AN APPROACH FOR QUANTIFYING PAIRED COMPARISONS AND RANK ORDER[1]

By Louis Guttman

Cornell University and War Department

1. Summary. Research for the Army demobilization point system evolved a new approach to paired comparisons and rank order. Each of N individuals compares or ranks n things; the problem is to determine a numerical value for each of the n things that will best represent the comparisons in some sense. The new criterion adopted is that the numerical values be determined so as best to distinguish between those things judged higher and those judged lower *for each individual*. Least-squares is employed in the analysis, and the solution appears in the form of the latent vector associated with the largest root of a matrix obtained from the comparisons or rankings.

This approach applies to the conventional problem of ordinary comparisons, the numerical solution being easily obtainable by simple iterations; the conventional use of hypothetical variables and unverified hypotheses is avoided. The Army point system is an example of a new and more complicated class of problems; the same principle for the solution applies here, only more details occur in the derivations and computations.

2. Introduction. The problem of paired comparisons arises when it is desired to obtain numerical values for a set of n things, with respect to one characteristic, such that these values will represent the judgments of a population of N individuals.

One procedure for obtaining the judgments is to have the individuals compare the things two at a time and to judge for each comparison which of the two things should be given the higher rank. An alternative procedure is to have each individual rank all the n things simultaneously. Such a ranking implies judging all the $n(n - 1)/2$ comparisons at once; hence, the two procedures are substantially equivalent. Two noteworthy differences between the procedures are: (a) comparing two things at a time allows inconsistencies to appear within judgments of an individual, and (b) it is sometimes harder in practice for people to judge n things simultaneously than to compare them two at a time.

The problem of quantification, of course, is identical for both procedures, so we do not distinguish between them in this paper. The judgments vary from person to person (and possibly within a person), and the problem is to determine a set of numerical values for the things being compared that will in some sense best represent or average the judgments of the whole population.

---

[1] Adapted from Report D-3, "An approach for quantifying paired comparisons," Research Branch, Information and Education Division, Headquarters Army Service Forces, Washington, D. C., 1945.

In some situations, the things being compared may be single items or objects; this we shall call the case of *ordinary* comparisons. In other situations, the things may be *combinations* of items or objects.

This paper is devoted to the presentation of a general approach to quantifying comparisons or rank orders, with particular application to ordinary comparisons and to the comparison of combinations of two things. It seems to differ from previous approaches in at least two important respects: (a) it is based on but one simple principle, namely, that the quantification shall be the one best able to *reproduce the judgment of each person in the population on each comparison;* and, as a consequence, (b) the approach yields solutions not only to the traditional case of ordinary comparisons, but also to more complex cases that do not seem to have been discussed previously.

An example of a major practical use of this approach is with respect to the demobilization score card of the United States Army. The problem was to determine the number of points to assign each of the variables on the score card according to the opinions of the soldiers themselves. The research on this was based on a form of paired comparisons more complicated than the ordinary one, and had additional complications of curvilinearities of various sorts in the data. Our approach handles such problems as well as the problem of ordinary comparisons.

Let us describe the score card problem in somewhat more detail. In a survey of enlisted men throughout the world by means of a questionnaire administered by field teams of the Research Branch, it was found that there were five variables that the men thought should receive consideration on the score card to determine order of demobilization: length of time in the Army, length of time overseas, amount of combat, age, and number of children.

The problem now was to determine how much weight to give each of these variables in obtaining total scores. According to ordinary paired comparisons, one would ask, for example, "Who should get out first after the war: a man who has two children or a man who has been in two battles?" But respondents refuse to judge such a comparison because the battle experience of the first man is not specified, nor is the number of progeny of the second man, so that there is insufficient basis for judgment.

Therefore, in the actual research, judgments were asked on each of ten comparisons put in the following form:

"Here are three men of the same age, all overseas the same length of time. Check the one you would want to have let out first:

—— A single man . . . . through two campaigns of combat

—— A married man with no children . . . . through one campaign of combat

—— A married man with two children . . . . not in combat."

Each variable was compared with every other one in this fashion.

The equations were derived for computing the relative number of points to assign to each month in the army, each month overseas, etc., which would be most consistent according to our principle. These are essentially the equations developed in section 6 of this paper.

The results showed strong curvilinearities in the men's judgments. Amount of combat received one amount of emphasis when compared with age, and another amount of emphasis when compared with number of children. Since the score card would be too complicated in practice if curvilinear scoring were used, equations were derived for the *linear* scoring scheme that would be most consistent according to our principle. These are essentially the equations derived in section 7. The weights arising out of the research were computed from such equations.

The variable age received a slight negative weight, which justified dropping it from the score card. The weights the Army finally adopted for the remaining factors were modified from the research weights, but yield essentially the same results as the research weights. Demobilization scores obtained from the one system of weights correlate very highly with scores obtained from the other.

It can now be revealed that the Army's modification was essentially to reverse the weights for children and battles. In subsequent attitude surveys on how well the soldiers liked the point system [8], a major complaint was found to be that battles got too little weight compared with babies!

3. The basic principle. Our basic principle in deriving numerical values—let us call them "x-values"—for the things being compared requires that the x-values of things a given person judges higher than other things should be as different as possible from the x-values of the things he judges to be lower than other things. This will be achieved if we make the x-values of things judged higher as homogeneous as possible among themselves, and the x-values of things judged lower as homogeneous as possible among themselves, for each individual. In the language of analysis of variance, our principle calls for *minimizing the variation within individuals*, compared with that within the group as a whole.[2] The resulting x-values will tend to be the best for reproducing the judgment of each individual on each comparison with a minimum overall proportion of errors of reproduction [3, pp. 342–343]. The smaller this overall proportion of error, the better the quantification represents the data. Least squares is used for convenience for measuring variation in deriving the equations.

The previous literature, on ordinary paired comparisons,[3] seems to have concentrated largely on the problem of estimating the differences between means of hypothetical variables assumed to underlie the judgments. Thurstone has shown that by using assumptions of normality of distribution, equality of variances, and zero correlations among hypothetical variables, it is possible to estimate relative distances between means for some kinds of data.

---

[2] This principle for quantification was suggested by previous work on scale analysis; see [3]. This theory has been developed further by the definition of a perfect scale in [4]. The equations for the perfect scale have interesting properties that may be related to paired comparisons; these equations are being prepared for publication. The referees have called my attention to related work on quantification by R. A. Fisher in [1, p. 283].

[3] A good survey of the previous work, including that of Thurstone, is given in [2, pp. 217–243]. For more recent work, see [7].

The problem of estimating differences between means is not identical with that of reproducing individual judgments. For example, it can be shown, within the same framework of hypothetical variables conventionally used, that if variances are unequal and/or correlations are unequal then the means of the hypothetical variables are *not* in general the best quantification for reproducing individual judgments; the principal axis of certain product-moments of raw scores is the best quantification. It is in the special case where variances are equal, and where correlations are equal—not even necessarily equal to zero—that the principal axis *is* the set of means. Proof of this is given in the appendix.

The approach of this paper does not use hypothetical variables, but inquires directly as to what numerical values can be derived from the observations that will best reproduce those observations.

In the next section is treated the case of ordinary comparisons. The more complicated problem of the demobilization score card is formalized in section 5, and the equations for its unrestricted solution are derived in section 6. Since the unrestricted solution brings out curvilinearities that may be present, and since the score card in practice required a linear scoring scheme, equations for the most consistent linear quantification are derived in section 7. These are essentially the equations used in the research on the weights for the score card.

The appendix shows a distinction between the conventional principle of estimating mean differences of hypothetical variables and the present principle of representing the comparisons of each individual.

4. The case of ordinary comparisons. Paired comparisons as treated in the literature seem concerned largely with the ordinary case where separate things are compared, rather than where combinations of things are compared. Our principle covers the ordinary case as well as more complex cases, and we shall treat the ordinary case first since it involves less details.

Let $O_1, O_2, \cdots, O_n$ be the n things to be compared, where the assigning of subscripts is arbitrary. Each of N individuals is asked to make judgments of the form that O_j is higher than (or lower than) O_k. For convenience, we assume the rules of the experiment to exclude judgments of equality. We shall also assume that all people compare all the pairs. Hence, there are N sets of $n(n-1)/2$ comparisons. Considering each comparison as comprising two judgments—one of "higher than" for one object and one of "lower than" for the other—there is a total of $Nn(n-1)$ judgments in the experiment.

The judgments of all the individuals on all the comparisons can be represented compactly as follows. Let

$$e_{ijk} \equiv \begin{matrix} 1 \text{ if individual } i \text{ judges } O_j > O_k \\ 0 \text{ if individual } i \text{ judges } O_j < O_k \\ 0\; j = k. \end{matrix} \tag{4.1}$$

The ranges of subscripts, whether free or dummy, will always be:

$$i = 1, 2, \cdots, N \qquad (4.2)$$
$$j, k = 1, 2, \cdots, n,$$

so that the ranges will not be explicitly stated again.

Definition (4.1) implies that if $e_{ijk} = 1$, then $e_{ikj} = 0$, and that

$$e_{ijk} + e_{ikj} \doteq 1, \qquad (j \neq k). \qquad (4.3)$$

Let f_{ij} be the number of things individual i judged to be lower than O_j, and let g_{ij} be the number of things he judged to be higher than O_j. Then

$$f_{ij} \equiv \sum_k e_{ijk}, \qquad g_{ij} \equiv \sum_k e_{ikj}. \qquad (4.4)$$

From (4.3) and (4.4), we have

$$f_{ij} + g_{ij} \equiv n - 1. \qquad (4.5)$$

Let F be the total number of comparisons made by each person; then

$$F = n(n-1)/2 \equiv \sum_k f_{ik} \equiv \sum_k g_{ik}. \qquad (4.6)$$

Let c be the number of times each O_j was judged in the whole experiment, and let C be the total number of judgments in the experiment:

$$c = N(n-1) \equiv \sum_i (f_{ij} + g_{ij}), \quad C = Nn(n-1). \qquad (4.7)$$

Both c and C count each comparison as two judgments, one of "lower than" and one of "higher than."

The means and variances to be considered are defined as follows. Let x_j be the numerical value to be derived for O_j on the basis of the comparisons. Let t_i be the mean of the x-values of the things individual i ranked *higher* than the other things, weighted by the respective frequencies of the judgments, and et y_i be the sum of squares of deviations from their mean of these x-values:

$$t_i \equiv \frac{1}{F} \sum_k x_k f_{ik}, \qquad (4.8)$$

$$y_i \equiv \sum_k (x_k - t_i)^2 f_{ik} \equiv \sum_k x_k^2 f_{ik} - t_i^2 F. \qquad (4.9)$$

Similarly, let u_i and z_i be the mean and sum of squares respectively for the x-values of the things individual i ranked *lower* than other things:

$$u_i \equiv \frac{1}{F} \sum_k x_k g_{ik}. \qquad (4.10)$$

$$z_i \equiv \sum_k (x_k - u_i)^2 g_{ik} \equiv \sum_k x_k^2 g_{ik} - u_i^2 F. \qquad (4.11)$$

Let V be the mean of all the x-values in the experiment, and let W be the sum of squares of deviation from their mean of the x-values:

$$V = \frac{1}{C} \sum_k x_k c = \frac{1}{n} \sum_k x_k, \qquad (4.12)$$

$$W = \sum_k (x_k - V)^2 c = c \sum_k x_k^2 - V^2 C. \qquad (4.13)$$

W is the total sum of squares for the experiment. Let R be the sum of squares *between* individuals, and let S be the sum of squares *within* individuals:

$$R = \sum_i [(t_i - V)^2 + (u_i - V)^2]F = F\sum_i (t_i^2 + u_i^2) - V^2 C, \tag{4.14}$$

$$S = \sum_i (y_i + z_i) = W - R. \tag{4.15}$$

Our principle is to quantify the judgments by obtaining the x-values that will *minimize the variation within individuals* compared to that of the group as a whole. This means making S as small as possible compared with W, which is equivalent to making R as large as possible compared with W.

Therefore, if we define the correlation ratio E by

$$E^2 = 1 - S/W, \tag{4.16}$$

the problem is to determine the x_j that will maximize E^2.

A convenient formula for E^2 is, from (4.15) and (4.16),

$$E^2 = R/W. \tag{4.17}$$

Since E^2 is invariant with respect to translations of the x-values, we can without loss of generality set

$$V = 0. \tag{4.18}$$

Then we can write from (4.14) and (4.13), respectively,

$$R = F\sum_i (t_i^2 + u_i^2) \tag{4.19}$$

$$W = c\sum_k x_k^2 . \tag{4.20}$$

To find the maximizing values x_j for E^2, we differentiate the right member of (4.17) with respect to the x_j, set the derivatives equal to zero, and obtain the stationary equations

$$\frac{\partial R}{\partial x_j} = E^2 \frac{\partial W}{\partial x_j} . \tag{4.21}$$

The derivatives of R can be evaluated by differentiating the right member of (4.19) with the aid of (4.8):

$$\frac{\partial R}{\partial x_j} = \frac{2}{F} \sum_k x_k \sum_i (f_{ij} f_{ik} + g_{ij} g_{ik}). \tag{4.22}$$

From (4.20), the derivatives of W are

$$\frac{\partial W}{\partial x_j} = 2cx_j . \tag{4.23}$$

If we let

$$H_{jk} \equiv \frac{1}{cF} \sum_i (f_{ij} f_{ik} + g_{ij} g_{ik}), \tag{4.24}$$

then (4.21) can be re-written from (4.22), (4.23), and (4.24) as:

$$\sum_k x_k H_{jk} = E^2 x_j . \tag{4.25}$$

Equations (4.25) are the equations to be solved numerically for the maximizing x_j.

Before indicating a procedure for the numerical solution, let us first verify that a solution of (4.25) will satisfy (4.18). Summing both members of (4.25) over j, and using (4.24) and relations among the notation previously defined, we get

$$\sum_k x_k = E^2 \sum_j x_j ,$$

or, from (4.12),

$$(1 - E^2) V = 0. \tag{4.26}$$

Therefore, if $E^2 \neq 1$, we must have $V = 0$. Since a perfect correlation ratio will not in general occur in practice, condition (4.18) will in general be satisfied by a solution of (4.25).

There is always a trivial solution of (4.25) for which E^2 is formally equal to unity. This is $x_j \equiv 1$. For this trivial solution, $t_i \equiv u_i \equiv 1$; $R = W = C$; $E^2 = 1$; and (4.25) is satisfied. Of course, E is not an actual correlation ratio for this trivial solution.

The non-trivial solution of (4.25) can be carried out with the aid of matrix algebra. Let $\boldsymbol{x}$ be a row vector of the n elements x_j, and let $\boldsymbol{H}$ be the $n \times n$ symmetric matrix $\| H_{jk} \|$. $\boldsymbol{H}$ is not only symmetric but Gramian, since its elements are product sums. Now (4.25) becomes the matric equation

$$\boldsymbol{x}\boldsymbol{H} = E^2 \boldsymbol{x} . \tag{4.27}$$

Equation (4.27) shows that x is a latent vector of H, and E^2 is a latent root to which this vector corresponds. Since we want the largest possible correlation ratio, we seek the largest of the non-trivial roots. If the two largest non-trivial roots are not equal, which should be the general case in practice, then there is a unique vector associated with the largest root which is the solution to our problem.

The numerical solution of (4.27) can be carried out by the simple iterative technique for latent roots and vectors (see, for example [6]). The iterations converge in general to the vector associated with the largest root. To avoid convergence to the trivial solution (which formally has the largest root), the trial vectors should be adjusted to satisfy (4.18); then they will converge in general to the vector associated with the largest non-trivial root.

A good way to choose a first trial vector is first to guess what the rank order of the x-values will be. Let r_j be the guessed rank of x_j, the r_j comprising the integers from one to n. If n is odd, then as the first trial x_j use $r_j - (n + 1)/2$. If n is even, then as the first trial x_j use $2 r_j - n - 1$.

A marginal check on the internal consistency of the judgments of the population is to compare each difference $(x_j - x_k)$ with the corresponding difference $(\sum_i e_{ijk} - \sum_i e_{ikj})$. If the population's judgments are sufficiently consistent, the signs of the two differences will be alike for all the comparisons. $\sum_i e_{ijk}$ is the frequency with which O_j is judged greater then O_k, and can be used as a basis for guessing the ranks of x_j and x_k.

5. Comparing combinations of two things. The problem of the score card is but one example of a class of problems that can be formalized as follows. Consider a set of n items, where the jth item has m_j categories. Let O_{jp} be the pth category of the jth item, $(p = 1, 2, \cdots, m_j; j = 1, 2, \cdots, n)$. The O_{jp} may be either qualitative or quantitative, and the order of subscripts assigned the categories can be arbitrary.

Each of N individuals is asked to make judgments of the form that the combination (O_{jp}, O_{kr}) is greater than (or less than) the combination (O_{jq}, O_{ks}). We shall assume that all people compare each of the pairs of combinations, and that the rules of the experiment exclude judgments of equality.

The judgments of all the individuals on all the comparisons can be represented compactly as follows. Let

$$e_{ijk/pr,qs} \equiv \begin{cases} 1 \text{ if individual } i \text{ judges } (O_{jp}, O_{kr}) > (O_{jq}, O_{ks}) \\ 0 \text{ otherwise.} \end{cases} \tag{5.1}$$

Here and throughout this paper the ranges of subscripts, whether free or dummy, will always be as follows:

$$\begin{aligned} i &= 1, 2, \cdots, N \\ j, k &= 1, 2, \cdots, n \\ p, q, r, s &= 1, 2, \cdots, m_j, \text{ (or } m_k \text{, as the case may be),} \end{aligned} \tag{5.2}$$

so that the ranges will not be explicitly stated again.

Definition (5.1) implies the symmetry

$$e_{ijk/pr,qs} \equiv e_{ikj/rp,sq}, \tag{5.3}$$

and that

$$e_{ijk/pr,qs} + e_{ijk/qs,pr} \equiv \begin{cases} 0 \text{ if individual } i \text{ omits the comparison of } (O_{jp}, O_{kr}) \text{ with } (O_{jq}, O_{ks}) \\ 1 \text{ if he judges these two combinations to be unequal.} \end{cases} \tag{5.4}$$

Additional notation is defined as follows. Let $a_{ijk/pr}$ be the number of combinations individual i judged to be lower than (O_{ip}, O_{kr}), and let $b_{ijk/pr}$ be the number of combinations he judged to be higher than (O_{ip}, O_{kr}):

$$a_{ijk/pr} \equiv \sum_q \sum_s e_{ijk/pr,qs} \equiv a_{ikj/rp} \tag{5.5}$$

$$b_{ijk/pr} \equiv \sum_q \sum_s e_{ijk/qs,pr} \equiv b_{ikj/rp} . \tag{5.6}$$

Let $c_{jk/pr}$ be the number of comparisons for all individuals involving (O_{ip}, O_{kr}):

$$c_{jk/pr} \equiv \sum_i (a_{ijk/pr} + b_{ijk/pr}) \equiv c_{kj/rp} . \tag{5.7}$$

Let f_{ijp} be the number of times that O_{jp} occurred in combinations that were judged to be higher than other combinations by individual i, and let g_{ijp} be the number of times O_{jp} occurred in combinations judged lower than others:

$$f_{ijp} \equiv \sum_k \sum_r a_{ijk/pr} \equiv \sum_k \sum_r a_{ikj/rp} , \tag{5.8}$$

$$g_{ijp} \equiv \sum_k \sum_r b_{ijk/pr} \equiv \sum_k \sum_r b_{ikj/rp} . \tag{5.9}$$

Let A_{jp} be the total number of times in the entire experiment that O_{jp} was judged:

$$A_{jp} \equiv \sum_i (f_{ijp} + g_{ijp}) \equiv \sum_k \sum_r c_{jk/pr} \tag{5.10}$$

Let F be the total number of comparisons made by each person, and let C be the total number of judgments in the entire experiment (a comparison comprises two judgments, one of "higher than" and one of "lower than"):

$$F \equiv \sum_j \sum_p f_{ijp} \equiv \sum_j \sum_p g_{ijp} , \tag{5.11}$$

$$C = \sum_j \sum_p A_{jp} = 2NF. \tag{5.12}$$

The means and variances required for the problem are defined as follows. Let x_{jp} be the numerical value to be derived for O_{jp} from the judgments. Let t_i be the mean of the x-values of the combinations individual i judged to be *higher* than other combinations, weighted by the respective frequencies of such judgments, and let u_i be the analogous mean of combinations judged lower than others:

$$t_i \equiv \frac{1}{F} \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr}) a_{ijk/pr} \equiv \frac{2}{F} \sum_k \sum_r x_{kr} f_{ikr} , \tag{5.13}$$

$$u_i \equiv \frac{1}{F} \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr}) b_{ijk/pr} \equiv \frac{2}{F} \sum_k \sum_r x_{kr} g_{ikr} . \tag{5.14}$$

Let y_i be the sum of squares of deviations from their mean of these "higher than" x-values, and let z_i be the analogous sum of squares for the "lower than" x-values:

$$y_i \equiv \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - t_i)^2 a_{ijk/pr} \tag{5.15}$$
$$\equiv \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 a_{ijk/pr} - t_i^2 F,$$

$$z_i \equiv \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - u_i)^2 b_{ijk/pr} \tag{5.16}$$
$$\equiv \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 b_{ijk/pr} - u_i^2 F.$$

Let V be the mean of all x-values, weighted by their respective frequencies in the entire experiment, and let W be the sum of squares of deviations from their mean of these x-values:

$$V = \frac{1}{C} \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr}) c_{jk/pr} = \frac{2}{C} \sum_k \sum_r x_{kr} A_{kr}, \tag{5.17}$$

$$W = \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - V)^2 c_{jk/pr} \tag{5.18}$$
$$= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 c_{jk/pr} - V^2 C.$$

W is the total sum of squares for the experiment. Let R be the sum of squares *between individuals* for the experiment, and let S be the sum of squares *within individuals*:

$$R = \sum_i [(t_i - V)^2 + (u_i - V)^2]F = F \sum_i (t_i^2 + u_i^2) - V^2 C, \tag{5.19}$$

$$S = \sum_i (y_i + z_i) = W - R. \tag{5.20}$$

Our principle for quantifying the judgments is to derive the x-values that will *minimize the variation within individuals* compared with that within the group as a whole. This means making S as small as possible compared with W.

Therefore, if we define the correlation ratio E by

$$E^2 = 1 - S/W, \tag{5.21}$$

our problem is to determine the x_{jp} that will maximize E^2.

A convenient formula for E^2 is, from (5.20) and (5.21),

$$E^2 = R/W. \tag{5.22}$$

Since E^2 is invariant with respect to translations of the x-values, we can without loss of generality set

$$V = 0. \tag{5.23}$$

Then we can write, from (5.19) and (5.18) respectively,

$$R = F \sum_i (t_i^2 + u_i^2) \tag{5.24}$$

$$W = \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 c_{jk/pr}\,. \tag{5.25}$$

6. The unrestricted maximum. To find the maximizing x-values for E^2, we differentiate the right member of (5.22) with respect to the x_{ip} and set the derivatives equal to zero. This yields the stationary equations

$$\frac{\partial R}{\partial x_{jp}} = E^2 \frac{\partial W}{\partial x_{jp}}. \tag{6.1}$$

To evaluate the partial derivatives of R, we differentiate the right member of (5.24), using (5.13) and (5.14), and obtain

$$\frac{\partial R}{\partial x_{jp}} = \frac{8}{F} \sum_k \sum_r x_{kr} \sum_i (f_{ijp} f_{ikr} + g_{ijp} g_{ikr}). \tag{6.2}$$

Similarly for W, we differentiate the right member of (5.25) and obtain

$$\frac{\partial W}{\partial x_{jp}} = 4(x_{jp} A_{jp} + \sum_k \sum_r x_{kr} c_{jk/pr}). \tag{6.3}$$

From (6.2) and (6.3), (6.1) can be written as

$$\sum_k \sum_r x_{kr} h_{jk/pr} = \tfrac{1}{2} E^2 (x_{jp} A_{jp} + \sum_k \sum_r x_{kr} c_{jk/pr}), \tag{6.4}$$

where

$$h_{jk/pr} \equiv \frac{1}{F} \sum_i (f_{ijp} f_{ikr} + g_{ijp} g_{ikr}). \tag{6.5}$$

The numerical solution of the x-values is to be obtained from (6.4).

Before showing a procedure for the numerical solution, let us verify that a solution of (6.4) will also satisfy (5.23). Summing both members of (6.4) over j and p, and using (6.5) and relations among the notation laid down in the previous section, we get

$$\sum_k \sum_r x_{kr} A_{kr} = \tfrac{1}{2} E^2 (\sum_j \sum_p x_{jp} A_{jp} + \sum_k \sum_r x_{kr} A_{kr})$$

or

$$(1 - E^2) \sum_k \sum_r x_{kr} A_{kr} = 0. \tag{6.6}$$

From (5.17), this can be written as

$$(1 - E^2)\, V = 0. \tag{6.7}$$

Therefore, if $E^2 \neq 1$, we must have $V = 0$. Hence, any solution of (6.4) which does not yield a perfect correlation ratio must have a weighted mean of zero for the x-values. Since a perfect correlation ratio will not in general occur in practice, condition (5.23) will in general be satisfied and is no restriction.

It should be noted that there is always a trivial solution for which E^2 is formally equal to unity. The trivial solution is to set $x_{jp} \equiv 1$. Then $t_i \equiv u_i \equiv 2$; $R = W = 4C$; $E^2 = 1$; and (6.4) is satisfied since it reduces to (6.7). For this trivial solution, E is of course not an actual correlation ratio.

The non-trivial numerical solution of (6.4) can be carried out in practice with the aid of matrix algebra. Instead of regarding the x_{ip} as elements of a table with n rows with m_j elements in the jth row, consider the rows of such a table placed end to end to form a single row of $M = \sum_j m_j$ elements. Denote this as the row vector $\boldsymbol{x}$. Correspondingly, consider the values $h_{jk/pr}$ arranged to form the elements of a symmetric matrix $\boldsymbol{H}$ of M rows and columns; consider the M values A_{jp} to be the *diagonal* elements of an $M \times M$ diagonal matrix $\boldsymbol{A}$; and consider the values of $c_{jk/pr}$ arranged to form an $M \times M$ symmetric matrix $\boldsymbol{C}$. Let $\lambda = \frac{1}{2}E^2$. Then (6.4) becomes in matric form:

$$\text{(6.8)} \qquad \boldsymbol{xH} = \lambda(\boldsymbol{xA} + \boldsymbol{xC}) = \lambda \boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C}).$$

In the next paragraph it is shown that, in general, $(\boldsymbol{A} + \boldsymbol{C})$ is non-singular, so that it has an inverse by which the members of (6.8) can be postmultiplied, yielding

$$\text{(6.9)} \qquad \boldsymbol{xH}(\boldsymbol{A} + \boldsymbol{C})^{-1} = \lambda \boldsymbol{x}.$$

This shows that $\boldsymbol{x}$ is a latent vector of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$, and λ is the latent root to which this vector corresponds. Since we want the largest possible correlation ratio, we seek the largest of the non-trivial latent roots. If the two largest non-trivial roots are not equal, which should ordinarily be the case in practice, then there will be a unique latent vector associated with the largest root.

It is of interest to show that all the latent roots of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$ are real and non-negative, and that all the latent vectors are real. First, we notice that $\boldsymbol{H}$ is Gramian, for its elements are product sums. To see that $\boldsymbol{A} + \boldsymbol{C}$ is Gramian, we notice that from (5.18) and (5.10),

$$\text{(6.10)} \qquad W = 2\sum_j \sum_p x_{jp}^2 A_{jp} + 2\sum_j \sum_k \sum_p \sum_r x_{jp} x_{kr} c_{jk/pr} - V^2C,$$

or, in matric notation, and transposing members,

$$\text{(6.11)} \qquad 2\boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C})\boldsymbol{x}' = W + V^2C.$$

Since W is a sum of squares, the right member is clearly non-negative; and hence

$$\text{(6.12)} \qquad \boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C})\boldsymbol{x}' \geqq 0,$$

for all $\boldsymbol{x}$. Thus, $\boldsymbol{A} + \boldsymbol{C}$ is nonnegative-definite, or Gramian. Furthermore, $\boldsymbol{A} + \boldsymbol{C}$ is in general nonsingular, because according to (5.17) and (5.18), V and W cannot vanish simultaneously unless

$$\text{(6.13)} \qquad (x_{jp} + x_{kr})c_{jk/pr} \equiv 0.$$

If $n \geqq 3$, then (6.13) will ordinarily imply that $x_{jp} \equiv 0$, that is, the equality in (6.12) will hold if and only if $\boldsymbol{x} = 0$. In such a case, $\boldsymbol{A} + \boldsymbol{C}$ is *positive*-definite, or is nonsingular as well as Gramian, and possesses an inverse.

As is well known, the inverse of a Gramian matrix is Gramian (see [5, p. 71], for example), so that $(\boldsymbol{A} + \boldsymbol{C})^{-1}$ is Gramian. That the latent roots of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$ are all nonnegative follows from a general theorem that the latent roots of the product of two Gramian matrices are always nonnegative [5, p. 116]. The proof of this is brief, and will be repeated here in a little different variation in order to prove in addition that the latent vectors are all real. Let G be a symmetric square root of $A + C$, so that $G^2 = A + C$. If we postmultiply both members of (6.9) by G, we can write the results as:

$$(xG)(G^{-1}HG^{-1}) = \lambda(xG). \tag{6.14}$$

This shows that xG is a latent vector of $G^{-1}HG^{-1}$ corresponding to the root λ. But $G^{-1}HG^{-1}$ is symmetric, and in fact Gramian, for it can be written in the form $(G^{-1}K)(G^{-1}K)'$, where $KK' = H$. Hence, each λ is nonnegative, and each xG is real, whence each x is real.

The numerical solution of (6.9) can be carried out by the simple iterative technique for latent roots and vectors (see, for example, [6]). The iterations converge in general to the vector associated with the largest root. To avoid convergence to the trivial solution (which formally has the largest root), the trial vectors should be adjusted to satisfy (5.23); then they will in general converge to the vector associated with the largest non-trivial root.

A marginal indication of the internal consistency of the judgments is the agreement in sign of

$$(x_{jp} + x_{kr}) - (x_{jq} + x_{ks})$$

with

$$\sum_i e_{ijk/pr,qs} - \sum_i e_{ijk/qs,pr} ,$$

for each of the comparisons. If one combination is judged higher by more people in comparison with another, then its x-values should exceed those of the other for marginal consistency.

7. The maximum under certain linear restrictions. In the previous section, no restrictions were placed on the x_{jp} in maximizing E^2. For some problems, the O_{jp} may be quantitative, and it may be desired within each item to keep the distances between the x_{jp} proportionate to the distances between the O_{jp}. This was the case for the score card, where a linear system of weighting had to be used to be practicable for the army. It was necessary to derive a constant multiplier for length of service, a constant multipler for time overseas, etc., even though there were curvilinearities in the judgments.

Our principle enables us to handle such restrictions just as well as the unrestricted case. We shall derive the set of multipliers which is most consistent for the judgments in the sense of least squares. The ordering of categories within an item will no longer be considered arbitrary. Instead, subscripts will be assigned in a fashion to make $(O_{jp} - O_{jq})$ proportional to $(p - q)$ *within* each item. For convenience, the subscripts can be assigned beginning from zero for each item.

The linear restriction is to determine x-values in the form

$$x_{jp} = \xi_j + p\eta_j , \tag{7.1}$$

where the ξ_j and the η_j are now the basic unknowns to be solved for to maximize E^2. It is the η_j that are of interest, for they will be the multipliers; but the ξ_j have to be used in the analysis to help determine the multipliers even though they are only additive constants that will not affect the order of total scores of people.

To maximize E^2 under the linear restrictions, we differentiate the right member of (5.22) with respect to the ξ_j and the η_j, set the derivatives equal to zero, and obtain the stationary equations

$$\frac{\partial R}{\partial \xi_j} = E^2 \frac{\partial W}{\partial \xi_j} \tag{7.2}$$

$$\frac{\partial R}{\partial \eta_j} = E^2 \frac{\partial W}{\partial \eta_j} . \tag{7.3}$$

In order to evaluate the indicated derivatives, it is helpful to introduce some more notations. Let:

$$l_{0,ik} \equiv \sum_r f_{ikr} , \qquad m_{0,ik} \equiv \sum_r g_{ikr} \tag{7.4}$$

$$l_{1,ik} \equiv \sum_r r f_{ikr} , \qquad m_{1,ik} \equiv \sum_r r g_{ikr} \tag{7.5}$$

$$d_{a,jk} \equiv \sum_p \sum_r p^a c_{jk/pr} \tag{7.6}$$

$$d_{11,jk} \equiv \sum_p \sum_r prc_{jk/pr} \equiv d_{11,kj} \tag{7.7}$$

$$D_{a,j} \equiv \sum_k \sum_p \sum_r p^a c_{jk/pr} \equiv \sum_k d_{a,jk} \tag{7.8}$$

$$h_{0,jk} \equiv \frac{1}{F} \sum_i (l_{0,ij} l_{0,ik} + m_{0,ij} m_{0,ik}) \tag{7.9}$$

$$h_{1,jk} \equiv \frac{1}{F} \sum_i (l_{1,ij} l_{0,ik} + m_{1,ij} m_{0,ik}) \tag{7.10}$$

$$h_{2,ik} \equiv \frac{1}{F} \sum_i (l_{1,ij} l_{1,ik} + m_{1,ij} m_{1,ik}). \tag{7.11}$$

It is important to notice that $d_{0,jk} \equiv d_{0,kj}$, but that $d_{1,jk} \not\equiv d_{1,kj}$, Similarly, $h_{0,jk} \equiv h_{0,kj}$ and $h_{2,jk} \equiv h_{2,kj}$, but $h_{1,jk} \not\equiv h_{1,kj}$.

To evaluate the derivatives of R, it is helpful to re-write the right members of (5.13) and (5.14) by means of (7.1), (7.4), and (7.5):

$$t_i = \frac{2}{F} \sum_k (\xi_k l_{0,ik} + \eta_k l_{1,ik}) \tag{7.12}$$

$$u_i = \frac{2}{F} \sum_k (\xi_k m_{0,ik} + \eta_k m_{1,ik}). \tag{7.13}$$

Differentiating the right member of (5.24) with respect to the ξ_j and the η_j respectively with the aid of (7.12) and (7.13), and using (7.9), (7.10), and (7.11), yields

$$\frac{\partial R}{\partial \xi_j} = 8\sum_k (\xi_k h_{0,jk} + \eta_k h_{1,kj}) \tag{7.14}$$

$$\frac{\partial R}{\partial \eta_j} = 8\sum_k (\xi_k h_{1,jk} + \eta_k h_{2,jk}). \tag{7.15}$$

For the derivatives of W, we re-write (5.25) using (7.1):

$$W = \sum_j \sum_k \sum_p \sum_r (\xi_j + p\eta_j + \xi_k + r\eta_k)^2 c_{jk/pr} . \tag{7.16}$$

Differentiating with respect to the ξ_j and η_j respectively, we obtain, using (7.6), (7.7), and (7.8),

$$\frac{\partial W}{\partial \xi_j} = 4[\xi_j D_{0,j} + \eta_j D_{1,j} + \sum_k (\xi_k d_{0,jk} + \eta_k d_{1,kj})] \tag{7.17}$$

$$\frac{\partial W}{\partial \eta_j} = 4[\xi_j D_{1,j} + \eta_j D_{2,j} + \sum_k (\xi_k d_{1,jk} + \eta_k d_{11,jk})] . \tag{7.18}$$

The stationary equations (7.2) and (7.3) can now be re-written by means of (7.14), (7.15), (7.17), and (7.18) as:

$$\sum_k (\xi_k h_{0,jk} + \eta_k h_{1,kj}) = \tfrac{1}{2}E^2[\xi_j D_{0,j} + \eta_j D_{1,j} + \sum_k (\xi_k d_{0,jk} + \eta_k d_{1,kj})] \tag{7.19}$$

$$\sum_k (\xi_k h_{1,jk} + \eta_k h_{2,jk}) = \tfrac{1}{2}E^2[\xi_j D_{1,j} + \eta_j D_{2,j} + \sum_k (\xi_k d_{1,jk} + \eta_k d_{11,jk})]. \tag{7.20}$$

These are the equations to be solved numerically for the maximizing ξ_j and η_j.

Before showing a procedure for the numerical solution, let us verify that a solution of (7.19) and (7.20) will satisfy (5.23). From (7.1), (5.17), and (7.8),

$$V = \frac{2}{C} \sum_k (\xi_k D_{0,k} + \eta_k D_{1,k}) . \tag{7.21}$$

Summing both members of (7.19) over j shows that

$$(1 - E^2) \sum_k (\xi_k D_{0,k} + \eta_k D_{1,k}) = 0,$$

or, from (7.21),

$$(1 - E^2)V = 0.$$

Hence, if $E^2 \neq 1$, the corresponding solution will satisfy the condition that $V = 0$.

As in the unrestricted case, there is always a trivial solution that will yield an E^2 formally equal to unity. This trivial solution is $\xi_j \equiv 1$, $\eta_j \equiv 0$, which makes $x_{ip} \equiv 1$ as in the previous case. These values satisfy (7.19) and (7.20), and have $E^2 = 1$. Of course, E is again not an actual correlation ratio for this trivial solution.

To obtain a non-trivial solution, it is convenient to write (7.19) and (7.20) in matric notation. Let

$$z = \| [\xi_i] \quad [\eta_i] \|. \tag{7.22}$$

z is a row vector of $2n$ elements, the first n elements being the ξ_i and the last n elements being the η_i. Let

$$h = \begin{Vmatrix} [h_{0,ik}] & [h_{1,ik}] \\ [h_{1,ki}] & [h_{2,ik}] \end{Vmatrix}. \tag{7.23}$$

h is $2n \times 2n$ and is symmetric; in fact it is also Gramian, since its elements are product sums. Let δ_{jk} be Kronecker's delta, and let

$$c = \begin{Vmatrix} [D_{0,j}\,\delta_{jk} + d_{0,jk}] & [D_{1,j}\,\delta_{jk} + d_{1,jk}] \\ [D_{1,j}\,\delta_{jk} + d_{1,kj}] & [D_{2,j}\,\delta_{jk} + d_{11,jk}] \end{Vmatrix}. \tag{7.24}$$

c also is $2n \times 2n$, symmetric, and Gramian. Again let

$$\lambda = \tfrac{1}{2}E^2. \tag{7.25}$$

Equations (7.19) and (7.20) can now be stated as a single matric equation:

$$zh = \lambda zc. \tag{7.26}$$

In general, c will be nonsingular, so that it will have an inverse by which both members of (7.26) can be postmultiplied to yield

$$zhc^{-1} = \lambda z. \tag{7.27}$$

Therefore z is a latent vector of hc^{-1}, and λ is a latent root. Since we want the largest correlation ratio, we seek the largest of the non-trivial latent roots. The largest root in practice will ordinarily be unique. There is then a unique latent vector corresponding to this root, and the elements of this vector provide the most consistent ξ_i and η_i for the population in the sense of least squares.

That c is Gramian and in general nonsingular, that the latent roots of hc^{-1} are all nonnegative, and that the latent vectors of hc^{-1} are all real, requires only proofs analogous to those for the corresponding properties of $A + C$ and $h(A + C)^{-1}$ in the previous section, which need not be repeated here.

As in the previous section, the final numerical steps can be carried out by iterations according to (7.27). Again, the trial vectors should be adjusted to conform to (5.23) to prevent convergence to the trivial solution.

A marginal indication of the consistency of the quantification is the agreement in sign of

$$(p - q)\eta_j + (r - s)\eta_k$$

with

$$\sum_i e_{ijk/pr,qs} - \sum_i e_{ijk/qs,pr}\,,$$

for all comparisons.

Appendix: A distinction between the conventional principle and the present principle. The relationship between the conventional principle of estimating means of hypothetical distributions and the present principle of reproducing the comparisons of each individual will be analyzed here for the case of ordinary comparisons. Only the *principles* will be contrasted here.

In the conventional approach, it is assumed that each of the N individuals has a numerical value for each of the O_j. Let s_{ij} be such a value of O_j for the ith individual. The hypothesis is that person i makes the judgment $O_j > O_k$ if $s_{ij} > s_{ik}$; and the conventional problem is to estimate from the judgments what the relative distances are between the means μ_j, where

$$\mu_j \equiv \frac{1}{N} \sum_i s_{ij} . \tag{A.1}$$

The ranges of the subscripts are: $i = 1, 2, \cdots, N$; $j, k, l = 1, 2, \cdots, n$; and will not be explicitly indicated.

According to the approach of this paper, *if we are to consider hypothetical variables*, the problem would be to determine for each O_j a numerical value x_j such that the differences $(x_j - x_k)$ will best approximate the $(s_{ij} - s_{ik})$ for each individual in the sense of least squares. This will separate "higher than" x-values from "lower than" x-values. If we let

$$Z = \sum_i \sum_j \sum_k [(s_{ij} - s_{ik}) - w_i (x_j - x_k)]^2 , \tag{A.2}$$

where w_i is a constant of proportionality to be determined for each individual separately, then the problem is to determine the x_j and the w_i which will minimize Z.

Differentiating Z with respect to the w_i and x_j respectively, and setting the derivatives equal to zero, yields the stationary equations

$$\sum_i w_i[(s_{ij} - \bar{s}_i) - w_i (x_j - \bar{x})] = 0 \tag{A.3}$$

$$\sum_k (x_k - \bar{x})(s_{ik} - w_i x_k) = 0 , \tag{A.4}$$

where

$$\bar{s}_i \equiv \frac{1}{n} \sum_k s_{ik} , \qquad \bar{x} = \frac{1}{n} \sum_k x_k . \tag{A.5}$$

Since Z is invariant with respect to translations of the x_j (also to translations of the s_{ij}), the origin of the x_j is arbitrary, and there is no loss in generality in setting

$$\bar{x} = 0. \tag{A.6}$$

Then if we let

$$\alpha = \sum_i w_1^2 , \qquad \beta = \sum_k x_k^2 . \tag{A.7}$$

equations (A.3) and (A.4) can be re-written respectively as

$$\sum_i w_i (s_{ij} - \bar{s}_i) = \alpha x_j , \tag{A.8}$$

$$\sum_k x_k s_{ik} = \beta w_i . \tag{A.9}$$

By summing both members of (A.8) over j, we see that

$$\alpha \sum_j x_j = 0. \tag{A.10}$$

Therefore, since in general $\alpha > 0$, we must have $\bar{x} = 0$; and a solution of (A.8) will necessarily be consistent with (A.6).

Using (A.9) in (A.8) yields the stationary equations for the x_j alone:

$$\sum_k x_k \sum_i s_{ik}(s_{ij} - \bar{s}_i) = \alpha\beta x_j . \tag{A.11}$$

This shows that the x_j are elements of a latent vector corresponding to a latent root $\alpha\beta$ of the $n \times n$ matrix defined by the elements S_{jk} , where

$$S_{jk} \equiv \sum_i s_{ik}(s_{ij} - \bar{s}_i) \equiv \sum_i s_{ij} s_{ik} - \frac{1}{n} \sum_l \sum_i s_{ik} s_{il} . \tag{A.12}$$

To determine which one of the latent roots provides the minimum Z, we first notice—by multiplying both members of (A.9) by w_i , summing over i, and using (A.7)—that

$$\sum_i \sum_k x_k s_{ik} w_i = \alpha\beta. \tag{A.13}$$

Then expanding the right member of (A.2) with the aid of (A.9) and (A.13), we obtain

$$Z/2n = \sum_i \sum_j (s_{ij} - \bar{s}_i)^2 - \alpha\beta . \tag{A.14}$$

Clearly, Z will be minimized if we use the largest $\alpha\beta$. Therefore, we seek the latent vector associated with the largest latent root of $\| S_{jk} \|$.

To examine the relation of the elements of this minimizing latent vector to the means μ_j of the hypothetical variables, denote the variances and correlations of the hypothetical variables by:

$$\sigma_j^2 \equiv \frac{1}{N} \sum_i (s_{ij} - \mu_j)^2 \equiv \frac{1}{N} \sum_i s_{ij}^2 - \mu_j^2 \tag{A.15}$$

$$\rho_{jk} \equiv \frac{\sum_i (s_{ij} - \mu_j)(s_{ik} - \mu_k)}{N \sigma_j \sigma_k} \equiv \frac{\frac{1}{N} \sum_i s_{ij} s_{ik} - \mu_j \mu_k}{\sigma_j \sigma_k} . \tag{A.16}$$

Then

$$\sum_i s_{ij} s_{ik} = N(\sigma_j \sigma_k \rho_{jk} + \mu_j \mu_k) . \tag{A.17}$$

From (A.17) and the last member of (A.12), we can write

$$\text{(A.18)} \qquad \frac{1}{N} S_{jk} \equiv \sigma_j \sigma_k \rho_{jk} + \mu_j \mu_k - \frac{1}{n} \sum_l (\sigma_k \sigma_l \rho_{kl} + \mu_k \mu_l).$$

The elements of the matrix of which the x_j are a latent vector are now expressed in terms of the means, variances, and correlations of the hypothetical variables, according to the right member of (A.18). It is clear that in general, the μ_j are not elements of a latent vector of $\| S_{jk} \|$, so that our approach is in general not equivalent to the conventional approach.

In the special case of equal variances and correlations, such as is often assumed in the conventional approach,[4] we can now see that the μ_j do define a latent vector. For this case, let the common variance be σ^2, and let the common correlation coefficient be ρ. Then

$$\text{(A.19)} \qquad \rho_{jk} \equiv \rho + \delta_{jk}(1 - \rho),$$

where δ_{jk} is Kronecker's delta; and (A.18) becomes

$$\text{(A.20)} \qquad \frac{1}{N} S_{jk} \equiv \sigma^2(1 - \rho)\left(\delta_{jk} - \frac{1}{n}\right) + (\mu_j - \bar{\mu})\mu_k,$$

where

$$\text{(A.21)} \qquad \bar{\mu} = \frac{1}{n} \sum_j \mu_j.$$

From (A.20) and (A.12), (A.11) becomes converted to

$$\text{(A.22)} \qquad [\gamma - \sigma^2(1 - \rho)]\, x_j = (\mu_j - \bar{\mu}) \sum_k \mu_k x_k,$$

where

$$\text{(A.23)} \qquad \gamma = \alpha\beta/N.$$

Multiplying both members of (A.22) by x_j and summing over j shows that

$$\text{(A.24)} \qquad (\sum_j \mu_j x_j)^2 = \beta[\gamma - \sigma^2(1 - \rho)].$$

From (A.22) and (A.24) we obtain the elements of the minimizing latent vector for Z to be, in normalized form,

$$\text{(A.25)} \qquad \frac{x_j}{\sqrt{\beta}} = \frac{\mu_j - \bar{\mu}}{\sqrt{\gamma - \sigma^2(1 - \rho)}}.$$

That this is the minimizing vector follows from the fact that the remaining latent roots must all have $\gamma = \sigma^2(1 - \rho)$ in order to have vectors distinct from (A.25); (A.25) does correspond to the largest nontrivial root, since for it the

[4] More specifically, zero correlations are assumed, but this is not necessary for our purpose.

root satisfies the inequality $\gamma > \sigma^2(1 - \rho)$. (The remaining latent vectors are not uniquely defined, for they all correspond to equal roots.) Therefore, the means of the hypothetical variables are a linear function of the elements of the minimizing latent vector for the case of equal variances and correlations.

As a final comment, it should be pointed out that paired comparisons are insufficient to estimate the hypothetical values. Two persons with widely different hypothetical values will make the same judgments provided only that their values have the same rank order. Therefore, hypotheses about variables presumed to underlie the comparisons cannot be completely tested only on the basis of the comparisons.

Psychologically, it may or may not be proper to assume that judgments of the type $O_j > O_k$ can be expressed as a function of differences $s_{ij} - s_{ik}$. Perhaps, psychologically, comparisons may operate on some more complicated principle. The approach presented in the body of this paper does not assume anything about underlying variables, but simply seeks a set of numerical values that will best help reproduce the observed data for each individual.

REFERENCES

[1] R. A. Fisher, *Statistical Methods for Research Workers*, Eighth Edition, Oliver and Boyd, 1941.

[2] J. P. Guilford, *Psychometric Methods*, McGraw-Hill, 1936.

[3] Louis Guttman, "The quantification of a class of attributes: a theory and method of scale construction," in P. Horst, et al., *The Prediction of Personal Adjustment*, Soc. Sci. Res. Council, New York, 1941.

[4] Louis Guttman, "A basis for scaling qualitative data," *Amer. Sociological Review*, Vol. 9 (1944), pp. 139–150.

[5] Louis Guttman, "Properties of Gramian matrices," in *The Prediction of Quantitative Variates by Factor Analysis*, Dissertation, Univ. of Minnesota, 1942.

[6] Harold Hotelling, "Some new methods in matrix calculation," *Annals of Math. Stat.*, Vol. 15 (1943), pp. 1–34.

[7] C. C. Lienau, "Discrete bivariate distribution in certain problems of statistical order," *Amer. Jour. of Hygiene*, Vol. 33 (1941), pp. 65–85.

[8] "What the soldiers think of the point discharge plan in operation," *What the Soldier Thinks*, War Department, No. 16, 1945.