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An Approach to Fuzzy Noncooperative Nash Games[1]

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Abstract. Systems that involve more than one decision maker are often optimized using the theory of games. In the traditional game theory, it is assumed that each player has a well-defined quantitative utility function over a set of the player decision space. Each player attempts to maximize/minimize his/her own expected utility and each is assumed to know the extensive game in full. At present, it cannot be claimed that the first assumption has been shown to be true in a wide variety of situations involving complex problems in economics, engineering, social and political sciences due to the difficulty inherent in defining an adequate utility function for each player in these types of problems. On the other hand, in many of such complex problems, each player has a heuristic knowledge of the desires of the other players and a heuristic knowledge of the control choices that they will make in order to meet their ends.

In this paper, we utilize fuzzy set theory in order to incorporate the players' heuristic knowledge of decision making into the framework of conventional game theory or ordinal game theory. We define a new approach to N-person static fuzzy noncooperative games and develop a solution concept such as Nash for these types of games. We show that this general formulation of fuzzy noncooperative games can be applied to solve multidecision-making problems where no objective function is specified. The computational procedure is illustrated via application to a multiagent optimization problem dealing with the design and operation of future military operations.

Key Words. Noncooperative games, fuzzy games, Nash equilibrium.

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1. Introduction

The theory of multicriteria decision making and games (Ref. 1) is concerned with situations in which a number of decision makers or players must take into account criteria, each of which depends on the decisions of all the decision makers. When there are many decision makers who do not cooperate in a decision process, we are in the realm of noncooperative games. The solution of such a game is defined generally in terms of the rationale that each player adopts as a means of describing optimality. One of the most commonly known rationales is the Nash equilibrium strategy (Ref. 2), which safeguards each player against attempts by any one player to improve further on his/her individual performance criterion. However, the set of objective functions in the game may have uncertain values which will affect the result of the decision making. In a game, the uncertainty is due entirely to the unknown decisions of the other players and is inherent in the model. The degree of uncertainty is reduced through the assumption that each player knows the desires of the other player and the assumption that they will take whatever actions which will aid them in attaining their goals. In order to deal with the uncertainty of the utility (or payoff) functions, Cruz and Simaan (Ref. 3) proposed a theory of ordinal games where, instead of a payoff function, the players are able to rank-order their decision choices against the choices by the other players. Another way to deal with the uncertainties associated with the payoff functions is to use the concept of fuzzy games. The theory of noncooperative fuzzy games started with the work of Butnariu (Ref. 4). In his paper, Butnariu declares a game to be fuzzy when the players have fuzzy preferences. On the other hand, cooperative games with fuzzy coalitions had been introduced by Aubin (Ref. 5). Billot (Ref. 6) studied the equilibrium points of fuzzy games and fuzzy economic equilibrium; he proved the existence of a general fuzzy equilibrium.

Inspired by the theory of ordinal games (Refs. 5, 7) and the fact that the decisions made by rational players may be imprecise owing to the players doubts and shades, their hesitations, and their differences, we use fuzzy-set theory (Refs. 8–9) to further blur the definitions used for softening in ordinal game theory. The result is a new approach to noncooperative fuzzy games in which the players use partly crisp and partly fuzzy strategies, as well as partly crisp and partly fuzzy preferences to play the game. We develop a Nash solution concept for these types of games. We show that this general formulation of fuzzy noncooperative games can be applied to solve multidecision-making problems where no objective function is specified. The computational procedure is illustrated via application to a multiagent optimization problem dealing with the design and operation of future military operations.

2. Fuzzy Matrix Games: Preliminaries and Formulation

In this paper, we consider a finite two-person fuzzy noncooperative game (a bimatrix game). Analogously to the design of fuzzy controllers (Ref. 10), we divide the process of playing the fuzzy game into three processes: fuzzification, inference, and defuzzification (Refs. 8–9), which are defined for each of the players fuzzy preference matrix. These processes help us to automate the selection of fuzzy strategies chosen by the players and their corresponding fuzzy preferences needed to play a fuzzy game.

Before we proceed further in defining the fuzzy game in terms of the above-mentioned processes, we introduce first the necessary notations. Let X be the universe of discourse associated with the decision space for all players. Let U be the universe of discourse associated with the players preferences of various options available to them. In a fuzzy system, any input and output variable with range X and U (e.g., denoting a set of the players strategies or players preferences) is represented by a set of linguistic terms $L(X)$ and $L(U)$(large, medium, and small).

Let P be a set of players with cardinality q (e.g., $1, 2, \ldots, s, \ldots, q$). Let Q denote the total number of strategies available to the sth player. That is, for each player $s \in P$, let

$$X_s = \{X_1^s, X_2^s, \ldots, X_Q^s\}$$

be a nonempty compact convex set of the players strategies X_Q^s. Let

$$U_s = \{J_1^s, J_2^s, \ldots, J_Q^s\}$$

be a nonempty compact convex set of the players preferences of various options available to them. The sets X_s and U_s are subsets of locally convex[4] topological vector spaces $X = \prod_{s \in P} X_s$ and $U = \prod_{s \in P} U_s$, respectively.

Let $F(X_s)$ and $F(U_s)$ denote the collections of all fuzzy sets over X_s and U_s, respectively, and let μ_{Xs} and g_s be fuzzy convex mappings representing a process of fuzzification and inference, while δ_s denotes a continuous mapping (defuzzification) that produces a crisp numerical value from a fuzzy set, a cardinal measure of the player preference.

Without loss of generality, we consider a bimatrix fuzzy noncooperative game. Suppose that we have the two sets of players strategies

$$X_1 = \{X_1^1, X_2^1, \ldots, X_l^1\}, \qquad l = 1, 2, \ldots, N, \tag{1}$$

$$X_2 = \{X_1^2, X_2^2, \ldots, X_h^2\}, \qquad h = 1, 2, \ldots, M, \tag{2}$$

where N is the number of strategies of the P1 player, and M is the number of strategies of the P2 player. First, we define the mappings μ_s, g_s, δ_s.

[4]A topological space X is locally convex if, any neighborhood of X is convex $\forall x \in X$.

The fuzzification process is used by both players to characterize the imprecision or uncertainty of the players decisions to proceed within a specified capacity of the chosen strategy. In order to characterize any measurement over X symbolically, let $L(X)$ be a set of linguistic terms. For example, the set $L_s(X) = \{\text{small, medium, large}\}$ could be used to represent the symbolic values of a specific capacity of the chosen strategy by the sth player. The meaning over X of a symbol $L_s \in L_s(X)$ is characterized, for all $X_Q^s \in X$, by its membership function, denoted $\mu_{T(X,L)}(X_Q^s)$, where the mapping $T(X, L)$ associates any symbol L_s of $L_s(X)$ with a subset X_s of X. The fuzzy meanings of small, medium, large, etc. are represented by the membership functions A_i^l, $i = 1, 2, \ldots, n$, for player P1 and by the membership functions B_j^h, $j = 1, 2, \ldots, m$, for player P2. The subscripts n and m represent the numbers of membership functions used to cover the universe of discourse of the lth strategy for player P1 and the hth strategy for player P2, respectively. For example, the set of symbols for the first ($l = 1$) strategy of player P1 is $A_i^1 = \{\text{small, medium, large}\}$, $i = 1, 2, 3$.

Definition 2.1. The fuzzification of the players strategies is defined by the mapping $\mu_{T(X,L)}(X_Q^s)$: $X \rightarrow [0, 1]$ by which each X_Q^s is assigned a number in [0, 1] indicating the extent to which X_Q^s has the attribute $T(X_s, L_s)$.

Fuzzy inference depends on the representation of the set of rules, called rule base. It is a finite set of linguistic statements that allows each player to incorporate his/her heuristic knowledge of a possible intent of the other player.

Definition 2.2. Given two sets X and U, inference is defined as a mapping from the set $F(X)$ associated with the players decision space to a set of fuzzy subsets associated with the players preferences $F(U)$ and denoted by g_s: $F(X) \rightarrow F(U)$.

Denote the symbols for the set of preferences of players P1 and P2, associated with the pair of strategies $\{X_l^1, X_h^2\}$, $l = 1, 2, \ldots, N$ and $h = 1, 2, \ldots, M$, by $U_k^{l,h}$ and $V_k^{l,h}$, $k = 1, 2, \ldots, r$, respectively, where r represents the number of membership functions used to cover the universe of discourse of the player preferences. For example,

$$U_k^{l,h} = \{\text{very small; small; medium; medium high; high; very high}\}, \quad k = 1, 2, \ldots, 6.$$

For our later convenience, suppose that we use a set of linguistic-numeric descriptions for the output membership functions, rather than the linguistic descriptions we used until now. The linguistic-numeric values associated with each membership function on the normalized universe of discourse for an output of a fuzzy preference system are simply taken to be the centers of the these membership functions,

$$U_k^{l,h} = \{0; 0.2; 0.4; 0.6; 0.8; 1\}.$$

The set of rules, described in a natural language, defines the relation between the elements X_l^1 of X_1, X_h^2 of X_2 and the elements $J_{l,h}^1 \in U_1, J_{l,h}^2 \in U_2$ of U. The sets of rules for the P1 and P2 player preferences are defined as follows:

Rule 1. If $X_l^1 \in T(A_i^l)_{i=1,2,\ldots,n}$ and $X_h^2 \in T(B_j^h)_{j=1,2,\ldots,m}$, then $J_{l,h}^1 \in T(U_k^{l,h})_{k=1,2,\ldots,r}$.
Rule 2. If $X_l^1 \in T(A_i^l)_{i=1,2,\ldots,n}$ and $X_h^2 \in T(B_j^h)_{j=1,2,\ldots,m}$, then $J_{l,h}^2 \in T(V_k^{l,h})_{k=1,2,\ldots,r}$.

The membership functions for the computational rule of inference from Rule 1 and Rule 2 are defined as

$$\begin{aligned}
&\mu^1_{\Gamma(i,j,k)\in K}(X_l^1, X_h^2, J_{l,h}^1) \\
&= \min(\mu^1_{T(A(i))}(X_l^1), \mu^1_{T(B(j))}(X_h^2)) \cdot \mu^1_{T(U(k))}(J_{l,h}^1) \\
&= \mu^1_{p(i,j,k)} \cdot \mu^1_{T(U(k))}(J_{l,h}^1), \qquad (3)\\
&\mu^2_{\Gamma(i,j,k)\in K}(X_l^1, X_h^2, J_{l,h}^2) \\
&= \min(\mu^2_{T(A(i))}(X_l^1), \mu^2_{T(B(j))}(X_h^2)) \cdot \mu^2_{T(V(k))}(J_{l,h}^2) \\
&= \mu^2_{p(i,j,k)} \cdot \mu^2_{T(V(k))}(J_{l,h}^2), \qquad (4)
\end{aligned}$$

where Γ is either a subset or a fuzzy subset of the Cartesian product $X \times U$.

Definition 2.3. The defuzzification process is a mapping from the set $F(U)$ associated with the output (player preference) to the set U and is denoted by

$$\delta_s: F(U) \to U.$$

The defuzzification process produces a crisp numerical value from a fuzzy subset of the universe of discourse of the players preferences. The crisp numerical value is a cardinal measure of a player preference. It evaluates which strategy is most possible according to the other player preferences. Using the classical method of the center of gravity for the defuzzification process, we obtain the cardinal measures of a player preference as

$$J^1_{(l,h)} = \sum_{i,j,k} \mu^1_{T(U(k))} \cdot \mu^1_{p(i,j,k)} / \sum_{i,j,k} \mu^1_{p(i,j,k)}, \qquad \sum_{i,j,k} \mu^1_{p(i,j,k)} \neq 0, \tag{5}$$

and

$$J^2_{(l,h)} = \sum_{i,j,k} \mu^2_{T(V(k))} \cdot \mu^2_{p(i,j,k)} / \sum_{i,j,k} \mu^2_{p(i,j,k)}, \qquad \sum_{i,j,k} \mu^2_{p(i,j,k)} \neq 0. \tag{6}$$

Following the procedure outlined above, each of the two players can design its own fuzzy preference matrix. By superimposing a player fuzzy preference matrix, we get the fuzzy bimatrix game.

Definition 2.4. The fuzzy bimatrix noncooperative game is defined as a pair of $N \times M$ matrices J^1 and J^2 obtained using the mappings μ_s, g_s, δ_s, respectively. The lth, hth entries of J^1 and J^2 represents the cardinal measure of the player preference of the decision pair $\{X^1_l, X^1_h\}$, $l = 1, 2, \ldots, N$ and $h = 1, 2, \ldots, M$.

Example 2.1. As a simple example to illustrate this idea, consider the following situation in the context of planning and conducting a military operation. The military operation is to be conducted between the Blue Force (friendly) and the Red Force (enemy) on the Red Force territory. The Blue Commander is presented with two different teams of unmanned air vehicles (UAVs) to address a task to destroy the specified list of targets that appear to be of high interest to the Blue Force. All choices of team compositions appear reasonable from the point of view of accomplishing the task. The Blue Commander anticipates that, for each choice of team composition that he/she is presented with, the Red Commander has several options of defensive force deployment packages to defend targets (two teams of air defense systems).

The framework for such a game is as follows:

(i) There are two players, Blue Force and Red Force.
(ii) Each player has a choice of two alternatives (strategies): Blue Team 1 (BT1) and Blue Team 2 (BT2); Red Team 1 (RT1) and Red Team 2 (RT2). The four elements of set X are described by $X = \{\text{BT1}, \text{BT2}, \text{RT1}, \text{RT2}\}$.
(iii) The play of the game consists of a single move: Blue and Red simultaneously and independently choose one of the two alternatives available to each of them. However, they are allowed to use

the different capacities of these alternatives (teams). This yields four possible scenarios:

(a) Blue considers the following two possible choices: use Team BT1 to attack the targets; use Team BT2 to attack the targets;
(b) Red considers the following two possible choices: use Air Defense Team RT1 to defend the targets; use Air Defense Team RT2 to defend the targets.

For each choice of team capacity by the Blue Commander and a corresponding choice of defensive force capacity deployment by the Red Commander, we can compute a fuzzy preference or objective using the theory of fuzzy sets and the earlier outlined procedure that involves three processes (fuzzification, inference, defuzzification). We define first a rule base which contains a fuzzy logic quantification of the Blue or Red linguistic description of the payoffs for every pair of strategies (the Blue and Red players fuzzy preference matrices, Tables 1 and 2). Let $U_k^{l,h}$ denote the (l, h) rule in the set of the Blue Commander preference rules for a pair of strategies $\{\mathrm{BT_l}, \mathrm{RT_h}\}$. Let $V_k^{i,j}$ be the (l, h) rule in the set of Red Commander preference rules for a pair of strategies $\{\mathrm{BT_l}, \mathrm{RT_h}\}$. As can be seen from Tables 1 and 2, for the sake of simplicity we have assigned to each of the players strategies three linguistic terms,

$$L(X) = \{\text{small, medium, large}\},$$

that is,

$$L_1(X_1^1) = \{\text{small, medium, large}\} = \{A_1^1, A_2^1, A_3^1\},$$

$$L_2(X_1^2) = \{\text{small, medium, large}\} = \{B_1^1, B_2^1, B_3^1\}),$$

Table 1. Blue fuzzy preferences matrix: S = small; M = medium; L = large; 1 = very high preference; 0.8 = high preference; 0.6 = medium high preference; 0.4 = medium preference; 0.2 = small preference; 0 = very small preference.

		BT1			BT2		
		$A_1^1 \triangleq S$	$A_2^1 \triangleq M$	$A_3^1 \triangleq L$	$A_1^2 \triangleq S$	$A_2^2 \triangleq M$	$A_3^2 \triangleq L$
RT1	$B_1^1 \triangleq S$	$U_5^{1,1} \triangleq 0.8$	$U_6^{1,1} \triangleq 1$	$U_6^{1,1} \triangleq 1$	$U_4^{2,1} \triangleq 0.6$	$U_5^{2,1} \triangleq 0.8$	$U_6^{2,1} \triangleq 1$
	$B_2^1 \triangleq M$	$U_4^{1,1} \triangleq 0.6$	$U_5^{1,1} \triangleq 0.8$	$U_6^{1,1} \triangleq 1$	$U_3^{2,1} \triangleq 0.4$	$U_4^{2,1} \triangleq 0.6$	$U_5^{2,1} \triangleq 0.8$
	$B_3^1 \triangleq L$	$U_3^{1,1} \triangleq 0.4$	$U_4^{1,1} \triangleq 0.6$	$U_5^{1,1} \triangleq 0.8$	$U_2^{2,1} \triangleq 0.2$	$U_3^{2,1} \triangleq 0.4$	$U_4^{2,1} \triangleq 0.6$
RT2	$B_1^2 \triangleq S$	$U_4^{1,2} \triangleq 0.6$	$U_4^{1,2} \triangleq 0.6$	$U_5^{1,2} \triangleq 0.8$	$U_5^{2,2} \triangleq 0.8$	$U_6^{2,2} \triangleq 1.0$	$U_6^{2,2} \triangleq 1.0$
	$B_2^2 \triangleq M$	$U_3^{1,2} \triangleq 0.4$	$U_4^{1,2} \triangleq 0.6$	$U_4^{1,2} \triangleq 0.6$	$U_4^{2,2} \triangleq 0.6$	$U_5^{2,2} \triangleq 0.8$	$U_6^{2,2} \triangleq 1.0$
	$B_3^2 \triangleq L$	$U_2^{1,2} \triangleq 0.2$	$U_3^{1,2} \triangleq 0.4$	$U_4^{1,2} \triangleq 0.6$	$U_3^{2,2} \triangleq 0.4$	$U_4^{2,2} \triangleq 0.6$	$U_5^{2,2} \triangleq 0.8$

Table 2. Red fuzzy preferences matrix: S = small; M = medium; L = large; 1 = very high preference; 0.8 = high preference; 0.6 = medium high preference; 0.4 = medium preference; 0.2 = small preference; 0 = very small preference.

		BT1			BT2		
		$A_1^1 \triangleq S$	$A_2^1 \triangleq M$	$A_3^1 \triangleq L$	$A_1^2 \triangleq S$	$A_2^2 \triangleq M$	$A_3^2 \triangleq L$
RT1	$B_1^1 \triangleq S$	$V_6^{1,1} \triangleq 1$	$V_5^{1,1} \triangleq 0.8$	$V_5^{1,1} \triangleq 0.8$	$V_6^{2,1} \triangleq 1$	$V_6^{2,1} \triangleq 1$	$V_6^{2,1} \triangleq 1$
	$B_2^1 \triangleq M$	$V_6^{1,1} \triangleq 1$	$V_6^{1,1} \triangleq 1$	$V_5^{1,1} \triangleq 0.8$	$V_5^{2,1} \triangleq 0.8$	$V_6^{2,1} \triangleq 1$	$V_6^{2,1} \triangleq 1$
	$B_3^1 \triangleq L$	$V_6^{1,1} \triangleq 1$	$V_6^{1,1} \triangleq 1$	$V_6^{1,1} \triangleq 1$	$V_4^{2,1} \triangleq 0.6$	$V_5^{2,1} \triangleq 0.8$	$V_6^{2,1} \triangleq 1$
RT2	$B_1^2 \triangleq S$	$V_3^{1,2} \triangleq 0.4$	$V_2^{1,2} \triangleq 0.2$	$V_1^{1,2} \triangleq 0$	$V_4^{2,2} \triangleq 0.6$	$V_3^{2,2} \triangleq 0.4$	$V_2^{2,2} \triangleq 0.2$
	$B_2^2 \triangleq M$	$V_4^{1,2} \triangleq 0.6$	$V_3^{1,2} \triangleq 0.4$	$V_2^{1,2} \triangleq 0.2$	$V_5^{2,2} \triangleq 0.8$	$V_5^{2,2} \triangleq 0.8$	$V_3^{2,2} \triangleq 0.4$
	$B_3^2 \triangleq L$	$V_4^{1,2} \triangleq 0.6$	$V_4^{1,2} \triangleq 0.6$	$V_3^{1,2} \triangleq 0.4$	$V_6^{2,2} \triangleq 1$	$V_5^{2,2} \triangleq 0.8$	$V_4^{2,2} \triangleq 0.6$

where each of these linguistic terms is specified with a membership function of trapezoidal or triangular shape (symmetrically distributed on the universe of discourse of the player strategy). The shape and distribution of the membership functions on the universe of discourse of the player strategy can be used to reflect certain constraints and rules imposed upon that strategy. On the other hand, we have assigned six linguistic terms for the player fuzzy preferences,

$$L(U) = \{\text{very small; small; medium; medium high; high; very high}\},$$

that is,

$$U_k^{l,h} = \{0; 0.2; 0.4; 0.6; 0.8; 1\},$$

where each of these linguistic terms is specified with a membership function of triangular shape (symmetrically distributed).

Table 1 describes the Blue Commander heuristic knowledge of the other player intentions and possible actions, as well as the Blue Commander preferred response. For example, Table 1 can be read as follows: If (the Blue Commander decides to use BT1 with medium capacity) and if (the Red Commander uses RT1 with large capacity), then (this pair of strategies has medium high preference (0.6) for the Blue Commander).

Similarly, Table 2 reflects the Red Commander heuristic knowledge of his opponent intentions. The fuzzy preference matrices (Tables 1 and 2) account inherently for the uncertainties associated with the lack of complete knowledge of the other player actions. The rule bases for the two players (Tables 1 and 2) can be updated according to the information obtained from the battlefield.

Suppose that the Blue Commander and the Red Commander use the following team configurations for a single move in the fuzzy game: (a) the

Blue Commander can use 70% of BT1 resources against 60% of Red Defense RT1; (b) the Blue Commander can use 50% of BT2 resources against 60% of Red Defense RT1; (c) the Blue Commander can use 70% of BT1 resources against 50% of the Red Defense RT2; (d) the Blue Commander can use 50% of BT2 resources against 50% of the Red Defense RT2. We compute first the Blue Commander fuzzy preferences for the pair of strategies in (a) by executing the three processes of fuzzification, inference, and defuzzification. The fuzzification process amounts to finding the values of the input membership functions for inputs BT1 = 70% of Blue Team 1 capacity and RT1 = 60% Red Team 1 capacity. Using Fig. 1a, we see that

$$\mu^{\text{Blue}}_{\text{Medium}}(\text{BT1}) = \mu^{\text{B}}_{\text{M}}(\text{BT1}) = 0.3, \qquad \mu^{\text{Blue}}_{\text{Large}}(\text{BT1}) = \mu^{\text{B}}_{\text{L}}(\text{BT1}) = 0.7,$$

$$\mu^{\text{Red}}_{\text{Medium}}(\text{RT1}) = \mu^{\text{R}}_{\text{M}}(\text{RT1}) = 0.6, \qquad \mu^{\text{Red}}_{\text{Large}}(\text{RT1}) = \mu^{\text{R}}_{\text{L}}(\text{RT1}) = 0.4.$$

Inference Mechanism: Determining Which Rules to Use. We quantify each of the rules given in Table 1 with fuzzy logic; we find that the rules that are on are the following:

(a) If BT1 is medium and RT1 is medium, then this pair of strategies has high preference (0.8).
(b) If BT1 is medium and RT1 is large, then this pair of strategies has medium high preference (0.6).
(c) If BT1 is large and RT1 is medium, then this pair of strategies has very high preference (1).
(d) If BT1 is large and RT1 is large, then this pair of strategies has high preference (0.8).

Note that we have at most two membership functions overlapping in this example. We will never have more than four rules at one time. In order to quantify the logic and operation that combines the meaning of two linguistic terms, we use the minimum operation and we denote this certainty by μ_p, Eqs. (6) and (7), e.g., $\mu_p(\text{BT1, RT1})$.

Inference Mechanism: Determining Conclusions. The conclusion reached by rule (a), referred to as rule (1); using the minimum to represent the premise and the product operation to represent the implication of the fuzzy rule, we have

$$\begin{aligned} \mu^{\text{B}}_{(1)}(J^{\text{B}}_{1,1}) &= \mu^{\text{B}}_{p(1)}(\text{BT1, RT1})\mu^{\text{B}}_{0.8}(J^{\text{B}}_{1,1}) \\ &= \min\{\mu^{\text{B}}_{\text{M}}(\text{BT1}), \mu^{\text{B}}_{\text{M}}(\text{RT1})\}\mu^{\text{B}}_{0.8}(J^{\text{B}}_{1,1}) \\ &= \min\{0.3, 0.6\}\mu^{\text{B}}_{0.8}(J^{\text{B}}_{1,1}) = 0.3\mu^{\text{B}}_{0.8}(J^{\text{B}}_{1,1}). \end{aligned}$$

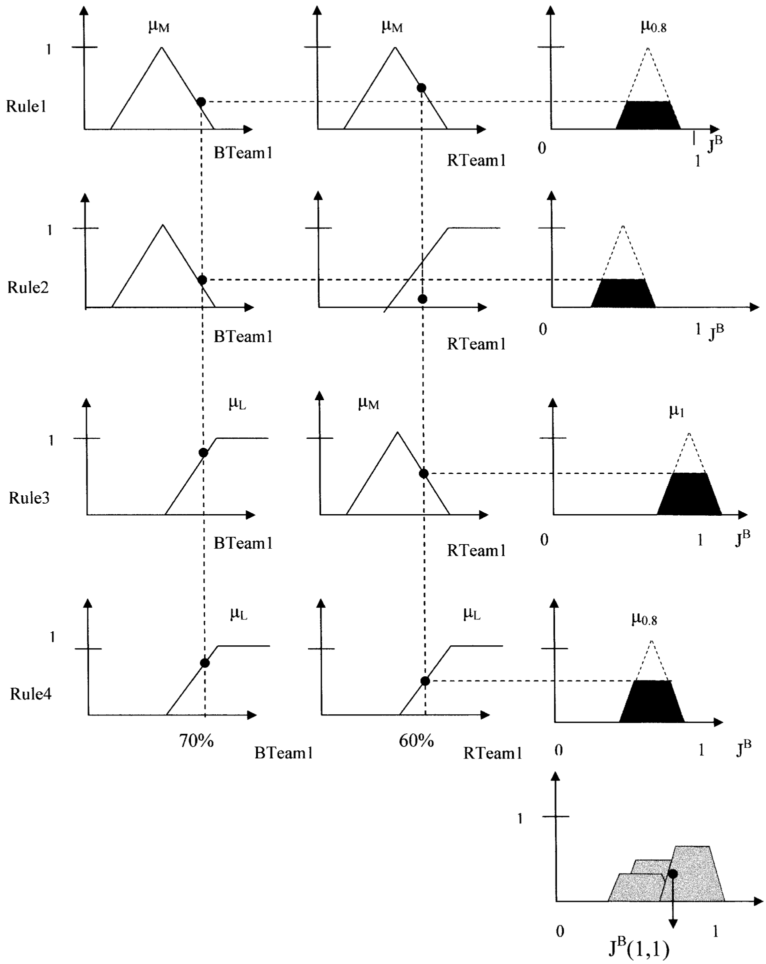

Fig. 1a. Architecture of Blue's fuzzy-logic preferences, Example 2.1, Rule (a).

This membership function defines the conclusion that is implied by fuzzy rule (a) and it is a mapping g: $F(Y) \to F(Z)$ between the fuzzy subset represented by the premise membership function [e.g. $\mu^{B}_{p(1)}$ (BT1,RT1)] and the fuzzy subset represented by the membership function defined on the universe of discourse of the players preferences,

$$J = \{J^{B}, J^{R}\} = \{\{J^{B}_{1,1}, J^{B}_{2,1}, J^{B}_{1,2}, J^{B}_{2,2}\}, \{J^{R}_{1,1}, J^{R}_{2,1}, J^{R}_{1,2}, J^{R}_{2,2}\}\} \in Z \subset U.$$

Here $J^{B}_{l,h}$ and $J^{R}_{l,h} l = 1, 2, \ldots, N$ and $h = 1, 2, \ldots, M$, are the preference ranking of the Blue Commander with respect to the Red Commander and the preference ranking of the Red Commander with respect to the Blue Commander, respectively.

The conclusions reached by rules (b), (c), (d) are

$$\begin{aligned}
\mu^{B}_{(2)}(J^{B}_{1,1}) &= \min\{\mu^{B}_{M}(\mathrm{BT1}), \mu^{B}_{L}(\mathrm{RT1})\}\mu^{B}_{0.6}(J^{B}_{1,1}) \\
&= \min\{0.3, 0.4\}\mu^{B}_{0.6}(J^{B}_{1,1}) = 0.3\mu^{B}_{0.6}(J^{B}_{1,1}), \\
\mu^{B}_{(3)}(J^{B}_{1,1}) &= \min\{\mu^{B}_{L}(\mathrm{BT1}), \mu^{B}_{M}(\mathrm{RT1})\}\mu^{B}_{1}(J^{B}_{1,1}) \\
&= \min\{0.7, 0.4\}\mu^{B}_{1}(J^{B}_{1,1}) = 0.4\mu^{B}_{1}(J^{B}_{1,1}), \\
\mu^{B}_{(4)}(J^{B}_{1,1}) &= \min\{\mu^{B}_{L}(\mathrm{BT1}), \mu^{B}_{L}(\mathrm{RT1})\}\mu^{B}_{0.8}(J^{B}_{1,1}) \\
&= \min\{0.7, 0.6\}\mu^{B}_{0.8}(J^{B}_{1,1}) = 0.6\mu^{B}_{0.8}(J^{B}_{1,1}).
\end{aligned}$$

Defuzzification operates on the implied fuzzy sets produced by the inference mechanism and combines their effects to provide the most preferred outcome a preferred pair of strategies. We use the center of average defuzzification method to compute the fuzzy preference for the pair of strategies $\{\mathrm{BT1}, \mathrm{RT1}\}$,

$$\begin{aligned}
&J^{B}_{1,1}(\mathrm{BT1} = 70\%, \mathrm{RT1} = 60\%) \\
&= \sum_i b_i \mu_{p(i)} / \sum_i \mu_{p(i)} \\
&= [(0.8)(0.3) + (0.6)(0.3) + (1)(0.4) + (0.8)(0.6)]/(0.3 + 0.3 + 0.4 + 0.6) \\
&= 0.8125.
\end{aligned}$$

The fuzzy preferences for the pair of strategies $\{\mathrm{BT2}, \mathrm{RT1}\}$ are (Fig. 1a)

$$\begin{aligned}
\mu^{B}_{(1)}(J^{B}_{2,1}) &= \min\{\mu^{B}_{M}(\mathrm{BT2}), \mu^{B}_{L}(\mathrm{RT1})\}\mu^{B}_{0.4}(J^{B}_{2,1}) \\
&= \min\{1, 0.2\}\mu^{B}_{0.4}(J^{B}_{2,1}) = 0.2\mu^{B}_{0.4}(J^{B}_{2,1}), \\
\mu^{B}_{(2)}(J^{B}_{2,1}) &= \min\{\mu^{B}_{M}(\mathrm{BT2}), \mu^{B}_{M}(\mathrm{RT1})\}\mu^{B}_{0.6}(J^{B}_{2,1}) \\
&= \min\{1, 0.8\}\mu^{B}_{0.6}(J^{B}_{2,1}) = 0.8\mu^{B}_{0.6}(J^{B}_{2,1}), \\
&J^{B}_{2,1}(\mathrm{BT2} = 50\%, \mathrm{RT1} = 60\%) \\
&= \sum_i b_i \mu_{p(i)} / \sum_i \mu_{p(i)}, \\
&= [(0.4)(0.2) + (0.6)(0.8)]/(0.8 + 0.2) \\
&= 0.56.
\end{aligned}$$

The fuzzy preferences for the pair of strategies $\{\mathrm{BT1}, \mathrm{RT2}\}$ are

$$\begin{aligned}
\mu^{B}_{(1)}(J^{B}_{1,2}) &= \min\{\mu^{B}_{M}(\mathrm{BT1}), \mu^{B}_{M}(\mathrm{RT2})\}\mu^{B}_{0.6}(J^{B}_{1,2}) \\
&= \min\{0.2, 1\}\mu^{B}_{0.6}(J^{B}_{1,2}) = 0.2\mu^{B}_{0.6}(J^{B}_{1,2}),
\end{aligned}$$

$$\mu^{B}_{(2)}(J^{B}_{1,2}) = \min\{\mu^{B}_{L}(\mathrm{BT1}), \mu^{B}_{M}(\mathrm{RT2})\}\mu^{B}_{0.4}(J^{B}_{1,2})$$
$$= \min\{0.8, 1\}\mu^{B}_{0.4}(J^{B}_{1,2}) = 0.8\mu^{B}_{0.4}(J^{B}_{1,2}),$$
$$J^{B}_{1,2}(\mathrm{BT1} = 70\%, \mathrm{RT2} = 50\%)$$
$$= \sum_i b_i \mu_{p(i)} / \sum_i \mu_{p(i)}$$
$$= [(0.6)(0.2) + (0.4)(0.8)]/(0.8 + 0.2) = 0.44.$$

The fuzzy preferences for the pair of strategies {BT2, RT2} are (Fig. 1b)

$$\mu^{B}_{(1)}(J^{B}_{2,2}) = \min\{\mu^{B}_{M}(\mathrm{BT2}), \mu^{B}_{M}(\mathrm{RT2})\}\mu^{B}_{0.8}(J^{B}_{2,2})$$
$$= \min\{1, 1\}\mu^{B}_{0.8}(J^{B}_{2,2}) = 1\mu^{B}_{0.8}(J^{B}_{2,2}),$$
$$J^{B}_{2,2}(\mathrm{BT2} = 50\%, \mathrm{RT2} = 50\%) = \sum_i b_i \mu_{p(i)} / \sum_i \mu_{p(i)}$$
$$= (0.8)(1)/1$$
$$= 0.8.$$

Similarly, applying the above outlined procedure, we compute the Red Commander fuzzy preferences $J^{R} = \{J^{R}_{1,1}, J^{R}_{2,1}, J^{R}_{1,2}, J^{R}_{2,2}\} \subset U$; see Fig. 2. The

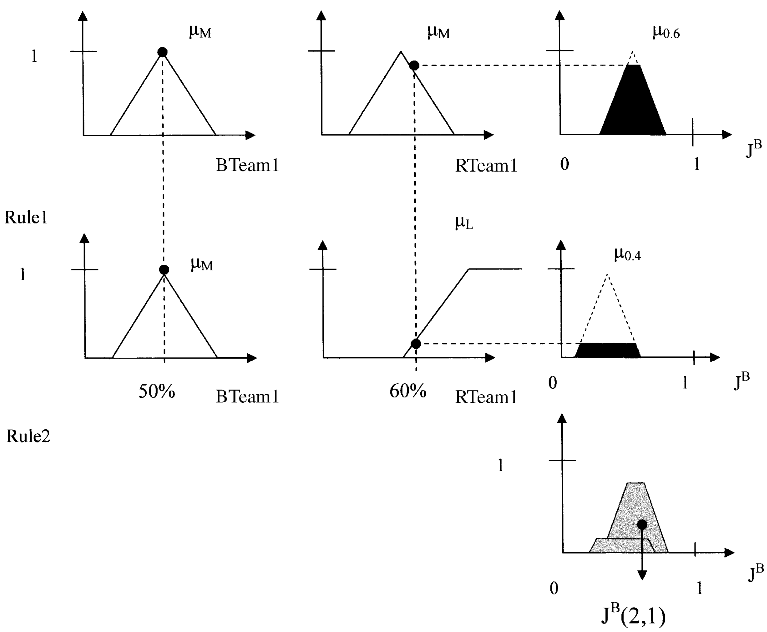

Fig. 1b. Architecture of Blue's fuzzy-logic preferences, Example 2.1, Rule (b).

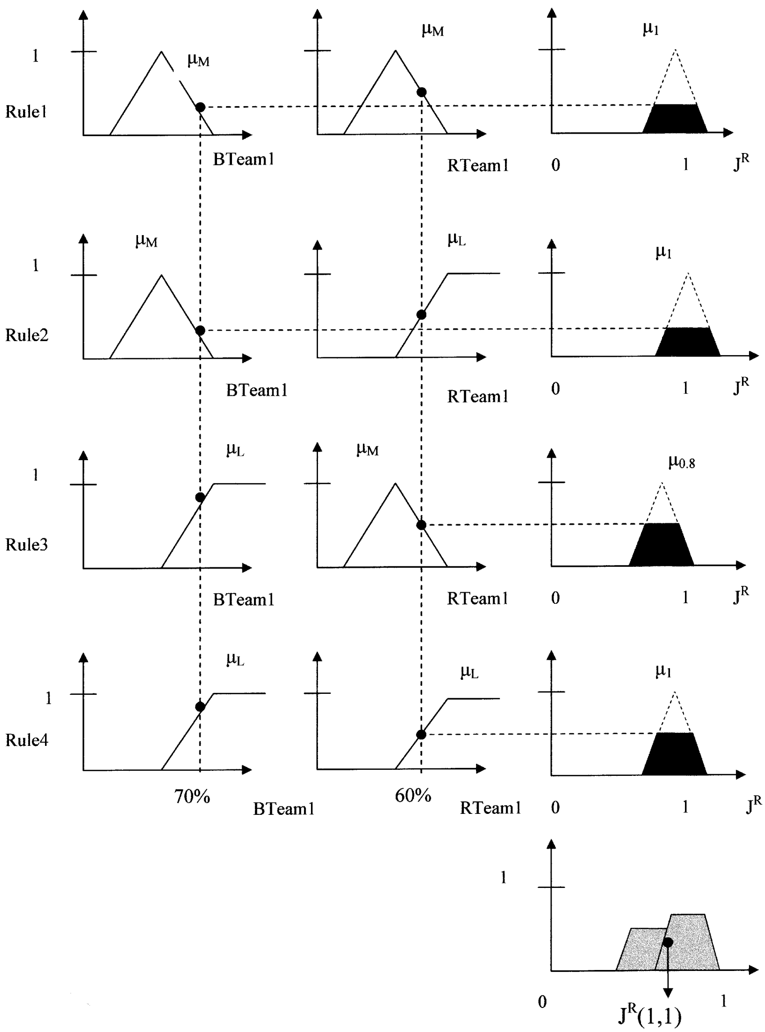

Fig. 2. Architecture of Red's fuzzy-logic preferences, Example 2.1, Rule (a).

players fuzzy preference matrices J^{B} and J^{R} are given as

$$J^{\mathrm{B}}=\begin{bmatrix}0.82 & 0.44\\ 0.56 & 0.80\end{bmatrix}, \qquad J^{\mathrm{R}}=\begin{bmatrix}0.95 & 0.24\\ 0.96 & 0.60\end{bmatrix}.$$

Superimposing the players fuzzy preference matrices, we get the fuzzy bimatrix game shown in Table 3.

Table 3. Blue and Red's fuzzy game for Example 2.1.

	RT1	RT2
BT1	0.82 / 0.95	0.44 / 0.24
BT2	0.56 / 0.96	0.8 / 0.6

3. Nash Equilibrium Solutions for Fuzzy Games

The Nash solution represents an equilibrium point when each player reacts to the other by choosing the option that gives him/her the largest preference function (Ref. 8).

Definition 3.1. A pair of strategies {BT*, RT*} for the fuzzy matrix game (J^B, J^R) given in Table 3 is said to be the Nash solution if, for any other strategies BT and RT,

$$J^B(\text{BT}^*, \text{RT}^*) \geq J^B(\text{BT}, \text{RT}^*), \qquad \forall \text{BT} \in \text{Blue Force} \subset X, \tag{7a}$$

$$J^R(\text{BT}^*, \text{RT}^*) \geq J^R(\text{BT}^*, \text{RT}), \qquad \forall \text{RT} \in \text{Red Force} \subset X. \tag{7b}$$

Example 3.1. In the fuzzy game from Table 3, {BT1 = 70%, RT1 = 60%} is the only pair of options that satisfies inequalities (7). To show this, we have

$$J^B(\text{BT1} = 70\%, \text{RT1} = 60\%) = 0.82,$$

$$J^R(\text{BT1} = 70\%, \text{RT1} = 60\%) = 0.95,$$

since

$$J^B(\text{BT1}^*, \text{RT1}^*) \geq J^B(\text{BT2}, \text{RT1}^*) = 0.56,$$

$$J^R(\text{BT1}^*, \text{RT1}^*) \geq J^R(\text{BT1}^*, \text{RT2}) = 0.24.$$

Then, {BT1 = 70%, RT1 = 60%} is the only Nash solution for the fuzzy game.

Example 3.2. Suppose that the Blue Commander and the Red Commander use the following team configurations for a single move in the fuzzy game: (a) the Blue Commander can use 100% of BT1 resources against 5% of Red Defense RT1; (b) the Blue Commander can use 70% of BT2 resources against 5% of Red Defense RT1; (c) the Blue Commander can use

100% of BT1 resources against 90% of the Red Defense RT2; (d) the Blue Commander can use 70% of BT2 resources against 90% of the Red Defense RT2. Then, applying the procedure outlined in Example 2.1, we obtain the players' fuzzy preference matrices J^{B} and J^{R},

$$J^{\mathrm{B}} = \begin{bmatrix} 1 & 0.6 \\ 1 & 0.7 \end{bmatrix}, \qquad J^{\mathrm{R}} = \begin{bmatrix} 0.6 & 0.4 \\ 1 & 0.7 \end{bmatrix}.$$

Superimposing the players' fuzzy preference matrices, we get the fuzzy bimatrix game shown in Table 4.

In the fuzzy game from Table 4, there are two different Nash equilibria. The pair {BT1 = 100%, RT1 = 5%} is a Nash solution with a cardinal measure of the players' preference

$$(J^{\mathrm{B}}(\mathrm{BT1}, \mathrm{RT1}), J^{\mathrm{R}}(\mathrm{BT1}, \mathrm{RT1})) = (1, 0.6), \tag{8}$$

which states the following: when BT1 reacts with 100% of its capacity to the choice of the Red Commander to use RT1 with only 5% of its capacity, then this play has a very high preference for the Blue Commander, $J^{\mathrm{B}} = 1$. On the other hand, when RT1 reacts with 5% of its capacity to the choice of the Blue Commander to use BT1 with 100%, then this play has a lower preference for the Red Commander, $J^{\mathrm{R}} = 0.6$. The second Nash solution for the fuzzy game from Table 4 is a pair of strategies {BT2 = 70%, RT1 = 5%} with a cardinal measure of the player's preference

$$(J^{\mathrm{B}}(\mathrm{BT2}, \mathrm{RT1}), J^{\mathrm{R}}(\mathrm{BT2}, \mathrm{RT1})) = (1, 1) \tag{9}$$

By comparison between the cardinal measures of the players' preferences given by (8) and (9), it is clear that the Nash solution {BT2, RT1} is more desirable than the Nash solution {BT1, RT1}.

Proposition 3.1. For any fuzzy matrix game, a solution exists always.

Proof. It can be shown that the fuzzy sets are nonempty, compact, bounded, and locally convex and that the fuzzy mappings (μ_s, g_s) are convex

Table 4. Blue and Red's fuzzy game for Example 3.2.

	RT1	RT2
BT1	1 0.6	0.6 0.4
BT2	1 1	0.7 0.7

(Refs. 14, 11); since clearly the transformation in Eqs. (5) and (6) is continuous, then the fixed-point theorem holds (Ref. 11). This implies that the fuzzy game G must have at least one Nash solution (Refs. 4, 6). □

4. Conclusions and Recommendations

In this paper, we have developed a new theoretical approach to fuzzy static noncooperative games. Analogously to the design of fuzzy controllers, we divided the process of playing the fuzzy game into three processes, called fuzzification, inference, and defuzzification, which are defined for each of the players fuzzy preference matrices. These processes helped us to automate the selection by the players of the fuzzy strategies chosen and their corresponding fuzzy preferences needed to play a fuzzy game.

We showed that this general formulation of fuzzy noncooperative games can be applied to solve multidecision-making problems where no objective functions are specified. We proved the existence of at least one Nash solution for a fuzzy matrix game.

Recommendations for future work include the development of a procedure to extend the concept of fuzzy static games to fuzzy dynamic games. Furthermore, incorporating the concept of fuzzy constraints into the definition of fuzzy noncooperative games would allow us to make the core of the game smaller and ultimately yield a set of solutions as small as possible, which is desirable for situations involving complex multidecision-making problems.

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