms of each deterministic equivalent stated above. A Monte Carlo simulation of this empirical problem was then performed. The results of this simulation were used as a standard with which to evaluate the results of each deterministic equivalent. The objective of this chapter is to specify the characteristics of the experimental model. Initially the specific empirical problem used to generate the experimental results is stated. Next, the experimental design is reviewed. Emphasis is placed upon the selection of the various initial stochastic fommulations of the empirical problem that were analyzed by the experimental
model. The three deterministic equivalents are then presented. Special attention here is placed upon the procedure used to include these deterministic equivalents in the experimental model. The chapter concludes with an appendix which contains the flow chart for computation of each deterministic equivalent and the flow chart and FORTRAN program of the entire experimental model.

## The Problem Used in the Experimental Model

 The empirical problem used in this study to generate the experimental results is a modification of an agricultural production problem which has appeared in the literature concerning stochastic linear programming. ${ }^{1}$ The problem, as it appears in the literature, contains five variables and five constraints. Because of the numerous ways in which uncertainty can be introduced into the parameters of the problem, the dimensions of the problem are reduced for present purposes. The modified problem contains three variables and three constraints. With a (3 x 3) problem there are sixty-three initial formulations of the problem which can be developed, in terms of combinations of parameters which are taken to be stochastic.[^12]The empirical problem used in this study can be stated as

$$
\begin{array}{lrl}
\text { Maximize: } & Z & =C^{\prime} X \\
\text { subject to: } & A X & \leq B \quad \text { and } \\
& X & \geqq 0, \tag{1}
\end{array}
$$

where the expected values of the parameters in [1] are:

$$
\begin{align*}
& A=\left(\begin{array}{lll}
0.31772 & 0.96956 & 0.27870 \\
0.02274 & 0.92490 & 0.02770 \\
0.02555 & 0.21186 & 0.07523
\end{array}\right), \\
& B=\left(\begin{array}{r}
1800 \\
148 \\
234
\end{array}\right), \text { and } \quad C=\left(\begin{array}{l}
1.56 \\
3.81 \\
0.84
\end{array}\right), \tag{2}
\end{align*}
$$

The variables of the problem are the quantities (in. bushels) of the agricultural products corn, flax, and oats which are to be produced subject to the available resources of capital, land, and labor. These resources are respectively expressed in dollars, acres, and man-hours.

This particular ( $3 \times 3$ ) problem was selected from the original (5 x 5) problem after an evaluation of the optimum solutions of the various ( $3 \times 3$ ) problems that can be formed from the original problem. In [2] above the $C$ vector contains the expected values of the profit margin of the more important variables from the original problem, while the $B$ vector contains the expected values of the more important resources available. The A matrix contains the technological coefficients nelating the amount of any particular resource required to produce
any product. These values were assumed to be constant throughout the entire experiment.

## Significance of the expected

## value solution

In addition to being one of the deterministic equivalents evaluated, the expected value solution was used in a number of ways in the experimental model. Initially the expected value solution was used to aid in the design of the experiment. The optimum solution vector of the problem described in [1] and [2], when all the parameters are assumed constant and equal to their expected values, contains non-zero values for the variables $X_{1}, X_{5}, X_{6}$. The variable $X_{1}$ refers to the first product, while the variables $X_{5}$ and $X_{6}$ are slack variables found in the second and third constraints respectively. The amount of slack, $X_{5}$, in the second constraint was found to be proportionally less than the amount of slack, $X_{6}$, in the third constraint.

Based on this expected value solution, it was decided that $X_{1}$ is the most important variable in the problem since it is the only non-zero decision variable which appeass in the optimum solution vector of the expected value model. Since the first constraint is the only constraint which has no slack in the optimum solution, then $b_{I}$ was viewed as the most important resource in the problem. Of the remaining variables $X_{2}$ has the larger
profit margin. This factor placed it as the next most important variable in the problem. Correspondingly, $b_{2}$ was the next most important resource since the second constraint has relatively less slack than the third constraint.

The ranking of the variables and the resources of the problem was important when one considers that not all the initial formulations of the problem were to be analyzed. There are seven ways in which uncertainty can be introduced into either the b or the c vectors and forty-nine ways in which uncertainty can be introduced into both vectors simultaneously. This is a total of sixty-three initial formulations that can be analyzed. Each formulation may also be considered under different assumptions concerning the standard deviation of the stochastic parameters which appear. Specifically in this study each initial formulation was analyzed under six different values of the standard deviation of the stochastic parameters. In view of this fact there are then 378 ( $63 \times 6$ ) initial formulations which can be analyzed.

The expected value solution, by providing a means for ranking the variables and the resources of the problem, was used to determine which of the 378 initial formulations are significant enough to be analyzed, and thus, to omit from analysis those judged to be insignificant. The significant formulations were considered to be those which
contain as stochastic parameters either the amount available of the most significant resource or the profit margin of the most significant variable.

An alternative experimental
problem
The experimental problem as it is stated in [l] and [2] is not a tightly constrained problem. This can be determined from a consideration of the optimum solution vector of the problem when the expected values of the parameters are used to determine a solution. The solution vector, $X_{E V}$, of the expected value approach indicates that 5665.4 units of $X_{I}$ are to be produced, that 19.2 units of slack are present in the second constraint, and that 89.2 units of slack are present in the third constraint.

It is highly probable, with these large amounts of slack in the last two constraints of the problem, that the optimum solutions generated assuming either a stochastic $b_{2}$ or $b_{3}$ are insensitive to the changing values of these parameters. Consider the third constraint where 89.2 units of slack ane available. If the coefficient of variation of $b_{3}$ is as large as .30 then the standard deviation, $\sigma_{b_{3}}$, of $b_{3}$ is 70.2. A value of $b_{3}$ which would eliminate all the slack in this third constraint must be at least $1.27 \sigma_{\mathrm{b}_{3}}$ units less than the mean of $\mathrm{b}_{3}$. Since $b_{3}$ is normally distributed, then the probability of generating a value of $b_{3}$ which would satisfy this condition is .1120. When the coefficient of variation of $b_{3}$
is smaller than .30 , then this probability decreases. For small values of the coefficient of variation (say .05 or . 10 ) it is highly improbable that this constraint would ever be a binding constraint. A similar argument can be presented concerning the second constraint, although the amount of slack in this case is much less and, therefore, the probability of generating a value of $b_{2}$ which would cause this constraint to be binding is somewhat greater.

Due to the slack present in the experimental problem stated in [1] and [2], an alteration of the problem is desirable so that less slack is present in the expected value solution. In order to make the first experimental problem a tightly constrained problem the following changes were made in the expected values of the parameters given in [2]. The profit margin, $c_{3}$, was increased to 1.50 . This change made the production of $X_{3}$ more desirable and increased the possibility of $X_{3}$ entering the optimum solution vector of the expected value approach. In addition $a_{12}$ was changed from 0.96956 to 0.66956 and. $a_{22}$ was changed from 0.9249 to 0.0549 . In the first formulation of the experimental problem $X_{2}$. required the use of relatively more resources than was justified by its profit margin. These changes in the A matrix made the production of $X_{2}$ more desirable and caused this variable to enter the optimum solution vector which was derived from the expected value approach.

The optimum solution vector of the expected value approach to the modified experimental problem indicates that 4475.1 units of $X_{1}$ and 564.8 units of $X_{2}$ are to be produced. In addition there are 15.2 units of slack in the second constraint, while the first and the third constraints contain no slack units.

This modified experimental problem is more tightly constrained than the initial experimental problem. Two sets of experimental results were generated by the experimental model, one set for the slightly constrained initial problem and the other set for the more tightly constrained modified problem. These problems are referred to respectively as experimental problem $A$ and experimental problem B.

## The Experimental Design

The experimental procedure was divided into three phases. In the first phase only the $B$ vector was considered to be stochastic, in the second phase only the $C$ vector was stochastic, and in the third phase uncertainty was introduced into both vectors simultaneously. In the first and the second phases, where seven initial formulations are each possible, only four formulations each were considered; while in the third phase only nine of the forty-nine possible formulations were analyzed. This is a total of seventeen formulations which were analyzed under different conditions.

With regand to the first phase, the first formulation treated the most significant resource ( $b_{1}$ ) as stochastic; the second formulation considered the two most significant resources $\left(b_{1}, b_{2}\right)$ as stochastic; the third formulation assuned the most and the least significant resources $\left(b_{1}, b_{3}\right)$ to be stochastic; and the fourth formulation treated all resources $\left(b_{1}, b_{2}, b_{3}\right)$ as stochastic. The reader should observe that each formulation contains $b_{1}$ as a stochastic element. The other resource values, since they are of lesser importance, were included as stochastic parameters only in combination with a stochastic $\mathrm{b}_{1}$ 。

Each of these four formulations was in turn analyzed under six different assumptions concerning the standard deviations of the stochastic parameters which appeared in the formulations. The four formulations described above, with the coefficient of variation ( $\sigma / \mu$ ) of each stochastic parameter equal to 0.5 , comprised the first four experiments that were run. The next four experiments (numbers 5 to 8 ) combined the same four formulations with the coefficients of variation of the stochastic parameters equal to .l0. To complete all the experiments in the first phase, the coefficients of variation of the stochastic parameters which appear were allowed, in turn, to equal .15, .20, .25, and .30. There were, then, twenty-four experiments (numbers 1 to 24) in this first phase.

The twenty-four experiments of the next phase (experiment numbers 25 to 48 ) were analogous to the corresponding set of experiments in the first phase. The first formulation in this phase treated the profit margin $c_{1}$ of the most significant variable as stochastic; the second formulation considered $c_{1}$ and $c_{2}$ as stochastic parameters, the third formulation considered $c_{1}$ and $c_{3}$ as stochastic parameters, and the fourth formulation treated all the profit margins $c_{1}, c_{2}$, and $c_{3}$ as stochastic. The coefficients of variation for each of the first four experiments (number 25 to 28 ) was . 05 . The coefficients of variation of the stochastic parameters were allowed, in turn, to equal .05, .10, .15, . $20, .25$, and . 30 , and each formulation described above was repeated six times.

When both the $B$ and the $C$ vectors contain stochastic parameters there are forty-nine formulations which can be analyzed. Nine representative formulations were selected to be included in this phase of the experiment. Again each of the nine formulations was analyzed with the six different values assumed by the coefficients of variation of the stochastic parametens which appeared in any formulation. In effect there were fifty-four experiments (numbers 49 to 102) included in phase three of the experiment. The parameters which were stochastic in these nine formulations are as follows.

Formulation

1

2

3

4

5

6

7
8

9

Stochastic Parameters
$b_{1}, c_{1}$
$b_{1}, c_{1}, c_{2}$
$b_{1}, b_{2}, c_{1}$
$b_{1}, c_{1}, c_{2}, c_{3}$
$b_{1}, b_{2}, b_{3}, c_{1}$
$b_{1}, b_{2}, c_{1}, c_{2}$
$b_{1}, b_{2}, c_{1}, c_{2}, c_{3}$
$b_{1}, b_{2}, b_{3}, c_{1}, c_{2}$
$b_{1}, b_{2}, b_{3}, c_{1}, c_{2}, c_{3}$

In the first nine experiments (numbers 49-57) of this phase the coefficients of variation of the stochastic parameters were .05. These nine formulations in [3] were repeated with the coefficient of variation varying up to .30 as was the case in the first two phases.

In summary this experiment was divided into three phases. The first phase and the second phase each contained four different formulations, while the third phase contained nine different formulations. Each of these seventeen formulations was analyzed under six different assumptions concerming the magnitude of the standard deviation of the stochastic parameters which appeared in the different formulations. This provided a total of 102 experiments which were analyzed by the simulation model.

## The Experimental Procedure ${ }^{2}$

The simulation model is programed in three parts. In the main program the definitions and assumptions pertaining to the study are presented; the input and output of data is controlled; and each deterministic equivalent is programmed. The first subroutine in the program (RAND) contains the pseudorandom number generator which has been adjusted to produce a normally distributed variable with a mean of zero and a standard deviation of one. The second subroutine (SIMPLX) is a simplex program which is designed to determine the optimum value of the objective function of any linear programming model. These different subroutines were called by the main program whenever their use was required.

The data pertaining to each of the experiments explained in the last section were read into the main program. For any particular experiment these data include (1) the number of stochastic parameters in the experiment, (2) the expected values of the stochastic parameters (this identifies which parameters are stochastic in the experiment), and (3) the coefficients of variation of each stochastic parameter.

[^13]Each experiment was iterated 100 times. At the start of each iteration a value was determined fon each stochastic parameter which appeared in the particular experiment. Since each stochastic parameter is assumed to be normally and independently distributed, these values were obtained by utilizing the subroutine RAND and the given mean and standard deviation of each of the parameters. The parameters in the model which were not assumed to be stochastic in any particular experiment were treated as constant at their expected values.

On each iteration, once the value of each parameter was determined, the problem was then adjusted, in turn, to conform to the specifications of the respective deterministic equivalents. The optimum value of the objective function of the problem was then determined, in turn, by utilizing each of the detemministic equivalents. The specific procedure involved in obtaining these optimum objective function values is explained in the remainder of this chapter.

## The one-stage expected

Value approach (ZEXPC)
The one-stage expected value approach utilized in this experiment can be represented in the following way

Maximize: $\quad Z=E[C] X$
subject to: $A X \leq E[B]$, and

$$
\begin{equation*}
x \geq 0 \tag{4}
\end{equation*}
$$

This deterministic equivalent replaces each stochastic parameter with its expected value. The optimum value of the objective function of the model in [4], assuming the expected values, is called ZEXPC. Since all the parameters were assumed to be constant with this approach, the ZEXPC value was constant for all the 102 experiments which were performed.

The simulation approach (ZSIM)
The simulation approach was next performed on each of the experimental formulations. Each formulation was iterated 100 times. The procedure involves: (1) a determination of which parameters are stochastic and the properties of those parameters; (2) for each stochastic parameter, subroutine RAND is used to determine a specific value of the parameter to use for each iteration; and (3) given the values of all the parameters as generated in (2), the subroutine SIMPLX is called and the conditional optimum value of the objective function is determined. The optimum value of the objective function on the $i t h$ iteration is $Z S T M_{i}$. Once all the iterations were completed, then the mean (ZSIMBR) and standard deviation (SDZSIM) of these optimum objective function values were computed.

## The two-stage slack approach (ZTWS)

The two-stage slack approach used in this study can be represented in the following form

Maximize: $\quad C^{\prime} X-E\left[F^{\prime} Y\right]$
subject to: $A X+\left(Y^{+}-Y^{-}\right) \leq B$,

$$
\begin{equation*}
X, Y \geq 0 \tag{5}
\end{equation*}
$$

where the second term in the objective function refers to the additional cost which may result from either excess slack or infeasibility in the final solution. In the set of constraints above when $B<A X$ then $Y^{+}=B-A X$ and $Y^{-}=0$. This is the case of excess slack. The infeasible situation arises when $B<A X$, then $Y^{+}=0$ and $Y^{-}=A X-B$.

This two-stage approach is based upon the premise that the investigator first solves the stochastic linear programming problem substituting an estimate for each of the stochastic parameters in the model. Once this initial solution is obtained, then the value of each of the stochastic parameters is observed and an adjustment made in the final solution reflecting any cost which may result from the development of either excess slack or infeasibility.

This procedure was reflected in the experimental model in the following way. The first stage solution was assumed to be the same as the expected value solution ZEXPC. It is assumed that the expected value of each stochastic parameter is the most reliable estimate the investigator has of that parameter. In accordance with this assumption the expected value solution is an
appropriate initial solution. Each iteration of each experiment, therefore, began with ZEXPC as the optimum value of the objective function.

The adjustments made in the second stage depended upon which parameters were stochastic. In the FORTRAN program of the model this two-stage approach followed after the simulation approach. The stochastic parameters had been identified and a specific value for each had been obtained. The first step in the two-stage approach was to make an adjustment in ZEXPC if any of the parameters in the $C$ vector were stochastic. This was required since ZEXPC had been determined by utilizing the expected values of the elements in the $C$ vector. If some of the elements in the $C$ vector were stochastic then the profit realized would depend upon the actual observed values of these stochastic elements. In the case where the stochastic elements were confined only to the $B$ vector, then the profit realized, which was as yet unadjusted for any additional cost of excess slack or infeasibility, was the same as the expected profit ZEXPC.

Having adjusted ZEXPC for the effects produced by the stochastic elements in the $C$ vector, the next step involved adjusting for the effects produced by any stochastic elements which may be present in the B. vector. Only constraints which contained a stochastic $b_{i}$ value had to be analyzed. If the solution vector of the expected
value solution, $X_{E V}$, is premultiplied by $A$, the matrix of fixed technological coefficients, then this product $A X_{E V}$ is a ( $3 \times 1$ ) vector which indicates the amount of the resources available which are accounted for in the expected value solution. In the quantity $A X_{E V}$ an expected amount of slack associated with the different constraints is also accounted for. For each constraint which was stochastic the corresponding rows of $A X_{E V}$ and $B$ had to be compared. For example if the ith constraint is stochastic, then $\sum_{j=1}^{3} a_{i j} x_{j}$ must be compared with $b_{i}$, where $b_{i}$ is the observed value of the stochastic parameter which was determined at the beginning of the current iteration. Infeasibility resulted when the sumnation above was greater than $b_{i}$ while excess slack resulted When the summation was less than $b_{i}$. Excess slack for any stochastic constraint is defined as the slack in the constraint over and above the expected slack which was indicated in the expected value solution. In the case where both quantities were equal then there was no additional cost to be deleted from the ZEXPC value.

The cost of infeasibility and the cost of excess slack for the constraints are exogenous parameters which must be defined in the experimental model. The constraints of the problem used in this study assume the nesources to be capital, land, and labor. In the case of excess slack
the interpretation is that interest, rent, and wages were contracted for but were not required. When infeasibility results the interpretation is that additional amounts of these resources' must be contracted for. In the latter case the unit costs of the different resources are assumed to be two times their regular unit costs.

The optimum value of the objective function determined with the two-stage slack approach is directly dependent upon the cost parameters which are used in the second stage of the approach to adjust the initial solution. This factor required an adjustment of the experimental model such that the two-stage slack approach was analyzed under alternative assumptions concerning the values of these cost parameters.

Three different sets of adjustment cost coefficients were used in the experimental model. The initial set of cost coefficients are

$$
\text { COSLK }=\left(\begin{array}{r}
.03 \\
20.00 \\
1.50
\end{array}\right) \quad \text { and } \quad \text { COINF }=\left(\begin{array}{r}
.06 \\
40.00 \\
3.00
\end{array}\right) \quad[6]
$$

The second and third sets are obtained by multiplying the set in [6] first by two and then by four. The vector COSLK in [6] is the vector of cost coefficients relating to excess slack which may result in each of the constraints. The cost coefficients relating to infeasibility in the constraints are contained in the vector COINF.

The optimum value of the objective function for the two-stage approach is called ZTWS. As is the case with
the simulation approach, each experiment was repeated 100. times with the mean (ZTWSBR) and the standard deviation (SDZTWS) of the optimum objective function values computed for each experiment.

The active approach (ZACT)
The active approach to stochastic linear programming is specifically designed to deal with stochastic elements which appear only in the constraints of the problem. Whenever stochastic elements appear only in the $C$ vector, then the active approach must be applied to the dual problem which can be formed from the stochastic primal problem. In practice, when both the $B$ and the $C$ vectors contain stochastic elements, the active approach is applied to the problem after first substituting for each stochastic element in the $C$ vector its expected value. If this procedure were carried out in the experimental model, then phase three of the experiment would be the same as phase one. Due to this fact, the active approach was only analyzed for the first two phases of the experiment.

The model of the active approach to linear programming under risk can be stated as

Maximize: $\quad Z=C^{\prime} X$.
subject to: $A X \leqq B U$,

$$
\begin{equation*}
x \geq 0 \tag{7}
\end{equation*}
$$

where in the constraints X is a n -dimensional diagonal matrix, $B$ is a m-dimensional diagonal matrix, and $U$ is a ( $m \times n$ ) matrix. The allocation matrix $U$ is such that $u_{i j}$ is the proportion of the ith resource to be allocated to the production of the $j$ th product. The active approach in this form assumes that all m constraints in the model contain stochastic elements. When some subset of the constraints of a problem contain stochastic elements, then the dimensions of the $A, B$, and $U$ matrices are reduced. For example when $r$ constraints contain stochastic elements, where $r<m$, then the dimensions of $A$ are ( $r \times n$ ), the dimensions of $B$ are ( $r \times r$ ), and the dimensions of $U$ are ( $\mathrm{n} \times \mathrm{n}$ ). The ( $\mathrm{m}-\mathrm{r}$ ) non-stochastic constraints are then added to the model in the same form as they would appear in a deterministic linear programming problem. For example in the ( $3 \times 3$ ) problem of this study if only $b_{I}$ is stochastic, then the constraints in the model of the active approach can be stated as

$$
\begin{gather*}
\left(a_{11} a_{12} a_{13}\right)\left(\begin{array}{ccc}
x_{1} & 0 & 0 \\
0 & x_{2} & 0 \\
0 & 0 & x_{3}
\end{array}\right) \leq b_{1}\left(u_{11} u_{12} u_{13}\right) \\
a_{21} x_{1}+a_{22} x_{2}+a_{23} x_{3} \leq b_{2} \\
a_{31} x_{1}+a_{32} x_{2}+a_{33} x_{3} \leqq b_{3} \\
x_{1}, x_{2}, x_{3} \geq 0 \tag{8}
\end{gather*}
$$

This set of constraints can be reduced to

$$
\begin{align*}
a_{11} x_{1} & \leq b_{1} u_{11} \\
a_{12} x_{2} & \leq b_{1} u_{12} \\
a_{13} x_{3} & \leq b_{1} u_{13} \\
a_{21} x_{1}+a_{22} x_{2}+a_{23} x_{3} & \leq b_{2} \\
a_{31} x_{1}+a_{32} x_{2}+a_{33} x_{3} & \leq b_{3} \tag{9}
\end{align*}
$$

The number of constraints in the final formulation of the active approach increases as the number of stochastic constraints increases. When there are two stochastic constraints the final formulation contains seven constraints; while with three stochastic constraints there are nine constraints in the final formulation.

When the stochastic elements are Iimited to the $C$ vector then the dual model can be stated as

Minimize: $\quad Z=B^{\prime} W$
subject to: $A^{\prime} W \geqq C V$,

$$
\begin{equation*}
W \geqq 0 \tag{10}
\end{equation*}
$$

where $W$ is a m-dimensional diagonal matrix, $C$ is a n-dimensional diagonal matrix, and $V$ is a ( $n \times m$ ) matrix analogous to the allocation matrix U. The general statement of the active approach in [10] assumes that all $n$ elements of $C$ are stochastic. In the case where only some subset of the elements of the $C$ vector are stochastic, then the dimensions of the matrices $A^{\prime}, C$, and $V$ can be reduced. The procedure involved in arriving at the final
formulation of the constraints in this dual case is analogous to the procedure described in the primal case. The only exception to the procedure is that in the dual model the sense of the constraints in the final formulation is "greater than or equal to" rather than "less than or equal to."

The programming of the active approach in the experimental model required that an assumption be made concerning the allocation ratios in $U$ and $V$. These ratios are exogenous values which must be read into the program. The expected value solution to the experimental problem was again used to detemine the values of these exogenous parametens. The expected value solution indicated that $X_{l}$ was the most significant variable in the problem. In fact $X_{I}$ is the only non-zero decision variable included in the optimum solution of the expected value approach. The importance of this variable can be reflected in the $U$ matrix by placing increased weight upon the allocation ratios in the first column of the matrix. Even though increased weight is placed upon the allocation ratios related to the production of $X_{1}$, the production of the other products $\left(X_{2}, X_{3}\right)$ must not be prohibited. The allocation ratios relating to all products must be greater than zero, so that the model allows for the possible production of all products.

Since the experimental results were dependent upon the specific values of the various allocation ratios which were used, the active approach was analyzed under two different sets of values fon the allocation natios of the U matrix. The allocation matrix $U$ which was used in the first run of the experimental model is

$$
U=\left(\begin{array}{lll}
.90 & .05 & .05  \tag{I1}\\
.90 & .05 & .05 \\
.90 & .05 & .05
\end{array}\right)
$$

The ratios in this matrix are designed to allocate 90 per cent of each resource to the production of $X_{I}$, while at the same time allowing lessex amounts of the different resources for the production of the remaining products.

The allocation matrix used in the second run of the experimental model was obtained by replacing each element of the first column of $U$ in [11] with. 75 and replacing each of the elements of the second and third columns with .125. In this second allocation matrix only 75 per cent of each resource is to be allocated to the producted of $X_{1}$. The sum of the ratios for each row in U must be one since each resource is to be fully allocated to the production of the different products.

When the $C$ vector contained stochastic elements and the dual problem was dealt with by the active approach, then the determination of the ratios in the matrix $V$ depended upon the importance of the constraints. in the
original primal problem. From the expected value solution of the specific example problem used it was determined that the first constraint is the most significant. The other two constraints each possessed some slack in the optimum solution. The importance of the first constraint in the primal problem was reflected in the dual probliem by placing the importance upon the first dual variable $W_{1}$. As was the case with the primal problem, the experimental results were dependent upon the specific values assigned to the allocation ratios of the $V$ matrix. Two different sets of values for the allocation ratios of the $V$ matrix were used. In the first run of the experimental model the allocation matrix used for the dual problem is

$$
V=\left(\begin{array}{lll}
.90 & .05 & .05  \tag{12}\\
.90 & .05 & .05 \\
.90 & .05 & .05
\end{array}\right)
$$

The changes made in the allocation matrix $V$ in the second run of the model were similar to the changes made in the allocation matrix $U$. Again, as was the case with the $U$ matrix, all the allocation ratios in the $V$ matrix must be greater than zero and the sum of the ratios for each row must equal one.

The FORTRAN program of the active approach followed the program of the two-stage approach in the experimental model. Initially the noutine embodied in the program determined which parameters in the experimental problem were stochastic. When the stochastic parameters were
confined to the $C$ vector the program converted the problem to its dual before the final formulation of the constraints was determined. The program proceeded directly to the final fomulation of the constraints in phase one where the stochastic parameters appeaned only in the B vector. When both vectors were stochastic the program omitted this section.

In phases one and two of the experiment the number of constraints in the final formulation ranged from five to nine depending upon the number of stochastic parameters in the particular experiment. In the second phase, when the problem was converted to its dual and the sense of the constraints was reversed, both a slack and an artificial variable had to be added to each constraint before using the simplex procedure. Since as many as nine constraints may be involved in any given application of the simplex routine, the number of variables in the simplex tableau can range up to twenty-one. In phase one the maximum number of variables encountered in the simplex procedure was twelve.

Having adapted the particular experimental problem to the requirements of the active approach the program then called the simplex subroutine (SIMPLX) and determined the optimum value of the objective function (ZACT), given that deterministic equivalent. Each experiment was repeated 100 times. The mean (ZACTBR)
and the standard deviation (SDZACT) of the optimum objective function values were then determined.

Summary of the Experimental Model
The experimental model developed in this study analyzed 102 different initial formulations of the specific problem used in the experiment. Each formulation was analyzed using a simulation approach (ZSIM), a two-stage approach (ZTWS), an active approach (ZACT), and an expected value approach (ZEXPC). One hundred iterations were performed for each initial formulation with the optimum objective function values recorded for each of the approaches. The differences between the results of the simulation approach and the results of each deterministic equivalent were also determined. The mean and the variance of these differences were then calculated for each deteministic equivalent. These respective means and variances were then used to evaluate the relative efficiency of each of the different deterministic equivalents.

## APPENDIX

## FLOW CHART OF

THE ACTIVE APPROACH


FLOW CHART OF THE TWO-STAGE APPROACH



FLOW CHART OF THE EXPERIMENTAL MODEL



C
FORTKAN PROGRAM UF THE EXPEKIFFNTAL MODEL

```
            DUNBLE PRECISIUN VAOBFU,A,R,C,AIJ,CJ,DAES,X,FOI,BI
            DUUBLE PKECISION XBAR,PCT, XEV,SAX,AX,STOD,ZSTD,CCJ
            DOUBLE PFECISION /SIM, ZTWS, ZEXPC,PHEF,SFFOF,COSINF
                DGUBLE PKECISION COSSLK,SUML,SOZ, KKLINHT,AL,GRL,CL,U,V
                OIMENSION XLV(0),XBAP(6),STOD(6),AX(3),YPLUS(3),Y(4)
            OIMEINSION X(4),SN/(3),SUMZ(3),/(4),CSL.(3),CIF(3)
```



```
            DIMENSIUN SENSE(21),N(21,21), B(2I),C(O1),JASIS(21)
            DIMENSION AIJ(21, 21),BI{21),CJ(21),BARX(6),STD(6)
            DIMENSION ZIM(100), ZNS(100), IACT(100),ZCT(100)
            DIMENSIUN IBVA(2),U(3,3),V(3,3)
            DIMENSION D(100,3),NS:N(100,3),SUDX(3),SUDSEx(3)
            DJMFNSIDN IDVNR(102,3),DHAR(1C2,3)
            COMMON A,AJJ,B,GI,C,CJ,VAGBFU,ZSTD,GASIS,SENSE,IX,MIN,
            IM,N,NN, IERSWT
C INITILIZINGVARIABLES
```



```
        DO100I=1,21
        SENSE(I)=0.C
        B(I)=0.0
        BI(I)=0.0
        BBI(J)=C.O
        C(I)=0.0
        CJ(I)=0.0
        100 CONTINUE
        DO1011=1,21
        00101J=1,21
        N(I,J)=0.0
        AIJ(I,J)=0.0
    101 CUNTINUE
        IER.SWT=1
C READING IN UATA PERTAINING TO THE PPOGRNAMINC NCDEL
```



```
    KCOUNT=100
    DO2 I= 1,11
    IX(1)=0
    READ(5,90) IX(I)
    90 FORMAT(110)
    2 CONT INUE
        KEAD(5,1010) MIN
    10LO FORMAT (II)
    READ(5,1000)M,N
    1000 FORMAT(213)
    DO1O2I=1,M
    KEAC(5,1001)SENSE(I)
    1001 FORMAT(F2.O)
102 CONTINUE
103 READ(5,1002)I,J,XX
1002 FORMAT(2I 5,D15.7)
    IF(I.EQ.99999)GOTO104
    A(I,J)=XX
```

```
            AIJ(l;J)=A(1,J)
            GOTO103
    l04 REAO(5,1003)I,XX
    1003 FURMAT(15,015.7)
        IF(I.GT.M)GOTO105
        R(I)=XX
        BI(I)=B(1)
        BDI(I)=B(I)
        GuTO104
    105 READ(5,10031J,XX
        IF(J.GT.N)GOTG199
            C(J)=XX
            CJ(J)=C(J)
            GOTO105
        199 REAC(5,10121I,J,XX,YY
    1012 FGRMAT(2I5,2D10.G)
        IF(I.E0.99999) CO TO 200
        U(I,J)=XX
        V(I,J)=YY
        GO TO 199
    200 REAL(5,102C)I,XX,YY
    1020 FORMAT(I5,2F10.4)
        It(I.EO.99909) GO TO 201
        CSL(I)=XX
        CIF(I)=YY
        GO TO 2CO
C CALLING SImPlEX ROUTINE FOR EXPECTED VAluE SOLUTIGN
```



```
    201 CALL SIMPLX
    DD1199I=1,NN
    XEV(I)=0.0
1199 CONTINUE
    DO119H K=1,NN
    AK=K
    D01198I=1,M
    IF(BASIS(I).NE.AK)GO TO 11g&
    XEV(K)=BI(I)
1198 CONTINUE
    ZEXPC=VAOBFU
    001400IEXPI=1,102
    BSWIT=0.
    CSWIT=0.
    MIN=0
    M=3
    N=3
    DO1013I=1,3
    IBVA(I)=0
1013 CONTINUE
    DO1409 I= 1,KCOUNT
    ZSIM(I)=0.
    ZTWS(I)=0.
    ZACT(I)=0
1409 CONTINUE
    DO1118I=1,6
```

```
    XBAR(I)=0.
    STDO(I)=0.
    BBT(I)=B(I)
    CCJ(I)=C(I)
    1118 CUNTINUE
C KEADING DNTA PERTAINIPIG TO THE VARICUS EXPERIMENTS
```



```
    REAC(5,11CO)KX
    1100 FOKMAT(I1)
            DO1200IT = 1,KX
            KEAD(5,11C1)XBAR(JI),PCT
    1LOL FORMAT(DI 5.7.05.3)
            STDD{II)=PCT*XBAR(II)
    1200 CONTINUF
C STARTING THE ITERATIONS FOR THE SIMULNTION MCDEL
```



```
    DO1300ISIM=1, KCOUNT
    DO1215I=1,21
    DO1215J=1,21
    AId(I,J)=A(I,J)
    L215 CONTINUE
    DO1216 I= 1,21
    3I(J)=0(I)
    CJ(I)=C(I)
    1216 CONTINUE
    BS*IT=0.
    CSWIT=0.
    M|N=0
    M=3
    N=3
    DO1218I=1 +21
    SENSE(I)=1.
    1218 CONTINUE
C UETERMINING THE STOCHASTIC ELEMEMTS FIIF EACH
```



```
    DO1220 II = 1,KX
    DO1201I=1,3
    IF((DAHS(XBAR(II)-B(I))).LT.L.) GO TO 1202
    IF((OA\SS(XBAR(II)-C(1))).LT. .OI) GO TO 1203
    1201 CONTINUE
    IERSWT=7
    GO TO 1401
    1202 BSWIT=1.
    IGVA(1)=1
    CALL RANL
    BI(I)=B(I)+ZSTD*STDD(II)
    BRI(I)=RI(1)
    GO TO 1220
2203 CSWIT=1.
    IBVA(I)=l
    CALL RAND
    CJ(I)=C(I)+ZSTD仵STDO(II)
    CCJ(I)=CJ(I)
1220 CONTINUE
```

```
C CALIING THF SIMPLEx ROIITINE FGH THIE SI:AULATIGN
```



```
    CALL SIMPLX
    IF(IERSWT.GT.I)ZSIM(ISIM)=0.O
    IT(IERSWT.GT.l) GO TO 1401
    ZSIM(ISIM)=VAOBFU
C DETERMINING THE TAO-STAGE SOLUTIDN--ZTGS
```



```
        IF(CSNIT.EQ.1.) 60 TO 1204
        PROF=ZEXPC
        GOITO 1206
    1204 PROF=0.
        D[1205K=1,NN
        SPROF=CJ(K)#XEV(K)
        PROF=PROF + SPROF
    1205 CONTINUF
    1206 IF(BSWJT.EQ.1.) GO TO 1207
        ZTWS(ISIM)=PROF
        GO T0 1500
    1207 001210I=1,M
        AX(I)=0.
    1210 CONTINUE
        DO12091=1,M
        DO1209J=1,N
        SAX=A(I;J)*XEV(J)
        AX(I)=AX(I)+SAX
    1209 CONTINUF
        CUSLK=0.
        COINF=0.
        001211I=1,M
        IF(AX(I)-B11{I))1212,1213,1214
    1212 YPLUS(I)=BEI(I)-AX(I)-XEV(I+M)
    IF(YPLUS(I).LT.C.) YPLUS(I)=0.
    COS=CSLII)*YPLUS(I)
    COSLK=CCSLK+COS
    GO TO 1211
    1213 YPLUS(I)=0.
    GO TO 1211
    1214 YPLUS(1)=AX(1)-3BI(I)
        COS=CIF(I)*YPLUS(I)
        COINF=COINF+COS
    1211 CONTINUF
        COSINF=COINF
        COSSLK=COSLK
        ZTWS(ISIM\=PROF-COSSLK-COSINF
C DETERMINING ThE ACTIVE RESULT--ZACT
```



```
1500 DO1520I=1,21
    BI(I)=B(I)
    DO1520J=1,21
    CJ(J)=C(J)
    AIJ(I,J)=A(1,J)
1520 CONTINUE
    ILDOP=0
```

```
IF((CSWIT*GSWIT).EO.L.) G日 TO 1411
[F(KX-2)1501,15C2,1b03
1501 1 \(P=0\)
GO TO 1503
1502 I \(P=3\)
1503 IFIBSWIT.FQ.1.) GO TO 1510
IF(MIN.EQ.C) GO Tü 1511
MIN=0
GOTOLS12
\(1511 \mathrm{MIN}=1\)
1512 DO15141=1, M
If(SENSE(I).EO.1.) GO TO 1513
SENSE(I)=1.
GO TO 1514
1513 SENSE(J)=2.
1514 CONTINUE
DO151 \(\mathrm{Cl}=1\), M
CJ(I)=B(I)
1516 GONTINUE
\(\mathrm{F}=\mathrm{N}\)
\(N=M\)
\(M=1=\)
\(15100015041=1,3\)
IF(IBVA(1).EO.1) ©U TU 1506
\(1 P=1 P+1\)
\(M M=M+[P\)
D01505J=1,3
IF(CSNIT.EO.1.) GO TU 1523
\(\operatorname{AIJ}(M M, J)=A(I, J)\)
GO TO 1505
1523 AlJ (M.4, J) = A (J,I)
1505 CONTINUE:
IF(CSNIT.EO.L.).GU TO 1524
EI(MM)=R(1)
SENSE(IAM)=SENSE(I)
GO TO 15O4
\(1524 \mathrm{BI}\left(\mathrm{MM}_{\mathrm{M}}\right)=\mathrm{C}(\mathrm{I})\)
SENSE (MA)=SENSE(I)
GO TO 1504
1506 D01507II=1,N
IF(ILOOP.EO.O) \(10=0\)
IF(ILOOP.FQ.1) IQ
IF(ILOOP.EO.2) [iv=2*, 4
[ \(M=10+11\)
IF(CSWIT.EQ.1.) GU TO 1521
BI(IM )=BBI(I)*U(I,II)
GO TO 1522
1521 BI(IM)=CCJ(I)*V(I,II)
1522 SENSE(IM )=SENSE(I)
DO1507J=1,N
IF(II.EQ.J) GO TO 1508
AIJ(IM, J) \(=0\).
GO TO 1507
1508 IFICSWIT.EQ. 1.1 GO TU 1525
```

```
    AIJ{IM,J)=^(1,J)
    G0 T0 1507
    1525 AlJ(IM,J)=A{J,I)
    1507 CONTINUE
        ILOUP=ILOOP+1
    1504 GONTINUE
        IF(KX.EQ.3)MM=9
        M=MM
        CALL SIMPIX
        ZACT(ISIM)=VAOBFU
        IF(IERSWT.GT.1) LNGT\ISIM|=0.0
    1411 IF(IERSWT.EO.1) GO TO 1300
C EXITS TO BF TAKLN WHEN AN ERPOR RESILTS IN THF SIMPIEX
```



```
    1401 GO TO (1410,1403,1404,1405,1406,1407,1408,1402),1FRSWT
    1402 WRITE(6,1004)
    1004 FGRMATIIK,'ERROE IN INITIAL TABLEAU O NO PDSITIVE ONE'
        1,' APPEARS IN THIS POW')
            GO TO 1300
    1403 WRITE(t,0,1005)
    LOO5 FORMAT(IX,'t:RROR IAS I'vITIAL IABLEAU O THIS COLUMN (SL',
        I'ACK DP ARTIFICIAL) HAS MORE THAN ONE UNIT RLEMLNT'J
        GO TO 1300
    1404 WRITE(6,1 102)
    1102 FORMAT(IX,'ERKOR O MORE VARIABLES ARE IN BASIS THAN'.
        I'THERE ARE CDNSTRAINTS')
        GO TO 1300
    1405 WRITE(6,5020)
    5020 FORMAT(IX,'SALUTIUN IS UNBCUNDED O NO AIJ(I,JK) IS',
        l'POSITIVE'l
        GO TO 1300
    1406 WNITE(6,5032)
    5032 FURMAT(IX,'PERTURBED CONSTPAINTS ARE STILL TIED....,
        l'CONSTRAINTS ARE LINFARLY DEPENDENT';
        GO TO 1300
    1407 WRITE(0,5040)
    5040 FORMATIIX,'PROGRAM MAY BE CYCLING VARIARLE HAS ENTERED
        1 10 TIMES')
        GO TO 13C0
    1408 WRITE(6.1006)
    2006 FERMATI/IX,'FRROR NU PARAMETERS ARE STOCHASTIC'I
        GO TO 1300
    1410 WRITE(6,1009)
    LOO9 FORMATI/IX,'ERROR. SNITCH SHOIULD NOT BRANCH TO 140I
        l WHEN IERSWT= 1')
    1300 CONTINUE
C DETFR:MNING THE MEAN AND THE varIANCES gF THF
```



```
    DO1903I=1,3
    sunx(I)=0.0
    SUDSOX(I) =0.0
    1903 CONTINUE
        DO19OOI=1,KCDUNT
    D(I, L)=ZSIM(I)-ZTWS(I)
```



```
            D(I, 3)=7.SIM(I)-7FXPC
            OSO(I, 1)=[(1,1)x*2
            OSQ(I,2)=0(1,2)*s%2
            0SO(I,3)=L(1,3)***2
    1900 CGNTINUE
    01j1901J =1,3
    0119011=1,KC(1UNT
    SUDX(J)=S|OX(J)+0(I,J)
    SUDSOX(J)=SUOSOX(J)+OS\(I,J)
    1901 CUNTINUE
            001902J=1,3
            OBAR(IEXPI,J)=SUDX(J)/LOO.
            OVAK(IEXPI,J)=(100.*SUBSivX{J)-SUCX(J)wx2)/(100.*99.)
    2902 CONTINUE
C DETERMINING THE MEAN, VARINNCE, ANO STANOARD DEVIATIGN
```



```
    001301 I= 1,3
    SOl(I)=0.
    SUMI(I)=0.
    1301 CUNTINUE
    DOL 3021=1,KCOUNT
    SUN7.(1)=SUML(1)+151.11()
    SUMZ(2)=SUMZ(2)+ZTWS(1)
    SUMZ(3)=SUMZ(3)+ZACT(1)
    SQZ(1)=50/(1)+2S1H(1)x+2
    SOZ(2)=S0Z(2)+LTWS(1)**2.
    S0<(3)=SO7(3)+2ACT(1)*:%2
    1302 CONTINIJF
    KKOUNT=KCOUNT
    ZSIMBK=SUMZ(1)/KKUUNT
    ZTWSBR=SUMZ(2)/KO゙GUNT
    ZACTBR=SUM2(3)/KKLUNT
    VRZSIM=(KKGUNT**SOZ(1)-SUM7(1))*22)/KKOUNT**2
    VRZTWS=(KKOUNT :SOZ(2)-SUMZ(2)**2)/KKOUNT:
    VKZACT=(KKOUNT*SU2(3)-SUMZ(3)**2)/KKOUNT*2%2
    SOZSIM=SQRT(VRZSIM)
    SDZTWS=SDRT(VRZTWS)
    SOLACT=SORT(VRLACT)
C THE CHI-SOUARF TEST OF NURMALITY GN ZSIM
```



```
    AL=ZSIMF;R-.6745*SD7SIM
    BL = 7SINBR
    CL=ZSIMARR+.6745*SU7SIM
    DO1309I=1.4
    Y(I)=0.
1309 CONTINUE
    DO1307I=1,KCOUNT
    IF(ZSIM(I).LT.AL) GO TO 1303
    IF{ZSIM(I).LT.BL) GO TO 1304
    IF(ZSIM(I).LT.CL) GO TO 23O5
    J=4
    GO TO 1306
1303 J=1
```

```
    CU TO 1306
    1304 J=2
        GO TO 130%
    1305 J=3
    1306 Y(J) =Y(J)+1.
    1307 CUNTINUE
        CHISUM=O.
        D013081=1,4
        C.HISUM=(.HISUM + (Y\I)-25.) %%2
    1308 CGNTINUE
        CHISSIM=.04%CHIS UM
C THE CHI-SOUARE TEST OF NDRNALITY ISN ZTHS
```



```
    NL=ZTWSAR - . 6745*SOTTWS
    BL=ZTWSBR
    CL=ZTWSBR+.6745*SOZT*S
    001310I=1,4
    X(I)=0.
    1310 CONT1NUE
    D[11311I=1,KCDUNT
    IF(7THS(I).LT.AL) GO TO 13L2
    IF(ZTAS(I).LT.3L) GU TO 1313
    IF(ZTKS(I).LT.CL) GQ TO 1314
    J=4
    GUTO 1315
    1312 J=1
    GO TO 1315
    1313 J=2
    GO TO 1315
    1314 J=3
    13l5 X(J)=X(J)+1.
    1311 CONTINUE
            CHISUM=O.
            DO131SI=1.4
            CHISUM=CHISUM + (X(I)-25. )**2
    1316 CONTINUE
            CHITHS=.04索CHISUM
C CHI-SOUARE TEST UF NIRMALITY ON ZACT
```



```
    AL=ZACTBR-.0745*SUZACT
    BL=ZACTBR
    CL=ZACTBR+.6745%SOZACT
    DO1317I=1,4
    Z(I)=0.
1317 CONTINUE
    DO1318I=1,KCOUNT
    IF(ZACT(I).LT.AL) GO TO 1319
    IF(ZACT(I).LT.BL) GO TO 1320
    IF(ZACT(I).LT.CL) GO TO 1321
    J=4
    GO TO 1322
1319 J=1
    GO TO 1322
    1320
    J=2
```

```
        00 ro 1322.
    1321 J=3
    1322.2(J)=2(J)+1
    1318 CINTINUE
        CHISUM=O.
        0013231=1,4
        CIII SUM=CHISUM+(7.(I)-25.1**2
    1323 ciNNTINUE
        CHIACT=.O4*C.ITISUM
C PRINT OHT STSTEMENTS
```



```
        WRIT[(G,1007)IEXPT,KX
    1007 FORNAT(/19X, 'FXPEKIHENT NUMBFR',3X,T3,17X, "NUMBFR OF',
    1' STOCHASTIC PARAMETERS',3X,12)
        WRETE(6,1103)
    1103 FURMAT(///13X,'PROPFRTIES DF STUCHASTIC PARA,METERS')
        WRITE(6,1104)
    1104 FGRMAT!/25X,'PARAMETER',5X,'MEAN',7X,'STANDARD',
    l' DEVIATION')
        DO1008I=1,KX
        BARX(I)=XBAR(I)
        STD(I)=STCD(I)
1008 C.ONTINUE
        WRITE(G,1105)(BARX(I),STD (I),I=1,KX)
1105 FORMAT(/37X,F12.'t.7X,F12.4)
        WRITF(6,1110)
1110 FORMAT(///23X,'PROPLRTIES GF THE ALTERNATIVE',
    1' DETER:AINISTIC EOUIVALENTS')
        WRITE{G,1111}
1l11 FURMAT(/30X,'SIMULATION',6X,'TWO-STAG[',1UX,'ACTIVE',
        16X,'EXPECTEO VALUE'।
        WRITE(6,1112)
1112 FGRMAT(31X,'APPROACH',8X,'APPRGACI',9X,'APPRGACH',9X,
    1'APPRGACH')
        ZEXG=7EXPC
        WRITE(G,Il13)ZSIMBR,ZTKSBR,ZACTPR, ZEXC
1113 FORMAT(/17X,'MEAN',9X,F10.4,6X,F10.4,6X,F10.4,6X,
    1F10.4)
        WRITE(G,1114)VRZSIM,VRZTWS,VRZACT
1114 FORMAT(/17X,'VARIANCE',4X,F11.3,5X,F11.3,5X,F11.3,1GX,
    1'NA')
        WRITE(6,1115)SD%SIM,SDZTWS,SDLACT
1115 FORMAT(/17X,'STD. DEV',5X,F10.3,0X,F10.3,6X,F10.3,1GX,
    1'NA"।
        WRITE(6,1116) CHISIM,CHITWS,CHIACT
1116 FORMAT(/17X,'CHI SU TEST',2X,F10.5,6X,F1O.5,6X,F10.5,
    110X,(NA')
        WRITE{6,1117)
1117 FORMAT(/17X,'ALPHA =.05 7.81473 AL=.01 11.3449*)
        WRITE(6,1106)
```




```
        WRITE(G,1920)DHAR(IEXPI,1),DBAR(IEXPI,2),DGAR(IEXPI,3)
1920 FORMAT(22X,Fll.2,1lX,F11.2,11X,Fll.2)
```



```
    WRITE(6,192O)OVAK([IXPI,1),0VAP(IFXPI,2),0VAR(IEXPI,j)
    NRITE(7, 1 O2O)DVAR(IEXP[ , 1),DVAR(IFXPI,2),DVAFIIEXPI,3)
    1400 CONTINUE
        STOP
        FND
C
C THE RANITIM NUMBFR GENEIRATOR SUBFODTINL PIRIDUCING A
```



```
    SUBRDUTINE PANI
    DOUBLE PRECISION /STU,A,AIJ,i},RI,C,CJ,VAOHFU
    OIMENSIUN IX(11),R(2),NIJ(21,21),0I(21),CJ(21)
    DIMENSION SFNSE(21),A(2L,21),H(21),C(21),BASIS(21)
    COMMON A,AIJ,[G,HI,C,CJ,VAUBFU,ZSTD,BASIS,SENSE,IX,MIN,
    LM,N,NN, IFRSWT
    DO22 II= L,2
    I=11
20 IIX=1X(I)
    IY=IIX** 46351
    IF(IY)5,6,6
    5 I Y=IY +21474&3647+1
    6 YFL=IY
    IX(I)=IY
    YFL=YFL\.4656613E-9
    IF(I.NE.11)GU TO 21
    IE(YFL.LT..1)GO TO ]L
    IF(YFL.LT..2) F(I TO 12
    IF(YFL.LT..3) GUTG) 13
    IF(YFL.LT..4)GO TO 24
    IF(YFL.LT..5) GO TO 15
    IF(YFL.LT..6) GO TO 16
    IF(YFL.LT..T) GU TD 17
    IF(YFL.LT..8) GO TO 18
    IF(YFL.LT..g) GOTO 19
    I=10
    GO TO 20
11 I = L
    GO TO 20
12 I=2
    GO TO 20
13 I = 3
    GO TO 20
14 I=4
    G0 TO 20
15 I=5
    GO TO 20
16 I=6
    GO TO 20
17 I=7
    GO TO 20
18 I=8
    G0 T0 20
19 I =9
    GO TO 20
```

```
    21 R(II)=YFL
    22 CONTINUE
        7.STI)=SQRT(-2.*ALUG(i*(1)1)*60S(2.*3.142857%R(2))
        RETURN
        t:ND
r
C
C
    THE SIMPLFX SURROIJTINE IDR SOIVIAG A LITFAR PRCORGMMING
```



```
    SUBRIJUTINE SIMPIXX
    DOUBLE PRECISION VAOHFU,ENTR,BIPYM,PAIJ,SUPAID,PISI
    DCIURLE PRECISIOM A,AIJ,R,BI,C,CJ,DASCJ,ZJCJ,ZJ,SUAZd
    DOUBLE PRECISIUN AIJPI,GIPI,SUBENT,ALABGE,SMALL,DABS
    DOUBLE PRECISION ZSTD,PIVIJT,VALU,SUVALU,X
    DIMENSION S[NSE(2l),A(21,21),B(21),C(2l),BASIS(21)
    DIMENSION ZJCJ(21),ZJ(21),SUM/J(21),FR:TK(21), QIPYM(21)
    DIMENSICN PAIJ{2l),SUPAIJ(2l),PBI(21),CYCLE(2l)
    DIMENSIUN AIJPI(21,21), fIPI(21),AIJ(21,21),BI(2I)
    DIMENSIPN HASCJ(21),CJ(21),IX(1)1)
    COMMUN A,ATJ,R,BI,C,CJ,VAOBFU,ZSTO,HASIS,SENSE,IX,MIN,
1M,N,NN,TERSWT
```



```
    DO1001=1,21
    BASIS(I)=0.0
    BASCJ(I)=0.0
    ZJCJ(I)=0.0
    7J(1)=0.0
    SUMZJTI}=0.0
        - BIPYM(I)=0.0
        PAIJ(I)=0.0
        SUPAIJ(I)=0.0
        PBI(I)=0.0
        CYCLE(I)=0.0
    100 CONTINUE
        I ERSWT=1
        NS=0
        NA=0
```



```
199 DO2101=1,M
    K=1
    IH{SENSE(I)-2.)200,201,202
200 N.S=NS+1
    AIJ(1,N+NS+NA)=1.O
    CJ(N+NS+NA)=0.0
    GUT0209
201 NS=NS+1
    AIJ(I,N+NS+NA)=-1.0
    CJ(N+NS+NA)=0.0
    NA=NA+1
    IF(MIN.EQ.I) GO TO 720
    AIJ(I,N+NS+NA)=1.0
    CJ(N+NA+NS) =-9909.
    G0TO209
720 AIJ(I,N+NS+NA)=1.0
    CJ(N+NS+NA)=9999.
```

```
        G0 T0 20G
    202 NA=NA+1
        IF(M|N.E.O.l) Gil TH 72l
        AIJ(I,N+NS+NA)=1.0
        C.J(N+NA+NS)=-90Gg.
        GO TO 209
        721 AlJ(1,N+NS+NA)=1.0
            C.J(N+NS+NA)=999%.
    209 IF(I.GF.MIGGTO300
    210 CONTINUE
    300 NN=N+NS+NA
```



```
    JJ=1
    DO32OL=1,M
    I = L
    J=N+1
    301 IF(AJJ(I,J).EQ.1.IGOTO3C2
        J=J+1
    [F(J.Lt.NN) GO TO 3Cl
    IERSWT=B
    GU TO 998
    302 1F(I.EQ.I)GGTO303
        KK=I
        I=0
    311 I= I +1
        IF(I.NE.KK)GOTO 304
        C0TO305
    303 I= I + 1
    304 IF(A(I, J).EQ.0.C)GOTD 305
        I ERSWT=2
        GO TO 998
    305 IF(I.GE.M)GOTO308
        IF(L.EQ.1)GOTO3O3
        G0r0311
    308 AJ=J
        IF(JJ.GT.M)GOTO 312
        BASIS[JJ)=AJ
        JJ=JJ+1
        GuTO320
    312 IERSWT=3
        GO TO 99R
    320 CONTINUE
C FINLJING CUST VECTGR FOR BASIS VARIABLFS ***************
400 DO4OLJJ=1,M
        JJK=BASIS(JJ)
        BASCJ(JJ)=CJ(JJK)
    401 CUNTINUE
C FINDING ZJCJ(J).INDIGATOR ROW AND VALUE OF OEJECTIVE
```



```
    603 DO 604 J=1,NN
        ENTR(J)=0.0
        ZJ(J) =0.0
        SUMZJ(J) =0.0
        ZJCJ(J) =0.0
```

```
    604 C.UNTINUF
    007401=1,M
    BIPYM(I)=559909.
    PAIJ(l)=0.0
    SUPAIJ(I)=C.O
    PRI(I)=0.0
    BIP[(I)=0.0
    7 4 0 ~ C O N T I N U E
    KMARK=0
    00741I=1,M
    00741J=1, NN
    AIJPI(I,J)=0.0
    741 CONTINUE
    D0402J=1,NN
    D04021=1,N
    LJ(J)=AIJ(I,J)*igASCJ(I)
    SUMZJ(J)=SUMZJ(J)+7J(J)
    4 0 2 ~ C O N T I N U E ~
        D0403J=1, NN
        ZJGJ(J)=SUMZJ(J)-CJ(J)
    403 CONTINUE
        VAEIBFU = 0.0
        DO420I=1,M
        VALU=BI(I)*UASCJ(I)
        VADBFU = VACBFU + VALU
    420 CONTINUE
```



```
        IF(MIN .EQ.I) GO TO }73
        1004G4J=1,NN
        IF(7JCJ(J).LT.0.0)ENTR(J)=2JCJ(J)
    404 CONTINUE
            GO TO 732
        730 00 731 J=1,NN
            IF (ZJCJ(J).GT.C.0) ENTP(J)=ZJCJ(J)
        731 CONTINUE
    732 DO 405 J=1.NN
        IF(ENTR(J).NE.0.0)GOTO406
    405 CONTINUE
            IERSWT=1
        998 RETURN
    40G ALARGE = 0.0
C
        TESTING FOR FNTERING VARIABLE ********)
        D0407J=1,NN
        SUBENT = ENTR(J)
        IF(DABS(ALARGE).GE.DABS(SUBENT)) GO TO 407
        ALARGE =DABS(SUBENT)
        JK =J
    407 CONTINUE
```



```
    MARKER=0
    D0503[=1,M
    [F(AIJ(1,JK).GT.0.0)GOTO501
    BIPYM(I)=999799.
    GOT0503
```

```
    501 MARKFiR=1
        BIPYM(I)=1.I(II/AIJ(I,JK)
    503 CUNTINUF
    IF(PIARKElZ.F(j. 1)GOTO504
    IFRSWT=4
    G(T) TO 9G8
    504 SMALL=ISIPYM(1)
        IP=1
        D0506I=2,M
        IF(SMALL.LE.BIPYM(I))GINTOSOG
        SMALL=BIPYM(I)
        IR=I
506 CONTINUE
    005051=1,M
    It(I.FO.IR) COJ TO 5C5
    IF((CABS(SMALL-BIPYM(I))).LY. .COCOOCO) KMAFK=1
    505 CONTINUE
    IF(KMA&K.NE.1IGOTOS10
    IRK = IR
    DO5091=1,N
    IF(I.EG.IRK) GO TO 509
    IF((DABS(SMALL-RIPYM(I))).GT. .OOCOCCO) GO TO 509
    WRITE(G,5030)
    5030 FERNAF(IX,'SOLUTICN IS DEGFNERATE')
```



```
    E=.01
    DO 750 K=1:M
    PAIJ(K)=0.0
    SUPAIJ(K)=0.0
    PBI(K)=0.0
    750 CONTINUE
    DO508J=1,NN
    PAIJ(IR)=E%*|#AIJ(IR,J)
    SUPAIJ(IR)=SUPAIN(IR)+PAIJ(IR)
    PAIJ(I)=E 2**)
    SUPAIJ(I)=SUPAIJ(I)+PAIJ(I)
508 CONTINUE
    PBI(IR)=BI(IP)+SUPAIJ(IR)
    PBI(I)=0I(I)+SUPAIJ(I)
    BIPYM(I) = PBI(I)/AIJ(I,JK)
    RIPYM(IR) = PEI(IR)/AIJ(IR,JK)
    IF((DABS(BIPYM(I)-BIPYM(IR)\).LT.1.00-17) GO TO 5II
    IF(BIPYM(I).LT.BIPYM(IR)) GO TO 760
    KKK=I
    CO TO 509
    760 KKK=IR
        IR=I
509 CONTINUE
    GU TO 510
    511 IERSWT=5
    GO TO G98
    510 BAS(S(IR)=JK
    BASCJ(IK)=CJ(JK)
    CYCLE(JK)=CYCLE(JK)+1.0
```

```
            IF(GYCLE(JK).LK.10.)GWTLIGÖG
            IFRSWT=6
            GO 「O 90%
```



```
    600 PIVI)T = AJJ(IR,JK)
            DOGO1J=1,NN
            AlJPl(IK,J)=AIJ(IK,N)/PIVOT
GOL CONTINUE
            GIPI(IR)=BI(IR)/PIVOT
            DO 700 I= 1,M
            IF(I.EO.IR)GOTO70O
            BIPI(I) = BI(I)-(AIJ(I,JK)*BIPI(IR))
    700 C.ONT INUE
            IF(KMARK.[Q.1) BIPI(KKK)=0.0
            DD6O2 I =1,M
            IF(I.EW.IR) GO TO 6C?
            DI? 7OL J=L,NN
            AIJPI(I,J)=AIJ(I,J)-{AIJ(I,JK)*AIJPI(IR,J))
    701 CONTINUE
602 CUNTINUE
            DU 7C5 J=L,NN
            D0 705 I = 1,M
            AIJ(I,J)=AIJPI(I,J)
    705 CENTINUE
            DO 7CG I= L,M
            B1(I)=BIPI(I)
    700 CGNTINUE
            GOTU603
            FND
```


## CHAPTER V

## ANALYSIS OF THE EXPERIMENTAL RESULTS

## Introduction

The objective of this study, as it is stated in the first chapter, has been to determine the efficiency of using various alternative deterministic equivalents in order to solve a stochastic programming model. To achieve this objective a simulation of the stochastic model was used as a standard for comparison. For each set of initial conditions the expectations of the optimum objective function values which have been determined by utilizing each deteministic equivalent were compared to the optimum objective function values determined from a simulation of the model.

This chapter presents an analysis of the results generated by the experimental model used in this study. These results are separated into two parts, reflecting the two sets of results obtained for the two problems used.

The next section of this chapter includes a statement of the statistical test which has been performed on the results of the study. In this section special emphasis is placed upon the organization of the results
of the experiments, the statement of the hypotheses which are tested, and the assumptions upon which the statistical tests are based. The following section presents analyses of the results of the statistical tests, given the various assumptions made concerning the initial formulations of the model. Each deterministic equivalent of the stochastic programming model was tested in each phase of the experiment as the coefficients of variation of the stochastic parameters changed. The final section of this chapter states the conclusions which have been drawn from this study and indicates some aspects of the problem which require further experimentation.

In the appendix to the chapter the experimental results are presented in tabular form. Tables 2 through 4 present, for problem $A$, the sample means of the distributions of the optimum objective function values for each deterministic equivalent as well as for the simulation approach. Tables 5 through 7, present a similar set of results for problem B. In Table 8 and Table 9 the results from the application of the statistical tests are presented for problems $A$ and $B$ respectively. Tables 10 through 12 summarize, for each deterministic equivalent, and for each phase of the experiment, the findings relevant to the cases when (1.) the solutions generated by each of the deterministic equivalents were on the average feasible and (2) when the results led to an acceptance of the
hypothesis of no difference between the simulation approach and the particular deterministic equivalent.

## The Statistical Tests

The characteristics of
the experimental results
In each phase of the experiment each initial formulation of the empirical problem has been used to generate a series of optimum objective function values for the simulation approach and for each deterministic equivalent. For each of the 102 initial formulations of the problem 100 iterations were performed. Thus 100 values of ZSIM, ZTWS (20), ZTWS(40), ZTWS(80), ZACT(.90), and ZACT (.75) were generated. ${ }^{I}$ For each of the 102 experiments the mean values ZSIMBR, ZTWSBR(20), ZTWSBR(40), ZTWSBR(80), $\operatorname{ZACTBR}(.90)$, and ZACTBR(.75) which were determined are presented in Tables 2 through 7 in the appendix. The optimum value of the objective function, ZEXPC, determined from the expected value approach is also included in the tables mentioned above.

For any given initial formulation of the problem, the sample of the optimum objective function values
$I_{\text {The }}$ term ZTWS (20) refers to the two-stage approach utilizing the set of cost vectors, which includes 20 as the first cost coefficient in the vector COSLK. The terms ZTWS (40) and ZTWS (80) respectively refer to the two-stage approach utilizing the second and thind sets of cost vectors COSLK and COINF. Similarly, ZACT(.90) and ZACT(.7.5) refer to the active approach utilizing the different sets of allocation natios indicated.
generated for each deterministic equivalent is not necessarily independent of the corresponding sample generated by the simulation approach for this initial formulation. . This statement is supported by the fact that on each iteration of the experiment, once the values for the stochastic parameters were determined, then those values were used to generate an optimum solution for each deterministic equivalent and for the simulation approach. In effect, then, dependent samples were used in the statistical tests performed in this study.

A statement of the hypothesis and the statistical test ${ }^{2}$

The statistical test performed in this study is designed to test the hypothesis of the equality of two population means under the conditions that the populations from which the samples were taken are normally distributed and that the individual samples are not independent.

For each deterministic equivalent tested in the experiments, $d_{i j}$ is defined as

$$
\begin{equation*}
\mathrm{d}_{i j}=\mathrm{ZSIM}_{j}-\mathrm{x}_{\mathrm{i} j} \tag{1}
\end{equation*}
$$

The value, $x_{i j}$, is the optimum objective function value determined on the jth iteration by using the ith

[^14]deterministic equivalent, while $Z_{S I M}^{j}$ is the optimum objective function value determined on this iteration by using the simulation approach. Since six variations of the different deterministic equivalents were considered the $i$ subscript ranged from 1 to 6 . The $j$ subscript which referred to the number of iterations ranged from l to 100. A value of $d_{i j}$ was determined for all deterministic equivalents on each iteration of the experiments.

For each experiment the mean and the variance of the $d_{i j}$ 's were defined as

$$
\begin{align*}
\mathrm{d}_{i h} & =\frac{\sum_{j=1}^{100} d_{i j}}{100}, \text { and } \\
s_{d_{i h}}^{2} & =\frac{100 \sum_{j=1}^{100} d_{i j}^{2}-\left(\sum d_{i j}\right)^{2}}{100 \cdot 99} \tag{2}
\end{align*}
$$

In the equations in [2] i refers to the different deterministic equivalents, $h$ refers to the different experiments which were performed, and $j$ refers to the number of iterations of each experiment.

The statistical tests were performed on the $\bar{d}_{i h}$ values, which are the means of the differences between the paired sample results. These mean values were determined by averaging the differences between the optimum objective function values generiated by the simulation approach and by each deteministic equivalent.

These optimum objective function values are assumed to be deperdent since on any iteration the same values of the stochastic parameters were used to. generate a solution for each approach.

The distributions of these different mean values are normal. The reader should reciall that through an application of the Central Limit Theorem it can be assumed that, as the size of the sample increases, the distribution of the means of a set of samples randomly and independently drawn from a given population will approach a normal distribution regardless of the shape of the parent population.

The general model of the statistical test can be stated as

$$
\begin{equation*}
d_{j}=x_{1 j}-x_{2 j}=\mu_{1}-\mu_{2}+\varepsilon_{j} \tag{3}
\end{equation*}
$$

where $\varepsilon_{j}$ is a normally distributed error term with a mean equal to zero and a standard deviation equal to the standard deviation of the population of the $d_{j}{ }^{i} s$.

The Z-statistic ${ }^{3}$ can be defined as

$$
\begin{equation*}
z_{i}=\frac{{\overline{\overline{d_{i}}}}_{i} \sqrt{n}}{s_{d_{t}}} \tag{4}
\end{equation*}
$$

${ }^{3}$ When the expected value approach is tested then $x_{i j}$ in equation [l] is constant for all values of $j$. In this case the statistic $Z$ in equation [4] is equivalent to

$$
z=\frac{(\bar{x}-a) \sqrt{n}}{s_{i}}
$$

where a is the constant value, ZEXPC, $\overline{\bar{x}}$ is the mean of the
where

$$
\begin{align*}
\overline{\mathrm{d}}_{i} & =\frac{\sum_{h=1}^{\mathrm{k}} \overline{\mathrm{~d}}_{i h}}{\mathrm{k}}, \\
\mathrm{~s}_{\mathrm{d}_{\mathrm{t}}}^{2} & =\sum_{h=1}^{\mathrm{k}} \mathrm{~s}_{\mathrm{d}_{i h}}^{2} \quad, \text { and where } \tag{5}
\end{align*}
$$

$\overline{\mathrm{d}}_{\text {ih }}$ and $s_{d_{i h}}^{2}$ are given in the equations in [2]. The second equation in [5] indicates that $s_{d_{t}}^{2}$ is a pooled variance Which is determined from the sample variances of the $d_{i j}{ }^{\prime} s$. Although the population variance is unknown, a $Z$ test is appropriate due to the large sample sizes resulting from each experiment.

In the first two phases of the experimental procedure four initial formulations of the experimental problems were evaluated as the coefficients of variation of the stochastic parameters changed. In the third phase nine initial formulations were tested. In the first two phases, the value of $h$ in equation [5] ranged from one to four and $n=400$; while in the third phase, the value of $h$ ranged from one to nine and $n=900$. For each experimental problem considered, six statistical tests were performed in each phase on the different variations of the deterministic equivalents which were evaluated.
$h$ values of $Z S I M B R$ and $s_{j}$ is the sample standard deviation of all $n$ ZSIM values included. In this case the nuIl hypothesis states that the mean of the population from which the samples of. ZSIMBR values are taken is equal to the constant value ZEXPC. When this null hypothesis cannot be rejected the conclusion is that the expected value approach yields a result which is statistically the same as the simulation approach.

The null hypothesis which was tested in each case is that the means of the two populations from which the samples were drawn are equal. . The alternative hypothesis is that they are unequal. When the null hypothesis is accepted, the conclusion to be drawn is that, given these conditions, the deterministic equivalent yields a result which is not statistically different from the result determined by the simulation approach.

## Analysis of the Experimental Results

For purposes of analysis the experimental results are sumarized in Tables 10 through 12 of the appendix. Each table refers to a different phase of the experiment. Within each table the performances of the different deterministic equivalents are summarized for each experimental problem which was used.

The two-stage approach is presented under three different assumptions concerning the cost coefficients in the vectors COSLK and COINF, while the active approach is presented in terms of the two different assumptions concerning the values of the allocation ratios in the $U$ and $V$ matrices.

For each value of the coefficient of variation of the stochastic parameters, an " $X$ " is placed in the appropriate columns of the table if the deterministic equivalent on the average yielded a feasible solution and if the null hypothesis cannot be rejected at either
the .05 or the .. 01 levels of significance. The ith deterministic equivalent is considered to yield a feasible solution "on the average" if the value of $Z_{i}$ in equation [4] is positive. The value $Z_{i}$ is positive when $\overline{\bar{d}}_{i}$ is positive. The reader should realize that even though $\overline{\bar{d}}_{i}$ is positive this does not exclude the possibility of an infeasible solution for the ith deterministic equivalent resulting on any particular iteration. The $\overline{\bar{d}}_{i}$ value is determined in the first two phases of the experiment by averaging $400 \mathrm{~d}_{\mathrm{ij}}{ }^{\prime} \mathrm{s}$ and in the third phase by averaging $900 \mathrm{~d}_{\mathrm{ij}}{ }^{\prime} \mathrm{s}$. These $\mathrm{d}_{i j}$ values can be positive or negative. The conditional probability of a feasible solution resulting on any iteration increases as the value of $Z_{i}$ increases from zero. As the value of $Z_{i}$ decreases from zero, the probability of an infeasible solution on any iteration is increased.

Consider, in Table 10 , the data which resulted from using the expected value approach in phase one of the experiment. From the table it can be seen that, given experimental problem $A$, for a coefficient of variation equal to .05 the expected value approach yielded an infeasible solution on the average which is not statistically different from the solution yielded by the simulation approach at either the .05 or the .01 levels of significance. For experimental problem B the interpretation is the same.

## Phase one

The results from phase $I$ of the experiment are summarized in Table 10. The three deterministic equivalents are referred to as ZEXPC, ZTWS, and ZACT. The last two detemministic equivalents mentioned above are presented under multiple assumptions concerning the exogenous parameters used in those approaches.

The expected value approach
In phase one the expected value approach yielded infeasible solutions, on the average, for all values of $V$ considered regardless of the experimental problem used. The results of the Z -test are also similar for both problems, with the only exception occurring when $V=.20$ and $\alpha=.01$. At both the .05 and the . 01 levels of significance, the null hypothesis was rejected in all. cases except where $V$ had small values. This is true for both experimental problems considered.

In summary the expected value approach in phase one yielded infeasible solutions which led to rejection of the hypothesis of no difference at both levels of significance for the largen values of $V$. In addition this approach was consistent in its results for both experimental problems.

The two-stage approach
The two-stage approach was considered under three different sets of exogenous cost coefficients. In phase
one where only, the B vector is stochastic the optimum value of the objective function in the two-stage approach is initially the same as the optimum value of the objective function for the expected value approach. . The costs resulting from either excess slack or infeasibility in the constraints are then deducted from this initial objective function value. This factor is represented in the table by the fact that the two-stage approach yielded feasible solutions for both problems for the smaller values of $V$ while the expected value approach did not. As the cost coefficients were increased, feasible solutions for both experimental problems resulted for the high values of $V$.

For the smallest set of cost coefficients the null hypothesis was not rejected at either level of significance for the smaller values of $V$. For the values of $V$ greater than or equal to .25 the null hypothesis was accepted only at a level of significance of $\alpha=.01$. When the adjustment cost coefficients were increased the results were feasible on the average for the higher level of $V$. However, as these adjustment cost coefficients were increased, this approach yielded inconsistent results with respect to the tests of the null hypotheses. For example, for the second set of adjustment cost coefficients considered, the null hypothesis was accepted at both levels of significance for all values of $V$. When the largest set of adjustment cost
coefficients were used, the null hypothesis was rejected in all cases except where $V=.05$ and $\alpha=. .01$. These results were consistent with respect to both of the experimental problems analyzed.

In phase one when the two smallest sets of cost coefficients were used the two-stage approach can be summarized as yielding results which were feasible on the average and not significantly different from the simulation results at the . 01 level of significance for all values of $V$. Fon the largest set of cost coefficients used the results were significantly different from the simulation results at both of the levels of significance. This approach also yielded very consistent results over both experimental problems considered.

The active approach
The active approach was evaluated under two different sets of allocation ratios. For each experimental problem and for each set of the allocation ratios the solutions using the active approach were feasible on the average for all values of $V$. For both of the problems analyzed all the tests led to a rejection of the null hypothesis of no difference from the simulation approach for all values of $V$.

Summary of phase one
In phase one the expected value approach yielded only infeasible solutions on the averiage. The two-stage
approach did yield feasible solutions on the average for the smaller values of $V$; and as the adjustment cost coefficients increased, feasible solutions resulted for the higher values of $V$. The active approach always yielded feasible solutions on the average. Each deterministic equivalent yielded consistent results in terms of the feasibility or infeasibility of the results generated for both experimental problems used.

The expected value approach yielded results which were not significantly different from the simulation approach at either level of significance for the smaller values of $V$. The two-stage approach, utilizing the second set of cost coefficients, yielded results which were not significantly different from the simulation approach at either level of significance for all values of V. For the smallest set of cost coefficients the null hypothesis was rejected at both levels of significance only for the larger values of $V$, while for the largest set of cost coefficients used, the null hypothesis was rejected at both levels of significance for all values of V. The active approach yielded results which were significantly different from the simulation approach at both levels of significance for all the values of $V$ considered.

## Phase two

In phase two of the experiment the stochastic parameters were confined to the C vector. The results
of this part of the experiment are presented in Table 11 of the appendix.

The expected value approach
In the seciond phase of the experiment the expected value approach yielded feasible results on the average in all except the cases dealing with problem $A$ for $V=.05$ and $V=$.l0. Only in two cases did this approach yield results leading to a rejection of the hypothesis. These exceptions occurred when dealing with problem $B$ for $V$ equal to .25 and $\alpha$ equal to either .05 or .01 . As was the case in phase one the results from this approach were consistent over both experimental problems considered.

The two-stage approach
The adjustment cost coefficients of the two-stage approach are related to the constraints of the problem under investigation. Since the stochastic parameters were present only in the $C$ vector in this phase, these adjustment costs do not affect the optimum objective function value which is initially determined. This initial value is determined by postmultiplying the $C$ vector, the elements of which have been randomly determined, by the optimum solution vector of the expected value approach. It was assumed in the experimental model that the investigator, in using this approach; based his planning upon the expected values of the stochastic parameters involved and
then later adjusted these plans at an additional cost. As was the case in phase one three sets of adjustment cost coefficients were used in the experimental model.

Since the adjustment cost coefficients are related directly to the constraints and since these constraints are deterministic, then it is expected that the experimental results should be independent of these adjustment costs. Table 11 indicates that this conclusion is correct. There was no change in the experimental results as the adjustment cost coefficients increased.

For any set of adjustment cost coefficients the results of the two-stage approach were feasible on the average for both of the experimental problems analyzed. The tests of the null hypothesis led to a distinct set of conclusions for the two problems considered. In dealing with the slightly constrained problem, the results of the two-stage approach led to a rejection of the null hypothesis at the .05 level of significance for values of V.greater than or equal to .20 , while at a level of significance of . 01 the results led to a rejection of the null hypothesis only when $V=.30$. The null hypothesis was rejected in all cases at both levels of significance when the tightly constrained problem was analyzed.

The active approach
The active approach in this phase was performed by first converiting the experimental problem to its dual
problem and then expanding the set of constraints of the dual according to the number of stochastic parameters involved in the formulation. As was the case in phase one the active approach was analyzed under two assumptions concerning the allocation ratios used in the model.

The results of the active approach were consistent with respect to the two sets of allocation ratios used and with respect to the two experimental problems analyzed. For all cases in this phase the results from the active approach were infeasible and led to a rejection of the null hypothesis at both the .05 and the .01 levels of significance.

Summary of phase two
In phase two the expected value approach yielded results which were generally consistent for both types of experimental problems considered. This approach yielded results which were also generally feasible on the average and not significantly different from the simulation results at either level of significance. The two-stage approach yielded feasible results on the average in all cases. These results were significantly different from the simulation results at both levels of significance when dealing with the tightly constrained problem. With the slightly constrained problem the results of this approach were significantly different from the simulation approach only. for the higher values of $V$ considered. The
active approach yielded results which were consistent for the two experimental problems analyzed. These results were always infeasible on the average and significantly different from the simulation results at both levels of significance for all the values of $V$ considered.

The table also indicates that the results of the two-stage approach were consistent with respect to the different adjustment cost coefficients that were used and that the results of the active approach were consistent with respect to the different allocation ratios used.

## Phase three

In phase three of the experimental procedure only the expected value approach and the two-stage approach were evaluated. The two-stage approach was analyzed for each of the three sets of adjustment cost coefficients. In Table 12 the results of the thind phase are summarized.

The expected value approach
The results generated by the expected value approach were feasible on the average for both experimental problems only when $V=.05$. For higher values of $V$ the results were not feasible. For both problems considered the experimental results led to a rejection of the hypothesis at the .05 level of significance only when $V$ equaled . 25 or .30 . At the Ievel of significance of $\alpha=.01$ the results for
the tightly constrained problem $B$ were not significantly different from the results of the simulation approach for any value of $V$, whereas for the slightly constrained problem the results were not significantly different for values of $V$ less than or equal to .20. The expected value approach was consistent in its results over the two experimental problems with the only exceptions resulting when $V=.25$ or .30 .

The two-stage approach
The solutions resulting from the two-stage approach in phase three were always feasible on the average for both experimental problems when the two largest sets of adjustment cost coefficients were used. For the smallest set of adjustment cost coefficients the feasibility of the results differed for the two experimental problems. In the slightly constrained problem A the solutions were feasible on the average only for $V=.05$, while in the tightly constrained problem the solutions were feasible on the average for all $V$ values except $V=.30$.

The two-stage model utilizing the smallest set of adjustment cost coefficients was consistent for each experimental problem with respect to the rejection of the null hypothesis at both levels of significance tested. The rull hypothesis, when problem $B$ was used, was not rejected at either Ievel of significance for any value of $V$ considered. The results when problem $A$ was used
led to a rejection of the null hypothesis at the .05 level of significance only when $V$ equaled . 30.

As the adjustment cost coefficients increased in this phase, the two-stage approach yielded inconsistent results. For example, when the second set of adjustment cost coefficients were used with problem A, the null hypothesis was accepted at both levels of significance for all. the values of $V$ except . 05 ; but when the langest set of adjustment cost coefficients were used with the same problem, the nuIl hypothesis was rejected at both of the levels of significance for all the values of $V$. Correspondingly the results also differed when the adjustment cost coefficients were increased in dealing with the tightly constrained problem. In this phase it appeared that the adjustment cost coefficients had a direct bearing upon the acceptance or the rejection of the null hypothesis for both of the types of problems considered.

Summary of phase three
In this phase the expected value approach genexally yielded results which were not feasible on the average. In addition, these results generally led to acceptance of the null hypothesis of no difference from the simulation results at both levels of significance for all values of $V$ except. for $V$ greater than or equal to .25 .

Fon the smallest set of adjustment cost coefficients the two-stage approach yielded results which generally led
to an acceptance of the null hypothesis at both level of significance for both problems considered. Only when dealing with the tightly constrained problem were the results generally feasible on the average for this set of cost coefficients. As the adjustment cost coefficients were increased the results became feasible on the average for both problems considered and the resulits generally led to a rejection of the null hypothesis in all cases except those dealing with the second set of cost coefficients with the slightly constrained problem.

## Conclusions

The development of an experimental model which can be used to evaluate proposed deterministic equivalents to the stochastic programming model was an important result of this study. The model as it was used in the study evaluated some of the linear deterministic equivalents to the stochastic programming model.

Before a statement of the findings from this study is presented it is necessary to briefly review the assumptions upon which the experimental model has been built. These major assumptions are as follows. (I) The stochastic parameters which appear in each formulation are assumed to be normally and independentiy distributed with known means and variances. (2) A specific empirical problem is used as a means of generating the results of each deteministic equivalent and of the
simulation approach. This initial problem is a slightly constrained problem. (3) A modified form of the initial problem is used as an example of a tightly constrained problem and another set of results are generated. (4) The expected value solution is used to determine a ranking of the constraints and of the variables of the experimental problem. These rankings are used to select appropriate formulations of the two experimental problems which are analyzed in the experimental model. (5) Each deterministic equivalent is evaluated as the positions of the stochastic parameters change and as the variances of the stochastic parameters change.

## Major findings

The major findings are summarized for each phase of the experiment. In the first phase the stochastic parameters are limited to the $B$ vector; in the second phase they are limited to the $C$ vector; and in the third phase the stochastic parameters appear in both vectors.

Phase one
The results indicate that the two-stage approach is the best deterministic equivalent to use in phase one. This approach is only slightly affected as the variances of the stochastic parameters increase. The results of this approach are also consistent for the two types of problems considered.

It should be pointed out that these conclusions are dependent upon the adjustment cost coefficients which are used. The results of the two-stage approach are affected by changes in the values of the adjustment cost coefficients.

The expected value approach does not on the average yield feasible solutions and becomes unreliable as the variances of the stochastic parameters increase. As a first approximation, however, the expected value approach does have some advantages, particularly when it is used to generate an initial solution in the two-stage approach.

The least desirable approach in this phase is the active approach. In all cases this approach yields feasible results on the average; however, these results are statistically different from the results generated by the simulation approach. This approach is conservative in that it limits the optimum value of the objective function by restricting the use of the resources through the allocation ratios. In addition the results of the active approach are consistent for each problem analyzed and for each set of allocation ratios used.

## Phase two

The expected value approach is the best approach to use when the stochastic parameters appean only in the C vector. This approach yields feasible solutions on the
average which are very reliable regardless of the type of problem analyzed. In addition the approach is not affected by increases in the variances of the stochastic parameters.

The two-stage approach is the next best approach in this phase. From the results it can be seen that the adjustment costs have no affect in this phase. This is to be expected since these costs are associated only with the constraints which are deterministic. The twostage approach always yields a feasible solution on the average and is more reliable when dealing with the slightly constrained problem than when dealing with the tightly constrained problem. When dealing with the tightly constrained problem this approach was affected somewhat by increasing the variances of the stochastic parameters.

The active approach is the least desirable approach in this phase since it always yields infeasible solutions on the average which are very unreliable regardless of the type of experimental problem considered.

Phase three
In the third phase, of the two deterministic equivalents evaluated, the two-stage approach is considered the better since it yields feasible solutions on the average in more cases than does the expected value approach. The results of this approach are generally statistically the same' as the results of the simulation approach. However
these results are affected by the different values of the adjustment cost coefficients which are used. As the cost coefficients increase the results tend to become feasible on the average but also tend to become significantly different from the results of the simulation approach. The results of the expected value approach are affected by increasing the variances of the stochastic parameters. For only the smallest set of values of the variances does this approach yield results which are feasible on the average. The results of this approach are generally the same as the results of the simulation approach except for the two largest sets of values of the variances of the stochastic parameters. In addition the approach is consistent with respect to the two problems considered.

## Areas of Further Research

During the development of the experimental model and the analysis of the results fnom the experiment a number of questions arose which can serve as the basis upon which additional experimentation can be performed. Some of the more important areas for further research are' as follows. (1) The deterministic equivalents can be studied assuming nonnormal distribution for the stochastic parameters. In addition formulations of an experimental problem can be studied where the different parameters are distributed according to different types
of distributions. (2) The distributions of the optimum objective function values which result from the application of the different deterministic equivalents can be analyzed to determine the properties of these distributions and the effects that the properties of the stochastic parameters have upon these distributions. (3) Additional detemministic equivalents, some of which are presented in the second chapter, can be analyzed by the experimental model to determine the effects that the positions and the properties of the stochastic parameters have upon the performance of these equivalents. (4) The possibility of combining different deterministic equivalents into one model in order to utilize the advantages of each can be investigated. For example the expected value approach is used in this way with the two-stage deterministic equivalent. Since the expeoted value approach appears to be a good first approximation, the possibility of combining it with other deterministic equivalents can be studied. (5) The results of the active approach are dependent upon the specific values of the allocation ratios used in the model. The effects that these ratios have upon the results generated by this deterministic equivalent can be analyzed. For example, the use of different sets of allocation natios for a tightly constrained problem and a slightly constrained problem can be studied. (6) The adjustment cost coefficients have an effect upon the resiults of the
two-stage approach. The effects that these coefficients have can be studied. Fon example in the finst and the third phases of the experiment the results of the two"stage approach varied significantly as the cost coefficients changed.

APPENDIX
table 2
The sample means of the oftimur objective function values FOR ALL EXPERIMEOTS IN PHASE I PROBLEM A

| $\begin{gathered} V \\ (\sigma / \mu) \end{gathered}$ | EXP. NO. | ZSI MER | ZEXPC | $\begin{gathered} \text { ZThiSth } \\ \text { (20) } \end{gathered}$ | $\begin{gathered} \text { ZTVIS BR } \\ (40) \end{gathered}$ | $\begin{gathered} 2 T h . S 8 R \\ (80) \end{gathered}$ | $\begin{array}{r} \text { ZACTBR } \\ 1.901 \end{array}$ | $\begin{gathered} \text { 2ACTBF } \\ (.75) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| .05 | 1 | 8842.14 | 3837.96 | 8834.97 | 8931.98 | 8825.98 | 8324.25 | 7429.30 |
|  | 2 | 8896.48 | 8837.96 | 8777.25 | 8716.54 | 8595.11 | 8261.65 | 7339.39 |
|  | 3 | 8708. Cl | 8837.96 | 8826.90 | 8815.83 | 8793.70 | 8089.88 | 7032.54 |
|  | 4 | 8845.02 | 9837.96 | 8774.53 | 8711.09 | 8584.29 | 8120.71 | 7034.26 |
| .10 | 5 | 8820.29 | 8837.96 | 8331.99 | 8826.01 | 3814.c6 | 8316.83 | 7422.90 |
|  | 6 | $8756 . \mathrm{C} 2$ | 8837.96 | 8677.09 | 8516.21 | 8194.45 | 8132.05 | 7196.11 |
|  | 7 | 8689.94 | 8837.96 | 8313.01 | 8788.05 | 8738.14 | 8092.89 | 7540.04 |
|  | 8 | 8791.73 | 8837.96 | 8662.68 | 8487.38 | 8130.80 | 8074.15 | 6957.82 |
| .15 | 5 | 8702.29 | 8837.96 | 8827.45 | 88.16 .52 | 8795.88 | 8266.22 | 7397.07 |
|  | 10 | 8700.42 | 8837.96 | 8552.79 | 8267.61 | 7697.25 | 8069.03 | 7172.07 |
|  | 11 | 8703.57 | 9837.96 | 8890.87 | 8763.62 | 8689.67 | 8147.72 | 7101.84 |
|  | 12 | 8704.07 | 88.37 .96 | 8537.06 | 8236.16 | 7634.36 | 7995.77 | 6933.33 |
| . 20 | 13 | 8702.75 | 8837.96 | 8824.16 | 881 C .35 | 8782.73 | 8355.50 | 7547.66 |
|  | 14 | 8294.43 | 9837.96 | 8450.21 | 8062.45 | 7266.94 | 7709.18 | 6831.32 |
|  | 15 | 8608.89 | 8837.96 | 8793.20 | 8748.44 | 6653.91 | 8122.30 | 7090.52 |
|  | 16 | 8332.28 | 8837.96 | 8361.82 | 7895.67 | 6933.38 | 7659.73 | 6650.91 |
| .25 | 17 | 5608.08 | 8837.96 | 8823.72 | 8009.47 | 678C.97 | 8258.96 | 7450.6 C |
|  | 15 | 7675.27 | 8837.96 | 8195.45 | 7558.94 | 6279.92 | 7141.65 | 0341.21 |
|  | 19 | 8141.15 | 8837.96 | 8719.66 | 8721.36 | 8604.77 | 7680.37 | 6775.84 |
|  | 20 | 7993.59 | 3837.96 | 8226.90 | 7615.83 | 6393.70 | 7356.45 | 6405.76 |
| .30 | 21 | 8328.71 | 9437.96 | 8819.35 | 890 C .73 | 8763.50 | 8016.16 | 7275.8 S |
|  | 22 | 7785.91 | 9837.96 | $81 \geqslant 3.21$ | $740 \mathrm{E.45}$ | 5978.93 | 7239.64 | 6419.97 |
|  | 23 | 8162.20 | 8837.96 | 8753.90 | 8675.75 | 8521.93 | 7728.02 | 6:49.71 |
|  | 24 | 7342.79 | 8837.96 | 7961.36 | 7084.75 | 5331.54 | 6761.45 | 5889.44 |

TABLE 3
the sample heans of the optinua objective function values FOR ALL EXPERIMENTS IN PHASE II PROBLEM A

| $\begin{gathered} V \\ (\sigma / \mu) \end{gathered}$ | $\begin{aligned} & \text { EXP. } \\ & \text { NO. } \end{aligned}$ | ZSIMER | ZEXPC | $\begin{gathered} \text { ZTKSBR } \\ (201 \end{gathered}$ | $\begin{gathered} 2 \text { Th. SER } \\ (401) \end{gathered}$ | ZThSBR ( 80 ) | $\begin{array}{r} \text { ZACTBR } \\ (.90) \end{array}$ | $\begin{array}{r} \text { ZACTBR } \\ (.75) \end{array}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 05 | 25 | 8908.52 | 3837.96 | 89.58 .52 | 8908.52 | 8003.52 | 9249.44 | 9760.82 |
|  | 26 | 8837.66 | 8837.96 | 8837.66 | 8837.66 | 8837.66 | 9175.87 | 9633.18 |
|  | 27 | 8782.91 | 8937.96 | 8782.91 | 8782.91 | 8782.91 | 9119.02 | 9623.19 |
|  | 28 | 8809.51 | 8837.96 | 8809.51 | 8805.51 | 8809.51 | 9146.63 | 9652.33 |
| . 10 | 29 | 8812.71 | 8837.96 | 8812.70 | 8812.70 | 8812.70 | 9149.95 | 9655.82 |
|  | 30 | 8726.16 | 8837.96 | 8726.05 | 8726.05 | 8726.05 | 9063.64 | 9568.10 |
|  | 31 | 8881.23 | 8837.96 | 8381.18 | 8381.18 | 8881.18 | 9221.05 | 9730.86 |
|  | 32 | 8822.34 | 8837.96 | 8322.15 | 8822.15 | 8822.15 | 9173.95 | 9678.00 |
| . 15 | 33 | 8783.38 | 8837.96 | 8782.64 | 8782.64 | 8782.64 | 9118.74 | 9622.8 c |
|  | 34 | 8853.25 | 8837.96 | 8852.C.4 | 8852.04 | 8852.04 | 9281.65 | 9774.64 |
|  | 35 | 9027.44 | 8837.96 | 9022.17 | 9022.17 | 9022.17 | 9385.21 | 9900.14 |
|  | 36 | 8816.38 | 8837.96 | 8807.80 | 8307.80 | 8807.80 | 926C. 22 | 9746.59 |
| . 20 | 37 | 8858.18 | 3837.96 | 8854.63 | 8854.63 | 8854.63 | 9194.65 | 9701.77 |
|  | 38 | 8933.11 | 8837.96 | 8929.59 | 8929.59 | 8929.59 | 9440.33 | 9924.75 |
|  | 39 | 8686.31 | 8837.96 | 8677.21 | 8677.31 | 3677.31 | 9035.24 | 5529.04 |
|  | 40 | 9141.80 | 8837.96 | 9131.85 | 9131.85 | 9131.85 | 9749.32 | 10228.55 |
| . 25 | 41 | 9292.91 | 8837.96 | 9278.13 | 9278.13 | 5278.13 | 9647.19 | 10169.52 |
|  | 42 | 9471.45 | 8837.96 | 9469.24 | 9469.24 | 9469.24 | 9986.00 | 10503.84 |
|  | 43 | 8756.77 | 8837.96 | 8714.73 | 8714.73 | 8714.73 | 9187.98 | 9664.88 |
|  | 44 | 9.917 .10 | 8837.96 | 9002.54 | 9002.54 | 9CC. 2.54 | 9623.71 | 10094.64 |
| . 30 | 45 | 8398.95 | 8837.96 | 8855.91 | 8355.91 | 8855.91 | 9251.43 | 0742.54 |
|  | 46 | 8952.70 | 8837.96 | 8924.36 | 8924.36 | 8924.30 | 9582.74 | 10125.79 |
|  | 47 | 8507.64 | 8837.96 | 8429.02 | 8429.62 | 8429.02 | 9002.52 | 9453.71 |
|  | 48 | 9344.43 | 8837.56 | 9285.29 | 9285.29 | 9285.29 | 10264.07 | 1064\%.10 |

table 4
the sample means of the optimum objective function values FOR ALL EXPFRIMEMTS IN PHASE III PROBLEM A

| $\begin{gathered} v \\ (\sigma / \mu) \end{gathered}$ | $\begin{aligned} & \text { EXP. } \\ & \text { NO. } \end{aligned}$ | 2SIMER | ZEXPC | $\begin{gathered} 2 \text { TNSBR } \\ (20) \end{gathered}$ | ZTVISRR $(40)$ | ZTWSBR ( 8 C ) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 05 | 49 | 8873. 36 | 3837.96 | 8824.23 | 882 C .44 | 8812.77 |
|  | 50 | 8906.91 | 8837.96 | 8795.74 | 8793.03 | 8787.61 |
|  | 51 | 8845.32 | 8837.96 | 8749.13 | 8682.32 | 8550.71 |
|  | 52 | 8853.83 | 8837.96 | 8876.89 | 8873.65 | 8867.17 |
|  | 53 | 8817.57 | 8837.96 | 8821.83 | 8754.07 | 861.8.54 |
|  | 54 | 8865.88 | 8837.96 | 8767.53 | 8712.57 | 8602.60 |
|  | 55 | 8866.89 | 8837.96 | 8338.43 | 8774.58 | 8646.88 |
|  | 50 | 8942.45 | 8837.96 | 8780.13 | 2705.55 | 8556.4 C |
|  | 57 | 8894.40 | 8837.96 | 8803.21 | 8744.73 | 8627.76 |
| . 10 | 58 | 88C5.82 | 8837.96 | 8802.38 | 8796.05 | 8783.38 |
|  | 59 | 8613.24 | 8837.96 | 8802.64 | 8795.65 | 8781.66 |
|  | 60 | 8837.61 | 3837.96 | 8740.24 | 8606.73 | 8339.71 |
|  | 61 | 8913.39 | 8837.96 | 8933.15 | 8931.93 | 8919.50 |
|  | 62 | 8475.67 | 5837.96 | 8485.83 | 8304.95 | 7943.20 |
|  | 63 | 8629.26 | 8837.96 | 8520.13 | 8456.69 | 8129.81 |
|  | 64 | 8750.25 | 8837.96 | 8709.48 | 8564.83 | 8275.52 |
|  | 65 | 8758.57 | 8837.96 | 8772.66 | 8590.67 | 8226.70 |
|  | 66 | 8617.03 | 2837.96 | 8563.73 | 8389.09 | 8039.80 |
| . 15 | E7 | 8879.04 | 8837.96 | 8723.76 | 8914.53 | 8897.27 |
|  | 68 | 8853.49 | 8837.96 | 8302.48 | 8792.95 | 8773.83 |
|  | 65 | 8699.75 | 8837.56 | 8530.36 | 8372.74 | 7857.50 |
|  | 7 C | 8866.36 | 8837.96 | 3944.83 | 8836.16 | 8818.74 |
|  | 71 | 8364.64 | E837.96 | 8585.95 | 8311.07 | 7761.29 |
|  | 72 | 8648.34 | 8037.06 | 8750.60 | 8494.27 | 7981.60 |
|  | 73 | 8582.41 | 8837.96 | 8471.21 | 8179.49 | 7596.06 |
|  | 74 | 8538.94 | 8837. 36 | 8673.26 | 8400.85 | 7846.04 |
|  | 75 | 834.9.65 | $\begin{aligned} & 8937.96 \\ & \text { (COMTINULD) } \end{aligned}$ | 8399.77 | 9095.27 | 7486.27 |

TABLE 4-CONTINUED

| $\begin{gathered} v \\ (\sigma / \mu) \end{gathered}$ | $\begin{aligned} & \text { EXP. } \\ & \text { ND. } \end{aligned}$ | ZSIMBr. | ZEXPC | ZTWSRR (2C) | $\begin{gathered} \text { ZTHSBR } \\ (40) \end{gathered}$ | 2.TKSBR ( 80 ) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 20 | 76 | 8313.55 | 8837.96 | 8706.69 | 8093.45 | 8666.97 |
|  | 77 | 8477.64 | 8837.96 | 8721.25 | 8706.84 | 8678.02 |
|  | 78 | 8097.c0 | 8837.96 | 8474.01 | 8044.57 | 7185.69 |
|  | 79 | 8311.57 | 8837.96 | 8803.37 | 8787.55 | 8755.92 |
|  | 80 | 8695.24 | 8837.96 | 8701.91 | 8168.57 | 7101.98 |
|  | 81 | 8777.83 | 8837.96 | 8724.51 | 8232.38 | 7248.13 |
|  | 82 | 8041.31 | 8837.96 | 7965.73 | 7442.17 | 6395.04 |
|  | 83 | 8414.81 | 8837.96 | 8564.41 | 8124.58 | 7244.92 |
|  | 84 | 8556.81 | 8837.96 | 8580.57 | 8079.46 | 7077.23 |
| . 25 | 85 | 8547.16 | 8837.96 | 8704.19 | 8747.14 | 8713.03 |
|  | 86 | 8501. 03 | 8837.96 | 8820.74 | 8305.32 | 8774.47 |
|  | 87 | 7554.36 | 8837.96 | 8047.45 | 7487.96 | 6368.97 |
|  | 88 | 8075.09 | 8837.95 | 8547.65 | 8530.70 | 8596.77 |
|  | 89 | 7585.52 | 9.837.96 | 7727.73 | 7.309 .38 | 5572.67 |
|  | 90 | 8582.09 | 8837.96 | 8510.31 | 7879.52 | 6617.92 |
|  | 91 | 8057.27 | 8837.96 | 8.984 .69 | 7526.49 | 6410.09 |
|  | 92 | 7929.35 | 3837.96 | 8075.74 | 7530.52 | 6590.09 |
|  | 93 | 7681.05 | 8837.96 | 7923.52 | 7299.07 | 6050.16 |
| .30 | 94 | 8691.c2 | 2837.96 | 8925.32 | 8905.86 | 8866.93 |
|  | S5 | 8666.53 | 8837.96 | 8889.29 | 8871.97 | 8837.35 |
|  | 96 | 7236.34 | 8837.96 | 8195.89 | 7372.88 | 5726.85 |
|  | 57 | 8596.45 | 3837.96 | 8903.33 | 8886.00 | 8851.32 |
|  | 98 | 7362.34 | 3837.96 | 7815.34 | 7006.61 | 5387.36 |
|  | 99 | 7664.05 | 8837.96 | 8098.05 | 7317.56 | 5756.59 |
|  | 100 | 7489.18 | 3837.96 | 7887.97 | 7218.98 | 5880.99 |
|  | 101 | 7392.61 | 8837.96 | 7735.71 | 6905.35 | 5244.63 |
|  | 102 | 7591.15 | 3237.95 | 8042.77 | 7354.28 | 5977.30 |

TABLE 5
the sample means df the optimum obijective function values FOR ALL EXPERIMENTS IN PHASE I PROBLEM B

| $\begin{gathered} v \\ (\sigma / \mu) \end{gathered}$ | $\begin{aligned} & \text { EXP. } \\ & \text { NO. } \end{aligned}$ | ZSIMRR | ZEXPC | $\begin{gathered} \text { ZThSER } \\ (20) \end{gathered}$ | $\begin{gathered} Z T K S B R \\ (40) \end{gathered}$ | $\begin{gathered} Z T W S 8 R \\ (80) \end{gathered}$ | $\begin{gathered} \text { ZACTBR } \\ (.90) \end{gathered}$ | $\begin{gathered} \text { ZACTBR } \\ (.75) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 05 | 1 | 9137.c1 | 9133.07 | 9130.07 | 9127.07 | 9121.09 | 8954.73 | 8972.04 |
|  | 2 | 9176.50 | 9133.07 | 9070.77 | 9008.48 | 8883.90 | 8907.00 | 8889.64 |
|  | 3 | 9013.87 | 9133.07 | 9111.93 | 9090. 80 | 9648.53 | 8283.73 | 7647.32 |
|  | 4 | 9112.39 | 9133.07 | 9048.32 | 8963.58 | 8794.09 | 8401.48 | 7736.17 |
| . 10 | 5 | 9098.79 | 9133.07 | 9127.69 | 9121.11 | 9109.16 | 8916.21 | 8894.00 |
|  | 6 | 8999.96 | 9133.67 | 8949.02 | 8764.97 | 8396.88 | 8759.00 | 3703.21 |
|  | 7 | 8983.73 | 9133.07 | 9082.34 | 9031.60 | $8 ¢ 30.14$ | 8286.31 | 7661.09 |
|  | 8 | 9044.78 | 9133.07 | 8916.13 | 8699.19 | 8265.32 | 8356.72 | 7704.21 |
| . 15 | 5 | 8963.83 | 9133.07 | 9122.54 | $9112 . \mathrm{C} 2$ | 9090.98 | 8750.91 | 8738.62 |
|  | 10 | $8 ¢ 44.93$ | 9133.67 | 8320.18 | 8507.29 | 7881.52 | 8719.43 | 8623.79 |
|  | 11 | 8993.25 | 9133.07 | 9058.59 | 8984.11 | 8835.14 | 8340.93 | 7732.50 |
|  | 12 | 8947.76 | 9133.07 | 8706.03 | 8399.00 | 7664.93 | 8278.11 | 7639.16 |
| . 20 | 13 | 8936.61 | 9133.07 | 9110.26 | 9105.45 | 9077.83 | 8766.54 | 8755.55 |
|  | 14 | 8547.74 | 9133.07 | 8705.19 | 8277.32 | 7421.57 | 8304.46 | 8201.88 |
|  | 15 | 8884.92 | 9133.07 | 0.047 .69 | 8962.32 | 8791.57 | 8291.51 | 7723.92 |
|  | 16 | 8582.68 | 9133.07 | 8564.27 | 7995.46 | 6857.86 | 7940.37 | 7352.51 |
| . 25 | 17 | 9854.46 | 9133.07 | 9118.82 | 9104.57 | 9070.07 | 8676.13 | 8563.84 |
|  | 18 | 7933.91 | 9133.07 | 8446.64 | 7760.21 | 0387.36 | 7693.04 | 7611.92 |
|  | 19 | 8408.93 | 9133.07 | 8997.26 | 8801.46 | 8589.85 | 7838.19 | 7317.58 |
|  | 20 | 8236.90 | 9133.07 | 8421.19 | 7709.31 | 6285.56 | 7651.61 | 7138.63 |
| . 30 | 21 | 8578.07 | 9133.07 | 9114.45 | 9095.83 | 9058.59 | 8390.73 | $837 \%$ \% 73 |
|  | 22 | 8523.11 | 9133.07. | 836.7 .60 | 7602.13 | 6071.19 | 7774.50 | 7557.19 |
|  | 23 | 8496.41 | 9133.87 | 8998.17 | 8863.27 | 8593.48 | 7899.69 | 7458.08 |
|  | 24 | 7605.27 | 9133.67 | 8117.04 | 7101.c2 | 50.08 .96 | 7030.23 | 6561.4 C |

table 6
the sample means of the optimua objective function values
FOR ALL EXPERIMENTS IN PHASE II PROBLEM B

| $\begin{gathered} V \\ (\sigma / \mu) \end{gathered}$ | EXP. <br> NO. | ZSIMBR | ZEXPC. | $\begin{gathered} \text { 2TWSER } \\ (20) \end{gathered}$ | $\begin{gathered} \text { ZTWSBR } \\ (40) \end{gathered}$ | $\begin{gathered} \text { ZTWSBR } \\ (80) \end{gathered}$ | ZACTBR (.90) | $\begin{array}{r} \text { ZACIBR } \\ (.75) \end{array}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| .05 | 25 | 9188.80 | 9133.07 | 9133.80 | 9188.80 | 9188.80 | 9326.35 | 9760.82 |
|  | 26 | 9141.82 | 9133.07 | 9123.65 | 9123.65 | 9123.65 | 10422.34 | 10749.05 |
|  | 27 | 9110.64 | 9133.07 | 9089.58 | 9089.58 | 9089.53 | 9958.81 | 10319.91 |
|  | 28 | 9142.28 | S133.07 | 9090.44 | 9090.44 | 9070.44 | 10490.00 | 10797.63 |
| . 10 | 29 | 9120.75 | 9133.07 | 9113.11 | 9113.11 | 9113.11 | 9345.34 | 9660.70 |
|  | 30 | 9083.36 | 9133.07 | 9006.10 | 9006.10 | 9006.10 | 10389.07 | 10713.46 |
|  | 31 | 9241.83 | 9133.07 | 9167.20 | 9167.20 | 9167.20 | 10182.07 | 10513.32 |
|  | 32 | 9232.64 | 9133.07 | 9085.87 | 9086.37 | 9085.87 | 10782.33 | 11063.26 |
| . 15 | 33 | 0112.01 | 9133.07 | 0089.36 | 9089.36 | 9089.36 | 9384.96 | $9678 . \mathrm{C} 2$ |
|  | 34 | 9255.62 | 9133.07 | 9118.85 | 9118.85 | 9118.85 | 10709.12 | 11023.52 |
|  | 35 | 9464.68 | 9133.07 | 9278.57 | 9278.57 | 9278.57 | 10666.33 | 10966.76 |
|  | 36 | 9270.64 | 9133.07 | 9060.18 | 9060.18 | 9060.13 | 11003.03 | 11279.23 |
| . 20 | 37 | 9198.57 | 9133.07 | 9146.23 | 7146.23 | 9146.23 | 9522.59 | 9808.45 |
|  | 38 | 9413.37 | 9133.07 | 9220.42 | 9220.42 | 9220.42 | 11070.95 | 11338.70 |
|  | 39 | 9220.60 | 9133.07 | 9006.16 | 9006.16 | 9006.16 | 10643.74 | 10352.49 |
|  | 40 | 967 C .14 | 9133.07 | 9409.00 | 9409.00 | 9409.00 | 11655.75 | 11918.40 |
| . $25^{\circ}$ | 41 | . 9590.25 | 9133.07 | 9480.75 | 9480.75 | 9430.75 | 9942.71 | 10305.32 |
|  | 42 | 9800.71 | 9133.C7 | 9560.49 | 956C.49 | 9560.49 | 11258.05 | 11620.91 |
|  | 43 | 9392.41 | 9133.07 | 9335.72 | 9035.72 | 9035.72 | 11159.91 | 1130.53 |
|  | 44 | 9636.59 | 9133.07 | 9205.24 | 9205.24 | 9205.24 | 11524.98 | 12040.11 |
| . 30 | 45 | 9313.63 | 9133.07 | 9147.24 | 9147.24 | 9147.24 | 9767.11 | 10030.36 |
|  | 46 | 9475.6 .6 | 9133.07 | 9193.78 | 9198.78 | 9198.78 | 11282.49 | 11522.05 |
|  | 47 | 9173.41 | 9133.07 | 8810.04 | 8810.04 | 3810.04 | 10941.08 | 11043.11 |
|  | 48 | 9937.57 | 9133.07 | 9462.86 | 9462.86 | 9452.86 | 12214.27 | 12457.14 |

table 7
the sample means df the optimur cbjective function values FOR ALL EXPERIMENTS IN PHASE III PRORLEM B

table 7-CCNTinued

| $\begin{gathered} v \\ (\sigma / H) \end{gathered}$ | $\begin{aligned} & \text { EXP. } \\ & \text { ND. } \end{aligned}$ | 2SIMER | ZEXPC | $\begin{gathered} \text { ZTWSBR } \\ (20) \end{gathered}$ | $\begin{gathered} Z T W S B R \\ (40) \end{gathered}$ | $\begin{gathered} \text { ZTHSBR } \\ (80) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| . 20 | 76 | 8678.95 | 9133.07 | 9026.59 | 9013.35 | 8986.38 |
|  | 77 | 8937.75 | 9133.07 | 9063.11 | 9048.70 | 9019.88 |
|  | 78 | 8425.00 | 9133.07 | 8716.59 | 8248.40 | 7312.02 |
|  | 79 | 8919.55 | 9133.07 | 9137.48 | 9121.66 | 9090.03 |
|  | 80 | 8979.48 | 9133.07 | 8817.58 | 8188.28 | 6929.68 |
|  | 81 | 9153.12 | 9133.07 | 8901.58 | 8366.95 | 7297.67 |
|  | 82 | 8623.34 | 9133.07 | 8258.45 | 7698.61 | 6578.94 |
|  | 83 | 8865.39 | 9133.07 | 8795.91 | 8258.57 | 7133.89 |
|  | 84 | 8968.09 | 9133.07 | 8716.42 | 8112.73 | 6905.34 |
| . 25 | 85 | 8931.69 | 9133.07 | 9071.21 | 9054.15 | 9020.05 |
|  | 86 | 9027.52 | 9133.07 | 9159.36 | 9143.94 | 9113.09 |
|  | 87 | 7577.75 | 9133.07 | 8332.95 | 7715.33 | 6480.03 |
|  | 88 | 8792.13 | 9133.07 | 8961.73 | 8944.78 | 3910.87 |
|  | 89 | 7975.56 | 9133.07 | 7994.39 | 7165.26 | 5507.00 |
|  | 90 | 9077.77 | 9133.07 | 8713.45 | 8032.23 | 6669.79 |
|  | 91 | 8672.82 | 9133.07 | 8367.05 | 7751.40 | 6520.11 |
|  | 92 | 8478.36 | 9133.047 | 8294.60 | 7676.04 | 6438.92 |
|  | 93 | 8344.67 | 9133.07 | 8170.89 | 7420.36 | 5919.27 |
| . 30 | 94 | 9048.35 | 9133.07 | 9197.98 | 9178.52 | 9139.59 |
|  | 95 | 9116.69 | 9133.07 | 9134.30 | 9117.49 | S082.87 |
|  | 96 | 7631.45 | 9133.07 | 8394.65 | 7513.30 | 5750.61 |
|  | 97 | 9244.91 | 9133.07 | 9227.49 | 9210.16 | 9175.43 |
|  | 98 | 7882.11 | 9133.07 | 8036.72 | 7108.86 | 5253.14 |
|  | 99 | 8237.15 | 9133.07 | 8282.29 | 744 C .78 | 5757.76 |
|  | 100 | 8291.58 | 9133.07 | 8227.70 | 7501.31 | 6048.52 |
|  | 101 | BClc.eo | 9133.07 | 7939.69 | 6993.68 | 5101.64 |
|  | 102 | 8251.77 | 9133.07 | 8239.64 | 7445.29 | 5856.57 |

TABLE 8
TEST RESULTS OA THE VARIDUS DETERMINISTIC EQUIVALENTS PROBLEM A

| $\stackrel{V}{(\sigma / \mu)}$ | EXPECTED <br> VAL!E | $\begin{gathered} \text { TWO-STAGE } \\ (20) \end{gathered}$ | $\begin{gathered} \text { TWO-STAGE } \\ (401) \end{gathered}$ | $\begin{gathered} \text { TWO-STAGE } \\ (80) \end{gathered}$ | $\begin{aligned} & \text { ACTIVE } \\ & 1.901 \end{aligned}$ | $\begin{gathered} \text { ACTIVE } \\ (.75) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PHASE |  |  |  |  |  |  |
| 1 |  |  |  |  |  |  |
| . 05 | -0.332 | 0.430 | 1.173 | 2.474 | 112.202 | 136.565 |
| .10 | - 0.899 | 0.226 | 1.340 | 3.212 | 60.582 | 77.799 |
| . 15 | -1.172 | 0.106 | $1.46 ?$ | 3.439 | 39.390 | 54.985 |
| . 20 | -2.405 | -0.852 | 0.730 | 3.309 | 28.383 | 40.511 |
| . 25 | -4.0.95 | -2.327 | -0.410 | 2.765 | 24.940 | 3r.306 |
| . 30 | -4.244 | -2.419 | -C.408 | 2.946 | 21.279 | 25.930 |
| PHASE |  |  |  |  |  |  |
| II |  |  |  |  |  |  |
| . 05 | -0.073 | C.C | 0.0 | 0.0 | -195.703 | -195.505 |
| . 10 | -0.324 | 1.36 c | 1.360 | 1.360 | -64.931 | -10.1.55 |
| . 15 | 0.251 | 1.462 | 1.462 | 1.462 | -18.924 | -48.488 |
| . 20 | 0.379 | 2.475 | 2.475 | 2.475 | -11.980 | -30.646 |
| . 25 | 1.395 | 2.269 | 2.259 | 2.269 | -10.310 | -24.943 |
| . 38 | 0.337 | 2.885 | 2.805 | 2.805 | -7.406 | -1t.947 |
| PHASE |  |  |  |  |  |  |
| 111 |  |  |  |  |  |  |
| .05 | 0.585 | 1.501 | 2.367 | 3.707 | NA | NA |
| - 1c | $-1.067$ | -0.0.84 | 1.256 | 3.295 | WA | NA |
| . 15 | -1.099 | -0.291 | 1.239 | 3.657 | NA | NA |
| - 20 | -1.3P? | -1.105 | 0.957 | 4.155 | NA | NA |
| . 25 | -2.834 | -1. 259 | 0.902 | 4.206 | NA | Na |
| - 30 | -3.099 | -1.779 | C.425 | 4.009 | NA | VA |

CRITICAL VALUES OF $7 \mathrm{FGR} \alpha=.05 \mathrm{ARE} \pm 1.96$
$F O R \alpha=. C 1$ ADE $\pm 2.53$

TABLE 9
TEST PESULTS ON THE VARIDUS DETERMINISTIC
EQUIVALENTS DROELFM P

| $\begin{gathered} v \\ (\sigma / \mu) \end{gathered}$ | EXPECTED VALUE | $\begin{gathered} \text { THO-stage } \\ 1201 \end{gathered}$ | $\begin{gathered} T H O-S T A G E \\ (4 O) \end{gathered}$ | $\begin{gathered} \text { THO-StAGE } \\ (80) \end{gathered}$ | $\begin{gathered} \text { ACTIVE } \\ (.90) \end{gathered}$ | $\begin{gathered} \text { ACTIVE } \\ (.75) \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PHASE |  |  |  |  |  |  |
| I |  |  |  |  |  |  |
| . 05 | -0.548 | 0.465 | 1.443 | 3.103 | 109.216 | 74.072 |
| - 10 | -1.339 | 0.179 | 1.708 | 4.184 | 59.418 | 47.929 |
| . 15 | -1.595 | 0.199 | 1.840 | 4.33 .3 | 31.488 | 33.072 |
| . 20 | -2.894 | -0.917 | 1.129 | 4.311 | 24.299 | 25.297 |
| . 25 | -4.430 | $-2.398$ | -0.002 | 3.797 | 21.464 | 20.00s |
| . 30 | -4.541 | $-2.387$ | C.042 | 3.900 | 20.459 | 18.239 |
| PHASE |  |  |  |  |  |  |
| II |  |  |  |  |  |  |
| . 05 | 0.359 | 4.083 | 4.088 | 4.083 | -22.6.09 | -39.345 |
| . 10 | C. 539 | 5.421 | 5.481 | 5.401 | -15.582 | -24.349 |
| . 15 | 1.375 | 5.691 | 5.691 | 5.691 | $-11.766$ | -17.847 |
| . 2 n | 1.645 | 6.123 | 6.123 | 6.123 | -11.538 | $-16.372$ |
| . 25 | 2.677 | 7.609 | 7.609 | 7.609 | $-10.835$ | $-15.395$ |
| . 30 | 1.535 | 7.310 | 7.310 | 7.310 | $-9.775$ | -13.08? |
| PHASE |  |  |  |  |  |  |
| I II |  |  |  |  |  |  |
| . 65 | 0.757 | 1.894 | 2.910 | 4.417 | NA | NA |
| .10 | -0.069 | 0.570 | 2.248 | 4.692 | NA | Na |
| . 15 | -0.9C5 | 0.646 | 2.521 | 5.347 | Na | NA |
| . 20 | $-1.454$ | 0.000 | 2.587 | 6.036 | Na | NA |
| . 25 | -2.223 | 0.135 | 2.673 | 5.165 | NA | Na |
| . 30 | $-2.515$ | -0.518 | 2.170 | 5.743 | NA | NA |
| GPITICAL VALUES OF 7 FOR $\alpha=.55$ ARE $\pm 1.96$ |  |  |  |  |  |  |

TABLE 10

## SUMMARY OF THE EXPERIMENTAL RESULTS - PHASE I




TABLE 12
SUMMARY OF THE EXPERIMENTAL RESULTS - PHASE III

| MODEL | $\begin{gathered} \mathrm{V} \\ (\sigma / \mu) \end{gathered}$ | PROBLEM A |  |  | PROBLEM B |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\begin{gathered} \text { FEASI- } \\ \text { BLE } \end{gathered}$ | ACCEPT | $\begin{aligned} & \mathrm{H}_{\circ} \\ & .01 \end{aligned}$ | $\begin{gathered} \text { FEASI- } \\ \text { BLE } \end{gathered}$ | $\begin{aligned} & \text { ACCEPT } \\ & .05 \end{aligned}$ | $\begin{aligned} & \mathrm{H}_{\mathrm{O}} \\ & .01 \end{aligned}$ |
| ZEXPC | . 05 | X | X | X | X | X | X |
|  | .10 |  | X | X |  | X | X |
|  | . 15 |  | X | $X$ |  | X | X |
|  | . 20 |  | X | X |  | X | X |
|  | . 25 |  |  |  |  |  | X |
|  | . 30 |  |  |  |  |  | X |
| ZTWS (20) | . 05 | X | X | X | X | X | X |
|  | . 10 |  | X | X | X | X | X |
|  | . 15 |  | X | X | X | X | X |
|  | . 20 |  | X | X | X | X | X |
|  | . 25 |  | X | X | X | X | X |
|  | . 30 |  |  | X |  | X | X |
| ZTWS (40) | . 05 | X |  | X | X |  |  |
|  | . 10 | X | X | X | X |  |  |
|  | . 15 | X | X | X | X |  | X |
|  | . 20 | X | X | X | X |  |  |
|  | . 25 | X | X | X | X |  |  |
|  | . 30 | X | X | X | X |  | X |
| ZTWS (80) | . 05 | X |  |  | X |  |  |
|  | . 10 | X |  |  | X |  |  |
|  | . 15 | X |  |  | X |  |  |
|  | . 20 | X |  |  | X |  |  |
|  | . 25 | X |  |  | X |  |  |
|  | . 30 | X |  |  | X |  |  |

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Candidate: Frank Paul Buffo

Major Field: Quantitative Methods
Title of Thesis: An Appraisal of the Efficiency of Alternative Deterministic Equivalents to the Stochastic Programming Model


EXAMINING COMMITTEE:


Date of Examination:

$$
\text { August 4, } 1970
$$


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    ${ }^{7}$ Charnes and Cooper, "Chance-Constrained Programming, " 73.
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[^1]:    ${ }^{18}$ Ibia., 29.

[^2]:    constraints in this form can be applied directly to the solution of specific models. In their appendix the authors utilize the constraints in the form in which they are presented in this text.
    ${ }^{2 I_{\text {Ibid. }}}$, 28-29.
    ${ }^{22}$ Ibi.a., 35-38.

[^3]:    ${ }^{25}$ Ibid., $30-33$. See also. Charnes and Kirby, "Some Special P-Models," 183-195.

[^4]:    ${ }^{39}$ G. Tintner, "A Note on Stochastic Linear Programming," Econometrica, XXVIII (April, 1960), 490.

    40 These two approaches are specified in Sengupta, Tintner, and Morrison, "Stochastic Linear Programming with Applications," 262-276; J. K. Sengupta, G. Tintner, and G. Millham, "On Some Theorems of Stochastic. Linear Programming with Applications," Management Science, X COctober, 1963), 143-159; and K. D. Cocks, "Discrete Stochastic Programming," Management Science, XV (September, 1968), 72-79.

[^5]:    ${ }^{4} 1_{\text {Sengupta, }}$ Tintner, and Millham, " Theorems of Stochastic Linear Programming," 145.

    42 Ibid.

[^6]:    ${ }^{44}$ Sengupta and Portillo-Campbell, "A Fractile Approach Under Risk,". 299-300.

[^7]:    ${ }^{45}$ Ibid., 299.
    46 Ibid., 300-301. See also: Geoffrion, "Stochastic Programming with Aspiration Criteria," 67.2-679: and Kataoka, "A Stochastic Programming Model," 181-196.

[^8]:    ${ }^{5}$ Naylor, Computer Simulation Techniques, pp. 43-67.

[^9]:    ${ }^{6}$ Ibid., p. 46.

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[^11]:    ${ }^{14}$ MacLaren, "Uniform Random Number Generators," 86-89; and T. E. Hull and A. R. Dobell, "Random Number Generators," SIAM Review, IV (July, 1962) 238-242.
    ${ }^{15}$ Naylor, Computer Simulation Techniques, pp. 49, 55-56.
    ${ }^{16}$ J. L. Alland, A. R. Dobell, and T. E. Hull, "Mixed Congruential Random Number Generators for Decimal Machines," Journal of the Association for Computing Machinery, $x$ (April, 1963); 131-141.

[^12]:    $l_{\text {M. M. Babbar, "Distributions of Solutions of a }}$ Set of Linear Equations (With an Application to Linear Programing)," Journal of the American Statistical Association, L (September, 1.955), 854-869; and J. K. Sengupta and J. H. Partillo-Campbell, "A Fractile Approach to Linear Progriaming Under Risk," Management Science, XVI (January, 1970), 298-308.

[^13]:    ${ }^{2}$ In the discussion of the experimental procedure the variable names used.in the FORTRAN program of the model are included where it would be beneficial to the reader to do so.

[^14]:    ${ }^{2}$ John E. Freund, Paul E. Livermore, and Irwin Miller, Manual of Experimental Statistics (Englewood Cliffs, $\bar{N} . \mathrm{J.;}$ Prentiae-Hali, Inc., 1960.), pp. 19-21; and I. M. Chakravarti, R. G. Laha, and J. Roy, Handbook of Methods of. Applied. Statistics (New York: John Wiley and Sons, Inc., 1967), pp.. 325-326.

