

# An Approximate Capacity Distribution for MIMO Systems

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**Abstract**—In this letter, we derive the exact variance of the capacity of a multiple-input multiple-output (MIMO) system. This enables an investigation of the accuracy of a Gaussian approximation to the capacity foreshadowed by various central limit theorems. We confirm recent results which state that the capacity variance appears to converge to a limit independent of absolute antenna numbers, but dependent on the ratio of the numbers of receive to transmit antennas. The Gaussian approximation itself is surprisingly good, even in the worst cases giving satisfactory results.

**Index Terms**—Capacity, central limit theorem, multiple-input multiple-output (MIMO), Rayleigh channel.

## I. INTRODUCTION

MULTIPLE-INPUT multiple-output (MIMO) systems have recently been a subject of intense research activity [1]–[3]. Our work focuses on MIMO capacity, and we take the well-known quasi-stationary channel approach [1], which leads to the concept of capacity as a random variable. From a system engineer's viewpoint, we would therefore like to know:

- 1) what the mean system capacity is, and how this varies with the number of transmit and receive antennas and receiver signal-to-noise-ratio (SNR);
- 2) what the variance of the capacity is and how this varies with the numbers of transmit/receive antennas and SNR;
- 3) what the probability density function (pdf) of the system capacity is so that percentages of time capacity below a certain threshold (known as capacity outage) may be estimated.

Telatar [3] has derived an exact expression for the mean system capacity of a MIMO system, and Rapajic and Popescu [4] have evaluated the limiting mean system capacity for large arrays. Results on the variance and the pdf of channel capacity are only recently emerging. Hence, in this letter, we show that:

- 1) the channel capacity of a MIMO system can be accurately modeled by a Gaussian random variable. The exact mean and variance of the capacity are given for any numbers of transmit and receive antennas;
- 2) the variance of the channel capacity is not sensitive to the number of antennas and is mainly influenced by the SNR.

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A closed-form formula for the variance is developed in the Appendix, in the form of a single numerical integral.

In summary, the Gaussian approximation to channel capacity is a simple and powerful tool to enable engineering estimates of system capacity, total throughput, and capacity outage probability. The rest of the letter is laid out as follows. In Section II, we give some background and review the relevant literature. In Section III, we discuss central limit theorems (CLTs) for the capacity and provide the methodology for the Gaussian approximation. In Section IV, results are given and in Section V, conclusions are presented.

## II. BACKGROUND

### A. Link and Channel Model

Consider a transmission system where each user transmits simultaneously via  $t$  antennas, and reception is via  $r$  antennas. We define  $m, n$  to be given by  $m = \min(r, t)$ ,  $n = \max(r, t)$ . The total power of the complex transmitted signal  $\mathbf{s}$  is constrained to  $P$ , regardless of the value of  $t$ . The received signal  $\mathbf{r}$  in this complex  $r$ -dimensional system is

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n} \quad (1)$$

where  $\mathbf{H}$  is a  $(r \times t)$  complex channel-gain matrix. For uncorrelated Rayleigh fading, the entries in  $\mathbf{H}$  are independent and identically distributed (i.i.d.), complex, zero-mean Gaussians with unit magnitude variance. In (1),  $\mathbf{n}$  is a complex  $r$ -dimensional additive white Gaussian noise (AWGN) vector, with statistically independent components of identical power  $\sigma^2$  at each of the  $r$  receive branches. We assume  $\sigma^2 = 1$  without loss of generality. Assuming a narrowband channel, the matrix channel response may be assumed constant over the band of interest, a frequency-flat channel. The relevant capacity for such a channel is expressed as [1]–[3]

$$C = \log_2 \left( \det \left[ \mathbf{I}_r + \left( \frac{P}{t} \right) \mathbf{H}\mathbf{H}^\dagger \right] \right) \text{ b/s/Hz} \quad (2)$$

where  $\dagger$  denotes transpose conjugate,  $\mathbf{I}_n$  denotes an  $n \times n$  identity matrix, and we assume equal power transmission on the  $t$  transmit antennas.

### B. Moments

In [3], Telatar has derived an exact expression for the mean of the system capacity given by

$$E(C) = \int_0^\infty \log \left( 1 + \frac{P\lambda}{t} \right) \sum_{k=0}^{m-1} \frac{k!}{(k+n-m)!} \times [L_k^{n-m}(\lambda)]^2 \lambda^{n-m} e^{-\lambda} d\lambda \quad (3)$$

where  $L_k^{n-m}(x)$  are generalized Laguerre polynomials of order  $k$ . In the limit as  $r, t \rightarrow \infty$ , and  $r/t$  is held constant, the mean capacity has been shown to converge to [4]

$$E(C) = n \left\{ \log \left( \frac{w_y}{\sigma^2} \right) + \frac{1-y}{y} \log \left( \frac{1}{1-v_y} \right) - \frac{v_y}{y} \right\} \quad (4)$$

where  $\sigma^2 = 1/P$ ,  $y = r/t$ ,  $0 < y \leq 1$ , and

$$w_y = \frac{1}{2} \left( 1 + y + \sigma^2 + \sqrt{(1 + y + \sigma^2)^2 - 4y} \right)$$

$$v_y = \frac{1}{2} \left( 1 + y + \sigma^2 - \sqrt{(1 + y + \sigma^2)^2 - 4y} \right).$$

Note that Rapajic and Popescu [4] also show how to interchange  $r$  and  $t$  so that (4) can always be used, whether  $r > t$  or  $r \leq t$ .

In terms of higher order moments, results are now appearing [5], [6] which give various limiting results for the variance. They show that the capacity variance converges to a constant as  $r, t \rightarrow \infty$ , and  $r/t$  is held constant. This limiting variance depends only on the ratio of  $r$  and  $t$ , and not on their individual values. However, to the best of our knowledge, no exact results are available for the variance. Hence, we derive the variance in Section III below.

### III. METHODOLOGY

We use relatively little-known CLTs for random matrices [7] which may be applied in the complex case to the capacity variable. Now it is known from [7, pp. 278–310] that a certain CLT exists which states that the distribution of the standardized capacity is asymptotically Gaussian as  $r \rightarrow \infty$ ,  $t \rightarrow \infty$  and  $r/t \rightarrow y$  for some constant  $y$ . The standardized capacity is simply the capacity shifted and scaled to have zero mean and unit variance. In other words, if  $C$  is the capacity variable with mean  $\mu$  and standard deviation  $\sigma$ , then the standardized capacity is  $(C - \mu)/\sigma$ . To implement the Gaussian approximation, we require  $E(C)$  and  $\text{Var}(C)$ . The exact mean was given in [3], see (3), and the limiting value in [4], see (4). The variance is derived here in the Appendix following Telatar's approach, and is given in two forms

$$\text{Var}(C) = m \int_0^\infty w^2(\lambda) p(\lambda) d\lambda$$

$$- \int_0^\infty \int_0^\infty K(\lambda_1, \lambda_2)^2 w(\lambda_1) w(\lambda_2) d\lambda_1 d\lambda_2 \quad (5)$$

$$\text{Var}(C) = m \int_0^\infty w^2(\lambda) p(\lambda) d\lambda$$

$$- \sum \sum \left[ \frac{(i-1)!(j-1)!}{[(i-1+n-m)!(j-1+n-m)!]} \right]$$

$$\times \left\{ \int_0^\infty \lambda^{n-m} \exp(-\lambda) L_{i-1}^{(n-m)}(\lambda) \right.$$

$$\left. \times L_{j-1}^{(n-m)}(\lambda) w(\lambda) d\lambda \right\}^2 \quad (6)$$

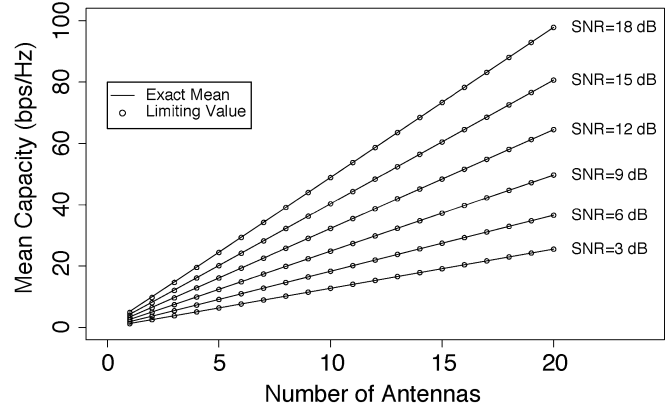


Fig. 1. Mean capacity versus antenna number for  $r = t$ .

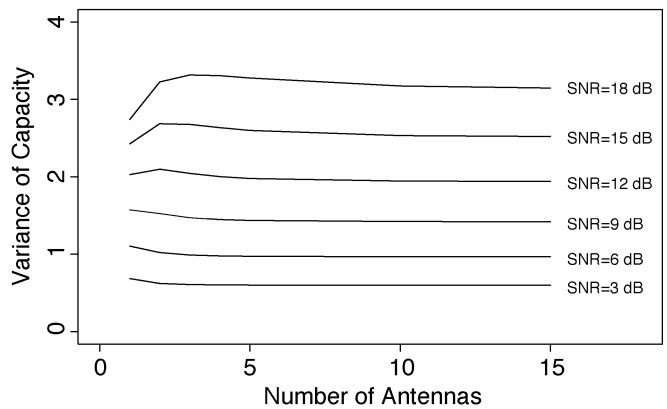


Fig. 2. Variance of capacity versus antenna number for  $r = t$ .

where  $w(\lambda) = \log(1 + P\lambda/t)$ ,  $L_i^{(r)}(x)$  is a generalized Laguerre polynomial, and

$$p(\lambda) = m^{-1} \sum_{i=1}^m (i-1)! [(i-1+n-m)!]^{-1} \lambda^{n-m}$$

$$\times \exp(-\lambda) L_{i-1}^{(n-m)}(\lambda)^2 \quad (7)$$

$$K(\lambda_1, \lambda_2) = m^{-1} \sum_{i=1}^m (i-1)! [(i-1+n-m)!]^{-1} (\lambda_1 \lambda_2)^{\frac{(n-m)}{2}}$$

$$\times \exp\left(-\frac{\lambda_1 + \lambda_2}{2}\right) L_{i-1}^{(n-m)}(\lambda_1)$$

$$\times L_{i-1}^{(n-m)}(\lambda_2). \quad (8)$$

Hence, the variance can be found by double numerical integration using (5), or several single numerical integrations via (6). In this letter, we have used (6) in all the results.

### IV. RESULTS

Fig. 1 shows the useful result that over the whole range of  $r, t$ , and SNR considered the Rapajic limiting mean value [4] is visually indistinguishable from Telatar's exact mean [3] (at least on this scale of plot). Also demonstrated is the well-known linear growth of  $E(C)$  with  $m$ . Fig. 2 shows the behavior of the capacity variance for  $r = t$  and various SNR values. It shows that the variance stabilizes as  $r = t$  increases for any SNR value,

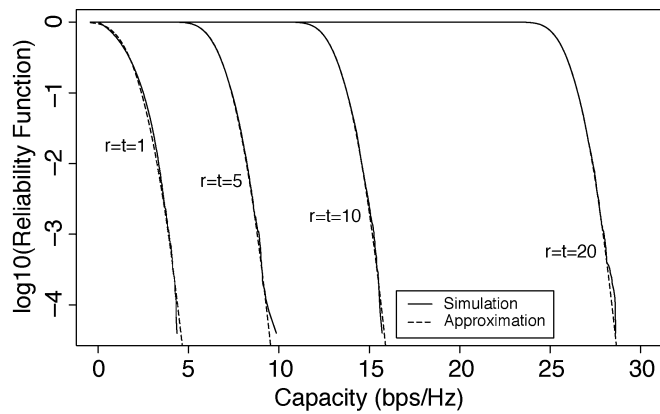


Fig. 3. Comparison of simulated capacity with a normal approximation (SNR = 3 dB).

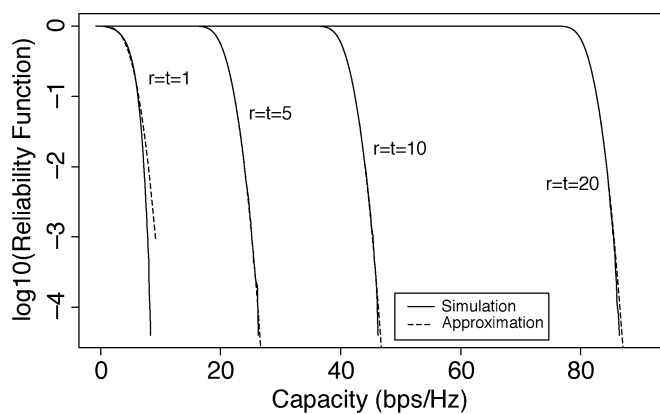


Fig. 4. Comparison of simulated capacity with a normal approximation (SNR = 15 dB).

although this stabilization occurs more rapidly for small SNR. These experimental results support the limiting variance results in [5] and [6]. Gaussian approximations to the capacity distribution can now be investigated, since we have results for the mean and variance. Note that the mean is straightforward to compute either by Rapajic's closed-form limiting value (4) or by a single well-behaved numerical integration (3). Figs. 3 and 4 show the accuracy of a Gaussian approximation to the reliability function or complementary cumulative distribution function ( $P(C > c_0)$ ). The Gaussian approximation does remarkably well over the whole range of  $r$  and  $t$  values, considering the CLT only offers Gaussianity as  $r, t \rightarrow \infty$ . When  $m \geq 5$ , the Gaussian approximation is virtually indistinguishable from the simulated curve, and accurately predicts the capacity percentiles. The worst fits occur for high SNR and low values of  $m$ . However, even the worst fit,  $r = t = 1$  in Fig. 4, is fairly respectable.

## V. CONCLUSIONS

We have derived the variance of the capacity of a MIMO system, allowing an investigation of the accuracy of a Gaussian approximation to capacity foreshadowed by various CLTs. We confirm recent results which state that the capacity variance appears to converge to a limit independent of absolute antenna

numbers, but dependent on the ratio  $r/t$ . The Gaussian approximation itself is surprisingly good, even in the worst cases giving satisfactory results.

## APPENDIX

### DERIVATION OF THE VARIANCE OF THE CAPACITY

We follow the derivation of the mean capacity given by Telatar [3] and extend this approach to the variance. Let  $\lambda_1, \lambda_2, \dots, \lambda_m$  denote the eigenvalues of  $\mathbf{H}\mathbf{H}^\dagger$  for  $r \leq t$  and  $\mathbf{H}^\dagger\mathbf{H}$  for  $r > t$ . Then from (1), we have

$$C = \sum_{i=1}^m \log_2 \left( 1 + \frac{P\lambda_i}{t} \right).$$

The variance of  $C$  is given by

$$\begin{aligned} \text{Var}(C) &= \sum_{i=1}^m \text{Var} \left( \log_2 \left( 1 + \frac{P\lambda_i}{t} \right) \right) \\ &\quad + 2 \sum_{i < j} \text{Cov} \left( \log_2 \left( 1 + \frac{P\lambda_i}{t} \right), \log_2 \left( 1 + \frac{P\lambda_j}{t} \right) \right) \\ &= m \text{Var} \left( \log_2 \left( 1 + \frac{P\lambda}{t} \right) \right) + m(m-1) \\ &\quad \times \text{Cov} \left( \log_2 \left( 1 + \frac{P\lambda_1}{t} \right), \log_2 \left( 1 + \frac{P\lambda_2}{t} \right) \right) \quad (9) \end{aligned}$$

where  $\lambda$  is a randomly selected eigenvalue, and  $(\lambda_1, \lambda_2)$  is a pair of randomly selected (distinct) eigenvalues. Using the notation  $\omega(\lambda) = \log_2(1 + P\lambda/t)$ , we have

$$\begin{aligned} \text{Var}(C) &= m \left[ E(\omega^2(\lambda)) - E(\omega(\lambda))^2 \right] \\ &\quad + m(m-1) \left[ E(\omega(\lambda_1)\omega(\lambda_2)) - E(\omega(\lambda))^2 \right]. \quad (10) \end{aligned}$$

The main difficulty in (10) is the evaluation of  $E(\omega(\lambda_1)\omega(\lambda_2))$ , for which we need the joint density of  $\lambda_1, \lambda_2$ . Telatar [3] gives the joint density of  $\lambda_1, \lambda_2, \dots, \lambda_m$  as

$$\begin{aligned} p_\lambda(\lambda_1, \lambda_2, \dots, \lambda_m) &= [m!]^{-1} \sum_{\alpha, \beta} (-1)^{\text{per}(\alpha) + \text{per}(\beta)} \\ &\quad \times \prod_{i=1}^m \phi_{\alpha_i}(\lambda_i) \phi_{\beta_i}(\lambda_i) \lambda_i^{n-m} e^{-\lambda_i} \quad (11) \end{aligned}$$

where the sum is over all possible permutations  $\alpha, \beta$  of  $\{1, 2, \dots, m\}$ ,  $\text{per}(\cdot)$  denotes the sign of the permutation, and  $\phi_k(\lambda)$  is given by

$$\phi_k(\lambda) = \left\{ \frac{(k-1)!}{(k-1+n-m)!} \right\}^{\frac{1}{2}} L_{k-1}^{(n-m)}(\lambda) \quad (12)$$

where  $L_{k-1}^{(n-m)}(\lambda)$  is a generalized Laguerre polynomial. Since  $\lambda_1, \lambda_2, \dots, \lambda_m$  are unordered, we can obtain the joint density of  $\lambda_1, \lambda_2$  by integrating (11) over  $\lambda_3, \lambda_4, \dots, \lambda_m$  and using the orthogonality relationship [3]

$$\int_0^\infty \phi_i(\lambda) \phi_j(\lambda) \lambda^{n-m} e^{-\lambda} d\lambda = \delta_{ij}.$$

This approach gives the joint density of  $\lambda_1, \lambda_2$  as

$$\begin{aligned} p(\lambda_1, \lambda_2) &= [m!]^{-1} \sum_{\alpha, \beta} (-1)^{\text{per}(\alpha) + \text{per}(\beta)} \\ &\quad \times \phi_{\alpha_1}(\lambda_1) \phi_{\alpha_2}(\lambda_2) \phi_{\beta_1}(\lambda_1) \phi_{\beta_2}(\lambda_2) \lambda_1^{n-m} \lambda_2^{n-m} \\ &\quad \times \exp(-\lambda_1) \exp(-\lambda_2) \prod_{i=1}^2 \delta_{\alpha_i \beta_i}. \quad (13) \end{aligned}$$

Identifying the nonzero terms in (13) (where  $\delta_{\alpha\beta} = 1$ ) and substituting (12) gives

$$\begin{aligned}
p(\lambda_1, \lambda_2) &= [m(m-1)]^{-1} \sum_{i=1}^m \sum_{j \neq i}^m (\lambda_1 \lambda_2)^{n-m} \\
&\times \exp(-(\lambda_1 + \lambda_2)) (i-1)!(j-1)! \\
&\times [(i-1+n-m)!(j-1+n-m)!]^{-1} \\
&\times \left\{ L_{i-1}^{(n-m)}(\lambda_1)^2 L_{j-1}^{(n-m)}(\lambda_2)^2 \right. \\
&\quad - L_{i-1}^{(n-m)}(\lambda_1) L_{j-1}^{(n-m)}(\lambda_1) \\
&\quad \left. \times L_{i-1}^{(n-m)}(\lambda_2) L_{j-1}^{(n-m)}(\lambda_2) \right\}. \quad (14)
\end{aligned}$$

With a little rearrangement, (14) can be rewritten as

$$p(\lambda_1, \lambda_2) = \frac{m}{m-1} p(\lambda_1) p(\lambda_2) - \frac{1}{m(m-1)} K(\lambda_1, \lambda_2)^2 \quad (15)$$

where  $p(\lambda)$  is the density of an arbitrary eigenvalue given by Telatar [3] as

$$\begin{aligned}
p(\lambda) &= m^{-1} \sum_{i=1}^m (i-1)! [(i-1+n-m)!]^{-1} \lambda^{n-m} \\
&\quad \times \exp(-\lambda) L_{i-1}^{(n-m)}(\lambda)^2
\end{aligned}$$

and

$$\begin{aligned}
K(\lambda_1, \lambda_2) &= \sum_{i=1}^m (i-1)! [(i-1+n-m)!]^{-1} (\lambda_1 \lambda_2)^{\frac{(n-m)}{2}} \\
&\quad \times \exp\left(-\frac{(\lambda_1 + \lambda_2)}{2}\right) L_{i-1}^{(n-m)}(\lambda_1) L_{i-1}^{(n-m)}(\lambda_2).
\end{aligned}$$

Now we can turn to the calculation of  $E(\omega(\lambda_1)\omega(\lambda_2))$  since

$$\begin{aligned}
&m(m-1)E(\omega(\lambda_1)\omega(\lambda_2)) \\
&= m(m-1) \int_0^\infty \int_0^\infty \omega(\lambda_1)\omega(\lambda_2)p(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2 \\
&= m^2 E(\omega(\lambda))^2 \\
&\quad - \int_0^\infty \int_0^\infty K(\lambda_1, \lambda_2)^2 \omega(\lambda_1)\omega(\lambda_2)d\lambda_1 d\lambda_2. \quad (16)
\end{aligned}$$

Substituting (16) in (10) gives

$$\begin{aligned}
\text{Var}(C) &= mE(\omega^2(\lambda)) \\
&\quad - \int_0^\infty \int_0^\infty K(\lambda_1, \lambda_2)^2 \omega(\lambda_1)\omega(\lambda_2)d\lambda_1 d\lambda_2
\end{aligned}$$

$$\begin{aligned}
&= m \int_0^\infty \omega^2(\lambda)p(\lambda)d\lambda \\
&\quad - \int_0^\infty \int_0^\infty K(\lambda_1, \lambda_2)^2 \omega(\lambda_1)\omega(\lambda_2)d\lambda_1 d\lambda_2. \quad (17)
\end{aligned}$$

The integrals in (17) appear to be intractable in closed form. The first single integral can be rewritten in terms of special functions, but the formulation as an integral is just as convenient, since the integrand is well behaved and numerical integration is straightforward. The double integral can also be evaluated numerically, or we can take the summations in  $K(\lambda_1, \lambda_2)^2$  outside the integrals to give

$$\begin{aligned}
\text{Var}(C) &= m \int_0^\infty \omega^2(\lambda)p(\lambda)d\lambda \\
&\quad - \sum_{i=1}^m \sum_{j=1}^m \left[ \frac{(i-1)!(j-1)!}{(i-1+n-m)!(j-1+n-m)!} \right] \\
&\quad \times \left\{ \int_0^\infty \lambda^{n-m} \exp(-\lambda) L_{i-1}^{(n-m)}(\lambda) \right. \\
&\quad \left. \times L_{j-1}^{(n-m)}(\lambda)\omega(\lambda)d\lambda \right\}^2. \quad (18)
\end{aligned}$$

Hence, we can either perform the single double-numerical integration in (17) or several single-numerical integrations in (18). Results in this letter were calculated using (18).

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