International Journal of Computational Intelligence Systems, Vol. 8, No. 4 (2015) 747-760

An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers

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Received 17 September 2014

Accepted 7 May 2015

Abstract

Motivated by intuitionistic fuzzy sets and fzzy linguistic approach, this article proposes the concept of linguistic intuitionistic fuzzy numbers (LIFNs) where membership and and nonmembership are represented as linguistic terms. In order to process the multiple attribute decision making (MADM) with LIFNs, we introduce the linguistic score index and linguistic accuracy index of the LIFN. Simultaneously, the operation laws for LIFNs are defined and the related properties of the operation laws are studied. Further, some aggregation operators are developed, involving the linguistic intuitionistic fuzzy weighted averaging (LIFWA) operator, linguistic intuitionistic fuzzy ordered weighted averaging (LIFOWA) operator and linguistic intuitionistic fuzzy hybrid averaging (LIFHA) operator, etc., which can be utilized to aggregate preference information taking the form of LIFNs. Based on the LIFWA and the LIFHA operators, we propose an approach to handle MADM under LIFNs environment. Finally, an illustrative example is given to verify the feasibility and effectiveness of the proposed method, which are then compared to other representative methods.

Keywords: Linguistic intuitionistic fuzzy numbers; Fuzzy linguistic approach; Linguistic intuitionistic fuzzy aggregation operator; Multiple attribute decision making.

1. Introduction

Atanassov [1] introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set. A prominent characteristic of an IFS is that it assigns to each element a membership degree and a nonmembership degree. Due to its capability of accommodating hesitation in human decision processes, IFSs have been widely applied to the field of decision making. For example, Chen and Tan [5] defined the score function to deal with the multiple attribute decision making (MACD) problems based on vague values [10], or equivalently, intuitionistic fuzzy numbers (IFNs), as pointed out by Deschrijver and Kerre [8]. Subsequently, Hong and Choi [11] proposed an accuracy function to furnish additional discrimination powers. Li [16] investigated a technique for solving the MADM problems where the attribute weights and attribute values are IFNs. On a basis of the multiplication operation by

Atanassov [2] and power operation by De and Biswas [7] on IFSs, Xu and Yager [26] developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric operator, the intuitionistic fuzzy ordered weighted geometric operator, and the intuitionistic fuzzy hybrid geometric operator. Further, Xu [27] proposed the intuitionistic fuzzy ordered weighted averaging operator, the intuitionistic fuzzy hybrid averaging operator and applied them to MACD problems.

Usually, in a quantitative setting, the information is expressed by means of numerical values. However, when we work in a qualitative setting, that is, with vague or imprecise knowledge, the information cannot be estimated with an exact numerical value. In that case, a more realistic approach may be to use linguistic assessments instead of numerical values [12,30], that is, to suppose that the variables which participate in the problem are assessed by means of linguistic terms. Several methods have been developed to solve the MACD problem with linguistic information. Bordogna et al. [3] developed the linguistic ordered weighted averaging operators. Liao et al. [17] presented a model for selecting an ERP system based on linguistic information processing. Zhang et al. [31] presented a method to handle fuzzy group decision making based on house of quality for multi-format and multi-granularity linguistic judgments in quality function deployment. Pei et al. [20] presented linguistic weighted aggregation operator to handle fuzzy risk analysis. Rodríguez et al. [21] presented a multicriteria linguistic decision making model in which experts provide their assessments by eliciting linguistic expressions. In order to effectively avoid the loss and distortion of information in linguistic information processing process, Herrera et al. [13,15] proposed 2-tuple linguistic representation model. Martínez et al. [19] made an overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges. Xu [24] adopted the virtual linguistic label to replace 2-tuple linguistic variable and proposed some new aggregation operators, such as linguistic weighted geometric averaging operator (LWGA), linguistic ordered weighted geometric averaging operator (LOWGA), and linguistic hybrid geometric averaging operator (LHGA). Based on the virtual linguistic label, Xu [25] further proposed the concept of uncertain linguistic variable (ULV) and developed uncertain linguistic ordered weighted averaging operator (ULOWA) and uncertain linguistic hybrid aggregation operator (ULHA).

An IFN is characterized by real-valued membership and nonmembership degree defined on $[0,1]$, and the hesitancy degree can be easily derived based on the aforesaid two values. However, under most conditions, decision information is usually uncertain or fuzzy due to the increasing complexity of the environment and the vagueness of the inherent subjective nature of human thought; thus, crisp values are inadequate or insufficient to model real-life decision problems; it might not be flexible or convenient for decision-makers to exactly quantify their opinions with crisp numbers [23]. A possible solution is to represent such membership degrees and nonmembership degree by linguistic variables. So, a new concept called linguistic intuitionistic fuzzy numbers (LIFNs) is established in this paper, which follows the membership degrees and nonmembership degree used by linguistic variables based on the given linguistic term set. LIFNs combine the advantages of both linguistic term sets and IFNs. Intuitively, extending from IFNs to LIFNs furnishes additional capability to handle vague or imprecise information because the membership and nonmembership degrees are only needed to be expressed as linguistic variables rather than exact values. To compare two LIFNs, we introduce the linguistic score index and linguistic accuracy index of a LIFN, which is able to differentiate any two LIFNs, and select the best alternative under LIFNs environment. Further, to process the MADM problem with LIFNs, some operations on LIFNs are defined and their properties are investigated. Simultaneously, some aggregation operators with LIFNs are developed, such as the linguistic intuitionistic fuzzy weighted averaging (LIFWA) operator, linguistic intuitionistic fuzzy ordered weighted averaging (LIFOWA) operator and linguistic intuitionistic fuzzy hybrid averaging (LIFHA) operator, etc.. Finally, we propose an approach to handle MADM under LIFNs environment, and an illustrative example is also given to verify the feasibility and effectiveness of the proposed method.

The rest of this paper is organized as follows. Section 2 briefly reviews the intuitionistic fuzzy sets and the linguistic approach. Section 3 develops the notion of LIFNs, gives the operation laws and analyzes the properties of the operation laws. Section 4 introduces several aggregation operators for LIFNs. Section 5 presents the MAGDM method with LIFNs assessments. A global supplier selection example is illustrated in Section 6. The comparison analyses with other methods are conducted in Section 7. Concluding remark is made in Section 8.

2. Preliminaries

For the convenience of analysis, some basic concepts and definitions on intuitionistic fuzzy numbers and the fuzzy linguistic approach are needed. They are stated as follows.

Definition 1. [29] Let X be a universe of discourse, a fuzzy set in X is defined as $A=\{<x,\mu_A(x)>|x\in X\}$, where $\mu_A:X\to[0,1]$ is the membership function of the fuzzy set A, and $0\leqslant\mu_A(x)\leqslant 1$.

Atanassov [1] introduced a generalized fuzzy set called intuitionistic fuzzy set (IFS), shown as follows:

Definition 2. [1] Let X be a universe of discourse, an intuitionistic fuzzy set in X is an expression: $A=\{<x,\mu_A(x),\nu_A(x)>|x\in X\}$, where the functions $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ define the degree of membership and the degree of nonmembership of the element $\forall x\in X$ to A, and $0\leqslant\mu_A(x)+\nu_A(x)\leqslant 1$.

Usually, $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ is called the intuitionistic fuzzy index of $x\in A$, representing the degree of indeterminacy or hesitation of x to A.

For an IFS A and a given x, Xu and Yager [26] called the pair $(\mu_A(x),\nu_A(x))$ an intuitionistic fuzzy number (IFN). For convenience, we denote an IFN by $\alpha=(\mu_\alpha,\nu_\alpha)$, where $\mu_\alpha\geqslant 0,\nu_\alpha\geqslant 0$ and $\mu_\alpha+\nu_\alpha\leqslant 1$.

For an IFN $\alpha=(\mu_\alpha,\nu_\alpha)$, Chen and Tan [5] introduced the score function $s_\alpha=\nu_\alpha-\mu_\alpha$ to get the score of α. Later, Hong and Choi [11] defined the accuracy function $h_\alpha=\nu_\alpha+\mu_\alpha$ to evaluate the accuracy degree of α.

Based on the score function and the accuracy function, Xu and Yager [26] gave an order relation between any two IFNs in the following:

Definition 3. [26] Let $\alpha=(\mu_\alpha,\nu_\alpha)$ and $\beta=(\mu_\beta,\nu_\beta)$ be two IFNs, s_α and s_β be the scores of α and β, respectively; and h_α and h_β be the accuracy degrees of α and β, respectively,

- If $S_\alpha>S_\beta$, then $\alpha>\beta$;
- If $S_\alpha=S_\beta$, then
  (1) if $h_\alpha=h_\beta$, then $\alpha=\beta$;
  (2) if $h_\alpha>h_\beta$, then $\alpha>\beta$.

To aggregate intuitionistic preference information, Xu and Yager [26] also introduced some operational laws of IFNs as follows:

Definition 4. [26] Let $\alpha=(\mu_\alpha,\nu_\alpha)$ and $\beta=(\mu_\beta,\nu_\beta)$ be two IFNs, then

(1) $\alpha\oplus\beta=(\mu_\alpha+\mu_\beta-\mu_\alpha\mu_\beta,\nu_\alpha\nu_\beta)$;
(2) $\alpha\otimes\beta=(\mu_\alpha\mu_\beta,\nu_\alpha+\nu_\beta-\nu_\alpha\nu_\beta)$;
(3) $\lambda\alpha=(1-(1-\mu_\alpha)^\lambda,\nu_\alpha^\lambda)$, $\lambda>0$;
(4) $\alpha^\lambda=(\mu_\alpha^\lambda,1-(1-\nu_\alpha)^\lambda)$, $\lambda>0$.

The linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables [9,14,25,30], it need to select the appropriate linguistic descriptors for the term set and its semantics.

Suppose that $S=\{s_i|i=0,1\cdots,t\}$ is a linguistic term set with odd cardinality, where t is a positive integer, s_i represents a possible value for a linguistic variable. For example, a set of seven linguistic terms S could be given as follows [18]:

$S=\{s_0=none,\ s_1=very\ low,\ s_2=low,\ s_3=medium,\ s_4=high,\ s_5=very\ high,\ s_6=perfect\}$

Typically, in such cases, the linguistic term set should have the following characteristics [13,18,14]:

- A negation operator: $neg(s_i)=s_j$ such that $j=t-i$;
- Be ordered: $s_i\leqslant s_j$ if and only if $i\leqslant j$;
- Max operator: $max(s_i,s_j)=s_i$ if $s_i\geqslant s_j$;
- Min operator: $min(s_i,s_j)=s_i$ if $s_i\leqslant s_j$.

To preserve all the given information, Xu [24] extend the discrete term set S to a continuous term set $S_{[0,t]}=\{s_\alpha|s_0\leqslant s_\alpha\leqslant s_t\}$, whose elements also meet all the characteristics above, and where, if $s_\alpha \in S$, then it is called the original term, otherwise, $s_\alpha \in S$ is called the virtual term.

In general, the decision maker uses the original linguistic term to evaluate attributes and alternatives, and the virtual linguistic terms can only appear in calculation.

3. Linguistic intuitionistic fuzzy numbers

3.1. *Notions for linguistic intuitionistic fuzzy numbers*

In reality, the degrees of membership and nonmembership for IFSs are sometimes difficult to be derived with exact values. So, we introduce the notion of linguistic intuitionistic fuzzy numbers where membership and nonmembership are represented as linguistic terms.

Definition 5. Let $s_\alpha, s_\beta \in S_{[0,t]}$ and $\gamma = (s_\alpha, s_\beta)$, if $\alpha + \beta \leqslant t$, then we call γ the linguistic intuitionistic fuzzy numbers defined on $S_{[0,t]}$. If $s_\alpha, s_\beta \in S$, then we call γ the original linguistic intuitionistic fuzzy numbers, otherwise, we call γ the virtual linguistic intuitionistic fuzzy numbers.

Remark 1. It should be noted that if $s_\alpha \in S_{[0,t]}$, then $(s_\alpha, neg(s_\alpha))$ is a linguistic intuitionistic fuzzy number (LIFN).

Remark 2. The uncertain linguistic variables [25] be converted into the LIFNs. Let $\widetilde{S} = [s_\alpha, s_\beta]$ be a uncertain linguistic variable (ULV), where $s_\alpha, s_\beta \in S_{[0,t]}$, s_α and s_β are the lower and the upper limits, respectively, then $(s_\alpha, s_{t-\beta})$ is a LIFN.

Let $\Gamma_{[0,t]}$ be the set of all LIFNs based on $S_{[0,t]}$ and (s_α, s_β), $(s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, then the union, intersection and complement operation for LIFNs are defined as follows:

- $(s_{\alpha_1}, s_{\beta_1}) \cup (s_{\alpha_2}, s_{\beta_2}) = (max(s_{\alpha_1}, s_{\alpha_2}), min(s_{\beta_1}, s_{\beta_2}))$;
- $(s_{\alpha_1}, s_{\beta_1}) \cap (s_{\alpha_2}, s_{\beta_2}) = (min(s_{\alpha_1}, s_{\alpha_2}), max(s_{\beta_1}, s_{\beta_2}))$;
- $(s_\alpha, s_\beta)^c = (s_\beta, s_\alpha)$.

According to the union, intersection and complement operation of LIFNs, the following Theorem 1 can be easily proven:

Theorem 1. *Let (s_α, s_β), $(s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, then the following equalities hold:*

(1) $(s_{\alpha_1}, s_{\beta_1}) \cup (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_2}, s_{\beta_2}) \cup (s_{\alpha_1}, s_{\beta_1})$;
(2) $(s_{\alpha_1}, s_{\beta_1}) \cap (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_2}, s_{\beta_2}) \cap (s_{\alpha_1}, s_{\beta_1})$;
(3) $(s_\alpha, s_\beta) \cup [(s_{\alpha_1}, s_{\beta_1}) \cup (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \cup (s_{\alpha_1}, s_{\beta_1})] \cup (s_{\alpha_2}, s_{\beta_2})$;
(4) $(s_\alpha, s_\beta) \cap [(s_{\alpha_1}, s_{\beta_1}) \cap (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \cap (s_{\alpha_1}, s_{\beta_1})] \cap (s_{\alpha_2}, s_{\beta_2})$;
(5) $(s_\alpha, s_\beta) \cup [(s_{\alpha_1}, s_{\beta_1}) \cap (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \cup (s_{\alpha_1}, s_{\beta_1})] \cap [(s_\alpha, s_\beta) \cup (s_{\alpha_2}, s_{\beta_2})]$;
(6) $(s_\alpha, s_\beta) \cap [(s_{\alpha_1}, s_{\beta_1}) \cup (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \cap (s_{\alpha_1}, s_{\beta_1})] \cup [(s_\alpha, s_\beta) \cap (s_{\alpha_2}, s_{\beta_2})]$;
(7) $[(s_{\alpha_1}, s_{\beta_1}) \cup (s_{\alpha_2}, s_{\beta_2})]^c = (s_{\alpha_1}, s_{\beta_1})^c \cap (s_{\alpha_2}, s_{\beta_2})^c$;
(8) $[(s_{\alpha_1}, s_{\beta_1}) \cap (s_{\alpha_2}, s_{\beta_2})]^c = (s_{\alpha_1}, s_{\beta_1})^c \cup (s_{\alpha_2}, s_{\beta_2})^c$.

In the following, we introduce the comparison of two LIFNs.

Definition 6. Let $\gamma = (s_\alpha, s_\beta) \in \Gamma_{[0,t]}$, denote

$$Ls(\gamma) = \alpha - \beta, \qquad Lh(\gamma) = \alpha + \beta, \tag{1}$$

then we call $Ls(\gamma)$ and $Lh(\gamma)$ the linguistic score index and the linguistic accuracy index of γ, respectively.

Definition 7. Let $\gamma_1 = (s_{\alpha_1}, s_{\beta_1}), \gamma_2 = (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$.

(i) If $Ls(\gamma_1) < Ls(\gamma_2)$, then γ_1 is smaller than γ_2, denoted by $\gamma_1 < \gamma_2$;
(ii) If $Ls(\gamma_1) > Ls(\gamma_2)$, then γ_1 is bigger than γ_2, denoted by $\gamma_1 > \gamma_2$;
(iii) If $Ls(\gamma_1) = Ls(\gamma_2)$,
  (a) and $Lh(\gamma_1) = Lh(\gamma_2)$, then γ_1 and γ_2 represent the same information, denoted by $\gamma_1 = \gamma_2$;
  (b) and $Lh(\gamma_1) < Lh(\gamma_2)$, then γ_1 is smaller than γ_2, denoted by $\gamma_1 < \gamma_2$;
  (c) and $Lh(\gamma_1) > Lh(\gamma_2)$, then γ_1 is bigger than γ_2, denoted by $\gamma_1 > \gamma_2$.

Obviously, we have $(s_0, s_t) \leqslant (s_\alpha, s_\beta) \leqslant (s_t, s_0)$ for any $(s_\alpha, s_\beta) \in \Gamma_{[0,t]}$.

Theorem 2. *Let $\gamma_1 = (s_{\alpha_1}, s_{\beta_1}), \gamma_2 = (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$. If $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_2$, then $\gamma_1 \leqslant \gamma_2$.*

Proof. By Definition 6, It is straightforward and thus omitted. □

Example 1. Let $\gamma_1 = (s_5, s_3), \gamma_2 = (s_5, s_2), \gamma_3 = (s_4, s_1) \in \Gamma_{[0,8]}$. It is easy to obtain that

$Ls(\gamma_1) = 2$, $Ls(\gamma_2) = 3$, $Ls(\gamma_3) = 3$,

$Lh(\gamma_1) = 8$, $Lh(\gamma_2) = 7$, $Lh(\gamma_3) = 5$.

According to Definition 6, we can conclude that $\gamma_2 > \gamma_3 > \gamma_1$.

3.2. Operation laws and properties for linguistic intuitionistic fuzzy numbers

Definition 8. Let $(s_\alpha, s_\beta), (s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, $\lambda > 0$, then the operation laws for the LIFNs are defined as follows:

(1) $(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}})$;

(2) $(s_{\alpha_1}, s_{\beta_1}) \otimes (s_{\alpha_2}, s_{\beta_2}) = (s_{\frac{\alpha_1\alpha_2}{t}}, s_{\beta_1+\beta_2-\frac{\beta_1\beta_2}{t}})$;

(3) $\lambda(s_\alpha, s_\beta) = (s_{t-t(1-\frac{\alpha}{t})^\lambda}, s_{t(\frac{\beta}{t})^\lambda})$;

(4) $(s_\alpha, s_\beta)^\lambda = (s_{t(\frac{\alpha}{t})^\lambda}, s_{t-t(1-\frac{\beta}{t})^\lambda})$.

Theorem 3. *Let* $(s_\alpha, s_\beta), (s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, $\lambda > 0$, *and* $\gamma_1 = (s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})$, $\gamma_2 = (s_{\alpha_1}, s_{\beta_1}) \otimes (s_{\alpha_2}, s_{\beta_2})$, $\gamma_3 = \lambda(s_\alpha, s_\beta)$, $\gamma_4 = (s_\alpha, s_\beta)^\lambda$, *then* $\gamma_i \in \Gamma_{[0,t]}, i = 1,2,3,4$.

Proof. Since $(s_\alpha, s_\beta), (s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, Definition 4 implies that $\beta \leqslant t - \alpha$, $\beta_1 \leqslant t - \alpha_1$ and $\beta_2 \leqslant t - \alpha_2$. By $\gamma_1 = (s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}})$, and $\gamma_3 = (s_{t-t(1-\frac{\alpha}{t})^\lambda}, s_{t(\frac{\beta}{t})^\lambda})$, one can have

$\alpha_1 + \alpha_2 - \frac{\alpha_1\alpha_2}{t} + \frac{\beta_1\beta_2}{t} \leqslant \alpha_1 + \alpha_2 - \frac{\alpha_1\alpha_2}{t} + \frac{(t-\alpha_1)(t-\alpha_2)}{t} = t$,

and

$t - t(1-\frac{\alpha}{t})^\lambda + t(\frac{\beta}{t})^\lambda \leqslant t - t(1-\frac{\alpha}{t})^\lambda + t(\frac{t-\alpha}{t})^\lambda = t$.

Hence, $\gamma_1, \gamma_3 \in \Gamma_{[0,t]}$.

Similarly, we can prove $\gamma_2, \gamma_4 \in \Gamma_{[0,t]}$. □

For any $(s_\alpha, s_\beta) \in \Gamma_{[0,t]}$, by Definition 8, we easily obtain

- $(s_\alpha, s_\beta) \oplus (s_t, s_0) = (s_t, s_0)$; $(s_\alpha, s_\beta) \oplus (s_0, s_t) = (s_\alpha, s_\beta)$.
- $(s_\alpha, s_\beta) \otimes (s_t, s_0) = (s_\alpha, s_\beta)$; $(s_\alpha, s_\beta) \otimes (s_0, s_t) = (s_0, s_t)$.

Example 2. Assume that $(s_4, s_2), (s_2, s_6) \in \Gamma_{[0,8]}$, and $\lambda = 0.5$, by Definition 8, we can obtain

(1) $(s_4, s_2) \oplus (s_2, s_6) = (s_{4+2-\frac{2\times4}{8}}, s_{\frac{2\times6}{8}}) = (s_5, s_{1.5})$;

(2) $(s_4, s_2) \otimes (s_2, s_6) = (s_{\frac{4\times2}{8}}, s_{2+6-\frac{2\times6}{8}}) = (s_1, s_{6.5})$;

(3) $\lambda(s_4, s_2) = (s_{8-8(1-\frac{4}{8})^{0.5}}, s_{8(\frac{2}{8})^{0.5}}) = (s_{2.343}, s_4)$;

(4) $(s_4, s_2)^\lambda = (s_{8(\frac{4}{8})^{0.5}}, s_{8-8(1-\frac{2}{8})^{0.5}}) = (s_{5.657}, s_{1.072})$.

In the following, we give the operation properties for LIFNs.

Theorem 4. *Let* $(s_\alpha, s_\beta), (s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$, $\lambda, \lambda_1, \lambda_2 > 0$, *then*

(1) $(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_2}, s_{\beta_2}) \oplus (s_{\alpha_1}, s_{\beta_1})$;

(2) $(s_{\alpha_1}, s_{\beta_1}) \otimes (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_2}, s_{\beta_2}) \otimes (s_{\alpha_1}, s_{\beta_1})$;

(3) $(s_\alpha, s_\beta) \oplus [(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \oplus (s_{\alpha_1}, s_{\beta_1})] \oplus (s_{\alpha_2}, s_{\beta_2})$;

(4) $(s_\alpha, s_\beta) \otimes [(s_{\alpha_1}, s_{\beta_1}) \otimes (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \otimes (s_{\alpha_1}, s_{\beta_1})] \otimes (s_{\alpha_2}, s_{\beta_2})$;

(5) $\lambda[(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = \lambda(s_{\alpha_1}, s_{\beta_1}) \oplus \lambda(s_{\alpha_2}, s_{\beta_2})$;

(6) $[(s_{\alpha_1}, s_{\beta_1}) \otimes (s_{\alpha_2}, s_{\beta_2})]^\lambda = (s_{\alpha_1}, s_{\beta_1})^\lambda \otimes (s_{\alpha_2}, s_{\beta_2})^\lambda$;

(7) $\lambda_1(s_\alpha, s_\beta) \oplus \lambda_2(s_\alpha, s_\beta) = (\lambda_1 + \lambda_2)(s_\alpha, s_\beta)$;

(8) $(s_\alpha, s_\beta)^{\lambda_1} \otimes (s_\alpha, s_\beta)^{\lambda_2} = (s_\alpha, s_\beta)^{(\lambda_1+\lambda_2)}$,

(9) $\lambda_1[\lambda_2(s_\alpha, s_\beta)] = \lambda_1\lambda_2(s_\alpha, s_\beta)$;

(10) $[(s_\alpha, s_\beta)^{\lambda_2}]^{\lambda_1} = (s_\alpha, s_\beta)^{\lambda_1\lambda_2}$.

Proof. (1) According to Definition 8, we have

$(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}}) = (s_{\alpha_2+\alpha_1-\frac{\alpha_2\alpha_1}{t}}, s_{\frac{\beta_2\beta_1}{t}}) = (s_{\alpha_2}, s_{\beta_2}) \oplus (s_{\alpha_1}, s_{\beta_1})$.

(2) It is similar to the proof of (1) and thus omitted.

(3) According to Definition 8, let

$(s_\alpha, s_\beta) \oplus [(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = (s_\alpha, s_\beta) \oplus (s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}}) = (s_u, s_v)$, then

$u = \alpha + (\alpha_1 + \alpha_2 - \frac{\alpha_1\alpha_2}{t}) - \frac{\alpha(\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t})}{t} = \alpha + \alpha_1 + \alpha_2 - \frac{\alpha_1\alpha_2}{t} - \frac{\alpha\alpha_1}{t} - \frac{\alpha\alpha_2}{t} + \frac{\alpha\alpha_1\alpha_2}{t^2}$,

$v = \frac{\beta(\frac{\beta_1\beta_2}{t})}{t} = \frac{\beta\beta_1\beta_2}{t^2}$.

On the other hand, let

$[(s_\alpha, s_\beta) \oplus (s_{\alpha_1}, s_{\beta_1})] \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha+\alpha_1-\frac{\alpha\alpha_1}{t}}, s_{\frac{\beta\beta_1}{t}}) \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{u'}, s_{v'})$, then we have

$u' = \alpha + \alpha_1 - \frac{\alpha\alpha_1}{t} + \alpha_2 - \frac{(\alpha+\alpha_1-\frac{\alpha\alpha_1}{t})\alpha_2}{t} = \alpha + \alpha_1 + \alpha_2 - \frac{\alpha\alpha_1}{t} - \frac{\alpha\alpha_2}{t} - \frac{\alpha_1\alpha_2}{t} + \frac{\alpha\alpha_1\alpha_2}{t^2} = u$,
$v' = \frac{(\frac{\beta\beta_1}{t})\beta_2}{t} = \frac{\beta\beta_1\beta_2}{t^2} = v$.

Therefore, $(s_\alpha, s_\beta) \oplus [(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = [(s_\alpha, s_\beta) \oplus (s_{\alpha_1}, s_{\beta_1})] \oplus (s_{\alpha_2}, s_{\beta_2})$.
(4) It is similar to the proof of (3) and thus omitted.
(5) According to Definition 8, we have
$\lambda[(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = \lambda(s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}}) = (s_{t-t(1-\frac{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}{t})^\lambda}, s_{t(\frac{\frac{\beta_1\beta_2}{t}}{t})^\lambda})$
$= (s_{t-t[(1-\frac{\alpha_1}{t})(1-\frac{\alpha_2}{t})]^\lambda}, s_{t(\frac{\beta_1\beta_2}{t^2})^\lambda})$.
On the other hand, let
$\lambda(s_{\alpha_1}, s_{\beta_1}) \oplus \lambda(s_{\alpha_2}, s_{\beta_2}) = (s_{t-t(1-\frac{\alpha_1}{t})^\lambda}, s_{t(\frac{\beta_1}{t})^\lambda}) \oplus (s_{t-t(1-\frac{\alpha_2}{t})^\lambda}, s_{t(\frac{\beta_2}{t})^\lambda}) = (s_u, s_v)$, then
$u = [t - t(1 - \frac{\alpha_1}{t})^\lambda] + [t - t(1 - \frac{\alpha_2}{t})^\lambda] - \frac{[t-t(1-\frac{\alpha_1}{t})^\lambda][t-t(1-\frac{\alpha_2}{t})^\lambda]}{t}$
$= 2t - t(1 - \frac{\alpha_1}{t})^\lambda - t(1 - \frac{\alpha_2}{t})^\lambda - [t - t(1 - \frac{\alpha_1}{t})^\lambda - t(1 - \frac{\alpha_2}{t})^\lambda + t(1 - \frac{\alpha_1}{t})^\lambda(1 - \frac{\alpha_2}{t})^\lambda]$
$= t - t[(1 - \frac{\alpha_1}{t})(1 - \frac{\alpha_2}{t})]^\lambda$,
$v = \frac{[t(\frac{\beta_1}{t})^\lambda][t(\frac{\beta_2}{t})^\lambda]}{t} = t(\frac{\beta_1\beta_2}{t^2})^\lambda$.
Hence, $\lambda[(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2})] = \lambda(s_{\alpha_1}, s_{\beta_1}) \oplus \lambda(s_{\alpha_2}, s_{\beta_2})$.
(6) It is similar to the proof of (5) and thus omitted.
(7) According to Definition 8, we have
$\lambda_1(s_\alpha, s_\beta) \oplus \lambda_2(s_\alpha, s_\beta) = (s_{t-t(1-\frac{\alpha}{t})^{\lambda_1}}, s_{t(\frac{\beta}{t})^{\lambda_1}}) \oplus (s_{t-t(1-\frac{\alpha}{t})^{\lambda_2}}, s_{t(\frac{\beta}{t})^{\lambda_2}})$
$= (s_{t-t(1-\frac{\alpha}{t})^{\lambda_1}+t-t(1-\frac{\alpha}{t})^{\lambda_2}-\frac{[t-t(1-\frac{\alpha}{t})^{\lambda_1}][t-t(1-\frac{\alpha}{t})^{\lambda_2}]}{t}}, s_{\frac{[t(\frac{\beta}{t})^{\lambda_1}][t(\frac{\beta}{t})^{\lambda_2}]}{t}})$
$= (s_{t-t(1-\frac{\alpha}{t})^{\lambda_1+\lambda_2}}, s_{t(\frac{\beta}{t})^{\lambda_1+\lambda_2}})$
$= (\lambda_1 + \lambda_2)(s_\alpha, s_\beta)$
(8) It is similar to the proof of (7) and thus omitted.
(9) From Definition 8, we have
$\lambda_1[\lambda_2(s_\alpha, s_\beta)] = \lambda_1(s_{t-t(1-\frac{\alpha}{t})^{\lambda_2}}, s_{t(\frac{\beta}{t})^{\lambda_2}}) = (s_{t-t[1-\frac{t-t(1-\frac{\alpha}{t})^{\lambda_2}}{t}]^{\lambda_1}}, s_{t[\frac{t(\frac{\beta}{t})^{\lambda_2}}{t}]^{\lambda_1}})$
$= (s_{t-t(1-\frac{\alpha}{t})^{\lambda_1\lambda_2}}, s_{t(\frac{\beta}{t})^{\lambda_1\lambda_2}}) = \lambda_1\lambda_2(s_\alpha, s_\beta)$
(10) It is similar to the proof of (9) and omitted. □

Theorem 5. *Let* $\gamma = (s_\alpha, s_\beta), \gamma_1 = (s_{\alpha_1}, s_{\beta_1}), \gamma_2 = (s_{\alpha_2}, s_{\beta_2}) \in \Gamma_{[0,t]}$.

(1) If $\alpha_1 \leqslant \alpha_2$ *and* $\beta_1 \geqslant \beta_2$, *then* $\gamma \oplus \gamma_1 \leqslant \gamma \oplus \gamma_2$;
(2) If $\alpha_1 \leqslant \alpha_2$ *and* $\beta_1 \geqslant \beta_2$, *then* $\gamma \otimes \gamma_1 \leqslant \gamma \otimes \gamma_2$.

Proof. By Definition 8, we have $\gamma \oplus \gamma_1 = (s_{\alpha+\alpha_1-\frac{\alpha\alpha_1}{t}}, s_{\frac{\beta\beta_1}{t}})$, $\gamma \oplus \gamma_2 = (s_{\alpha+\alpha_2-\frac{\alpha\alpha_2}{t}}, s_{\frac{\beta\beta_2}{t}})$. Hence, $Ls(\gamma \oplus \gamma_1) = \alpha + \alpha_1 - \frac{\alpha\alpha_1}{t} - \frac{\beta\beta_1}{t}$, $Ls(\gamma \oplus \gamma_2) = \alpha + \alpha_2 - \frac{\alpha\alpha_2}{t} - \frac{\beta\beta_2}{t}$, and
$Hs(\gamma \oplus \gamma_1) = \alpha + \alpha_1 + \frac{\alpha\alpha_1}{t} + \frac{\beta\beta_1}{t}$, $Hs(\gamma \oplus \gamma_2) = \alpha + \alpha_2 + \frac{\alpha\alpha_2}{t} + \frac{\beta\beta_2}{t}$. Since $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_2$. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\gamma \oplus \gamma_1 = \gamma \oplus \gamma_2$; otherwise, $\alpha_1 - \alpha_2 < \beta_1 - \beta_2$. So $Ls(\gamma \oplus \gamma_1) - Ls(\gamma \oplus \gamma_2) = \alpha_1 - \alpha_2 - \frac{\alpha\alpha_1}{t} - \frac{\beta\beta_1}{t} + \frac{\alpha\alpha_2}{t} + \frac{\beta\beta_2}{t} = \frac{(t-\alpha)(\alpha_1-\alpha_2)-\beta(\beta_1-\beta_2)}{t} < \frac{(t-\alpha-\beta)(\beta_1-\beta_2)}{t} \leqslant 0$, that is, $Ls(\gamma \oplus \gamma_1) < Ls(\gamma \oplus \gamma_2)$. Hence, we have $\gamma \oplus \gamma_1 < \gamma \oplus \gamma_2$.
Similarly, we can prove (2). □

4. Some aggregation operators with LIFNs

Based on the operational principle for LIFNs, we shall develop the linguistic intuitionistic fuzzy weighted averaging (LIFWA) operator, linguistic intuitionistic fuzzy ordered weighted averaging (LIFOWA) operator, and linguistic intuitionistic fuzzy hybrid averaging (LIFHA) operator.

Definition 9. Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy weighted averaging (LIFWA) operator is a mapping $\Gamma^n_{[0,t]} \to \Gamma_{[0,t]}$ such that:

$$LIFWA_\omega(\gamma_1, \gamma_2, \ldots, \gamma_n) = \omega_1\gamma_1 \oplus \omega_2\gamma_2 \oplus \cdots \oplus \omega_n\gamma_n \qquad (2)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\gamma_j (j = 1, 2 \ldots, n)$, satisfying that $0 \leqslant \omega_j \leqslant 1$, $\sum_{j=1}^n \omega_j = 1$.

Theorem 6. *Let* $\gamma_1 = (s_{\alpha_1}, s_{\beta_1}), \gamma_2 = (s_{\alpha_2}, s_{\beta_2}), \ldots, \gamma_n = (s_{\alpha_n}, s_{\beta_n}) \in \Gamma_{[0,t]}$, *and* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *be the weight vector of* $\gamma_j (j = 1, 2 \ldots, n)$, *satisfying that*

$0 \leqslant \omega_j \leqslant 1 (j = 1,2\ldots,n)$ *and* $\sum_{j=1}^{n} \omega_j = 1$. *Then*

$$LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_n) = (s_{t-t\prod_{j=1}^{n}(1-\frac{\alpha_j}{t})^{\omega_j}}, s_{t\prod_{j=1}^{n}(\frac{\beta_j}{t})^{\omega_j}}) \quad (3)$$

Proof. We use the mathematical inductive method to prove Theorem 6.

First, for $n = 2$, By Definition 8 and 9, we have
$LIFWA_{\omega}(\gamma_1,\gamma_2) = \omega_1\gamma_1 \oplus \omega_2\gamma_1$
$= (s_{t-t(1-\frac{\alpha_1}{t})^{\omega_1}}, s_{t(\frac{\beta_1}{t})^{\omega_1}}) \oplus (s_{t-t(1-\frac{\alpha_2}{t})^{\omega_2}}, s_{t(\frac{\beta_2}{t})^{\omega_2}})$

Let $LIFWA_{\omega}(\gamma_1,\gamma_2) = (s_{u_2}, s_{v_2})$, then
$u_2 = [t - t(1-\frac{\alpha_1}{t})^{\omega_1}] + [t - t(1-\frac{\alpha_2}{t})^{\omega_2}] - \frac{[t-t(1-\frac{\alpha_1}{t})^{\omega_1}][t-t(1-\frac{\alpha_2}{t})^{\omega_2}]}{t}$
$= t - t(1-\frac{\alpha_1}{t})^{\omega_1}(1-\frac{\alpha_2}{t})^{\omega_2} = t - t\prod_{j=1}^{2}(1-\frac{\alpha_j}{t})^{\omega_j}$,
$v_2 = \frac{[t(\frac{\beta_1}{t})^{\omega_1}][t(\frac{\beta_2}{t})^{\omega_2}]}{t} = t(\frac{\beta_1}{t})^{\omega_1}(\frac{\beta_2}{t})^{\omega_2} = t\prod_{j=1}^{2}(\frac{\beta_j}{t})^{\omega_j}$.

So the result is true for $n = 2$.

Secondly, we assume that the result is true for $n-1$, i.e.
$LIFWA_{\omega}(\gamma_1,\ldots,\gamma_{n-1}) = (s_{t-t\prod_{j=1}^{n-1}(1-\frac{\alpha_j}{t})^{\omega_j}}, s_{t\prod_{j=1}^{n-1}(\frac{\beta_j}{t})^{\omega_j}})$.

Then, for n,
$LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_n) = LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_{n-1}) \oplus \omega_n\gamma_n$
$= (s_{t-t\prod_{j=1}^{n-1}(1-\frac{\alpha_j}{t})^{\omega_j}}, s_{t\prod_{j=1}^{n-1}(\frac{\beta_j}{t})^{\omega_j}}) \oplus (s_{t-t(1-\frac{\alpha_n}{t})^{\omega_n}}, s_{t(\frac{\beta_n}{t})^{\omega_n}})$.

Let $LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_n) = (s_{u_n}, s_{v_n})$, then
$u_n = [t - t\prod_{j=1}^{n-1}(1-\frac{\alpha_j}{t})^{\omega_j}] + [t - t(1-\frac{\alpha_n}{t})^{\omega_n}] - \frac{[t-t\prod_{j=1}^{n-1}(1-\frac{\alpha_j}{t})^{\omega_j}][t-t(1-\frac{\alpha_n}{t})^{\omega_n}]}{t}$
$= t - t\prod_{j=1}^{n}(1-\frac{\alpha_j}{t})^{\omega_j}$,
$v_n = \frac{t\prod_{j=1}^{n-1}(\frac{\beta_j}{t})^{\omega_j}\cdot[t(\frac{\beta_n}{t})^{\omega_n}]}{t} = t\prod_{j=1}^{n}(\frac{\beta_j}{t})^{\omega_j}$.

Hence, the result is true for any n. □

From (5) of Theorem 4, if $\omega_j = 1/n$ $(j = 1,2,\ldots,n)$, then the LIFWA operator is reduced to the linguistic intuitionistic fuzzy averaging (LIFA) operator:

$$LIFA(\gamma_1,\gamma_2,\ldots,\gamma_n) = \frac{1}{n}(\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n) \quad (4)$$

Example 3. Assume $\gamma_1 = (s_5,s_3), \gamma_2 = (s_1,s_6), \gamma_3 = (s_4,s_3), \gamma_4 = (s_2,s_6) \in \Gamma_{[0,8]}$, and $\omega = (0.2,0.3,0.4,0.1)$, According to Theorem 6, we can obtain

$$\begin{aligned} &LIFWA_{\omega}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) \\ &= 0.2(s_5,s_3) \oplus 0.3(s_1,s_6) \oplus 0.4(s_4,s_3) \oplus 0.1(s_2,s_6) \\ &= (s_{8-8(1-\frac{5}{8})^{0.2}(1-\frac{1}{8})^{0.3}(1-\frac{4}{8})^{0.4}(1-\frac{2}{8})^{0.1}}, s_{8(\frac{3}{8})^{0.2}(\frac{6}{8})^{0.3}(\frac{3}{8})^{0.4}(\frac{6}{8})^{0.1}}) \\ &= (s_{3.349}, s_{3.959}) \end{aligned}$$

Based on Theorem 4, it can be easily proved that the LIFWA operator has the following properties:

(1) **Commutative**: Let $\gamma_j \in \Gamma_{[0,t]} (j = 1,2,\ldots,n)$, and $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, then

$$LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_n) = LIFWA_{\omega}(\gamma_{\sigma(1)},\gamma_{\sigma(2)},\ldots,\gamma_{\sigma(n)}). \quad (5)$$

(2) **Monotonic**: Let $\gamma_j = (s_{\alpha_j}, s_{\beta_j}), \gamma_j^* = (s_{\alpha_j^*}, s_{\beta_j^*}) \in \Gamma_{[0,t]}$, if $\alpha_j \leqslant \alpha_j^*$, and $\beta_j \geqslant \beta_j^* (j = 1,2,\ldots,n)$, then

$$LIFWA_{\omega}(\gamma_1,\ldots,\gamma_n) \leqslant LIFWA_{\omega}(\gamma_1^*,\ldots,\gamma_n^*). \quad (6)$$

(3) **Idempotency**: Let $\gamma_j = (s_{\alpha_j}, s_{\beta_j}) \in \Gamma_{[0,t]} (j = 1,2,\ldots,n)$, and for any j, always have $\gamma_j = \gamma$, then

$$LIFWA_{\omega}(\gamma_1,\gamma_2,\ldots,\gamma_n) = \gamma. \quad (7)$$

(4) **Bounded**: Let $\gamma_j = (s_{\alpha_j}, s_{\beta_j}) \in \Gamma_{[0,t]} (j = 1,2,\ldots,n)$, and $\alpha^- = \min\{\alpha_j\}, \alpha^+ = \max\{\alpha_j\}$, $\beta^- = \min\{\beta_j\}, \beta^+ = \max\{\beta_j\}$, then

$$(s_{\alpha^-}, s_{\beta^+}) \leqslant LIFWA_{\omega}(\gamma_1,\ldots,\gamma_n) \leqslant (s_{\alpha^+}, s_{\beta^-}). \quad (8)$$

Yager [28] introduced an ordered weighted averaging (OWA) operator, which is the reordering step. In the following we shall extend the OWA operator to accommodate the situations where the input arguments are LIFNs.

Definition 10. Let $\gamma_1,\gamma_2,\ldots,\gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy ordered weighted averaging (LIFOWA) operator is a mapping $\Gamma_{[0,t]}^n \to \Gamma_{[0,t]}$, which has associated with it a weighting vector $w = (w_1,w_2,\ldots,w_n)^T$, with $0 \leqslant w_j \leqslant 1, \sum_{j=1}^{n} w_j = 1$ such that

$$LIFOWA_w(\gamma_1,\gamma_2,\ldots,\gamma_n) = w_1\gamma_{\sigma(1)} \oplus w_2\gamma_{\sigma(2)} \oplus \cdots \oplus w_n\gamma_{\sigma(n)} \quad (9)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\gamma_{\sigma(j-1)} \geqslant \gamma_{\sigma(j)}$ for all $j = 2, \ldots, n$.

The feature of the LIFOWA operator is that w_j is only determined by the jth position in the aggregation process. So, w can be called the position weighted vector.

The LIFWA operator only considers the self-importance of each LIFN, and the LIFOWA operator only emphasizes position importance of each LIFN. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the linguistic intuitionistic fuzzy hybrid averaging (LIFHA) operator.

Definition 11. Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy hybrid averaging (LIFHA) operator is a mapping $\Gamma^n_{[0,t]} \to \Gamma_{[0,t]}$, which has associated with it a weighting vector $w = (w_1, w_2, \ldots, w_n)^T$, with $0 \leqslant w_j \leqslant 1$, $\sum_{j=1}^n w_j = 1$ such that

$$LIFHA_{\omega,w}(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ = w_1\gamma'_{\sigma(1)} \oplus w_2\gamma'_{\sigma(2)} \oplus \cdots \oplus w_n\gamma'_{\sigma(n)} \quad (10)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\gamma_j (j = 1, 2 \ldots, n)$, with $0 \leqslant \omega_j \leqslant 1$, $\sum_{j=1}^n \omega_j = 1$, and $\gamma'_j = n\omega_j\gamma_j$, n is the balancing coefficient, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\gamma'_{\sigma(j-1)} \geqslant \gamma'_{\sigma(j)}$ for all $j = 2, \ldots, n$.

Example 4. Assume $\gamma_1 = (s_7, s_1), \gamma_2 = (s_2, s_5), \gamma_3 = (s_4, s_3), \gamma_4 = (s_6, s_2) \in \Gamma_{[0,8]}$, and $\omega = (0.4, 0.1, 0.2, 0.3)$ is the weighting vector of the $\gamma_j (j = 1, 2, 3, 4)$, and $w = (0.2, 0.3, 0.3, 0.2)$ is the position weighted vector.
According to Definition 8, $\lambda(s_\alpha, s_\beta) = (s_{t-t(1-\frac{\alpha}{t})^\lambda}, s_{t(\frac{\beta}{t})^\lambda})$. Thus, we have

$\gamma'_1 = 4 \times 0.4(s_7, s_1) = (s_{7.713}, s_{0.287})$,
$\gamma'_2 = 4 \times 0.1(s_2, s_5) = (s_{0.870}, s_{6.629})$
$\gamma'_3 = 4 \times 0.2(s_4, s_3) = (s_{3.405}, s_{3.650})$,
$\gamma'_4 = 4 \times 0.3(s_6, s_2) = (s_{6.484}, s_{1.516})$

To rank these arguments, we calculate the linguistic score index and the linguistic accuracy index of each argument γ'_i:

$Ls(\gamma'_1) = 7.626$, $Ls(\gamma'_2) = -5.759$,
$Ls(\gamma'_3) = -0.145$, $Ls(\gamma'_4) = 4.968$

Then we rank the arguments $\gamma'_i (i = 1;2;3;4)$ in descending order in accordance with the values $Ls(\gamma'_i)(i = 1;2;3;4) : \gamma'_1 > \gamma'_4 > \gamma'_3 > \gamma'_2$. Thus, by Eq. (10) and (3), we can obtain
$LIFHA_{\omega,w}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$
$= 0.2\gamma'_1 \oplus 0.3\gamma'_4 \oplus 0.3\gamma'_3 \oplus 0.2\gamma'_2 = 0.2(s_{7.713}, s_{0.287}) \oplus 0.3(s_{6.484}, s_{1.516}) \oplus 0.3(s_{3.405}, s_{3.650}) \oplus 0.2(s_{0.870}, s_{6.629})$
$= (s_{5.935}, s_{1.900})$

Further, we shall propose the linguistic intuitionistic fuzzy weighted geometric (LIFWG) operator, linguistic intuitionistic fuzzy ordered weighted geometric (LIFOWG) operator, and linguistic intuitionistic fuzzy hybrid geometric (LIFHG) operator.

Definition 12. Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy weighted geometric (LIFWG) operator is a mapping $\Gamma^n_{[0,t]} \to \Gamma_{[0,t]}$ such that:

$$LIFWG_\omega(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma_1^{\omega_1} \otimes \gamma_2^{\omega_2} \otimes \cdots \otimes \gamma_n^{\omega_n} \quad (11)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\gamma_j (j = 1, 2 \ldots, n)$, satisfying that $0 \leqslant \omega_j \leqslant 1$, $\sum_{j=1}^n \omega_j = 1$.

Similar to Theorem 6, we can obtain following Theorem 7.

Theorem 7. *Let $\gamma_1 = (s_{\alpha_1}, s_{\beta_1}), \gamma_2 = (s_{\alpha_2}, s_{\beta_2}), \ldots, \gamma_n = (s_{\alpha_n}, s_{\beta_n}) \in \Gamma_{[0,t]}$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of $\gamma_j (j = 1, 2 \ldots, n)$, satisfying that $0 \leqslant \omega_j \leqslant 1 (j = 1, 2 \ldots, n)$ and $\sum_{j=1}^n \omega_j = 1$. Then*

$$LIFWG_\omega(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ = (s_{t\prod_{j=1}^n(\frac{\alpha_j}{t})^{\omega_j}}, s_{t-t\prod_{j=1}^n(1-\frac{\beta_j}{t})^{\omega_j}}) \quad (12)$$

From (6) of Theorem 4, if $\omega_j = 1/n$ $(j = 1, 2, \ldots, n)$, then the LIFWG operator is reduced to the intuitionistic fuzzy linguistic geometric (LIFG) operator:

$$LIFG(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_n)^{\frac{1}{n}} \quad (13)$$

Like LIFWA operator, LIFWG operator is also commutative, monotonic, bounded and idempotent.

Definition 13. Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy ordered weighted geometric (LIFOWG) operator is a mapping $\Gamma^n_{[0,t]} \to \Gamma_{[0,t]}$, which has associated with it a weighting vector

$w = (w_1, w_2, \dots, w_n)^T$, with $0 \leqslant w_j \leqslant 1$, $\sum_{j=1}^{n} w_j = 1$ such that

$$LIFOWG_w(\gamma_1, \gamma_2, \dots, \gamma_n) = \gamma_{\sigma(1)}^{w_1} \otimes \gamma_{\sigma(2)}^{w_2} \otimes \cdots \otimes \gamma_{\sigma(n)}^{w_n} \quad (14)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$, such that $\gamma_{\sigma(j-1)} \geqslant \gamma_{\sigma(j)}$ for all $j = 2, \dots, n$.

Definition 14. Let $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma_{[0,t]}$, The linguistic intuitionistic fuzzy hybrid geometric (LIFHG) operator is a mapping $\Gamma_{[0,t]}^n \to \Gamma_{[0,t]}$, which has associated with it a weighting vector $w = (w_1, w_2, \dots, w_n)^T$, with $0 \leqslant w_j \leqslant 1$, $\sum_{j=1}^{n} w_j = 1$ such that

$$LIFHG_{\omega,w}(\gamma_1, \gamma_2, \dots, \gamma_n) = (\gamma'_{\sigma(1)})^{w_1} \otimes (\gamma'_{\sigma(2)})^{w_2} \otimes \cdots \otimes (\gamma_{\sigma(n)'})^{w_n} \quad (15)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\gamma_j (j = 1, 2 \dots, n)$, with $0 \leqslant \omega_j \leqslant 1$, $\sum_{j=1}^{n} \omega_j = 1$, and $\gamma'_j = \gamma_j^{n\omega_j}$, n is the balancing coefficient, $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$, such that $\gamma'_{\sigma(j-1)} \geqslant \gamma'_{\sigma(j)}$ for all $j = 2, \dots, n$.

In the following, an example is used to illustrate the LIFHG operator.

Example 5. Assume $\gamma_1 = (s_7, s_1), \gamma_2 = (s_2, s_5), \gamma_3 = (s_4, s_3), \gamma_4 = (s_6, s_2) \in \Gamma_{[0,8]}$, and $\omega = (0.4, 0.1, 0.2, 0.3)$ is the weighting vector of the $\gamma_j (j = 1, 2, 3, 4)$, and $w = (0.2, 0.3, 0.3, 0.2)$ is the position weighted vector.
According to Definition 8, $(s_\alpha, s_\beta)^\lambda = (s_{t(\frac{\alpha}{t})^\lambda}, s_{t-t(1-\frac{\beta}{t})^\lambda})$. Thus, we have

$\gamma'_1 = (s_7, s_1)^{4\times0.4} = (s_{6.461}, s_{1.539})$,
$\gamma'_2 = (s_2, s_5)^{4\times0.1} = (s_{4.595}, s_{2.596})$
$\gamma'_3 = (s_4, s_3)^{4\times0.2} = (s_{4.595}, s_{2.507})$,
$\gamma'_4 = (s_6, s_2)^{4\times0.3} = (s_{5.665}, s_{2.335})$

To rank these arguments, we calculate the linguistic score index and the linguistic accuracy index of each argument γ'_i:

$Ls(\gamma'_1) = 4.922$, $Ls(\gamma'_2) = 1.999$,
$Ls(\gamma'_3) = 2.088$, $Ls(\gamma'_4) = 3.330$

Then we rank the arguments $\gamma'_i (i = 1;2;3;4)$ in descending order in accordance with the values $Ls(\gamma'_i)(i = 1, 2, 3, 4) : \gamma'_1 > \gamma'_4 > \gamma'_3 > \gamma'_2$. Thus, by Eq. (15) and (12), we can obtain

$LIFHG_{\omega,w}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$
$= (\gamma'_1)^{0.2} \otimes (\gamma'_4)^{0.3} \otimes (\gamma'_3)^{0.3} \otimes (\gamma'_2)^{0.2}$
$= (s_{6.461}, s_{1.539})^{0.2} \otimes (s_{5.665}, s_{2.335})^{0.3} \otimes (s_{4.595}, s_{2.507})^{0.3} \otimes (s_{4.595}, s_{2.596})^{0.2} = (s_{5.238}, s_{2.292})$

5. An approach to group decision making with linguistic intuitionistic fuzzy information

This section describes the multiple attribute group decision making problems with linguistic intuitionistic fuzzy assessments.

Let $X = \{x_1, x_2, \cdots, x_m\}$ be a discrete set of m possible alternatives and $A = \{a_1, a_2, \cdots, a_n\}$ be a finite set of n attributes, where x_i denotes the ith alternative and a_j denotes the jth attribute. Let $D = \{d_1, d_2, \cdots, d_t\}$ be a finite set of t experts, where d_k denotes the kth expert.

The expert d_k provides his/her assessment information of an alternative x_i on an attribute a_j as a LIFN $\gamma_{ij}^k (i = 1, 2, \dots, m; j = 1, 2, \cdots, n)$ according to a predefined linguistic term set S. Thus, the experts' assessment information can be represented by the linguistic intuitionistic fuzzy decision matrices $R_k = (\gamma_{ij}^k)_{m\times n} (k = 1, 2, \dots, t)$.

Suppose that $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is weight vector of the attributes, where ω_j denotes the weight of the attribute a_j such that $0 \leqslant \omega_j, \sum_{j=1}^{n} \omega_j = 1$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)^T$ is the weighting vector of the decision makers, where λ_j denotes the weight of the decision maker d_j such that $\lambda_j \leqslant 1, \sum_{j=1}^{t} \lambda_j = 1$.

The problem concerned in this paper is how to rank alternatives or select the most desirable alternative(s) among the finite set X on the basis of the linguistic intuitionistic fuzzy decision matrices and the weight information of attributes and experts. An algorithm and process of the multiple attribute group decision making problems with linguistic intuitionistic fuzzy information may be given as follows.

Step1: Utilize the decision information given in matrix R_k, and the LIFWA operator, the individual overall linguistic intuitionistic fuzzy preference value γ_i^k of the alternative x_i is derived as follows:

$$\gamma_i^k = LIFWA_\omega(\gamma_{i1}^k, \gamma_{i2}^k, \ldots, \gamma_{in}^k),\\ i = 1,2,\ldots,m, k = 1,2,\ldots,t \quad (16)$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is weight vector of the attribute.

Step2: According to the individual overall linguistic intuitionistic fuzzy preference value γ_i^k of alternative $x_i (i = 1,2,\ldots,m, k = 1,2,\ldots,t)$, then using LIFHA operator which has the associated weighting vector $w = (w_1, w_2, \cdots, w_t)^T$:

$$\gamma_i = LIFHA_{\lambda,w}(\gamma_i^1, \gamma_i^2, \ldots, \gamma_i^t),\\ i = 1,2,\ldots,m \quad (17)$$

to derive the collective overall linguistic intuitionistic fuzzy preference value γ_i of the alternative $x_i (i = 1,2,\ldots,m)$, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)^T$ is the weighting vector of the decision makers.

step3: By using Eq.(1), we calculate the linguistic score index $Ls(\gamma_i)$ and the linguistic accuracy index $Lh(\gamma_i)$ of the collective overall linguistic preference value $\gamma_i (i = 1,2,\ldots,m)$.

step4: By Definition 5, we rank the alternatives $x_i (i = 1,2,\ldots,m)$ and then select the best one(s).

6. An illustrative example

In this section, a problem of searching the best global supplier (adapted from Chan and Kumar [6]) is used to illustrate the multiple attribute group decision making with linguistic intuitionistic fuzzy information.

A manufacturing company desires to search the best global supplier for one of its most critical parts used in assembling process. Suppose that $X = \{x_1, x_2, x_3, x_4\}$ is a set of four potential global suppliers (i.e., alternatives) under consideration and $A = \{a_1, a_2, a_3, a_4, a_5\}$ is a set of attributes, where a_i $(i = 1,\ldots,5)$ stands for "overall cost of the product", "quality of the product", "service performance of supplier", "supplier's profile", "risk factor", respectively. The four alternatives $x_i (i = 1,\ldots,4)$ are to be evaluated using the LIFNs according to the linguistic term set:

$S = \{s_0 = extremely\ poor,\ s_1 = very\ poor,\ s_2 = poor,\ s_3 = slightly\ poor,\ s_4 = fair,\ s_5 = slightly\ good,\ s_6 = good,\ s_7 = very\ good,\ s_8 = extremely\ good\}$
by four decision makers $d_k (k = 1,\ldots,4)$ under the above five attributes, and construct the linguistic intuitionistic fuzzy decision matrices $R_k = (\gamma_{ij}^k)_{4\times 5} (k = 1,2,3,4)$ as listed in Tables 1−4, respectively.

Table 1. Decision matrix R_1

	a_1	a_2	a_3	a_4	a_5
x_1	(s_7,s_1)	(s_6,s_2)	(s_4,s_3)	(s_7,s_1)	(s_5,s_2)
x_2	(s_6,s_2)	(s_5,s_2)	(s_6,s_1)	(s_6,s_2)	(s_7,s_1)
x_3	(s_6,s_1)	(s_5,s_3)	(s_7,s_1)	(s_5,s_1)	(s_3,s_4)
x_4	(s_5,s_2)	(s_7,s_1)	(s_4,s_3)	(s_6,s_1)	(s_4,s_4)

Table 2. Decision matrix R_2

	a_1	a_2	a_3	a_4	a_5
x_1	(s_7,s_1)	(s_4,s_4)	(s_6,s_2)	(s_5,s_2)	(s_3,s_5)
x_2	(s_7,s_1)	(s_5,s_1)	(s_6,s_1)	(s_5,s_2)	(s_4,s_3)
x_3	(s_5,s_2)	(s_6,s_1)	(s_7,s_1)	(s_5,s_3)	(s_4,s_4)
x_4	(s_6,s_2)	(s_4,s_3)	(s_5,s_2)	(s_7,s_1)	(s_5,s_3)

Table 3. Decision matrix R_3

	a_1	a_2	a_3	a_4	a_5
x_1	(s_6,s_1)	(s_5,s_2)	(s_3,s_4)	(s_7,s_1)	(s_5,s_2)
x_2	(s_7,s_1)	(s_6,s_2)	(s_7,s_1)	(s_6,s_2)	(s_5,s_1)
x_3	(s_5,s_3)	(s_5,s_2)	(s_6,s_1)	(s_4,s_3)	(s_3,s_1)
x_4	(s_6,s_2)	(s_7,s_1)	(s_5,s_1)	(s_5,s_2)	(s_5,s_3)

Table 4. Decision matrix R_4

	a_1	a_2	a_3	a_4	a_5
x_1	(s_5,s_3)	(s_4,s_4)	(s_7,s_1)	(s_5,s_1)	(s_4,s_2)
x_2	(s_6,s_1)	(s_7,s_1)	(s_6,s_1)	(s_5,s_2)	(s_6,s_1)
x_3	(s_5,s_2)	(s_3,s_4)	(s_6,s_2)	(s_3,s_3)	(s_5,s_2)
x_4	(s_4,s_3)	(s_5,s_1)	(s_4,s_2)	(s_6,s_2)	(s_5,s_2)

In the following, we shall utilize the proposed approach in this paper getting the most desirable alternative(s):

Step1: Assume that the weight vector of attributes is $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^T = (0.25, 0.2, 0.15, 0.18, 0.22)^T$. Combine the decision matrix R_1 and the weight vector of attributes with the LIFWA operator, the individual overall preference value γ_1^1 of candidate x_1 is derived as follows:

$\gamma_1^1 = LIFWA_\omega(\gamma_{11}^1, \gamma_{12}^1, \gamma_{13}^1, \gamma_{14}^1, \gamma_{15}^1)$

$= 0.25(s_7,s_1) \oplus 0.2(s_6,s_2) \oplus 0.15(s_4,s_3) \oplus 0.18(s_7,s_1) \oplus 0.22(s_5,s_2) = (s_{6.199},s_{1.578})$

Likewise, we have

$\gamma_2^1 = (s_{6.138},s_{1.444})$, $\gamma_3^1 = (s_{5.428},s_{1.690})$,
$\gamma_4^1 = (s_{5.510},s_{1.902})$;
$\gamma_1^2 = (s_{5.458},s_{2.363})$, $\gamma_2^2 = (s_{5.715},s_{1.433})$,
$\gamma_3^2 = (s_{5.501},s_{1.966})$, $\gamma_4^2 = (s_{5.644},s_{2.093})$;
$\gamma_1^3 = (s_{5.598},s_{1.647})$, $\gamma_2^3 = (s_{6.343},s_{1.301})$,
$\gamma_3^3 = (s_{4.673},s_{1.842})$, $\gamma_4^3 = (s_{5.824},s_{1.716})$;
$\gamma_1^4 = (s_{5.129},s_{2.023})$, $\gamma_2^4 = (s_{6.127},s_{1.133})$,
$\gamma_3^4 = (s_{4.572},s_{2.471})$, $\gamma_4^4 = (s_{4.871},s_{1.927})$.

Step2: Assume that the weight vector of four experts is $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)^T = (0.25,0.3,0.2,0.25)^T$. Utilize the LIFHA operator which has an associated weighing vector $w = (w_1,w_2,w_3,w_4)^T = (0.15,0.35,0.35,0.15)^T$:

$$\gamma_i = LIFHA_{\lambda,w}(\gamma_i^1,\gamma_i^2,\gamma_i^3,\gamma_i^4) \quad i = 1,2,3,4$$

to aggregate the individual overall linguistic intuitionistic fuzzy preference values γ_i^k $(k = 1,2,3,4)$ and obtain the collective overall preference value γ_i of alternative $x_i (i = 1,2,3,4)$.

By $4\lambda_1\gamma_1^1 = (s_{6.199},s_{1.578})$, $4\lambda_2\gamma_1^2 = (s_{5.979},s_{1.852})$, $4\lambda_3\gamma_1^3 = (s_{4.945},s_{2.259})$, $4\lambda_4\gamma_1^4 = (s_{5.129},s_{2.023})$, and $4\lambda_1\gamma_1^1 > 4\lambda_2\gamma_1^2 > 4\lambda_4\gamma_1^4 > 4\lambda_3\gamma_1^3$, we can obtain

$\gamma_1 = LIFHA_{\lambda,w}(\gamma_1^1,\gamma_1^2,\gamma_1^3,\gamma_1^4)$
$= 0.15(s_{6.199},s_{1.578}) \oplus 0.35(s_{5.979},s_{1.852}) \oplus 0.35(s_{5.129},s_{2.023}) \oplus 0.15(s_{4.945},s_{2.259})$
$= (s_{5.610},s_{1.921})$

Similarly, we have

$\gamma_2 = LIFHA_{\lambda,w}(\gamma_2^1,\gamma_2^2,\gamma_2^3,\gamma_2^4)$
$= 0.15(s_{6.222},s_{1.024}) \oplus 0.35(s_{6.129},s_{1.133}) \oplus 0.35(s_{6.138},s_{1.444}) \oplus 0.15(s_{5.730},s_{1.871})$
$= (s_{6.092},s_{1.310})$

$\gamma_3 = LIFHA_{\lambda,w}(\gamma_3^1,\gamma_3^2,\gamma_3^3,\gamma_3^4)$
$= 0.15(s_{6.020},s_{1.485}) \oplus 0.35(s_{5.428},s_{1.690}) \oplus 0.35(s_{4.572},s_{2.471}) \oplus 0.15(s_{4.035},s_{2.471})$
$= (s_{5.082},s_{2.004})$

$\gamma_4 = LIFHA_{\lambda,w}(\gamma_4^1,\gamma_4^2,\gamma_4^3,\gamma_4^4)$
$= 0.15(s_{6.155},s_{1.601}) \oplus 0.35(s_{5.510},s_{1.902}) \oplus 0.35(s_{4.871},s_{1.927}) \oplus 0.15(s_{5.177},s_{2.335})$
$= (s_{5.372},s_{1.920})$

Step3: Calculate the linguistic score index $Ls(\gamma_i)(i = 1,2,3,4,5)$ of the collective overall preference value $\gamma_i(i = 1,2,3,4)$ as follows:

$Ls(\gamma_1) = 3.689$, $Ls(\gamma_2) = 4.782$, $Ls(\gamma_3) = 3.078$, $Ls(\gamma_4) = 3.452$

Then we rank γ_i in descending order in accordance with the values of $Ls(\gamma_i)(i = 1,2,3,4)$:

$$\gamma_2 > \gamma_1 > \gamma_4 > \gamma_3$$

Step4: Rank all the alternatives x_i in accordance with $\gamma_i(i = 1,2,3,4)$:

$$x_2 > x_1 > x_4 > x_3$$

Thus the best alternative is x_2.

7. A comparison analysis to MADM with uncertain linguistic information

Xu [25] proposed an approach to multiple attribute group decision making based on the ULOWA and the ULHA operators with uncertain linguistic information. In the following, we use the method proposed in this paper to solve the evaluating university faculty for tenure and promotion problem of [25] (adapted from Chan and Kumar [4]), and then conduct a comparison analysis.

In [25], a practical use involves the evaluation of university faculty for tenure and promotion. The attributes used at some universities are a_1: teaching, a_2: research, and a_3: service. Five faculty candidates (alternatives) x_j $(j = 1,2,3,4,5)$ are to be evaluated using the term set

$S = \{s_0 = extremely\ poor,\ s_1 = very\ poor,\ s_2 = poor,\ s_3 = slightly\ poor,\ s_4 = fair,\ s_5 = slightly\ good,\ s_6 = good,\ s_7 = very\ good,\ s_8 = extremely\ good\}$

by four decision makers $d_k(k = 1,2,3,4)$ (whose weight vector $\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)^T = (0.24,0.26,0.23,0.27)^T$) under these three attributes. The uncertain linguistic decision matrices see [25] in detail.

Xu [25] used the ULOWA operator which has the associated weighting vector $w = (0.3,0.4,0.3)^T$ to derive the individual overall preference value of alternative, and Utilize the ULHA operator which has an associated weighing vector $w' = (w'_1,w'_2,w'_3,w'_4)^T = (0.2,0.3,0.3,0.2)^T$ to obtain the collective overall preference value of alternative, then constructing a complementary matrix to obtain the ranking order of the alternatives is $x_3 > x_2 > x_1 > x_4 > x_5$.

To begin, applied the proposed method in this paper, the uncertain linguistic decision information in [25] should be firstly transformed into the LIFN forms (see Remark 2). For example, the uncertain linguistic variable $[S_6, S_7]$ in $\Gamma_{[0,8]}$ can be replaced by the LIFN (s_6, s_1). The values following conversion are shown as $R_k = (\gamma_{ij}^k)_{3\times5} (k = 1,2,3,4)$, where γ_{ij}^k takes the form of the LIFN, given by the decision maker d_k, for alternative x_j with respect to attribute a_i, the result is listed in Tables 5−8, respectively.

Table 5. Decision matrix R_1

	x_1	x_2	x_3	x_4	x_5
a_1	(s_7, s_0)	(s_5, s_2)	(s_4, s_3)	(s_7, s_0)	(s_7, s_0)
a_2	(s_5, s_2)	(s_6, s_1)	(s_7, s_0)	(s_3, s_3)	(s_5, s_1)
a_3	(s_5, s_1)	(s_6, s_0)	(s_6, s_0)	(s_5, s_2)	(s_6, s_1)

Table 6. Decision matrix R_2

	x_1	x_2	x_3	x_4	x_5
a_1	(s_5, s_2)	(s_4, s_2)	(s_6, s_1)	(s_6, s_0)	(s_7, s_0)
a_2	(s_7, s_0)	(s_5, s_1)	(s_5, s_2)	(s_5, s_2)	(s_6, s_1)
a_3	(s_4, s_3)	(s_6, s_0)	(s_6, s_0)	(s_6, s_1)	(s_4, s_1)

Table 7. Decision matrix R_3

	x_1	x_2	x_3	x_4	x_5
a_1	(s_6, s_1)	(s_4, s_2)	(s_7, s_0)	(s_6, s_1)	(s_5, s_2)
a_2	(s_7, s_0)	(s_5, s_2)	(s_6, s_1)	(s_4, s_3)	(s_5, s_1)
a_3	(s_5, s_2)	(s_6, s_1)	(s_5, s_2)	(s_7, s_0)	(s_4, s_1)

Table 8. Decision matrix R_4

	x_1	x_2	x_3	x_4	x_5
a_1	(s_5, s_2)	(s_7, s_0)	(s_6, s_0)	(s_4, s_2)	(s_4, s_2)
a_2	(s_6, s_0)	(s_6, s_0)	(s_6, s_1)	(s_6, s_1)	(s_5, s_1)
a_3	(s_5, s_1)	(s_6, s_1)	(s_7, s_0)	(s_5, s_2)	(s_4, s_3)

Next, repeating the same steps as in Section 5, we first use the LIFOWA operator which has the associated weighting vector $w = (0.3, 0.4, 0.3)^T$ to derive the individual overall preference value of alternative, then utilize the weight vector of experts $\lambda = (0.24, 0.26, 0.23, 0.27)^T$ and the LIFHA operator which has an associated weighing vector $w' = (0.2, 0.3, 0.3, 0.2)^T$ to obtain the collective overall preference value γ_j of alternative $x_j (j = 1,2,3,4,5)$ as follows:

$\gamma_1 = (s_{5.750}, s_0)$, $\gamma_2 = (s_{5.620}, s_0)$, $\gamma_3 = (s_{6.053}, s_0)$, $\gamma_4 = (s_{5.586}, s_0)$, $\gamma_5 = (s_{5.406}, s_0)$

Since $Ls(\gamma_1) = 5.750$, $Ls(\gamma_2) = 5.620$, $Ls(\gamma_3) = 6.053$, $Ls(\gamma_4) = 5.586$, $Ls(\gamma_5) = 5.406$, and $Ls(\gamma_3) > Ls(\gamma_1) > Ls(\gamma_2) > Ls(\gamma_4) > Ls(\gamma_5)$, the ranking is $x_3 > x_1 > x_2 > x_4 > x_5$, and the most desirable alternative is x_3.

It is easily seen that the ranking results obtained by the method proposed in this paper and the method [25] are slightly different. The difference is the ranking order of x_1 and x_2, i.e., $x_1 > x_2$ by the former while $x_2 > x_1$ by the latter, but the best alternative both is x_3. The main reasons are as follows:

(a) The operations of LIFNs defined in this paper are remarkably different from the operations of uncertain linguistic variables (ULVs) defined in [25]. For example, the addition operation of LIFNs is defined as $(s_{\alpha_1}, s_{\beta_1}) \oplus (s_{\alpha_2}, s_{\beta_2}) = (s_{\alpha_1+\alpha_2-\frac{\alpha_1\alpha_2}{t}}, s_{\frac{\beta_1\beta_2}{t}})$, and the addition operation of ULVs in [25] is defined as $[S_{\alpha_1}, S_{\beta_1}] \oplus [S_{\alpha_2}, S_{\beta_2}] = [S_{\alpha_1+\alpha_2}, S_{\beta_1+\beta_2}]$, where $S_{\alpha_1}, S_{\alpha_2}, S_{\beta_1}, S_{\beta_2} \in S_{[0,t]}$. It should be noted that the addition operation of ULVs is not closed, i.e., $S_{\alpha_1+\alpha_2}$ and $S_{\beta_1+\beta_2}$ may not belong to $S_{[0,t]}$.

(b) The ranking method of LIFNs in this paper is obtained by the linguistic score index and the linguistic accuracy index. However, The ranking method of ULVs in [25] is obtained by comparing each ULV with all ULVs and then constructing a complementary matrix.

8. Conclusions

Intuitionistic fuzzy set theory, originally proposed by Atanassov, has become an effective mathematical tool to deal with uncertainty. The linguistic approach represent qualitative aspects as linguistic values by means of linguistic variables, which can provide us with more degrees of freedom to characterize the uncertainty and the vagueness of the real world. In this paper, we first propose the concept of intuitionistic fuzzy linguistic variables by integrating intuitionistic fuzzy sets and the linguistic approach. We define some operations on intuitionistic fuzzy linguistic variables and give some properties. Furthermore, we develop some aggregation operators such as the intuitionistic fuzzy linguistic weighted averaging operator, the intuitionistic fuzzy linguistic weighted geometric averaging operator, and propose an approach to handle multiple attribute group decision making problems under intuitionistic fuzzy linguistic environment.

In this paper we do not make any conclusion about the determining method of the weighted vector correlating with the intuitionistic fuzzy linguistic aggregation operators and effectively determining the expert weights in the form of the numerical values or LIFNs, which will be investigated in the near future. In addition, the method for group decision-making based on multi-granularity intuitionistic fuzzy linguistic information are also worthy of consideration for future research.

Acknowledgment

This work is partially supported by the National Natural Science Foundation (61372187, 61473239) and the open research fund of key laboratory of intelligent network information processing, Xihua University (SZJJ2012-026, SZJJ2014-052).

References

1. K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, **20(1)**, 87-96(1986).
2. K. Atanassov, "New operations defined over the intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, **61(2)**, 137-142(1994).
3. G. Bordogna, M. Fedrizzi, G. Pasi, "A linguistic modelling of consensus in group decision making based on OWA operator," *IEEE Transactions on Systems, Man, and Cybernetics*, **27(1)**, 126-132(1997).
4. N. Bryson, A. Mobolurin, "An action learning evaluation procedure for multiple criteria decision making problems," *European Journal of Operational Research*, **96**, 379-386 (1995).
5. S.M. Chen , J.M. Tan, "Handling multicriteria fuzzy decision-making problems based on vague set theory," *Fuzzy Sets and Systems*, **67(2)**, 163-172 (1994).
6. F.T.S. Chan, N. Kumar, "Global supplier development considering risk factors using fuzzy extended AHP-based approach," *Omega*, **35(4)**, 417-431(2007).
7. S.K. De, R. Biswas, A.R. Roy, "Some operations on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, **114(3)**, 477-484(2000).
8. G. Deschrijver, E.E. Kerre, "On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision," *Information Sciences*, **177(8)**, 1860-1866(2007).
9. M. Delgado, F. Herrera, E. Herrera-Viedma, L. Martínez, "Combining numerical and linguistic information in group decision making," *Information Sciences*, **107**, 177-194(1998).
10. W.L. Gau, D.J. Buehrer, "Vague sets," *IEEE Transactions on Systems Man and Cybernetics*, **23(2)**, 610-614(1993).
11. D.H. Hong, C.H. Choi, "Multicriteria fuzzy decision-making problems based on vague set theory," *Fuzzy Sets and Systems*, **114**, 103-113(2000).
12. F. Herrera, E. Herrera-Viedma, "Aggregation operators for linguistic weighted information," *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, **27(5)**, 646-656(1997).
13. F. Herrera, L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Transactions on Fuzzy Systems*, **8(6)**, 746-752(2000).
14. F. Herrera, E. Herrera-Viedma, L. Martínez, "A fusion approach for managing multi-granularity linguistic term sets in decision-making," *Fuzzy Sets and Systems*, **114(1)**, 43-58(2000).
15. F. Herrera, L. Martínez, " A model based on linguistic 2-tuple for dealing with multi-granular hierarchical linguistic contexts in multi-expert decision making," *IEEE Transactions on Systems, Man, and Cybernetics-Part B: Cybernetics*, **31(2)**, 227-234(2001).
16. D.F. Li, "Multiattribute decision making models and methods using intuitionistic fuzzy sets," *Journal of Computer and System Sciences*, **70(1)**, 73-85(2005).
17. X.W. Liao, Y. Li and B. Lu, "A model for selecting an ERP system based on linguistic information processing," *Information Systems*, **32**, 1005-1017(2007).
18. J.M. Merigó, A.M. Gil-Lafuente, "Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making," *Information Sciences*, **236**, 1-16(2013).
19. L. Martínez and F. Herrera, "An overview on the 2-tuple linguistic model for Computing with Words in Decision Making: Extensions, applications and challenges," *Information Sciences*, **207**, 1-18 (2012).
20. Z. Pei and P. Shi, "Fuzzy risk analysis based on linguistic aggregation operators," *International Journal of Innovative Computing, Information and Control*, **7**, 7105-7118 (2011).
21. R.M. Rodríguez, L. Martínez and F. Herrera, "Hesitant Fuzzy Linguistic Term Sets for Decision Making," *IIEEE Transactions on Fuzzy Systems*, **20**, 109-119(2012).
22. S.P. Wan, "2-Tuple linguistic hybrid arithmetic aggregation operators and application to multi-attribute group decision making," *Knowledge-Based Systems*, **45**, 31-40(2013).
23. J.q. Wang, J.t. Wu, J. Wang, H.Y. Zhang, X.H. Chen,

"Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problemsg," *Information Sciences*, **288**, 55-72(2014).
24. Z. Xu, "A method based on linguistic aggregation operators for group decision making with linguistic preference relations," *Information Sciences*, **166**, 19-30(2004).
25. Z. Xu, "Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment," *Information Sciences*, **168**, 171-184(2004).
26. Z. Xu, R.R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, **35**, 417-433(2006).
27. Z.S. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, **15(6)**, 1179-1187(2007).
28. R.R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Transactions on Systems, Man, and Cybernetics*, **18(1)**, 183-190(1988).
29. L.A. Zadeh, "Fuzzy sets," *Information and Control*, **8(3)**, 338-353(1965).
30. L.A. Zadeh, "The concept of a linguistic variable and its applications to approximate reasoning-1," *Information Sciences*, **8(3)**, 199-249(1975).
31. Z.F. Zhang and X.N. Chu, "Fuzzy group decisionmaking for multi-format and multi-granularity linguistic judgments in quality function deployment," *Expert Systems with Applications*, **36**, 9150-9158(2009).