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An Approximate Deflection Function for Simply Supported Quadrilateral Thin Plate by Variational Approach

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Abstract. This paper presents more convenient deflection function that properly substitutes slow converging series function by Naiver for simply supported quadrilateral thin plate. The formulation of the deflection function was conducted by using variational approach. During formulation, Rayleigh-Ritz procedures was used and coordinate function that satisfies kinematic boundary condition and the minimum potential energy principle was developed. In addition to coordinate function, modification factor was applied to the main function for curve fitting purpose. To validate the current study, comparisons were done with Navier's solution and finite element analysis. The results shown that the deviation for mid-point deflection of current study was very small, i.e. around 0.03%, from Navier's solution. Furthermore, it has been investigated that finite element analysis also resulted closer result with current study.

INTRODUCTION

Numerical methods were the most commonly used approaches in engineering to provide approximate solution to the problems in question. In most cases, they provide very closer results no matter how accurate they are not. In some more complex instances, the exact solution by using different approach are not possible and too time consuming. However, for simple plate configuration such as simply supported plate, exact solution can be applied to analyze its elastic deformations under static loadings [1]. Even though, analytical solution are the most desirable, but not easily attainable due to the above mentioned boundary conditions [2]. For plate with different shaped cutouts and composites, which are common in civil, marine, mechanical, and aerospace structures, numerical approach is more suitable than analytical solution [3].

Currently Rayleigh-Ritz method is most efficient and famous method in plate analysis, which utilize energy concept. This methods solves the boundary value problems by choosing approximate displacement functions that minimized to functional involving energy equation[4] [5]. Finite Element Methods (FEM) is also the most popular and powerful in wide range structural with some modification from Ritz method. In FEM, plate is discretized into a finite number of elements and connected at their nodes and along inter element boundaries. At each node and along the boundaries, both the equilibrium and compatibility conditions must be satisfied to determine unknown parameters (deflections, slopes, internal forces, and moments) are determined at nodes [6].

This paper provide an approximate deflection solution for simply supported thin plate. The formulation was done by using Ritz method. Accordingly, function that satisfies boundary conditions and deflection patterns for the chosen plate problem was selected. In addition to this function, a modification factor was utilized for curve fitting. After all necessary computations were completed, the new deflection equation was cross checked with Navier's solution [7] and finite element outputs from Abaqus analysis. The comparative study shows that the deflection pattern and numerical results from current study resulted very close estimation with Navier's solution and finite element analysis outputs.

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GENERAL FORMULATION OF VARITIONAL APPROACH FOR PLATE BENDING

The formulation of Varitional method is based on the variation of total energy due to internal and external forces from initial state to deformed state in solid mechanics. And the general equation is given by Eqs.(1)-(6) [6].

$$\Pi = U - \Omega \tag{1}$$

Where: Π is the total potential energy, U is strain energy of deformation (internal potential energy), and Ω is external potential energy due to external forces. Explicitly the strain energy (U) for plates with constant and variable thickness is expressed in integral form over the thickness of the plate by:

$$U = \frac{1}{2}\iint_A D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-v)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dA \tag{2}$$

In which $D = \frac{Eh^3}{12(1-v^2)}$ is the plate rigidity, E, v, and h are Young's modulus, poison's ratio and plate thickness respectively. And the potential energy due to external force, which assumed unchanged (constant) during deformation, is given by:

$$\Omega = \Omega_p + \Omega_\Gamma \tag{3}$$

Where;

$$\Omega_p = -\left[\iint_A P(x,y) w dA + \sum_i P_i w_i + \sum_j M_j \vartheta_j\right] \tag{4}$$

In which; $\iint_A P(x,y) w dA$ is potentila energy due to distributed load

$\sum_i P_i w_i$ is potentila energy due to concentrated load

$\sum_i M_i \vartheta_i$ is potentila energy due to moment applied at the middle surface of plate

And the potential energy of edge loads Ω_Γ is given as:

$$\Omega_\Gamma = \oint_\Gamma \left(V_n w + m_n \frac{\partial w}{\partial n} + m_t \frac{\partial w}{\partial t}\right) ds \tag{5}$$

Where: V_n is transverse edge forces, m_n and m_t are edge moments.

The principle of varitional approach is based on conservation of energy which states that, "the strain energy stored in an elastic body is equal to the work done by the applied external load during the loading process, assuming that there are no thermal or inertial effect [6]. At extreme condition, Eq. (1) is solved by satisfying Eq. (6), which will be applied in the next section to calculate the unknown constant.

$$\partial \Pi = 0 \tag{6}$$

Hence the plate problems are solved by varitional approach by using the governing equations Eqs. (1) - (6).

DEFLECTION FUNCTION FORMULATION

For this study, an isotropic quadrilateral plate with all sides simply supported was considered. The configuration of the chosen plate was shown in Fig.1.By considering the boundary condition, deflection function w (see Eq. (7)) under uniformly distributed load $p(x,y)$ was selected. This function has two parts. The first part, which is equation in the first bracket in Eq. (7), represents coordinate function that satisfies kinematic boundary condition and the minimum potential energy principle. The constant C is determined from the minimum potential energy principle as stated in Ritz method [4][6]. And the second part, that is expression in the second bracket in Eq. (7)), is modification factor (λ).This modification factor was calculated after determining constant C. Numerical value of this modification factor was calculated from the ratio of mid span deflection from Navier's double trigonometric series solution to mid span deflection obtained from Eq.(14b).

The second part of the deflection equation was intended for curve fitting. Derivation and integration will be applied to calculate the force effects such as stress resultants, bending and twisting moments and shear resultants for the first part only. Since the second part is used as constant multiplier that will be multiplied with any force effect.

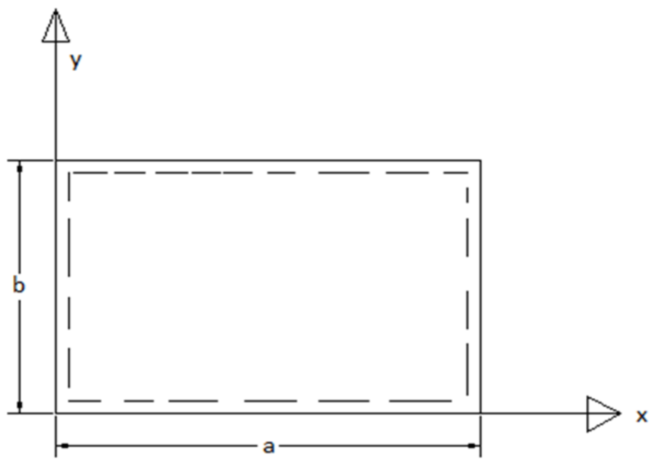

FIGURE 1.Simply supported plate at four edges

The following deflection function is developed for the formulation of new approximate solution.

$$w = \{C(x^2 - a * x)\sin(\pi y/b)\}\{\lambda\} \tag{7}$$

Where;

$C =$ Constant which can be calculated from Eq. (6) (see Eqs. (13))

, and

$\lambda =$ Modification factor (see Eq. (16))

By using Eq. (7) boundary conditions become;

$$\begin{aligned} w = 0, \quad & \frac{\partial^2 w}{\partial x^2} = 0 \quad for \quad x = 0 \; and \; x = a \\ w = 0, \quad & \frac{\partial^2 w}{\partial y^2} = 0 \quad for \quad y = 0 \; and \; y = b \end{aligned} \tag{8}$$

It can be seen from Eq. (8) that all boundary conditions are satisfied for simply supported plate. For simplicity, let's introduce new dimensionless coordinates as follows:

$$\eta = \frac{y}{b} \quad and \; x = \xi a \tag{9}$$

Hence by substituting Eq. (9) into Eq. (7), the expression of deflection function rewritten as:

$$w = \{C((\xi a)^2 - a^2\xi)\sin(\pi\eta)\}\{\lambda\} \tag{10}$$

From Eq. (2), the integral of the second terms will vanish during formulation. Since it is zero for fixed and simply supported conditions. Thus, by substituting Eq. (9) into Eq. (2), the strain energy can be rewritten as:

$$U = \frac{1}{2}Dab\int_0^1\int_0^1\left(\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\eta^2}\right)^2 d\xi d\eta \tag{11a}$$

$$U = \frac{C^2Da(\pi^4a^4 + 20\pi^2a^2b^2 + 120b^4)}{120b^3} \tag{11b}$$

In the same way, the second and third terms in Eq. (4) becomes zero due to the load considered in this study. In addition the potential energy of edge loads, i.e. Ω_Γ, will also vanish. Therefore, potential energy due to external force rewritten as:

$$\Omega = -ab\int_0^1\int_0^1 pwd\xi d\eta \tag{12a}$$

$$\Omega = \frac{CPa^3b}{3\pi} \tag{12b}$$

Upon substituting Eq. (11b) and Eq. (12b) into Eq. (1), and evaluating the derivative $\frac{\partial\pi}{\partial c} = 0$, the coefficient C can be determined as:

$$C = \frac{20Pa^2b^4}{D\pi(\pi^4a^4 + 20\pi^3a^2b^2 + 120b^4)} \tag{13a}$$

For square plate, in which $a = b$, the C value simplified as:

$$C = 0.015\frac{Pa^2}{D} \tag{13b}$$

Thus by substituting Eq. (13b) into Eq. (7), the expression in the first bracket for square plate becomes:

$$w_1 = 0.015\frac{Pa^2}{D}(x^2 - a * x)\sin(\pi y/b) \tag{14a}$$

Hence the maximum mid span deflection from Eq. (14a) will be:

$$w^*{}_1 = 0.00375\frac{(Pa^4)}{D} \tag{14b}$$

Note: Eq. (14b) was used just to calculate modification factor not to calculate maximum mid span deflection from this study. The expression to calculate maximum deflection from this study was given in Eq. (18) in the next page.

Thus the modification factor (λ) will be calculated as follows. Let β be maximum deflection ratio which is calculated as:

$$\beta = \frac{0.00416\frac{(Pa^4)}{D}}{0.00375\frac{(Pa^4)}{D}} = 1.109 \tag{15}$$

Where $0.00416\frac{(Pa^4)}{D}$ is maximum mid span deflection expression for square plate from Navier's solution [7]. By using β and the trigonometric function in the first bracket of Eq. (7), i.e. $\sin(\pi y/b)$), the modification factor that approximately fit the deflection curve was obtained as:

$$\lambda = \frac{\beta}{\sin\left(\frac{\pi y}{b}\right)} \quad where\ y \neq 0 \tag{16}$$

Note that the term $\frac{1}{\sin\left(\frac{\pi y}{b}\right)}$ in Eq. (16) will not be used during derivations to calculate force effects. Hence, it is used as constant multiplier for curve fitting. Finally upon substituting Eq. (14a) and Eq. (16) into Eq. (7), the new approximate deflection function for square plate deflection becomes:

$$w = \left\{0.015\frac{Pa^2}{D}(x^2 - a * x)\sin\left(\frac{\pi y}{b}\right)\right\}\left\{\frac{\beta}{\sin\left(\frac{\pi y}{b}\right)}\right\} \tag{17a}$$

Or by substituting the value of β from Eq. (15) into Eq. (17a) results:

$$w = \left\{0.016635\frac{Pa^2}{D}(x^2 - a * x)\sin\left(\frac{\pi y}{b}\right)\right\}\left\{\frac{1}{\sin\left(\frac{\pi y}{b}\right)}\right\} \tag{17b}$$

Hence for square plate, in which $a = b$, the maximum mid span deflection will be

$$w = -0.00415875\frac{Pa^4}{D} \tag{18}$$

Which is very close to Navier's solution, i.e. $w \approx \frac{4a^4p}{\pi^6 D} = 0.00416\ \mathrm{Pa}^2/D$. The negative sign in Eq. (18) indicates that the deflection is in negative z-axis or opposite to internal potential energy.

The modification factor in the second bracket of Eq. (17a) and (17b) is used as constant multiplier as stated above. And for general case, by substituting Eq. (13a) and Eq. (17) into Eq. (7) the plate deflection function calculated as:

$$w = \left\{0.1059\frac{Pa^2b^4}{D(\pi^4a^4 + 20\pi^3a^2b^2 + 120b^4)}(x^2 - a * x)\sin(\pi y/b)\right\}\left\{\frac{1}{\sin\left(\frac{\pi y}{b}\right)}\right\} \tag{19}$$

NUMERICAL ANALYSIS AND DISCUSSION

In this section, an isotropic square steel plate with material properties, load and geometry as specified in Table 1 was considered to validate Eq. (17). Numerical results were compared with Numerical solution by Navier's and finite element analysis using Abaqus software. For finite element analysis, section property is defined as homogeneous. And for mesh control, quadratic element shape with structured technique was used. Thus, by using those data's in Table 1, maximum response of the plate was studied. The maximum mid span deflection was discussed in Fig.2. The comparative study conducted in Fig.2 was based on the results from current study, Navier's solution, and finite element analysis.

TABLE 1. Data's used for thin plate in Abaqus

Material properties		**Uniform Load intensity**	**Dimensions**											
			sides (mm)											
E (N/mm2)	**V**	**(N/mm2)**	**a**	**b**	**Thickness, h (mm)**									
200,000	0.3	10	50	50	1	2	3	4	5	6	7	8	9	10

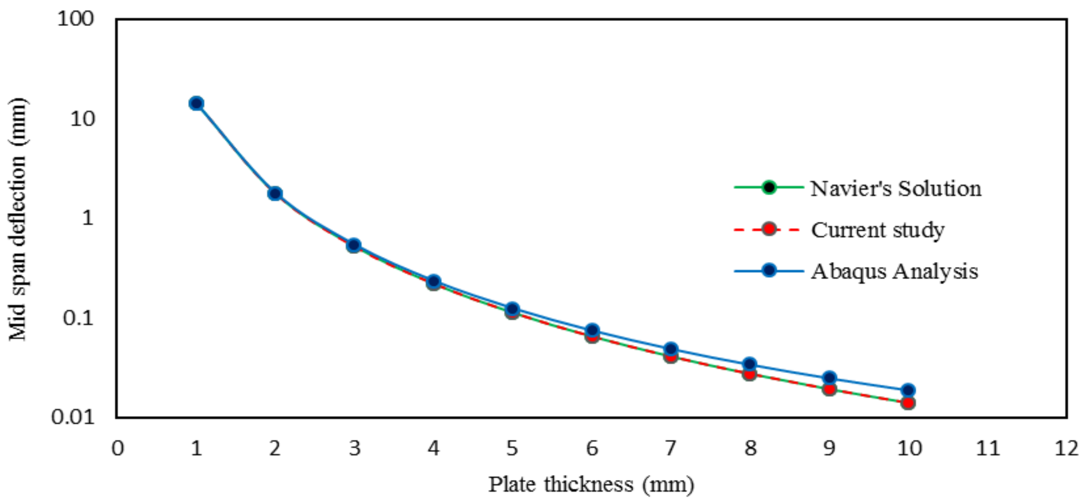

FIGURE 2. Mid span deflection for plate thickness

It can be clearly seen form Fig.2 for each plate thickness mentioned, the maximum mid span deflections for all cases were acceptable according to plate theory. Since deflection is inversely proportional to thickness of plate that is, as the thickness of the plate increases deflection becomes minimum. Interestingly, the deflection from current study and Navier's solution already overlap to each other. It has been calculated that the average deviation between Navier's solution and current study was about 0.03%, which can be considered as insignificant. However, there is some deviation from finite element analysis as the thickness of the plate increases. This can minimized by using appropriate meshing for thicker plates.

Figure 3 shows the deflection pattern for 1mm thick plate along mid strip. The comparison was conducted between finite element analysis using Abaqus software and current study. The comparison plot shows that results obtained from finite element analysis confirm the current study with minimum deviation. In addition, the given curve is symmetrical with reference to mid-point, which satisfies the curve pattern for uniform loading and homogenous plate section.

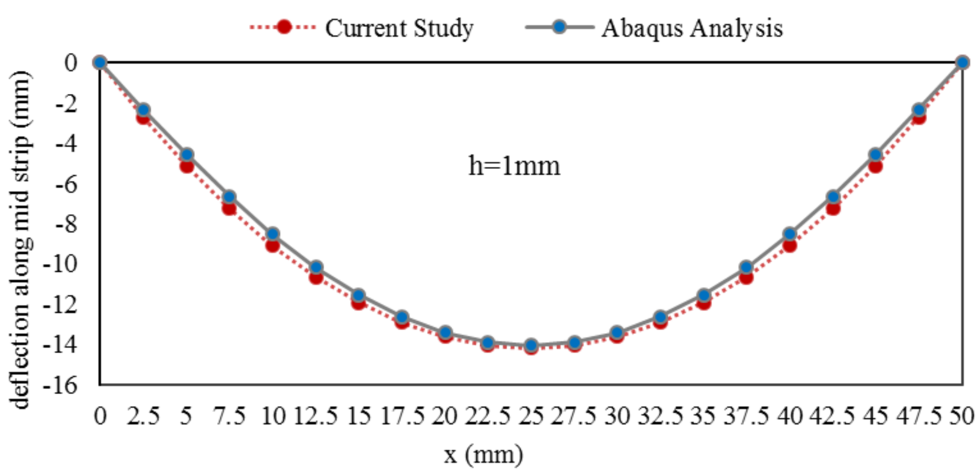

FIGURE 3. Comparison for deflected curve along mid strip.

Under Fig 4, finite element analysis for 1mm thick plate was presented. The deflection contour reveals that the Abaqus analysis also resulted slightly the same value with the current study. It can be clearly seen that from Fig.4, the maximum deflection appear at mid-point of the thin plate, i.e. 14.05mm.

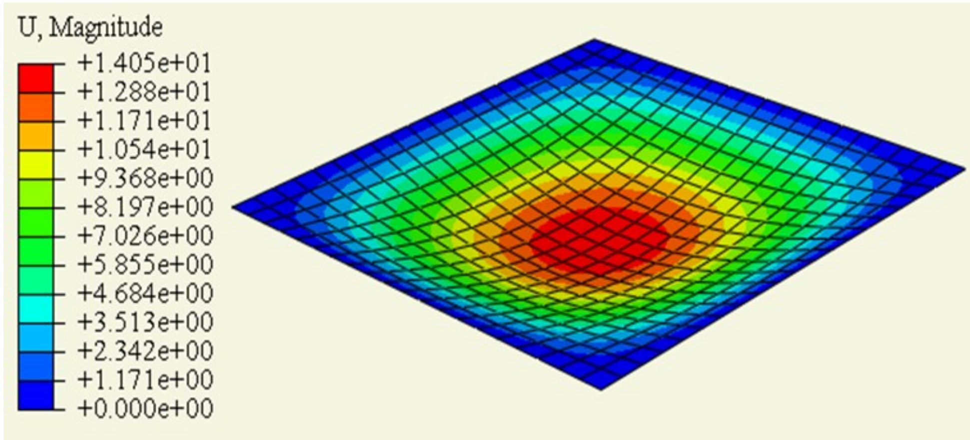

FIGURE 4. Deflection contour from finite element analysis

CONCLUSIONS

A great number of researches were conducted to solve plate problems by many scholars for different boundary conditions and different mode of loading. Nevertheless, some of them are not easy to handle. For instance Navier's solution for simply supported quadrilateral plate, which was commonly used expression in thin plate analysis, was not easy to calculate force effects due to long and slowly converging series function. However, this study will minimize the computation time by eliminating long and slowly converging series coordinate function in Navier's solution. Since the deflection function for simply supported thin plate in this study is convenient to calculate displacement and other force effects even manually at a desired point of plate cross section.

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