Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 394103, 13 pages
doi:10.1155/2008/394103

Research Article

An Approximate Solution for Boundary Value Problems in Structural Engineering and Fluid Mechanics

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Received 10 January 2008; Accepted 19 May 2008

Recommended by David Chelidze

Variational iteration method (VIM) is applied to solve linear and nonlinear boundary value problems with particular significance in structural engineering and fluid mechanics. These problems are used as mathematical models in viscoelastic and inelastic flows, deformation of beams, and plate deflection theory. Comparison is made between the exact solutions and the results of the variational iteration method (VIM). The results reveal that this method is very effective and simple, and that it yields the exact solutions. It was shown that this method can be used effectively for solving linear and nonlinear boundary value problems.

1. Introduction

This paper discusses the analytical approximate solution for fourth-order equations with nonlinear boundary conditions involving third-order derivatives. The general form of the equation for a fixed positive integer n, $n \geq 2$, is a differential equation of order $2n$:

$$y^{(2n)} + f(x, y) = 0 \tag{1.1}$$

subject to the boundary conditions

$$y^{(2j)}(a) = A_{2j}, \quad y^{(2j)}(b) = B_{2j}, \quad j = 0(1)n-1, \tag{1.2}$$

where $-\infty < a \leq x \leq b < \infty$, A_{2j}, B_{2j}, $j = 0(1)n-1$ are finite constants.

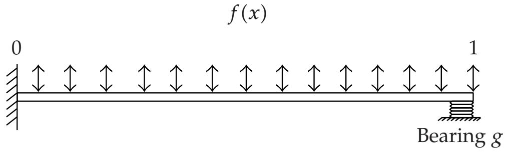

Figure 1: Beam on elastic bearing.

It is assumed that y is sufficiently differentiable and that a unique solution of (1.1) exists. Problems of this kind are commonly encountered in plate-deflection theory and in fluid mechanics for modeling viscoelastic and inelastic flows [1–3]. Usmani [1, 2] discussed sixth order methods for the linear differential equation $y^{(4)} + P(x)y = q(x)$ subject to the boundary conditions $y(a) = A_0$, $y''(A) = A_2$, $y(b) = B_0$, $y''(b) = B_2$. The method described in [1] leads to five diagonal linear systems and involves $p',\ p'',\ q',\ q''$ at a and b, while the method described in [2] leads to nine diagonal linear systems.

Ma and Silva [4] adopted iterative solutions for (1.1) representing beams on elastic foundations. Referring to the classical beam theory, they stated that if $u = u(x)$ denotes the configuration of the deformed beam, then the bending moment satisfies the relation $M = -EIu^{//}$, where E is the Young modulus of elasticity and I is the inertial moment. Considering the deformation caused by a load $f = f(x)$, they deduced, from a free-body diagram, that $f = -v^{/}$ and $v = M^{/} = -EIu^{///}$, where v denotes the shear force. For u representing an elastic beam of length $L = 1$, which is clamped at its left side $x = 0$, and resting on an elastic bearing at its right side x =1, and adding a load f along its length to cause deformations (Figure 1), Ma and Silva [4] arrived at the following boundary value problem assuming an $EI = 1$:

$$u^{(iv)}(x) = f(x, u(x)), \quad 0 < x < 1, \tag{1.3}$$

the boundary conditions were taken as

$$u(0) = u^{/}(0) = 0, \tag{1.4}$$

$$u^{//}(1) = 0, \qquad u^{///}(1) = g(u(1)), \tag{1.5}$$

where $f \in C([0,1] \times \mathbb{R})$ and $g \in C(\mathbb{R})$ are real functions. The physical interpretation of the boundary conditions is that $u^{///}(1)$ is the shear force at $x = 1$, and the second condition in (1.5) means that the vertical force is equal to $g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. Furthermore, since $u^{//}(1) = 0$ indicates that there is no bending moment at $x = 1$, the beam is resting on the bearing g.

Solving (1.3) by means of iterative procedures, Ma and Silva [4] obtained solutions and argued that the accuracy of results depends highly upon the integration method used in the iterative process.

With the rapid development of nonlinear science, many different methods were proposed to solve differential equations, including boundary value problems (BVPS). These two methods are the homotopy perturbation method (HPM) [5–7] and the variational iteration method (VIM) [8–17]. In this paper, it is aimed to apply the variational iteration method proposed by He [14] to different forms of (1.1) subject to boundary conditions of physical significance.

2. Basic idea of He's variational iteration method

To clarify the basic ideas of He's VIM, the following differential equation is considered:

$$L\left[u(t)\right] + N\left[u(t)\right] = g(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is an inhomogeneous term. According to VIM, a correction functional could be written as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau, \tag{2.2}$$

where λ is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$.

For fourth-order boundary value problem with suitable boundary conditions, Lagrangian multiplier can be identified by substituting the problem into (2.2), upon making it stationary leads to the following:

$$\begin{gathered}\frac{d^4}{d\tau^4}\lambda = 0,\\ -\lambda''' + 1|_{\tau=x} = 0,\\ \lambda''|_{\tau=x} = 0.\end{gathered} \tag{2.3}$$

Solving the system of (2.3) yields

$$\lambda = \frac{1}{6}(\tau - x)^3 \tag{2.4}$$

and the variational iteration formula is obtained in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3\left(u_n^{(4)}(\tau) + f\left(\tau, u_n, u_n', u_n'', u_n'''\right)\right)d\tau. \tag{2.5}$$

3. The applications of VIM method

In this section, the VIM is applied to different forms of the fourth-order boundary value problem introduced in through (1.1).

Example 3.1. Consider the following linear boundary value problem:

$$u^{(4)}(x) = 4e^x + u(x), \quad 0 < x < 1, \tag{3.1}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 2, \quad u(1) = 2e, \quad u'(1) = 3e. \tag{3.2}$$

The exact solution for this problem is

$$u(x) = (1+x)e^{x}. \tag{3.3}$$

According to (2.5), the following iteration formulation is achieved:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_n^{(4)}(\tau) - u_n(\tau) - 4e^{\tau}\right) d\tau. \tag{3.4}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \tag{3.5}$$

where $a,\ b,\ c$, and d are unknown constants to be further determined.

By the iteration formula (3.4), the following first-order approximation may be written:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_0^{(4)}(\tau) - u_0(\tau) - 4e^{\tau}\right) d\tau \\ &= ax^3 + bx^2 + cx + d + \int_0^x \frac{1}{6}(\tau - x)^3 \left(-a\tau^3 - b\tau^2 - c_{\tau} - d - 4e^{\tau}\right) d\tau \\ &= \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \frac{1}{120}cx^5 + \frac{1}{24}dx^4 + \left(-\frac{2}{3} + a\right)x^3 + (b-2)x^2 + (c-4)x + 4e^{x} + d - 4. \end{aligned} \tag{3.6}$$

Incorporating the boundary conditions (3.2), into $u_1(x)$, the following coefficients can be obtained:

$$a = -\frac{2289756}{301681} + \frac{916440}{301681}e, \qquad b = \frac{4575063}{301681} - \frac{1516680}{301681}e, \qquad c = 2, \qquad d = 1. \tag{3.7}$$

Therefore, the following first-order approximate solution is derived:

$$\begin{aligned} u_1(x) = {} & \left(-\frac{27259}{3016810} + \frac{1091}{301681}e\right)x^7 + \left(\frac{1525021}{36201720} - \frac{4213}{301681}e\right)x^6 \\ & + \frac{1}{60}x^5 + \frac{1}{24}x^4 + \left(-\frac{7472630}{905043} + \frac{916440}{301681}e\right)x^3 + \left(\frac{3971701}{301681} - \frac{1516680}{301681}e\right)x^2 - 2x - 3 + 4e^{x}. \end{aligned} \tag{3.8}$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 1, showing a remarkable agreement.

Similarly, the following second-order approximation is obtained:

$$\begin{aligned} u_2(x) &= u_1(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_1^{(4)}(\tau) - u_1(\tau) - 4e^{\tau}\right) d\tau \\ &= \frac{1}{6652800}ax^{11} + \frac{1}{1814400}bx^{10} + \frac{1}{362880}cx^9 + \frac{1}{40320}dx^8 + \left(\frac{1}{840}a - \frac{1}{1260}\right)x^7 \\ &\quad + \left(\frac{1}{360}b - \frac{1}{180}\right)x^6 + \left(-\frac{1}{30} + \frac{1}{120}c\right)x^5 + \left(-\frac{1}{6} + \frac{1}{24}d\right)x^4 \\ &\quad + \left(-\frac{4}{3} + a\right)x^3 + (b-4)x^2 + (c-8)x - 8 + 8e^{x} + d, \\ a &= -\frac{12706529114180}{681628862391} + \frac{85535681616000}{12042109902241}e, \quad c = 2, \\ b &= \frac{8416302814865}{227209620797} - \frac{157452726614400}{12042109902241}e, \quad d = 1. \end{aligned} \tag{3.9}$$

Table 1: Comparison of the first-order approximate solution with exact solution.

x	U_E	U_1	Error
0	1.000000000	1.000000000	$0.0000E+000$
0.1	1.215688010	1.215681524	$6.4860E-006$
0.2	1.465683310	1.465660890	$2.2420E-005$
0.3	1.754816450	1.754773923	$4.2527E-005$
0.4	2.088554577	2.088492979	$6.1598E-005$
0.5	2.473081906	2.473007265	$7.4641E-005$
0.6	2.915390080	2.915312734	$7.7346E-005$
0.7	3.423379602	3.423312592	$6.7010E-005$
0.8	4.005973670	4.005929404	$4.4266E-005$
0.9	4.673245911	4.673229891	$1.6020E-005$
1.0	$2e$	$2e$	$0.0000E+000$

Table 2: Comparison of the second-order approximate solution with exact solution.

x	U_E	U_2	Error
0	1.000000000	1.000000000	$0.0E+000$
0.1	1.215688010	1.215688008	$2.0E-009$
0.2	1.465683310	1.465683305	$5.0E-009$
0.3	1.754816450	1.754816444	$6.0E-009$
0.4	2.088554577	2.088554566	$1.1E-008$
0.5	2.473081906	2.473081902	$4.0E-009$
0.6	2.915390080	2.915390064	$1.6E-008$
0.7	3.423379602	3.423379600	$2.0E-009$
0.8	4.005973670	4.005973650	$2.0E-008$
0.9	4.673245911	4.673245930	$1.9E-008$
1.0	$2e$	$2e$	$0.0E+000$

Therefore, the second-order approximate solution may be written as

$$\begin{aligned}
u_2(x) = {} & \left(-\frac{57756950519}{20612456798703840}+\frac{12857095}{12042109902241}e\right)x^{11} \\
& +\left(\frac{1683260562973}{82449827194815360}-\frac{86779501}{12042109902241}e\right)x^{10}+\frac{1}{181440}x^{9} \\
& +\frac{1}{40320}x^{8}+\left(-\frac{731163797543}{31809346911580}+\frac{101828192400}{12042109902241}e\right)x^{7} \\
& +\left(\frac{7961883573271}{81795463486920}-\frac{437368685040}{12042109902241}e\right)x^{6}-\frac{1}{60}x^{5} \\
& -\frac{1}{8}x^{4}+\left(-\frac{13615367597368}{681628862391}+\frac{85535681616000}{12042109902241}e\right)x^{3} \\
& +\left(\frac{7507464331677}{227209620797}-\frac{157452726614400}{12042109902241}e\right)x^{2}-6x-7+8e^{x}.
\end{aligned} \tag{3.10}$$

Again, the obtained solution is of distinguishing accuracy, as indicated in Table 2 and Figure 2.

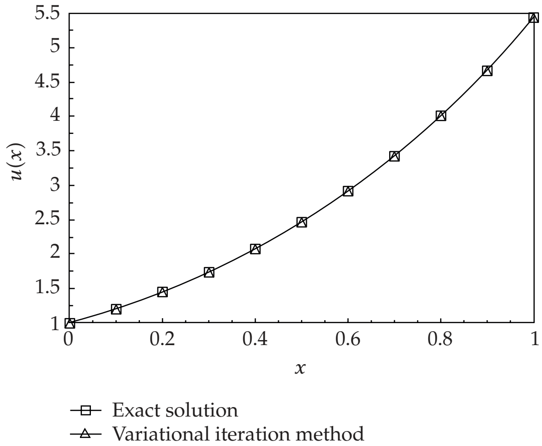

Figure 2: Comparison between different solutions.

Example 3.2. Consider the following linear boundary value problem:

$$u^{(4)}(x) = u(x) + u''(x) + e^{x}(x-3), \quad 0 < x < 1, \tag{3.11}$$

subject to the boundary conditions

$$u(0) = 1, \qquad u'(0) = 0, \qquad u(1) = 0, \qquad u'(1) = -e. \tag{3.12}$$

The exact solution for this problem is

$$u(x) = (1-x)e^{x}. \tag{3.13}$$

According to (2.5), the iteration formulation may be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3\Big(u_n^{(4)}(\tau) - u_n(\tau) - u_u''(\tau) - e^{\tau}(\tau-3)\Big)d\tau. \tag{3.14}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d. \tag{3.15}$$

Where $a,\ b,\ c$, and d are unknown constants to be further determined.

By the iteration formula (3.14), the following first-order approximation is developed:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau - x)^3\Big(u_0^{(4)}(\tau) - u_0(\tau) - u_0''(\tau) - e^{\tau}(\tau-3)\Big)d\tau \\ &= ax^3 + bx^2 + cx + d + \int_0^x \frac{1}{6}(\tau - x)^3\big(-a\tau^3 - b\tau^2 - (6a+c)\tau - 2b - d - e^{\tau}(\tau-3)\big)d\tau \\ &= \frac{1}{840}ax^7 + \frac{1}{360}bx^6 + \left(\frac{1}{20}a + \frac{1}{120}c\right)x^5 + \left(\frac{1}{12}b + \frac{1}{24}d\right)x^4 + \left(\frac{2}{3} + a\right)x^3 + \left(b + \frac{5}{2}\right)x^2 \\ &\quad + \Big(e^{x} + 6 + c\Big)x - 7e^{x} + 7 + d. \end{aligned} \tag{3.16}$$

Table 3: Comparison of the first-order approximate solution with exact solution.

x	U_E	U_1	Error
0	1.0000000000	1.0000000000	$0.0000000E+000$
0.1	0.9946538262	0.9947931547	$1.3932850E-004$
0.2	0.9771222064	0.9775949040	$4.7269760E-004$
0.3	0.9449011656	0.9457776230	$8.7645740E-004$
0.4	0.8950948188	0.8963297250	$1.2349062E-003$
0.5	0.8243606355	0.8258087440	$1.4481085E-003$
0.6	0.7288475200	0.7302919280	$1.4444080E-003$
0.7	0.6041258121	0.6053240800	$1.1982679E-003$
0.8	0.4451081856	0.4458625400	$7.5435440E-004$
0.9	0.2459603111	0.2462193000	$2.5898890E-004$
1.0	0.0000000000	0.0000000000	$0.0000000E+000$

Incorporating the boundary conditions (3.12), into $u_1(x)$, it can be written as

$$a=\frac{7904470}{323149}-\frac{2950080}{323149}e,\qquad b=-\frac{12770295}{323149}+\frac{4640400}{323149}e,\qquad c=0,\qquad d=1. \tag{3.17}$$

Therefore, the following first-order approximate solution is obtained:

$$\begin{aligned}u_1(x)=&\left(\frac{112921}{3877788}-\frac{3512}{323149}e\right)x^7+\left(-\frac{851353}{7755576}+\frac{12890}{323149}e\right)x^6\\&+\left(\frac{790447}{646298}-\frac{147504}{323149}e\right)x^5+\left(-\frac{25217441}{7755576}+\frac{386700}{323149}e\right)x^4\\&+\left(\frac{24359708}{969447}-\frac{2950080}{323149}e\right)x^3+\left(-\frac{23924845}{646298}+\frac{4640400}{323149}e\right)x^2\\&+\left(6+e^x\right)x+8-7e^x.\end{aligned} \tag{3.18}$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 3, again showing a clear agreement. Even higher accurate solutions could be obtained without any difficulty.

Similarly, the following second-order approximation can be written as

$$\begin{aligned}u_2(x)&=u_1(x)+\int_0^x\frac{1}{6}(\tau-x)^3\left(u_1^{(4)}(\tau)-u_1(\tau)-u_1''(\tau)-e^{\tau}(\tau-3)\right)d\tau\\&=\frac{1}{6652800}ax^{11}+\frac{1}{1814400}bx^{10}+\left(\frac{1}{362880}c+\frac{1}{30240}a\right)x^9+\left(\frac{1}{10080}b+\frac{1}{40320}d\right)x^8\\&\quad+\left(\frac{1}{5040}c+\frac{1}{420}a+\frac{1}{1260}\right)x^7+\left(\frac{1}{720}d+\frac{1}{144}+\frac{1}{180}b\right)x^6+\left(\frac{1}{12}+\frac{1}{20}a+\frac{1}{120}c\right)x^5\\&\quad+\left(\frac{1}{2}+\frac{1}{24}d+\frac{1}{12}b\right)x^4+(3+a)x^3+\left(b+\frac{21}{2}\right)x^2+\left(24+3e^x+c\right)x+27-27e^x+d.\end{aligned} \tag{3.19}$$

Incorporating the boundary conditions, (3.12), into $u_2(x)$, yields

$$a = \frac{381804789300110}{4289712004667} - \frac{140985028800000}{4289712004667}e, \qquad c = 0,$$
$$b = -\frac{629495301082065}{4289712004667} + \frac{230790037363200}{4289712004667}e, \qquad d = 1. \tag{3.20}$$

The following second-order approximate solution is then achieved in the following form:

$$\begin{aligned}
u_2(x) = {} & \left(\frac{3470952630001}{259441782042260160} - \frac{63575500}{12869136014001}e\right)x^{11} \\
& + \left(-\frac{41966353405471}{518883564084520320} + \frac{381597284}{12869136014001}e\right)x^{10} \\
& + \left(\frac{38180478930011}{12972089102113008} - \frac{13986610000}{12869136014001}e\right)x^{9} \\
& + \left(-\frac{2513691492323593}{172961188028173440} + \frac{22895837040}{4289712004667}e\right)x^{8} \\
& + \left(\frac{1149704079904997}{5405037125880420} - \frac{335678640000}{4289712004667}e\right)x^{7} \\
& + \left(-\frac{415373822050043}{514765440560040} + \frac{1282166874240}{4289712004667}e\right)x^{6} \\
& + \left(\frac{233372585584733}{51476544056004} - \frac{7049251440000}{4289712004667}e\right)x^{5} \\
& + \left(-\frac{1203224346103459}{102953088112008} + \frac{19232503113600}{4289712004667}e\right)x^{4} \\
& + \left(\frac{394673925314111}{4289712004667} - \frac{140985028800000}{4289712004667}e\right)x^{3} \\
& + \left(-\frac{1168906650066123}{8579424009334} + \frac{230790037363200}{4289712004667}e\right)x^{2} \\
& + \left(3e^{x} + 24\right)x + 28 - 27e^{x}.
\end{aligned} \tag{3.21}$$

The obtained solution is of evident accuracy, as shown in Table 4 and Figure 3.

Example 3.3. Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x), \quad 0 < x < 1, \tag{3.22}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = 1, \qquad u'(1) = 1, \tag{3.23}$$

where

$$g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48. \tag{3.24}$$

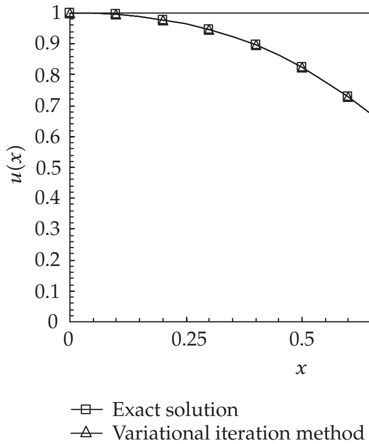

Figure 3: Comparison between different solutions.

Table 4: Comparison of the second-order approximate solution with exact solution.

x	U_E	U_2	Error
0	1.0000000000	1.0000000000	0.00000E + 000
0.1	0.9946538262	0.9946577580	3.93180E − 006
0.2	0.9771222064	0.9771357780	1.35716E − 005
0.3	0.9449011656	0.9449268900	2.57244E − 005
0.4	0.8950948188	0.8951321100	3.72912E − 005
0.5	0.8243606355	0.8244058800	4.52445E − 005
0.6	0.7288475200	0.7288945300	4.70100E − 005
0.7	0.6041258121	0.6041666500	4.08379E − 005
0.8	0.4451081856	0.4451352800	2.70944E − 005
0.9	0.2459603111	0.2459701300	9.81890E − 006
1.0	0.0000000000	0.0000000000	0.00000E + 000

The exact solution for this problem is

$$u(x) = x^5 - 2x^4 + 2x^2. \tag{3.25}$$

According to (2.5), the iteration formulation is written as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_n^{(4)}(\tau) - u_n^2(\tau) - g(\tau)\right) d\tau. \tag{3.26}$$

Now it is assumed that an initial approximation has the form

$$u_0(x) = ax^3 + bx^2 + cx + d, \tag{3.27}$$

where $a,\ b,\ c,$ and d are unknown constants to be further determined.

Table 5: Comparison of the first-order approximate solution with exact solution.

x	U_E	U_1	Error
0	0.0000000000	0.0000000000	$0.0000000E+000$
0.1	0.0198100000	0.0198624243	$5.2424300E-005$
0.2	0.0771200000	0.0773022107	$1.8221070E-004$
0.3	0.1662300000	0.1665781379	$3.4813790E-004$
0.4	0.2790400000	0.2795490972	$5.0909720E-004$
0.5	0.4062500000	0.4068747265	$6.2472650E-004$
0.6	0.5385600000	0.5392178270	$6.5782700E-004$
0.7	0.6678700000	0.6684511385	$5.8113850E-004$
0.8	0.7884800000	0.7888727023	$3.9270230E-004$
0.9	0.8982900000	0.8984356964	$1.4569640E-004$
1.0	1.0000000000	1.0000000000	$0.0000000E+000$

By the iteration formula (3.26), the following first-order approximation is obtained:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_0^{(4)}(\tau) - u_0^2(\tau) + \tau^{10} - 4\tau^9 + 4\tau^8 + 4\tau^7 - 8\tau^6 + 4\tau^4 - 120\tau + 48\right) d\tau \\ &= -\frac{1}{24024}x^{14} + \frac{1}{4290}x^{13} - \frac{1}{2970}x^{12} - \frac{1}{1980}x^{11} + \left(\frac{1}{5040}a^2 + \frac{1}{630}\right)x^{10} \\ &\quad + \frac{1}{1512}abx^9 + \left(-\frac{1}{420} + \frac{1}{1680}b^2 + \frac{1}{840}ac\right)x^8 + \left(\frac{1}{420}bc + \frac{1}{420}ad\right)x^7 \\ &\quad + \left(\frac{1}{180}bd + \frac{1}{360}c^2\right)x^6 + \left(1 + \frac{1}{60}cd\right)x^5 + \left(\frac{1}{24}d^2 - 2\right)x^4 + ax^3 + bx^2 + cx + d. \end{aligned} \tag{3.28}$$

Incorporating the boundary conditions (3.23), into $u_1(x)$, results in the following values:

$$a = -0.006871650809; \qquad b = 2.005929593; \qquad c = 0, \qquad d = 0. \tag{3.29}$$

The following first-order approximate solution is then achieved:

$$\begin{aligned} u_1(x) = &-4.162504162 \times 10^{-5}x^{14} + 2.331002331 \times 10^{-4}x^{13} \\ &- 3.367003367 \times 10^{-4}x^{12} - 5.050505050 \times 10^{-4}x^{11} \\ &+ 1.587310956 \times 10^{-3}x^{10} - 9.116433669 \times 10^{-6}x^9 \\ &+1.4139007 \times 10^{-5}x^8 + x^5 - 2x^4 - 6.871650809 \times 10^{-3}x^3 \\ &+2.005929593x^2. \end{aligned} \tag{3.30}$$

Comparison of the first-order approximate solution with exact solution is tabulated in Table 5, which once again shows an excellent agreement.

Similarly, the following second-order approximation may be written:

$$u_2(x) = u_1(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left(u_1^{(4)}(\tau) - u_1^2(\tau) + \tau^{10} - 4\tau^9 + 4\tau^8 + 4\tau^7 - 8\tau^6 + 4\tau^4 - 120\tau + 48\right) d\tau. \tag{3.31}$$

Table 6: Comparison of the second-order approximate solution with exact solution.

x	U_E	U_2	Error
0	0.0000000000	0.0000000000	0.000E + 000
0.1	0.0198100000	0.0198100068	6.800E − 009
0.2	0.0771200000	0.0771200239	2.390E − 008
0.3	0.1662300000	0.1662300464	4.640E − 008
0.4	0.2790400000	0.2790400692	6.920E − 008
0.5	0.4062500000	0.4062500874	8.740E − 008
0.6	0.5385600000	0.5385600961	9.610E − 008
0.7	0.6678700000	0.6678700906	9.060E − 008
0.8	0.7884800000	0.7884800670	6.700E − 008
0.9	0.8982900000	0.8982900292	2.920E − 008
1.0	1.0000000000	1.0000000012	1.200E − 009

Incorporating the boundary conditions, (3.23), into $u_2(x)$, yields

$$a = -8.269548014E-7; \qquad b = 2.000000763; \qquad c = 0, \qquad d = 0. \tag{3.32}$$

The following second-order approximate solution is obtained:

$$\begin{aligned}
u_2(x) = {} & -1.093855974 \times 10^{-9} x^9 + 1.817 \times 10^{-9} x^8 - 2x^4 - 1.117934793 \times 10^{-8} x^{21} \\
& + 1.463705892 \times 10^{-9} x^{20} + 6.586694874 \times 10^{-8} x^{19} + 2.000000763 x^2 \\
& - 8.269548014 \times 10^{-7} x^3 - 1.047931585 \times 10^{-7} x^{18} - 3.536760165 \times 10^{-8} x^{17} \\
& + 1.453571773 \times 10^{-7} x^{16} - 5.173598972 \times 10^{-13} x^{28} - 2.569735395 \times 10^{-14} x^{31} \\
& + 1.252296566 \times 10^{-13} x^{30} - 2.016131906 \times 10^{-13} x^{29} + 2.007605778 \times 10^{-15} x^{32} \\
& + 2.564345160 \times 10^{-12} x^{27} + 3.603899741 \times 10^{-9} x^{22} + 3.025 \times 10^{-13} x^{14} \\
& - 1.392179800 \times 10^{-10} x^{12} + 6.103539401 \times 10^{-10} x^{11} + x^5 + 9.879565106 \times 10^{-12} x^{24} \\
& - 2.268156651 \times 10^{-12} x^{26} - 5.281071651 \times 10^{-12} x^{25} - 3.917282540 \times 10^{-10} x^{23} \\
& - 1.335600908 \times 10^{-13} x^{15} - 6.059998643 \times 10^{-10} x^{10}.
\end{aligned} \tag{3.33}$$

The obtained solution is once again of remarkable accuracy, as shown in Table 6 and Figure 4.

4. Conclusion

This study showed that the variational iteration method is remarkably effective for solving boundary value problems. A fourth-order differential equation with particular engineering applications was solved using the VIM in order to prove its effectiveness. Different forms of the equation having boundary conditions of physical significance were considered. Comparison between the approximate and exact solutions showed that one iteration is enough to reach the exact solution. Therefore the VIM is able to solve partial differential equations using a minimum calculation process. This method is a very promoting method, which promises to find wide applications in engineering problems.

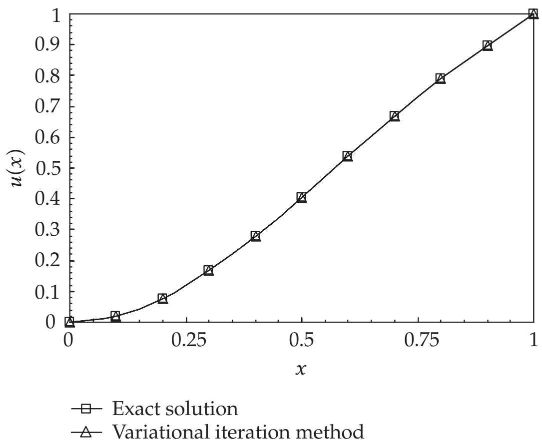

Figure 4: Comparison between different solutions.

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