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AN APPROACH TO COMPUTING INTERVAL-VALUED EGALITARIAN SHAPLEY VALUES OF INTERVAL-VALUED COOPERATIVE GAMES WITH COALITION MONOTONICITY-LIKE

Abstract. *There are not effective and practical methods for interval-valued (IV) cooperative games due to the invertible interval subtraction. This paper focuses on developing a fast and simplified approach to computing IV egalitarian Shapley values for such a rather large subclass of IV cooperative games. In this method, through adding some weaker coalition monotonicity-like conditions, it is proven that egalitarian Shapley value is monotonic and non-decreasing. Hence, the IV egalitarian Shapley value of IV cooperative games can be directly obtained by determining its lower and upper bounds, respectively. The method proposed in this paper does not use the Moore's interval subtraction and hereby can effectively avoid the issues resulted from it. Furthermore, some important properties of the IV egalitarian Shapley values of IV cooperative games are discussed. The feasibility and applicability of the method proposed in this paper are illustrated with real numerical examples.*

Keywords: *cooperative game, Shapley value, interval computing, fuzzy cooperative game.*

JEL Classification: C71

1. Introduction

Cooperative games with coalitions' payoffs represented by interval-values (IVs), which are called IV cooperative games for short, were first introduced by Branzei et al. (2003). Presently, there has been increasing research on IV cooperative games. Branzei et al. (2010) updated the results about IV cooperative games and reviewed various existing and potential applications of IV cooperative games in management situations. Much research investigated the IV core (Branzei

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et al., 2011; AlparslanGök et al., 2011) and the IV Shapley-like values of IV cooperative games by extending the Shapley value (Shapley, 1953) through using Moore's interval algebraic algorithms (Moore, 1979).Han et al. (2012) proposed the IV core and the IV Shapley-like value of IV cooperative games by defining new order relation of intervals. Liao (2012) characterized the IV Shapley value with the weak inessential games property and the individual rationality property. Liao (2014) provided several axiomatizations of the IV Shapley value based on the equivalence theorem. AlparslanGök et al. (2010) studied some properties of the IV Shapley value on the class of size monotonic IV games and an axiomatic characterization of the IV Shapley value on the given special subclass of IV cooperative games. AlparslanGök (2014) and Palanci et al. (2015) further investigated on characterizing the IV Shapley values by using some properties. Palanci et al. (2016) introduced the IV Shapley value of a transportation IV game. However, most of the aforementioned methods used the partial subtraction operator or Moore's interval subtraction (Moore, 1979) which is not invertible and usually enlarges uncertainty of the resulted interval. Therefore, it is a very necessary and an important research on how to solve the IV cooperative games without using the interval subtraction and the ranking of intervals. Thereby, the main purpose of this paper is to developing a fast and simplified method for computing IV egalitarian Shapley values of IV cooperative games under some weaker conditions. In this method, through adding some weaker conditions such as the size monotonicity-like, we prove that the egalitarian Shapley value of any cooperative game is a monotonic and non-decreasing function of coalition's payoffs. Thus, the IV egalitarian Shapley value of any IV cooperative game can be directly and explicitly obtained via determining its lower and upper bounds by using the lower and upper bounds of the IV coalitions' payoffs, respectively. Moreover, we prove that the derived IV egalitarian Shapley values of IV cooperative games possess some useful and important properties.

The rest of this paper is organized as follows. In the next section, we briefly review the concepts and notations of cooperative games, intervals, and IV cooperative games. Section 3 introduces the concept of IV egalitarian Shapley values of IV cooperative games and develops a fast and simplified method for computing IV egalitarian Shapley values under some conditions. Section 4 discusses several useful and important properties of the IV egalitarian Shapley values of IV cooperative games. In Section 5, the feasibility and applicability of the method proposed in this paper is illustrated with real examples. Conclusion is made in Section 6.

2.Interval and interval-valued cooperative games

2.1 Interval notation and arithmetic operations

An interval is denoted by $\bar{a}=[a_L,a_R]$, where $a_L \in \mathrm{R}$, $a_R \in \mathrm{R}$, $a_L \le a_R$, and R is the set of real numbers, a_L and a_R are called the lower bound and the upper bound of the interval $\bar{a}$, respectively. The set of all intervals is denoted by $\bar{R}$.

According to the interval arithmetic operations as follows (Moore, 1979), for any intervals $\bar{a}=[a_L,a_R]$ and $\bar{b}=[b_L,b_R]$, the equality of two intervals is defined as $\bar{a}=\bar{b} \Leftrightarrow a_L=b_L$ and $a_R=b_R$; the addition (or sum) of two intervals is defined as $\bar{a}+\bar{b}=[a_L+b_L, a_R+b_R]$; the scalar multiplication of a real number and an interval is defined as $\gamma\bar{a}=[\gamma a_L, \gamma a_R]$ if $\gamma \geq 0$ or $\gamma\bar{a}=[\gamma a_R, \gamma a_L]$ if $\gamma<0$, where $\gamma \in R$ is any real number.

2.2 Interval-valued cooperative games

A *n*-person IV cooperative game $\bar{\upsilon}$ is an ordered-pair $<N,\bar{\upsilon}>$, where $N=\{1,2,\cdots,n\}$ is the set of players and $\bar{\upsilon}$ is the IV characteristic function of players' coalitions, and $\bar{\upsilon}(\varnothing)=[0,0]$. Generally, for any coalition $S \subseteq N$, $\bar{\upsilon}(S)$ is denoted by the interval $\bar{\upsilon}(S)=[\upsilon_L(S),\upsilon_R(S)]$, where $\upsilon_L(S) \leq \upsilon_R(S)$. We usually write $\bar{\upsilon}(S\setminus i)$, $\bar{\upsilon}(S \bigcup i)$, $\bar{\upsilon}(i)$, and $\bar{\upsilon}(i,j)$ instead of $\bar{\upsilon}(S\setminus\{i\})$, $\bar{\upsilon}(S\bigcup\{i\})$, $\bar{\upsilon}(\{i\})$, and $\bar{\upsilon}(\{i,j\})$, respectively. In the sequent, a *n*-person IV cooperative game $<N,\bar{\upsilon}>$ is simply called the IV cooperative game $\bar{\upsilon}$. The set of *n*-person IV cooperative games $\bar{\upsilon}$ is denoted by $\bar{G}^n$.

For any IV cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$, according to the interval addition as stated in Subsection 2.1, $\bar{\upsilon}+\bar{\nu}$ is defined as an IV cooperative game with the IV characteristic function $\bar{\upsilon}+\bar{\nu}$, where

$$(\bar{\upsilon}+\bar{\nu})(S)=\bar{\upsilon}(S)+\bar{\nu}(S)=[\upsilon_L(S)+\nu_L(S),\upsilon_R(S)+\nu_R(S)] \quad (1)$$

for any coalition $S \subseteq N$. Usually, $\bar{\upsilon}+\bar{\nu}$ is called the sum of the IV cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$. Obviously, $\bar{\upsilon}+\bar{\nu}$ is also an IV cooperative game belonging to $\bar{G}^n$, i.e., $(\bar{\upsilon}+\bar{\nu}) \in \bar{G}^n$.

For any IV cooperative game $\bar{\upsilon} \in \bar{G}^n$, it is easy to see that each player should receive an IV payoff from the cooperation due to the fact that each coalition's value is an interval. Let $\bar{x}_i(\bar{\upsilon})=[x_{Li}(\bar{\upsilon}),x_{Ri}(\bar{\upsilon})]$ be the IV payoff which is allocated to the player $i \in N$ under the cooperation that the grand coalition is reached. Denote $\bar{\mathbf{x}}(\bar{\upsilon})=(\bar{x}_1(\bar{\upsilon}),\bar{x}_2(\bar{\upsilon}),\cdots,\bar{x}_n(\bar{\upsilon}))^{\mathrm{T}}$, which is the vector of the IV payoffs for all n players in the grand coalition N.

For an IV cooperative game $\bar{\upsilon} \in \bar{G}^n$, the efficiency and individual rationality of an IV payoff vector $\bar{\mathbf{x}}(\bar{\upsilon})=(\bar{x}_1(\bar{\upsilon}),\bar{x}_2(\bar{\upsilon}),\cdots,\bar{x}_n(\bar{\upsilon}))^{\mathrm{T}}$ can be

expressed as follows:

$$\sum_{i=1}^{n}\bar{x}_i(\bar{\upsilon})=\bar{\upsilon}(N)$$

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and

$$\bar{x}_i(\bar{\upsilon}) \geq \bar{\upsilon}(i) \quad (i = 1, 2, \cdots, n),$$

respectively.

3.Interval-valued egalitarian Shapley values

Inspired by the idea of the egalitarian Shapley value (Joosten, 1996), for any IV cooperative game $\bar{\upsilon} \in \bar{G}^n$,the IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\text{ESH}\zeta}(\upsilon) = (\bar{\phi}_1^{\text{ESH}\zeta}(\upsilon), \bar{\phi}_2^{\text{ESH}\zeta}(\upsilon), \cdots, \bar{\phi}_n^{\text{ESH}\zeta}(\upsilon))^{\text{T}}$ is defined as follows:

$$\bar{\boldsymbol{\Phi}}^{\text{ESH}\zeta}(\bar{\upsilon}) = (1-\zeta)\bar{\boldsymbol{\rho}}^{\text{ED}}(\bar{\upsilon}) + \zeta\bar{\boldsymbol{\Phi}}^{\text{SH}}(\bar{\upsilon}) \qquad (2)$$

where $\zeta \in [0,1]$ is any parameter which may be chosen by the players according to need in real situations, $\bar{\boldsymbol{\rho}}^{\text{ED}}(\bar{\upsilon})$ and $\bar{\boldsymbol{\Phi}}^{\text{SH}}(\bar{\upsilon})$ are the IV egalitarian value and the IV Shapley value, respectively.

For the IV cooperative game $\bar{\upsilon} \in \bar{G}^n$, we can define an associated cooperative game $\upsilon(\alpha) \in G^n$, where the set of players still is $N = \{1, 2, \cdots, n\}$ and the characteristic function $\upsilon(\alpha)$ of coalitions of players is defined as follows:

$$\upsilon(\alpha)(S) = (1-\alpha)\upsilon_L(S) + \alpha\upsilon_R(S) \quad (S \subseteq N) \qquad (3)$$

and $\upsilon(\alpha)(\emptyset) = 0$. The parameter $\alpha \in [0,1]$ is any real number, which may be interpreted as an attitude factor.

Thus, we can easily obtain the egalitarian Shapley value $\boldsymbol{\Phi}^{\text{ESH}\zeta}(\upsilon(\alpha)) = (\phi_1^{\text{ESH}\zeta}(\upsilon(\alpha)), \phi_2^{\text{ESH}\zeta}(\upsilon(\alpha)), \cdots, \phi_n^{\text{ESH}\zeta}(\upsilon(\alpha)))^{\text{T}}$ of the cooperative game $\upsilon(\alpha) \in G^n$, where

$$\phi_i^{\text{ESH}\zeta}(\upsilon(\alpha)) = (1-\zeta)\rho_i^{\text{ED}}(\upsilon(\alpha)) + \zeta\phi_i^{\text{SH}}(\upsilon(\alpha)) \quad (i = 1, 2, \cdots, n). \qquad (4)$$

Here, $\rho_i^{\text{ED}}(\upsilon(\alpha))$ and $\phi_i^{\text{SH}}(\upsilon(\alpha))$ are the egalitarian value (or equal division value) and the Shapley value of the cooperative game $\upsilon(\alpha) \in G^n$ .Specifically,

$$\rho_i^{\text{ED}}(\upsilon(\alpha)) = \frac{\upsilon(\alpha)(N)}{n} \quad (i = 1, 2, \cdots, n),$$

and

$$\phi_i^{\text{SH}}(\upsilon(\alpha)) = \sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}(\upsilon(\alpha)(S \cup i) - \upsilon(\alpha)(S)) \quad (i = 1, 2, \cdots, n).$$

Accordingly, using Eq. (3), it easily follows from Eq. (4) that

$$\phi_i^{\text{ESH}\zeta}(\upsilon(\alpha)) = (1-\zeta)\frac{(1-\alpha)\upsilon_L(N) + \alpha\upsilon_R(N)}{n} +$$
$$\zeta \sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}\{[(1-\alpha)\upsilon_L(S \cup i) + \alpha\upsilon_R(S \cup i)] - [(1-\alpha)\upsilon_L(S) + \alpha\upsilon_R(S)]\}$$
$$(i = 1, 2, \cdots, n),$$

where $\alpha \in [0,1]$.

Obviously, the egalitarian Shapley value $\boldsymbol{\Phi}^{\mathrm{ESH}\zeta}(\upsilon(\alpha))$ is a continuous function of the characteristic function value $\upsilon(\alpha)$ of coalitions in the cooperative game $\upsilon(\alpha)$. Note that $\upsilon(\alpha)$ is also a continuous function of $\alpha\in[0,1]$ due to Eq.(3). Accordingly, the egalitarian Shapley value $\boldsymbol{\Phi}^{\mathrm{ESH}\zeta}(\upsilon(\alpha))$ is a continuous function of the parameter $\alpha\in[0,1]$.

Theorem 1 For any IV cooperative game $\bar{\upsilon}\in\bar{G}^n$, if the following system of inequalities

$$\upsilon_R(S\cup i)-\upsilon_L(S\cup i)\geq\upsilon_R(S)-\upsilon_L(S)\;(i=1,2,\cdots,n;\;\;S\subseteq N\setminus i) \tag{5}$$

is satisfied, then the egalitarian Shapley value $\boldsymbol{\Phi}^{\mathrm{ESH}\zeta}(\upsilon(\alpha))$ of the cooperative game $\upsilon(\alpha)\in G^n$ is a monotonic and non-decreasing function of the parameter $\alpha\in[0,1]$.

Proof. For any $\alpha\in[0,1]$ and $\alpha'\in[0,1]$, it easily follows from Eqs. (3) and(4) that

$$\begin{aligned}\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha))-\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha'))&=(1-\zeta)\frac{\upsilon(\alpha)(N)-\upsilon(\alpha')(N)}{n}+\\&\quad\zeta\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}[(\upsilon(\alpha)(S\cup i)-\upsilon(\alpha')(S\cup i))-(\upsilon(\alpha)(S)-\upsilon(\alpha')(S))]\\&=(\alpha-\alpha')\{(1-\zeta)\frac{\upsilon_R(N)-\upsilon_L(N)}{n}+\zeta\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}[(\upsilon_R(S\cup i)-\upsilon_L(S\cup i))-(\upsilon_R(S)-\upsilon_L(S))]\}\end{aligned}$$

where $i=1,2,\cdots,n$.

If $\alpha\geq\alpha'$, noting that $\zeta\in[0,1]$, then it easily follows from Eq. (5) that

$$\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha))-\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha'))\geq 0\quad(i=1,2,\cdots,n),$$

i.e., $\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha))\geq\phi_i^{\mathrm{ESH}\zeta}(\upsilon(\alpha'))$ ($i=1,2,\cdots,n$),which mean that the egalitarian Shapley value $\boldsymbol{\Phi}^{\mathrm{ESH}\zeta}(\upsilon(\alpha))$ is a monotonic and non-decreasing function of the parameter $\alpha\in[0,1]$. Thus, we have completed the proof of Theorem 1.

Therefore, for any IV cooperative game $\bar{\upsilon}\in\bar{G}^n$, if it satisfies Eq. (5), then it is directly derived from Theorem 1 and Eq. (4) that the lower and upper bounds of the components (intervals) $\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})$ ($i=1,2,\cdots,n$) of the IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=(\bar{\phi}_1^{\mathrm{ESH}\zeta}(\bar{\upsilon}),\bar{\phi}_2^{\mathrm{ESH}\zeta}(\bar{\upsilon}),\cdots,\bar{\phi}_n^{\mathrm{ESH}\zeta}(\bar{\upsilon}))^{\mathrm{T}}$ are given as follows:

$$\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\phi_i^{\mathrm{ESH}\zeta}(\upsilon(0))=(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S))\quad(i=1,2,\cdots,n) \tag{6}$$

and

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$$\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\phi_i^{\mathrm{ESH}\zeta}(\upsilon(1))=(1-\zeta)\frac{\upsilon_R(N)}{n}+\zeta\sum_{S\subseteq N\backslash i}\frac{s!(n-s-1)!}{n!}(\upsilon_R(S\cup i)-\upsilon_R(S))$$
$$(i=1,2,\cdots,n).\ (7)$$

respectively.

Thus, the IV egalitarian Shapley values $\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=[\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon}),\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})]$ of the players i ($i=1,2,\cdots,n$) in the IV cooperative game $\bar{\upsilon}\in\bar{G}^n$ are directly and explicitly expressed as follows:

$$\begin{aligned}\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=[(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta\sum_{S\subseteq N\backslash i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S)),\\ (1-\zeta)\frac{\upsilon_R(N)}{n}+\zeta\sum_{S\subseteq N\backslash i}\frac{s!(n-s-1)!}{n!}(\upsilon_R(S\cup i)-\upsilon_R(S))]\end{aligned}\quad(8)$$

Obviously, if $\zeta=0$, then the IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$ is reduced to the IV equal division value $\bar{\boldsymbol{\rho}}^{\mathrm{ED}}(\bar{\upsilon})$, i.e., $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}0}(\bar{\upsilon})=\bar{\boldsymbol{\rho}}^{\mathrm{ED}}(\bar{\upsilon})$; if $\zeta=1$, then the IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$ is reduced to the IV Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{SH}}(\bar{\upsilon})$, i.e., $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}1}(\bar{\upsilon})=\bar{\boldsymbol{\Phi}}^{\mathrm{SH}}(\bar{\upsilon})$.

4.Some properties of interval-valued egalitarian Shapley values

It is obvious from Eq. (8) that the IV egalitarian Shapley value is a convex combination of the IV Shapley value and the IV equal division value (Li, 2016). Accordingly, we can easily obtain some useful and important properties of IV egalitarian Shapley values of IV cooperative games through studying the properties of IV Shapley values and IV equal division values.

Theorem 2 (Existence and Uniqueness) For an arbitrary IV cooperative game $\bar{\upsilon}\in\bar{G}^n$ and a given parameter $\zeta\in[0,1]$, if $\bar{\upsilon}$ satisfies Eq. (5), then there always exists an unique IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$, which is determined by Eq. (8).

Proof. Theorem 2 can be easily proven according to Eq. (8) (omitted).

Theorem 3 (Efficiency) For any IV cooperative game $\bar{\upsilon}\in\bar{G}^n$, if it satisfies Eq. (5), then its IV egalitarian Shapley value $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$ satisfies the efficiency, i.e., $\sum_{i=1}^{n}\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\bar{\upsilon}(N)$.

Proof. It follows from Eq. (6) that

$$\sum_{i=1}^{n}\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\sum_{i=1}^{n}[(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S))]$$
$$=(1-\zeta)\upsilon_L(N)+\zeta\sum_{i=1}^{n}\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S))$$

In the same way to that of the Shapley value (Driessen, 1988; Owen, 1982), we have

$$\sum_{i=1}^{n}\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S))=\upsilon_L(N)$$

Hence, we obtain

$$\sum_{i=1}^{n}\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=(1-\zeta)\upsilon_L(N)+\zeta\upsilon_L(N)=\upsilon_L(N)$$

i.e., $\sum_{i=1}^{n}\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\upsilon_L(N)$ $(i=1,2,\cdots,n)$.

Analogously, according to Eq. (7), we can easily prove that $\sum_{i=1}^{n}\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\upsilon_R(N)$. Combining with the aforementioned conclusion, according to the definition of the interval equality as stated in Subsection 2.1, we obtain

$$\sum_{i=1}^{n}\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\bar{\upsilon}(N).$$

Therefore, we have completed the proof of Theorem 3.

Theorem 4 (Additivity) For any IV cooperative games $\bar{\upsilon}\in\bar{G}^n$ and $\bar{\nu}\in\bar{G}^n$, if they satisfy Eq. (5), then $\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\nu})$ $(i=1,2,\cdots,n)$, i.e., $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\nu})$.

Proof. It is derived from Eqs. (1) and(6) that

$$\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=(1-\zeta)\frac{\upsilon_L(N)+\nu_L(N)}{n}+\zeta\{\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}[(\upsilon_L(S\cup i)+\nu_L(S\cup i))-(\upsilon_L(S)+\nu_L(S))]\}$$
$$=\{(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta[\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-\upsilon_L(S))]\}+$$
$$\{(1-\zeta)\frac{\nu_L(N)}{n}+\zeta[\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\nu_L(S\cup i)-\nu_L(S))]\}$$
$$=\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\nu})$$

i.e., $\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\nu})$.

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Analogously, it can easily be proven from Eq. (7) that $\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\nu})$. Accordingly, combining with the aforementioned conclusion, according to the definition of the interval equality as stated in Subsection 2.1, we obtain

$$\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\nu})\,(i=1,2,\cdots,n)$$

i.e., $\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon}+\bar{\nu})=\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\upsilon})+\bar{\boldsymbol{\Phi}}^{\mathrm{ESH}\zeta}(\bar{\nu})$. Therefore, we have completed the proof of Theorem 4.

Players $i\in N$ and $k\in N$ ($i\neq k$) are said to be symmetric in the IV cooperative game $\bar{\upsilon}\in\bar{G}^n$ if $\bar{\upsilon}(S\cup i)=\bar{\upsilon}(S\cup k)$ for any coalition $S\subseteq N\setminus\{i,k\}$.

Theorem 5(Symmetry) For any IV cooperative game $\bar{\upsilon}\in\bar{G}^n$, if it satisfies Eq. (5), and players $i\in N$ and $k\in N$ ($i\neq k$) are symmetric in $\bar{\upsilon}\in\bar{G}^n$, then $\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\bar{\phi}_k^{\mathrm{ESH}\zeta}(\bar{\upsilon})$.

Proof. It easily follows from the definition of player symmetry that

$$\bar{\upsilon}(S\cup i)=\bar{\upsilon}(S\cup k)$$

for any coalition $S\subseteq N\setminus\{i,k\}$. Namely, $\upsilon_L(S\cup i)=\upsilon_L(S\cup k)$ and $\upsilon_R(S\cup i)=\upsilon_R(S\cup k)$. Hence, it is easily derived from Eq. (6) that

$$\begin{aligned}\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon})&=[(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta[\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-(\upsilon_L(S))]\\&=(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta[\sum_{S\subseteq N\setminus k}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup k)-(\upsilon_L(S))]\\&=\phi_{Lk}^{\mathrm{ESH}\zeta}(\bar{\upsilon}).\end{aligned}$$

In the same way, it is easily derived from Eq. (7) that $\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\phi_{Rk}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$. Combining with the aforementioned conclusion, according to the definition of the interval equality as stated in Subsection 2.1, we obtain

$$\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})=\bar{\phi}_k^{\mathrm{ESH}\zeta}(\bar{\upsilon}).$$

Thus, Theorem 5 has been proven.

Let σ be any permutation on the set N. For an IV cooperative game $\bar{\upsilon}\in\bar{G}^n$, we can define the IV cooperative game $\bar{\upsilon}^\sigma\in\bar{G}^n$ with IV characteristic function $\bar{\upsilon}^\sigma$, where $\bar{\upsilon}^\sigma(S)=\bar{\upsilon}(\sigma^{-1}(S))$ for any coalition $S\subseteq N$.

Theorem 6 (Anonymity) For any IV cooperative game $\bar{\upsilon}\in\bar{G}^n$ and any permutation σ on the set N, if $\bar{\upsilon}$ satisfies Eq. (5), then $\bar{\phi}_{\sigma(i)}^{\mathrm{ESH}\zeta}(\bar{\upsilon}^\sigma)=\bar{\phi}_i^{\mathrm{ESH}\zeta}(\bar{\upsilon})$.

Proof. It follows from Eq. (6) that

$$\phi_{L\sigma(i)}^{\mathrm{ESH}\zeta}(\bar{\upsilon}^{\sigma})=[(1-\zeta)\frac{\upsilon_L^{\sigma}(N)}{n}+\zeta[\sum_{S\subseteq N\setminus\sigma(i)}\frac{s!(n-s-1)!}{n!}(\upsilon_L^{\sigma}(S\cup\sigma(i))-(\upsilon_L^{\sigma}(S))]$$
$$=[(1-\zeta)\frac{\upsilon_L(\sigma^{-1}(N)))}{n}+\zeta[\sum_{S\subseteq N\setminus\sigma(i)}\frac{s!(n-s-1)!}{n!}(\upsilon_L((\sigma^{-1}(S\cup\sigma(i)))-\upsilon_L((\sigma^{-1}(S))]$$
$$=[(1-\zeta)\frac{\upsilon_L(N)}{n}+\zeta[\sum_{S\subseteq N\setminus i}\frac{s!(n-s-1)!}{n!}(\upsilon_L(S\cup i)-(\upsilon_L(S))]$$
$$=\phi_{Li}^{\mathrm{ESH}\zeta}(\bar{\upsilon}).$$

Analogously, it is easily derived from Eq. (7)that $\phi_{R\sigma(i)}^{\mathrm{ESH}\zeta}(\bar{\upsilon}^{\sigma})=\phi_{Ri}^{\mathrm{ESH}\zeta}(\bar{\upsilon})$. Combining with the aforementioned conclusion, according to the definition of the interval equality as stated in Subsection 2.1, we obtain $\bar{\phi}_{\sigma(i)}^{\mathrm{ESH}\zeta}(\bar{\upsilon}^{\sigma})=\bar{\phi}_{i}^{\mathrm{ESH}\zeta}(\bar{\upsilon})\quad(i=1,2,\cdots,n)$ .Thus, we have completed the proof of Theorem 6.

Generally, IV egalitarian Shapley values of IV cooperative games do not satisfy the dummy player property, the null player property, and the invariance although IV Shapley values do. Moreover, IV egalitarian Shapley values do not always satisfy the individual rationality.

5. Computational results of real numerical examples and analysis

5.1Two real numerical examples

Example 1. Suppose that there are three salesmen(i.e., players)in a real estate brokerage company, denoted the set of players by $N'=\{1,2,3\}$. They plan to work together to sell houses for raising their gains. Due to the incomplete and uncertain information, they cannot precisely forecast their profits (or gains). Generally, they can estimate the ranges of their profits. In this case, the optimal allocation problem of profits for the salesmen may be regarded as a three-person IV cooperative game $\bar{\upsilon}'$. Thus, we suppose that if they sell houses by themselves, then their profits are expressed with the intervals $\bar{\upsilon}'(1)=[0,1]$, $\bar{\upsilon}'(2)=[1,3]$, and $\bar{\upsilon}'(3)=[2,4]$, respectively. Similarly, if any two salesmen work together, then their profits are expressed with the intervals $\bar{\upsilon}'(1,2)=[2,4]$, $\bar{\upsilon}'(1,3)=[1,4]$, and $\bar{\upsilon}'(2,3)=[2,5]$, respectively. If all three salesmen (i.e., the grand coalition N') work together, then the profit is expressed with the interval $\bar{\upsilon}'(N')=[2,7]$. We want to compute the IV egalitarian Shapley value of the IV cooperative game $\bar{\upsilon}'\in\bar{G}^3$ to determine the distribution of profits.

Obviously, the IV cooperative game $\bar{\upsilon}'\in\bar{G}^3$ satisfies Eq. (5). Thus, according to Eqs. (6) and (7) (or Eq. (8)), the lower and upper bounds of the IV egalitarian Shapley value of the salesman 1 can be obtained as follows:

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$$\phi_{L1}^{\text{ESH}\zeta}(\bar{\upsilon}')) = (1-\zeta)\frac{\upsilon_L'(N')}{3} + \zeta\sum_{S\subseteq\{2,3\}}\frac{s!(2-s)!}{3!}(\upsilon_L'(S\cup 1) - \upsilon_L'(S))$$
$$= (1-\zeta)\times\frac{2}{3} + \zeta[\frac{0!2!}{3!}(\upsilon_L'((1) - \upsilon_L'((\varnothing)) + \frac{1!1!}{3!}(\upsilon_L'((1,2) - \upsilon_L'((2)) +$$
$$\frac{1!1!}{3!}(\upsilon_L'(1,3) - \upsilon_L'(3)) + \frac{2!0!}{3!}(\upsilon_L'(N') - \upsilon_L'(2,3))]$$
$$= \frac{2(1-\zeta)}{3} + \zeta[\frac{1}{3}(0-0) + \frac{1}{6}(2-1) + \frac{1}{6}(1-2) + \frac{1}{3}(2-2)]$$
$$= \frac{2-2\zeta}{3}$$

and

$$\phi_{R1}^{\text{ESH}\zeta}(\bar{\upsilon}')) = (1-\zeta)\frac{\upsilon_R'(N')}{3} + \zeta\sum_{S\subseteq\{2,3\}}\frac{s!(2-s)!}{3!}(\upsilon_R'(S\cup 1) - \upsilon_R'(S))$$
$$= (1-\zeta)\times\frac{7}{3} + \zeta[\frac{0!2!}{3!}(\upsilon_R'(1) - \upsilon_R'(\varnothing)) + \frac{1!1!}{3!}(\upsilon_R'(1,2) - \upsilon_R'(2)) +$$
$$\frac{1!1!}{3!}(\upsilon_R'(1,3) - \upsilon_R'(3)) + \frac{2!0!}{3!}(\upsilon_R'(N') - \upsilon_R'(2,3))]$$
$$= \frac{7(1-\zeta)}{3} + \zeta[\frac{1}{3}(1-0) + \frac{1}{6}(4-3) + \frac{1}{6}(4-4) + \frac{1}{3}(7-5)]$$
$$= \frac{14-7\zeta}{6}$$

respectively, where $\zeta \in [0,1]$.

Similarly, the lower and upper bounds of the IV egalitarian Shapley value of the salesmen 2 and 3 can be obtained as follows: $\phi_{L2}^{\text{ESH}\zeta}(\bar{\upsilon}') = \frac{2+\zeta}{3}$, $\phi_{R2}^{\text{ESH}\zeta}(\bar{\upsilon}') = \frac{7+\zeta}{3}$, $\phi_{L3}^{\text{ESH}\zeta}(\bar{\upsilon}') = \frac{2+\zeta}{3}$, and $\phi_{R3}^{\text{ESH}\zeta}(\bar{\upsilon}') = \frac{14+5\zeta}{6}$, respectively.

Thus, the IV egalitarian Shapley value of the IVcooperative game $\bar{\upsilon}' \in \bar{G}^3$ is obtained as

$$\bar{\boldsymbol{\Phi}}^{\text{ESH}\zeta}(\bar{\upsilon}') = ([\frac{2-2\zeta)}{3}, \frac{14-7\zeta}{6}], [\frac{2+\zeta}{3}, \frac{7+\zeta}{3}], [\frac{2+\zeta}{3}, \frac{14+5\zeta}{6}])^{\mathrm{T}}.$$

Example 2. The economic situation is stated as in Example 1. We construct a new IV cooperative game $\bar{\upsilon}'' \in \bar{G}^2$, where the set of players is $N'' = \{1,2\}$, $\bar{\upsilon}''(1) = [0.3,1]$, $\bar{\upsilon}''(2) = [2,5]$, and $\bar{\upsilon}''(1,2) = [4,6]$.Let us discuss the IV egalitarian Shapley value of the IV cooperative game $\bar{\upsilon}''$.

It is easy to see from $\bar{\upsilon}''(2) = [2,5]$ and $\bar{\upsilon}''(1,2) = [4,6]$ that

$$\bar{\upsilon}''_R(1,2) - \bar{\upsilon}''_L(1,2) = 6 - 4 = 2 < \bar{\upsilon}''_R(2) - \bar{\upsilon}''_L(2) = 5 - 2 = 3 .$$

Thus, the IVcooperative game $\bar{\upsilon}''$ does not satisfy Eq. (5). In this case, if Eqs. (6) and (7) were used, then we have

$$\phi_{L1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'') = \frac{\upsilon_L(1,2) + \zeta[\upsilon_L(1) - \upsilon_L(2)]}{2} = \frac{4 + \zeta(0.3 - 2)}{2} = \frac{4 - 1.7\zeta}{2}$$

and

$$\phi_{R1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'') = \frac{\upsilon_R(1,2) + \zeta[\upsilon_R(1) - \upsilon_R(2)]}{2} = \frac{6 + \zeta(1 - 5)}{2} = \frac{6 - 4\zeta}{2} .$$

Clearly, $\phi_{L1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'') \le \phi_{R1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'')$ is not always true. For example, if $20/23 < \zeta < 1$,then we have $\phi_{L1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'') > \phi_{R1}^{\mathrm{ESH}\zeta}(\bar{\upsilon}'')$, which conflicts with the notation of intervals given in the previous subsection 2.1. Therefore, the IVcooperative game $\bar{\upsilon}''$ does not always have the IVegalitarian Shapley value defined by Eq. (8) (or Eqs. (6) and (7)). The reason is that the IVcooperative game $\bar{\upsilon}''$ is not size monotonic.

6. Conclusions

There are not effective and practical methods for IV cooperative games due to the invertible interval subtraction. We introduced the concept of IV egalitarian Shapley values for IV cooperative games and hereby develop a fast and simplified method for computing the IV egalitarian Shapley values for a subclass of IV cooperative games which satisfy some weaker condition of coalition monotonicity-like. This kind of IV cooperative games is rather large. Moreover, we have proven that the IV egalitarian Shapley values satisfy some useful and important properties such as the efficiency, additivity, the symmetry, and anonymity. In the future research, we will further study more effective and practical methods for computing the IV egalitarian Shapley values of IV cooperative games by relaxing the condition of coalition monotonicity-like.

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