An Approximate Inverse to the Extended Born Modeling Operator

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The Rice Inversion Project

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Outline

Full Waveform Inversion :

Given $d \in \mathcal{D}$, find $m \in \mathcal{M}$ so that

$$\mathcal{F}[m] \simeq d$$

- $\mathcal{M}$ =model space, $\mathcal{D}$ = data space
- $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{D}$ Forward Map

Least Squares formulation :

Given $d \in \mathcal{D}$, find $m \in \mathcal{M}$ to minimize

$$J_{LS} = ||\mathcal{F}[m] - d||^2 [+\text{regularizing terms}]$$

Strong nonlinearity, many local minima (descent methods fail)

Extended Model

$\mathcal{M}$ = physical model space

$\bar{\mathcal{M}}$ = bigger extended model space

$\bar{\mathcal{F}} : \bar{\mathcal{M}} \rightarrow \mathcal{D}$ extended modeling operator

Extension Property:

- $\mathcal{M} \subset \bar{\mathcal{M}}$
- $m \in \bar{\mathcal{M}} \rightarrow \bar{\mathcal{F}}[m] = \mathcal{F}[m]$

2D Constant Density Acoustic Wave equation:

$$\frac{1}{v^2(\mathbf{x})}\frac{\partial^2 u}{\partial t^2}(\mathbf{x},t) - \nabla^2 u(\mathbf{x},t) = f(\mathbf{x_s},t)$$

$$u \equiv 0, t \ll 0$$

Forward Modeling (Solve the wave equation) :

$$\mathcal{F}[v] = u(\mathbf{x_r}, t; \mathbf{x_s})$$

Linearization :

$$v(\mathbf{x}) = v_0(\mathbf{x}) + \delta v(\mathbf{x})$$

Then

$$\mathcal{F}[v] \approx \mathcal{F}[v_0] + F[v_0]\delta v$$

Born Approximation

$$F[v]\delta v(\mathbf{x_r}, t; \mathbf{x_s}) = \delta u(\mathbf{x_r}, t; \mathbf{x_s})$$

Born Modeling Operator $F[v]$

$$\left(\frac{1}{v(\mathbf{x})^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x_s}) = \delta(t)\delta(\mathbf{x} - \mathbf{x_s});$$

$$\left(\frac{1}{v(\mathbf{x})^2} - \nabla^2\right) \delta u(\mathbf{x_r}, t; \mathbf{x_s}) = \frac{2\delta v(\mathbf{x})}{v(\mathbf{x})^3} G(\mathbf{x}, t; \mathbf{x_s})$$

where G is Green's function, the implulse response of the medium

Assumption: Single scattering at points of discontinuity of impedance in the subsurface(No multiple scattering!)

Given smooth background velocity $v(\mathbf{x})$, seismic reflection data $d(\mathbf{x_r}, t; \mathbf{x_s})$, find perturbation model $\delta v(\mathbf{x})$ to fit the data:

$$F[v]\delta v \simeq d$$

Linearized Inverse Problem

Given smooth background velocity $v(\mathbf{x})$, seismic reflection data $d(\mathbf{x_r}, t; \mathbf{x_s})$, find perturbation model $\delta v(\mathbf{x})$ to fit the data:

$$F[v]\delta v \simeq d$$

Migration is an approximate solution of this linearized inverse problem

- Migration operator (producing image) is adjoint or transpose of modeling operator(Lailly, Tarantola, Claerbout(80's)).
- Migration operator can position reflectors correctly but with possibly incorrect amplitudes and wavelets.
- True amplitude migration is (pseudo) inverse

Born Modeling Operator

$$F[v]\delta v(\mathbf{x_r}, t; \mathbf{x_s}) = \frac{\partial^2}{\partial t^2} \int d\mathbf{x} \int d\tau \frac{2\delta v(\mathbf{x})}{v^3(\mathbf{x})} G(\mathbf{x}, \mathbf{t} - \tau; \mathbf{x_r}) G(\mathbf{x}, \tau; \mathbf{x_s})$$

Born Modeling Operator

$$F[v]\delta v(\mathbf{x_r}, t; \mathbf{x_s}) = \frac{\partial^2}{\partial t^2} \int d\mathbf{x} \int d\tau \frac{2\delta v(\mathbf{x})}{v^3(\mathbf{x})} G(\mathbf{x}, \mathbf{t} - \tau; \mathbf{x_r}) G(\mathbf{x}, \tau; \mathbf{x_s})$$

The adjoint of F (migration operator) is defined by

$$\int d\mathbf{x_s} d\mathbf{x_r} dt (F\delta v)(\mathbf{x_r}, t; \mathbf{x_s}) d(\mathbf{x_r}, t; \mathbf{x_s}) = \int d\mathbf{x} \delta v(\mathbf{x}) (F^* d)(\mathbf{x})$$

Born Modeling Operator

$$F[v]\delta v(\mathbf{x_r}, t; \mathbf{x_s}) = \frac{\partial^2}{\partial t^2} \int d\mathbf{x} \int d\tau \frac{2\delta v(\mathbf{x})}{v^3(\mathbf{x})} G(\mathbf{x}, \mathbf{t} - \tau; \mathbf{x_r}) G(\mathbf{x}, \tau; \mathbf{x_s})$$

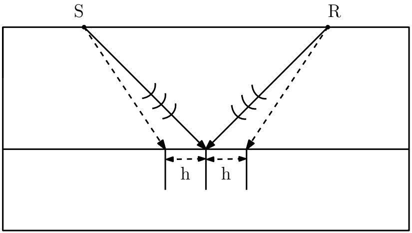
The adjoint of F (migration operator) is defined by

$$\int d\mathbf{x_s} d\mathbf{x_r} dt (F\delta v)(\mathbf{x_r}, t; \mathbf{x_s}) d(\mathbf{x_r}, t; \mathbf{x_s}) = \int d\mathbf{x} \delta v(\mathbf{x})(F^* d)(\mathbf{x})$$

Integration by parts leads to

$$F^* d(\mathbf{x}) = -\frac{2}{v^3(\mathbf{x})} \int d\mathbf{x_s} d\mathbf{x_r} dt d\tau \, G(\mathbf{x}, \tau; \mathbf{x_s}) \frac{\partial^2 d(\mathbf{x_r}, t; \mathbf{x_s})}{\partial t^2} G(\mathbf{x}, t - \tau; \mathbf{x_r})$$

Subsurface offset Extension



Subsurface Extension : $2\mathbf{h}$ = Difference between subsurface scatering points (subsurface offset)

Physical meaning : action at a positive distance

Extend the operator by permiting δv to also depend on (half) offset $\mathbf{h}$.

Extended Born Modeling and Migration Operator

$$\bar{F}[v]\delta v = \frac{\partial^2}{\partial t^2} \int d\mathbf{x} d\mathbf{h} d\tau \, G(\mathbf{x}+\mathbf{h}, t-\tau; \mathbf{x_r}) \frac{2\delta v(\mathbf{x},\mathbf{h})}{v^3(\mathbf{x})} G(\mathbf{x}-\mathbf{h}, \tau; \mathbf{x_s})$$

$$\bar{F}^* d = -\frac{2}{v^3(\mathbf{x})} \int d\mathbf{x_s} d\mathbf{x_r} dt d\tau \, G(\mathbf{x}-\mathbf{h}, \tau; \mathbf{x_s}) G(\mathbf{x}+\mathbf{h}, t-\tau; \mathbf{x_r}) \frac{\partial^2 d(\mathbf{x_r}, t; \mathbf{x_s})}{\partial t^2}$$

Extended Kirchhoff Operator

Fons ten Kroode (2012) constructed the inverse of the extended Kirchhoff Operator (in asymptotic sense) :

Fons ten Kroode,2012

$$\tilde{K}i = \frac{1}{2\pi}\int d\mathbf{x}d\mathbf{h}d\omega e^{-i\omega t} G(\mathbf{x_r},\mathbf{x}+\mathbf{h},\omega)\frac{\partial i(\mathbf{x},\mathbf{h})}{\partial z} G(\mathbf{x}-\mathbf{h},\mathbf{x_s},\omega)$$

$$\tilde{I}d = \frac{32}{\pi v^2(\mathbf{x})}\int d\mathbf{x_r}d\mathbf{x_s}d\omega(-i\omega)\frac{\partial G^*(\mathbf{x}+\mathbf{h},\mathbf{x_r},\omega)}{\partial z_r} d(\mathbf{x_r},\mathbf{x_s},\omega)\frac{\partial G^*(\mathbf{x_s},\mathbf{x}-\mathbf{h},\omega)}{\partial z_s}$$

(http://iopscience.iop.org/0266-5611/28/11/115013)

Can we construct a similar operator to extended Born Modeling Operator?

Construction of the Inverse Operator

Asymptotic Analysis of the Normal Operator $\bar{F}[v]^*\bar{F}[v]\delta v(\mathbf{x}, h)$

Extended Born Modeling Operator and its Adjoint

$$\bar{F}[v]\delta v = \frac{\partial^2}{\partial t^2}\int d\mathbf{x}d\mathbf{h}d\tau\, G(\mathbf{x}+\mathbf{h}, t-\tau; \mathbf{x_r})\frac{2\delta v(\mathbf{x},\mathbf{h})}{v^3(\mathbf{x})}G(\mathbf{x}-\mathbf{h}, \tau; \mathbf{x_s})$$

$$\bar{F}^*[v]d = -\frac{2}{v^3(\mathbf{x})}\int d\mathbf{x_s}d\mathbf{x_r}dtd\tau\, G(\mathbf{x}-\mathbf{h}, \tau; \mathbf{x_s})G(\mathbf{x}+\mathbf{h}, t-\tau; \mathbf{x_r})\frac{\partial^2 d(\mathbf{x_r}, t; \mathbf{x_s})}{\partial t^2}$$

- Step 1 High Frequency Approximation : in 2D

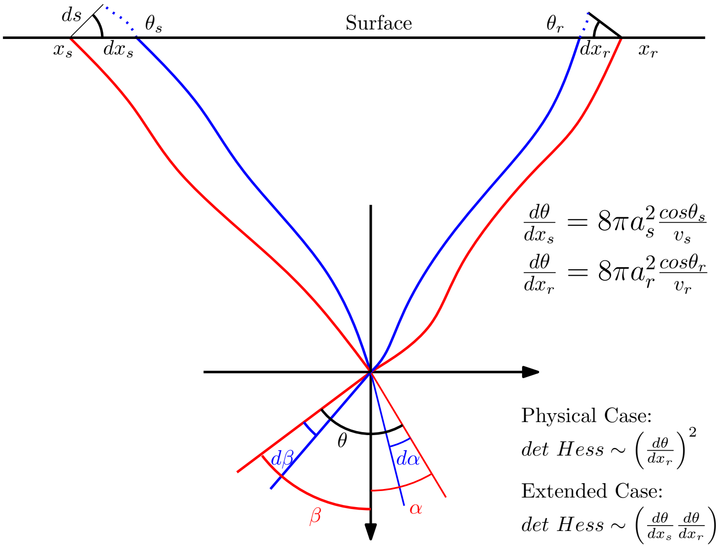
$$G(\mathbf{x_s}, \mathbf{x}, t) \cong a(\mathbf{x_s}, \mathbf{x})S(t-\tau(\mathbf{x_s}, \mathbf{x})),\ \ S(t) = t^{-1/2}H(t)$$

$$G(\mathbf{x}, \mathbf{x_r}, t) \cong a(\mathbf{x}, \mathbf{x_r})S(t-\tau(\mathbf{x}, \mathbf{x_r})),\ \ S(t) = t^{-1/2}H(t)$$

- Step 2 Principle of Stationary Phase $\left(\frac{a_s^2 a_r^2}{\sqrt{det\ \ Hess}}\right)$
- Step 3 Modify adjoint operator by some Velocity-independent Filters

Key point in the derivation

TRIP



$$\frac{d\theta}{dx_s} = 8\pi a_s^2 \frac{cos\theta_s}{v_s}$$

$$\frac{d\theta}{dx_r} = 8\pi a_r^2 \frac{cos\theta_r}{v_r}$$

Physical Case:

$$det\ Hess \sim \left(\frac{d\theta}{dx_r}\right)^2$$

Extended Case:

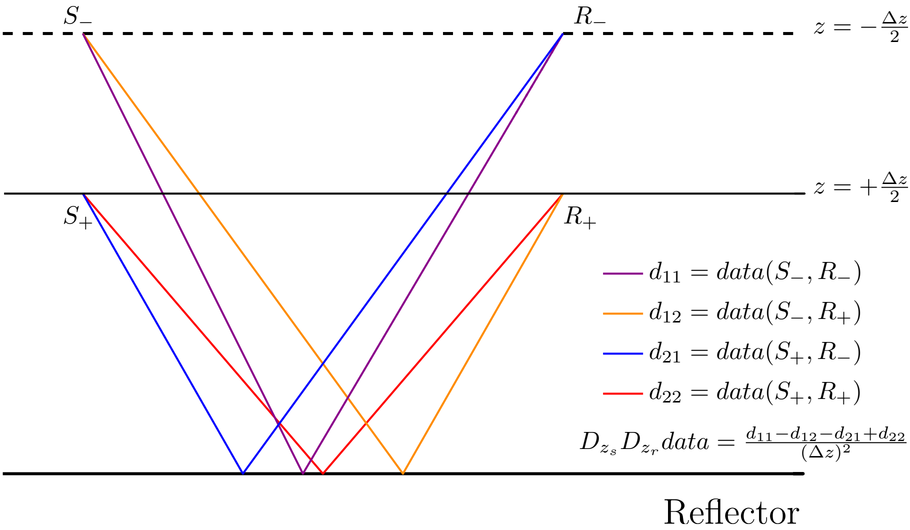
$$det\ Hess \sim \left(\frac{d\theta}{dx_s}\frac{d\theta}{dx_r}\right)^2$$

An Approximate Inverse

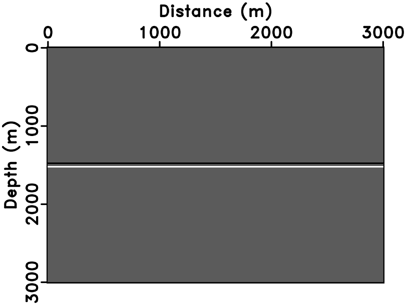
$$\bar{F}^{-1}[v_0]\delta d(\mathbf{x}, h) = -16|k||k'|v_0(\mathbf{x})^5 \bar{F}^*[v_0] I_t^4 D_{z_s} D_{z_r} \delta d(\mathbf{x}, h)$$

- $k = (k_x, k_z)$ and $k' = (k_h, k_z)$ are the wavenumbers
- I_t is the time integral
- D_{z_s}, D_{z_r} are the source and receiver depth derivative.

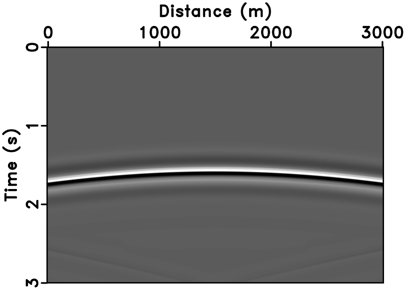
Apply $D_{z_s} D_{z_r}$



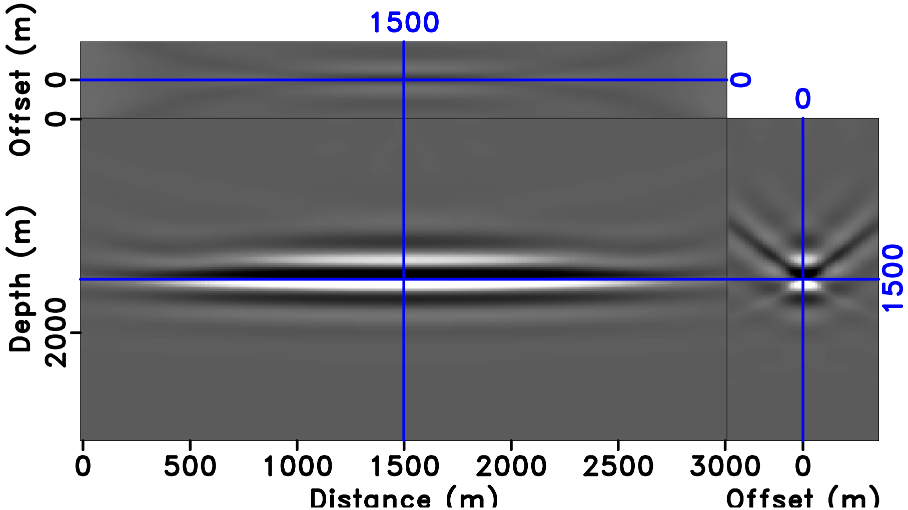
Numerical Test I



Velocity Model



One-shot Born Data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Offset (m)
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Depth (m)
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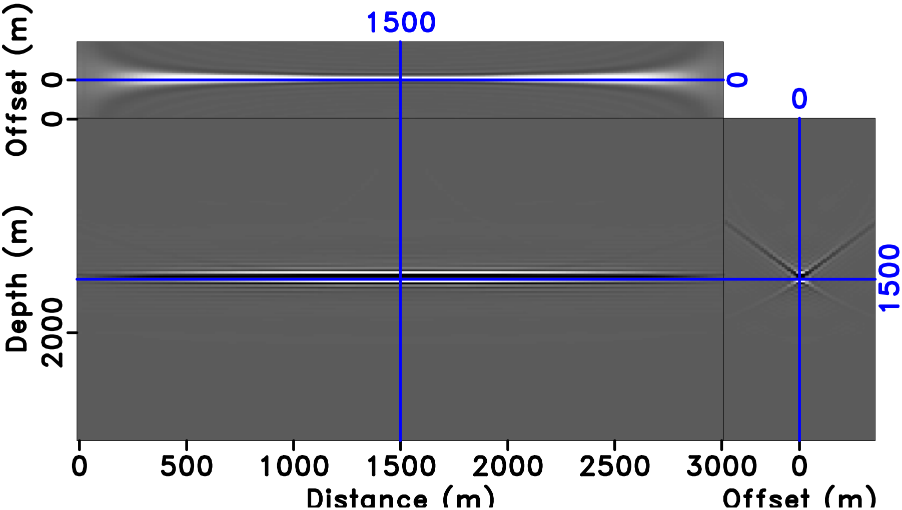
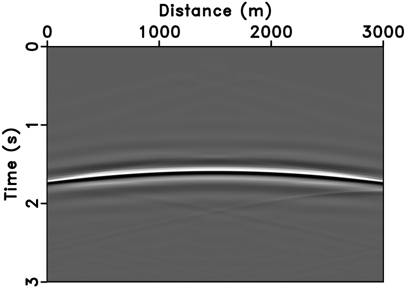
Distance (m)
Offset (m)

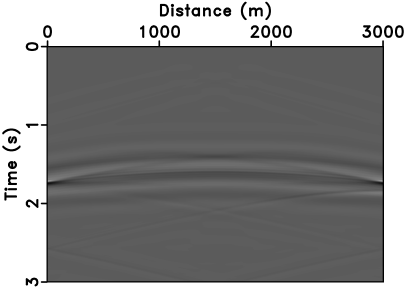
Extended Inversion Result



Resimulated Data



Resimulated Data



Data Residual
(= *6.34% ||observed data||*)

One trace Comparison

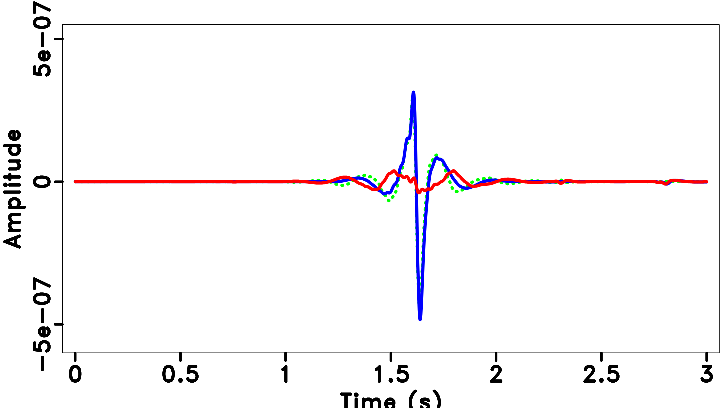
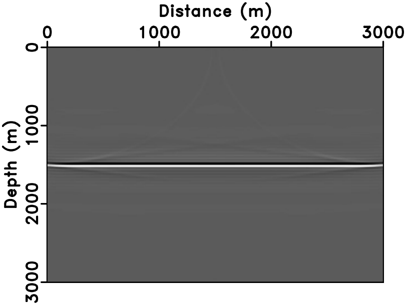
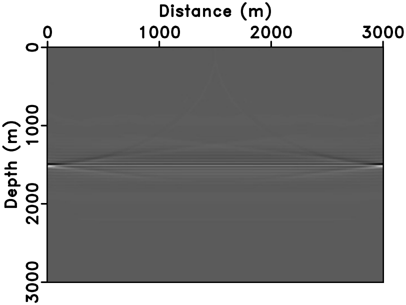


Figure: One trace (middle) comparison between the original data(blue) and resimulated data(green). The differnce is shown as the red line.

Non-extended Inversion Result



Non-extended Inversion Result



Model Residual
(= *9.74% ||model||*)

$$\delta v(\mathbf{x}) = \sum_h \delta v(\mathbf{x}, h)$$

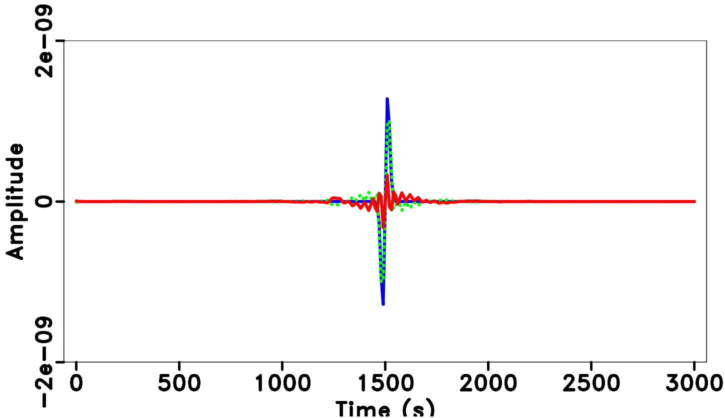
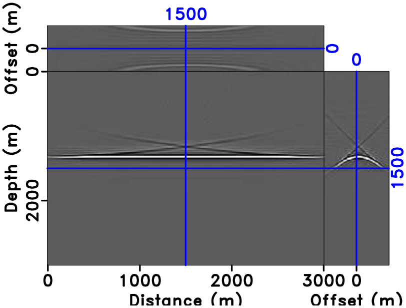
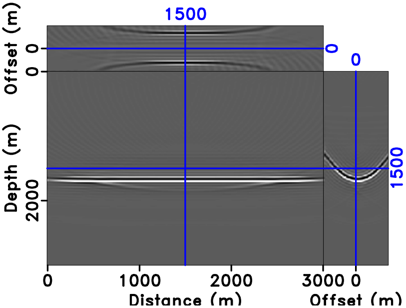


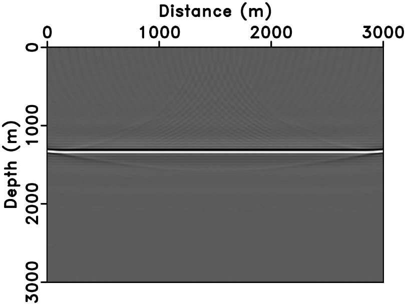
Figure: One trace (middle) comparison between the reflectivity model (blue) and non-extended inversion result (green). The differnce is shown as the red line.



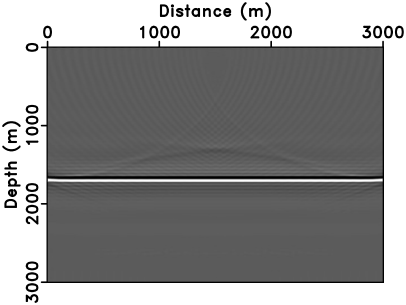
Background Velocity : $0.9v_0$



Background Velocity : $1.1v_0$

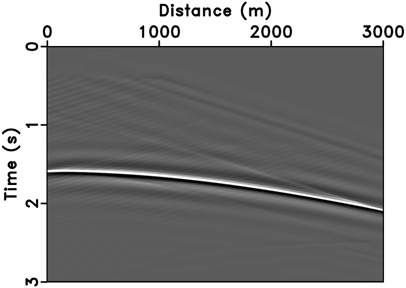


Background Velocity : $0.9v_0$

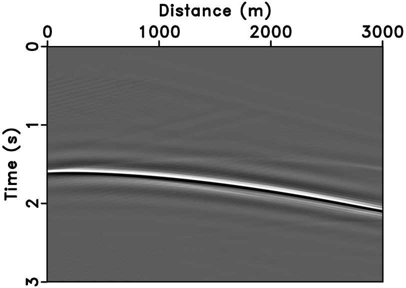


Background Velocity : $1.1v_0$

Resimulated Data for Extended Inversion



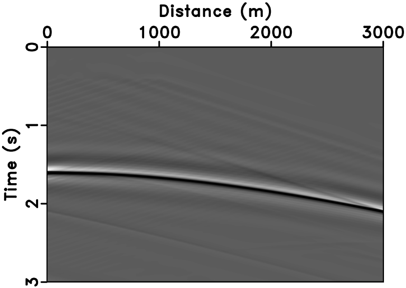
Background Velocity : $0.9v_0$



Background Velocity : $1.1v_0$

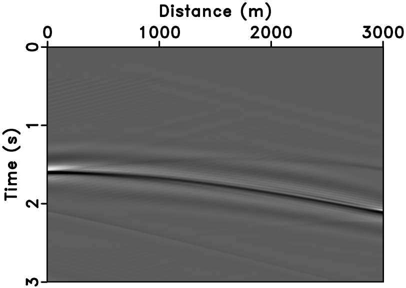
Data Residual for Extended Inversion

TRIP



Background Velocity : $0.9v_0$

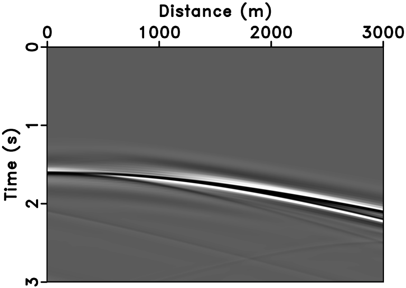
29.3%||original data||



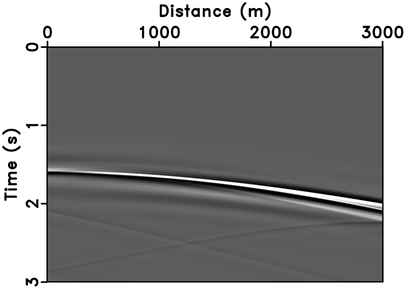
Background Velocity : $1.1v_0$

15.6%||original data||

Data Residual for Non-extended Inversion

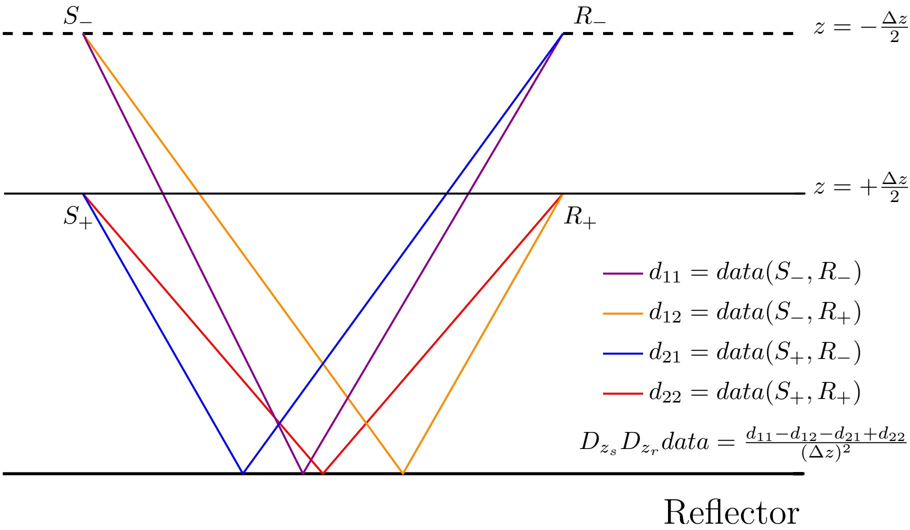


Background Velocity : $0.9v_0$
132.88%||original data||

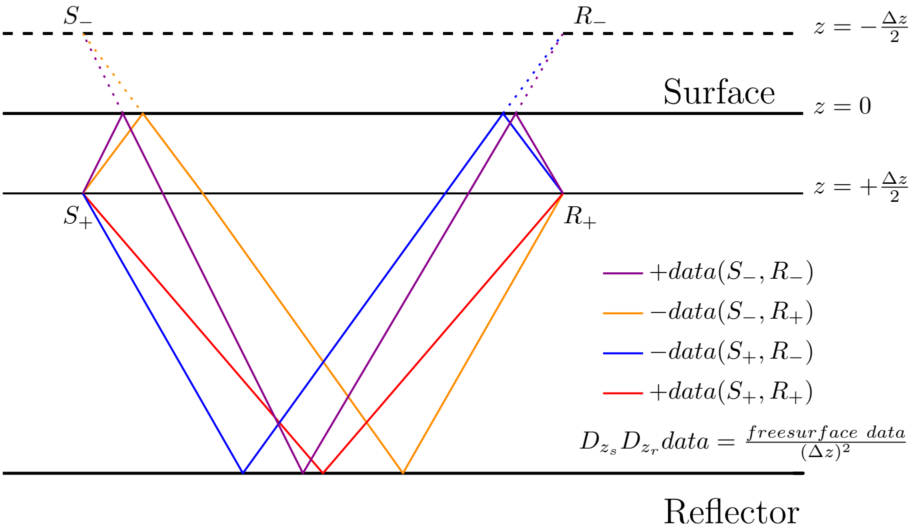


Background Velocity : $1.1v_0$
158.77%||original data||

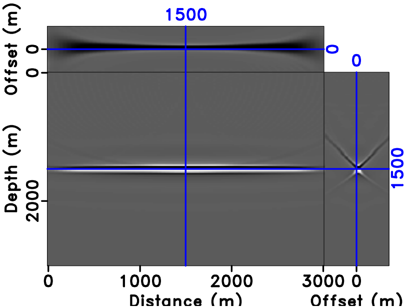
Apply $D_{z_s} D_{z_r}$-Naive Implementation



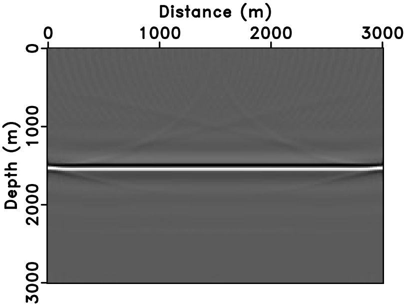
Apply $D_{z_s} D_{z_r}$-Free Surface Simulation



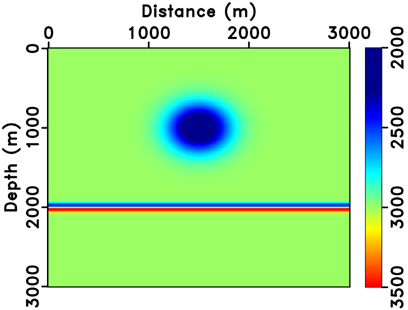
Inversion Result



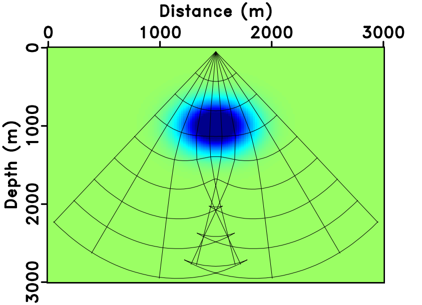
Extended Inversion



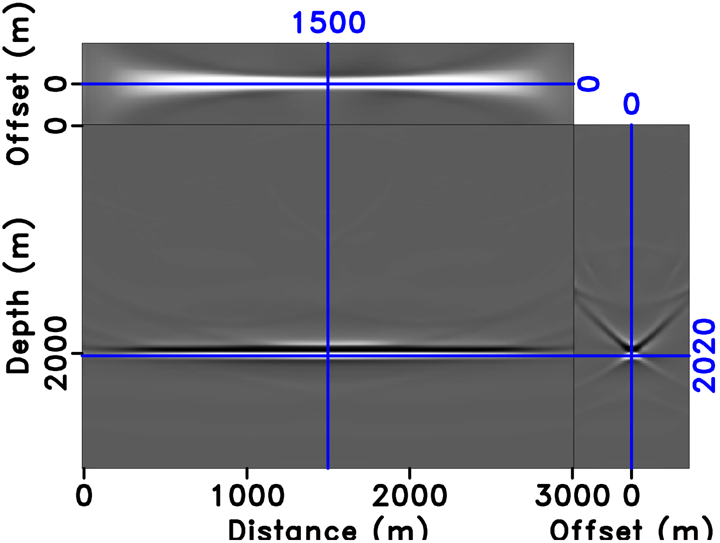
Non-extended Inversion



Velocity Model



Wavefronts and Rays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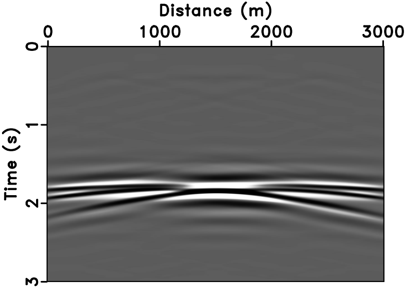
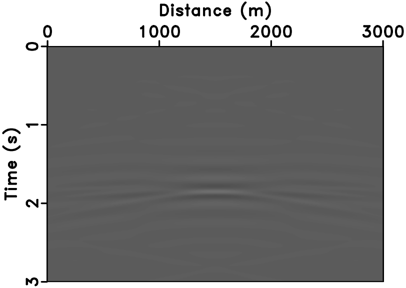


Resimulated Data



Data Difference
=10.4%||observed data||

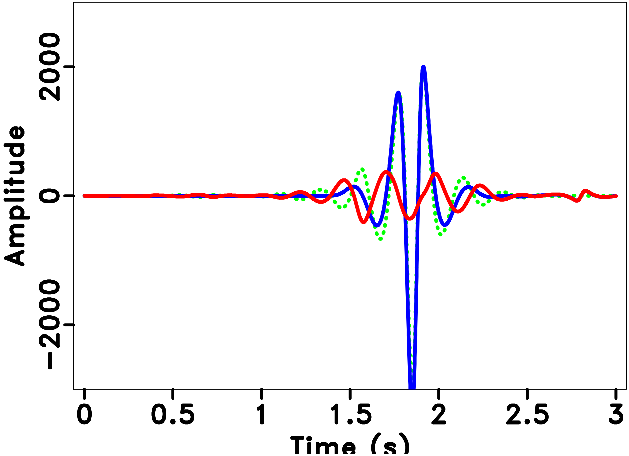
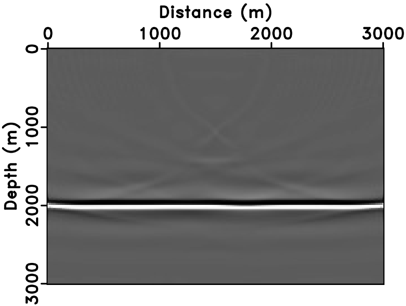
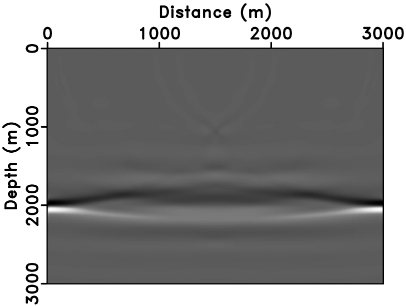


Figure: One trace (middle) comparison between the original data(blue) and resimulated data(green). The differnce is shown as the red line.

Non-extended Inversion



Non-extended Inversion Result



Model Difference

=21.3% ||model||

One trace Comparison

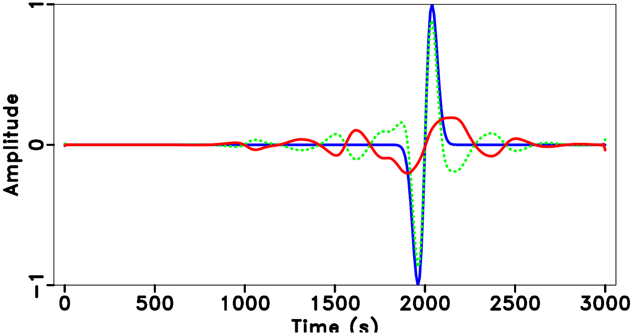
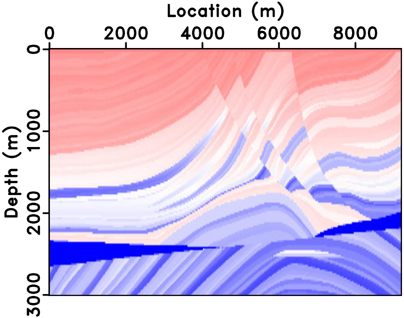
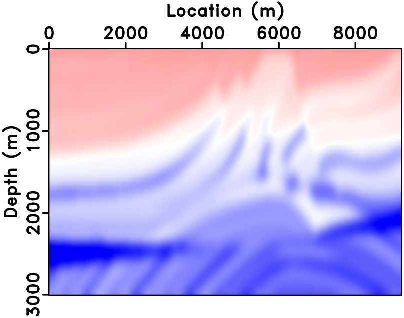


Figure: One trace (middle) comparison between the reflectivity model (blue) and non-extended inversion result (green). The differnce is shown as the red line.

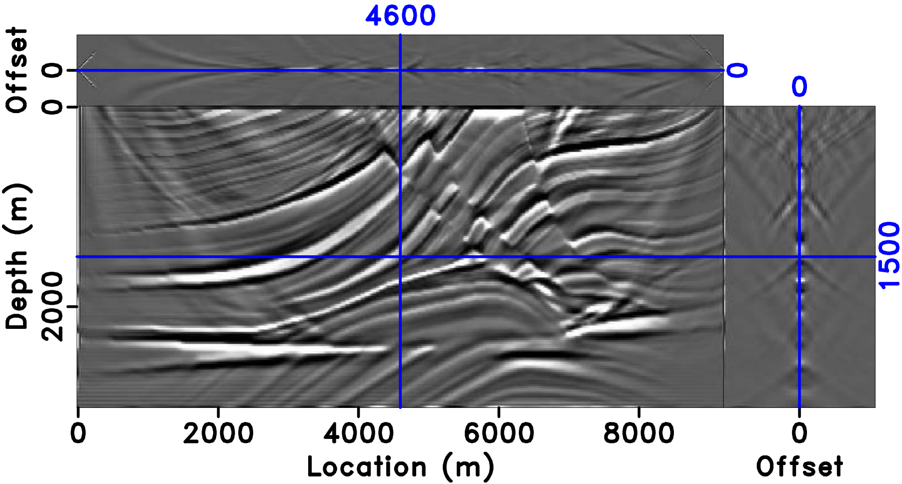


Marmousi Model

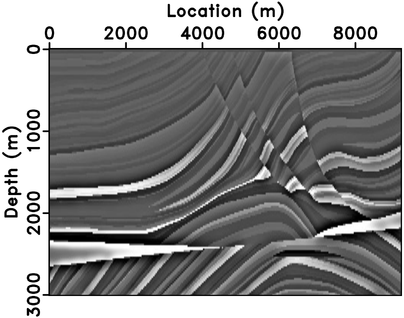


Background Velocity Model

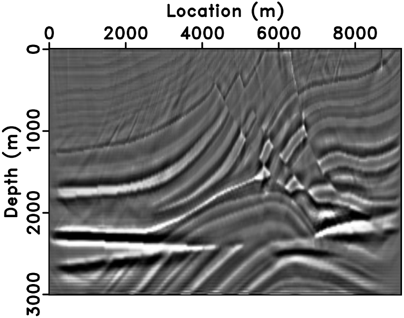
Extended Inversion Result



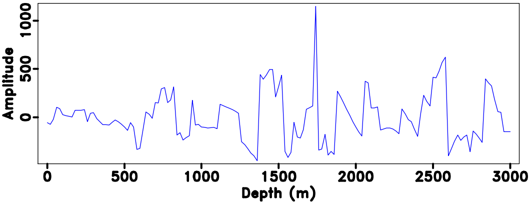
Marmousi Model



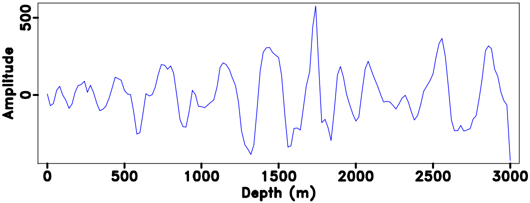
Reflectivity Model



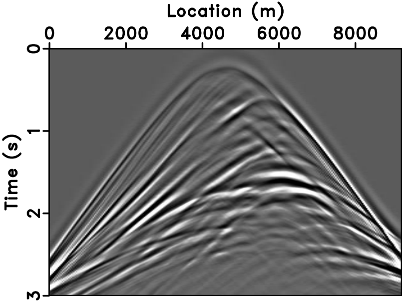
Non-extended Inversion Result



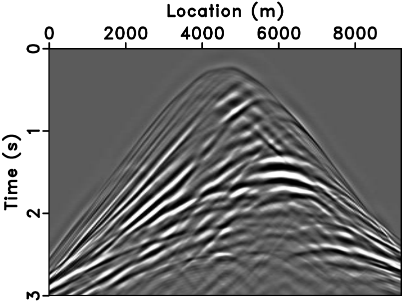
Reflectivity Model (middle trace)



Non-extended Inversion (middle trace)

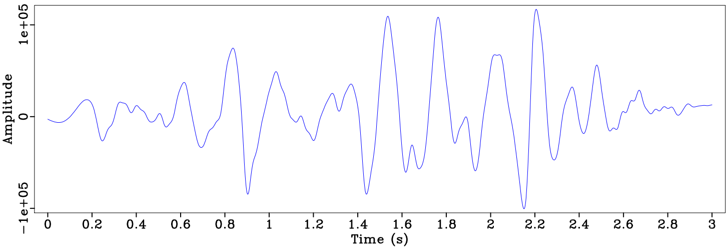


Original Data

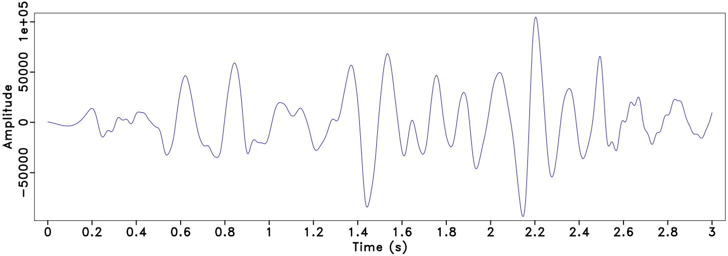


Resimulated Data

One Trace comparison



Original Data (middle trace)



Resimulated Data (middle trace)

Takeaway Messages

- Migration is a kinematic solution of the linearized inverse problem
- Subsurface offset extended RTM can be modified into an asymptotic inverse to the extended Born Modeling Operator
- The new inverse operator can approximate the least sqaure extended RTM solution
- The new inverse operator can also produce non-extended inversion, which can approximate least square RTM

Future Plans

- More Numerical Tests
- Replace D_{z_s}, D_{z_r} with respect to one-way operator
- Extension to 3D
- Apply this operator as a preconditioner to LSM and FWI

Acknowledgement

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