NASA TECHNICAL MEMORANDUM

NASA TM X-62,299
(Revised 8-28-78)

NASA TM X-62,299

(NASA-TM-X-62299) AN APPROXIMATE SOLUTION FOR THE FREE VIBRATIONS OF ROTATING UNIFORM CANTILEVER BEAMS (NASA) 43 p HC A03/MF A01 CSCL 20K

N78-33289

Unclas
G3/32 35830

AN APPROXIMATE SOLUTION FOR THE FREE VIBRATIONS OF ROTATING UNIFORM CANTILEVER BEAMS

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September 1973

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LIST OF SYMBOLS

a_{ij}	modal vectors
A_{ij}	Galerkin integrals
Ai,Bi,Gi	Airy's functions, related Airy's function
a,b,c,d,f	constants
[D]	Galerkin matrix
EI	bending stiffness $N\text{-}m^2$
GJ	torsion stiffness $N\text{-}m^2$
[I]	identity matrix
i,j	indices
k_A, k_M	polar radius of gyration for area, mass m
m	mass per unit length, kg/m
N	number of orthogonal functions
n	mode number
O( )	order of terms
P_ν, Q_ν	Legendre functions
r	distance along beam divided by R
$\bar{r}, \tilde{r}$	inner variables $r/\eta^{1/2}$, $(1 - r)/\eta^{1/3}$
R	beam length, m
$u, \bar{u}, \tilde{u}$	flap displacement divided by R
v	lead-lag displacement divided by R
x	independent variable $= r/\sqrt{1 + 2\gamma}$
α_n, β_n	modal parameters
γ	torsion stiffness parameter, $GJ/m\Omega^2R^2k_M^2$
ε	small parameter

ζ	dummy variable
η	bending stiffness parameter, $EI/m\Omega^2R^4$
θ	torsional deflection, rad
Λ	eigenvalues
ν	Legendre parameter
ϕ_n	nonrotating modes
χ_i	right hand side of outer expansion
Ω	rotational speed, rad/sec
ω	natural frequency divided by Ω
ω_{NR}	nonrotating natural frequency divided by Ω
$(\ )_L$	large η approximation
$(\ )_S$	small η approximation
$(\ )'$	$d(\ )/dr$

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SUMMARY

Approximate solutions are obtained for the uncoupled frequencies and modes of rotating uniform cantilever beams. The frequency approximations for flap bending, lead-lag bending, and torsion are simple expressions having errors of less than a few percent over the entire frequency range. These expressions provide a simple way of determining the relations between mass and stiffness parameters and the resultant frequencies and mode shapes of rotating uniform beams.

INTRODUCTION

Many theoretical analyses of hingeless rotors are performed for the ideal case of uniform untwisted rotor blades. These analyses need to be performed over a wide range of blade stiffness parameters requiring calculation of a broad spectrum of rotating blade natural modes and frequencies. In such theoretical investigations, the investigator often needs to translate some desired set of natural frequencies into blade properties and natural mode shapes which correspond to those frequencies. Then the results can be compared with other theories as well as experiments. There is, therefore, a practical requirement for a simple and accurate means of relating the frequencies and modes to the mass and stiffness characteristics of rotating uniform beams.

The usual methods for calculating frequencies and modes of rotating beams, however, are not well suited to the particular computations required in this type of analysis. First, traditional energy and finite element methods yield poor convergence in the practical range of flapping frequencies (and in the lower extremes of lead-lag and torsion frequencies). Thus, a moderate number of assumed modes (or finite elements) must be used. As pointed out in Reference 1, this poor convergence necessitates an involved numerical solution for the frequencies and modes, which must be repeated for each separate blade configuration. Second, because the solutions are numerical, a trial and error process is required to determine the blade properties which will result in a desired set of natural frequencies.

The purpose of this report is to provide simplified expressions for the frequencies and modes of rotating uniform beams. It will be shown that these expresisons are accurate through the entire blade stiffness range yet elementary enough to overcome the disadvantages of purely numerical solutions.

Equations of Motion

The uncoupled equations ("uncoupled" implies: zero twist, coincident mass center and elastic axis, beam parallel to plane of rotation, and warp included only in calculation of J) for the free vibrations of rotating uniform beams are[2]

$$\eta u'''' - \frac{1}{2}[(1 - r^2)u']' - \omega^2 u = 0 \qquad \text{(flapping)} \quad (1a)$$

$$\eta v'''' - \frac{1}{2}[(1 - r^2)v']' - (\omega^2 + 1)v = 0 \qquad \text{(lead-lag)} \quad (1b)$$

$$-\gamma\theta'' - \frac{1}{2}[(1 - r^2)\theta']' - (\omega^2 - 1)\theta = 0 \qquad \text{(torsion)} \quad (1c)$$

where u and v are the flapping and lead-lag displacements nondimensionalized on the blade radius R, r is the distance along the beam divided by R, ω is the natural frequency of the vibration divided by the rotational frequency Ω, and η and γ are stiffness parameters.

$$\eta = \frac{EI}{m\Omega^2 R^4} \quad , \quad \gamma = \frac{GJ}{m\Omega^2 R^2 k_M^2}$$

The first term in each line of Eq. (1) represents the elastic stiffness of the blade in bending or torsion. The ω^2 term in each equation represents the inertial force due to the harmonic motion. These terms dominate for large values of η and γ. The remaining terms are due to the centrifugal tension, and they dominate (with ω^2) for small values of η and γ. The boundary conditions corresponding to a cantilever beam are

$$u(0) = u'(0) = u''(1) = u'''(1) = 0 \qquad (2a)$$

$$v(0) = v'(0) = v''(1) = v'''(1) = 0 \qquad (2b)$$

$$\theta(0) = \theta'(1) = 0 \qquad (2c)$$

The r = 0 conditions are geometric boundary conditions which specify that the displacement (and slope in bending) must vanish at the root. The r = 1 conditions are natural boundary conditions which specify that the shears and moments vanish at the tip. Eq. (1) and Eq. (2) together form the uncoupled rotating beam equations.

Solution by Orthogonal Expansion

Eq. (1) is often solved using an expansion in orthogonal functions which satisfy the geometric boundary conditions. With application of an energy method (e.g. Galerkin or Rayleigh-Ritz), such expansions transform the characteristic differential equation into an algebraic eigenvalue determinant.[3] When the known, exact nonrotating mode shapes ϕ_n are chosen as the orthogonal functions, this determinant takes on the particularly simple form

$$\left| \left[A_{ij}\right] + \left[\diagdown \left(\omega^2_{NR}\right)_n \diagdown\right] - \Lambda\left[I\right] \right| \equiv \left| \left[D\right] - \Lambda\left[I\right] \right| = 0 \qquad (3)$$

where $\left(\omega^2{}_{NR}\right)_n$ are the nonrotating frequencies and

$$A_{ij} = \frac{1}{2}\int_0^1 \phi_i' \phi_j' (1 - r^2) dr, \Lambda = \begin{cases} \omega^2 & \text{(flapping)} \\ \omega^2 + 1 & \text{(lead-lag)} \\ \omega^2 - 1 & \text{(torsion)} \end{cases} \qquad (4)$$

In practice, [D] is trancated at a specified number of rows and columns. Its eigenvalues Λ_n and eigenvectors a_{ij} then give approximations for the rotating frequencies ω_n^2, Eq. (4), and for the rotating mode shapes.

$$\left.\begin{matrix} u_n \\ v_n \\ \theta_n \end{matrix}\right\} = \sum_{j=1}^{N} a_{jn} \phi_j$$

In Table I, formulas are given for the nonrotating modes ϕ_n, nonrotating frequencies ω_{NR}, and the Galerkin integrals A_{ij}. The formulas for nonrotating frequencies are specifically

$$\omega^2{}_{NR} = \beta_n^4 \eta \qquad \text{(flap, lead-lag)}$$

$$\omega^2_{NR} = \frac{\pi^2 (2n - 1)^2}{4} \gamma \qquad \text{(torsion)}$$

Whenever convenient, therefore, the stiffness parameters η and γ may be expressed solely in terms of nonrotating frequencies and universal constants.

Numerical Limitations

In Fig. 1, the value of the lowest frequency of the eigenvalue determinant is presented for various modal truncations of the flapping equation.[1] As ω_{NR} decreases, the number of orthogonal functions N must be increased in order to retain a specified accuracy. In fact, when $\omega_{NR} = 0$, no value of N will result in the known exact value of the first frequency, $\omega = 1.0$. This failure at $\omega_{NR} = 0$ can be explained in mathematical terms. When $\eta = 0$, Eq. (1a) is reduced from a fourth order equation to a singular second order equation, relaxing all but one of the boundary conditions of Eq. (2a). Thus, the orthogonal functions chosen to satisfy the boundary conditions of the fourth order equation are not applicable to the second order case. From a physical standpoint, the $\eta = 0$ modes (which have nonzero slopes at $r = 0$) cannot be formed from a linear combination of nonrotating cantilever beam modes (which have identically zero slope at the root).

Although this failure of the eigenvalue determinant at $\omega_{NR} = 0$ does not strictly hold in the range of practical flap frequencies ($0.05 < \omega_{NR} < 0.50$), such values of ω_{NR} are close enough to zero to create convergence difficulties in the eigenvalue determinant. Fig. 1 reveals that ten or more orthogonal functions may be required for satisfactory results. Thus, involved numerical calculations are required in order to find ω. Furthermore, the number of orthogonal functions required in this procedure is also an unknown which must be determined by iteration. These same drawbacks also apply to the lead-lag and torsion equations (although to a much lesser degree). It is the purpose of this paper to provide simple expressions for ω, valid at all values of ω_{NR}, so that the above drawbacks can be overcome. The flapping equation will be considered first, with the lead-lag solution following directly by replacing ω^2_{FLAP} with $1 + \omega^2_{LEAD\text{-}LAG}$ (see Eq. (1)); and then the torsion equation will be considered.

SMALL STIFFNESS EXPANSION FOR BENDING

The first step toward developing simple frequency expressions is the determination of the asymptotic behavior of ω^2(flapping) as ω_{NR} (i.e. η) approaches zero. A similar phenomenon, the asymptotic behavior of a statically loaded beam under tension as EI approaches zero, is a classic problem in the theory of matched asymptotic expansions.[4] That solution indicates that as EI approaches zero the beam takes on the shape of a loaded string ($EI = 0$, nonzero slope near the supports) except in a small region adjacent to the supports where the boundary condition (zero slope) is fulfilled. Since the rotating beam flapping equation is similar in character to this classic example, a matched asymptotic expansion promises to provide the desired small η solution.

Outer Expansion

General solution. A dimensional analysis of Eq. (1) indicates that the basic length dimension (divided by R) of the flapping equation is $\eta^{1/2}$. Thus, an expansion in powers of $\eta^{1/2}$ is a logical starting point for the analysis. Expanding u and ω^2 in $\eta^{1/2}$,

$$u = u_o + \eta^{1/2}u_1 + \eta u_2 + \eta^{3/2}u_3 + \ldots \quad (5a)$$

$$\omega^2 = \omega_o^2 + \eta^{1/2}\omega_1^2 + \eta\omega_2^2 + \eta^{3/2}\omega_3^2 + \ldots \quad (5b)$$

substituting into Eq. (1a), and collecting like powers of η yields the following equations for the flapping behavior away from the boundaries.

$$[(1 - r^2)u_o']' + 2\omega_o^2 u_o = 0 \equiv \chi_o$$

$$[(1 - r^2)u_1']' + 2\omega_o^2 u_1 = -2\omega_1^2 u_o \equiv \chi_1 \tag{6}$$

$$[(1 - r^2)u_i']' + 2\omega_o^2 u_i = 2u_{i-2}'''' - 2\sum_{j=1}^{i} \omega_j^2 u_{i-j} \equiv \chi_i \quad i = 2,3,\ldots$$

Eq. (6) is simply the well-known Legendre equation with a nonzero right-hand side. Its solution can be expressed as integrals of the Legendre functions, P_ν[5].

$$u_i = C_1 P_\nu + P_\nu \int^r \frac{C_2 + \int_o^r P_\nu \chi_i d\zeta}{P_\nu^2(1 - r^2)} dr \tag{7}$$

$$2\omega_o^2 = \nu(\nu + 1) , \quad \nu > 0$$

The constants C_1, C_2, and ν of Eq. (7) must be determined from the boundary conditions of the problem. The condition u_i finite on the interval $0 < r < 1$ requires: (1) that ν be an integer (in which case the P_ν reduce to Legendre polynomials), and (2) that

$$C_2 = - \int_o^1 P_\nu \chi_i d\zeta$$

(so that the indefinite integral remains bounded). The boundary condition $u_o(0) = 0$ requires that ν assume only odd values, because the even Legendre polynomials are nonzero at $r = 0$. Finally, the constant C_1 must be chosen so that the order 1 solution is not repeated in the higher order solutions, thus avoiding secular terms. Therefore, $C_1 = 1$ for $i = 0$ and $C_1 = 0$ for $i \neq 0$. The complete solutions for all orders of the outer expansion are correspondingly

$$\omega_o^2 = n(2n - 1) \quad , \quad \omega_i^2 \text{ as yet undetermined}$$

$$u_o = P_{2n-1} \quad , \quad u_i = -P_{2n-1} \int^r \frac{\int_r^1 P_{2n-1}\chi_i \, d\zeta}{P_{2n-1}^2 (1 - r^2)} dr \tag{8}$$

$$n = 1,2,3,. . . \quad i = 1,2,3,. . .$$

where n is the number of the desired mode, and the ω_i^2 are the yet undetermined frequency expansion coefficients.

Specific expressions. The first order approximation for the natural frequencies and modes of a low stiffness beam are given from Eq. (8) as

mode	ω_o^2	u_o
1	1	r
2	6	$\frac{1}{2}(5r^3 - 3r)$
n	n(2n - 1)	P_{2n-1}

The higher order corrections to $u(u_1, u_2, . . .)$ come from repeated applications of Eq. (8) using the properties of Legendre polynomials given in Table II. For example, for the first mode,

$$u_1 = \frac{2}{3}\omega_1^2 \, [- 1 + r\ell n(1 + r)] \tag{9a}$$

$$u_2 = \frac{2}{3}\omega_2^2 \, [- 1 + r\ell n(1 + r)] + \frac{4}{3}\omega_1^4 \left\{ -\frac{r}{3}\int_o^r \frac{\ell n(1 + \zeta)}{1 - \zeta} d\zeta + \frac{1}{3}\ell n(1 + r) \right.$$

$$\left. - \frac{1}{9} r\ell n(1 + r) + \frac{7}{9} - \frac{1}{3}\ell n(2)[2 + r\ell n(1 + r) - r\ell n(1 - r)] \right\} \tag{9b}$$

and for the second mode

$$u_1 = \omega_1^2 \left[\frac{4}{21} - \frac{5}{7}r^2 + \frac{6}{35}r^3 - \frac{3}{7}(r - \frac{5}{3}r^3) \, \ell n(1 + r)\right] \tag{9c}$$

In general, the higher order correction terms are combinations of polynomials and logarithms with ω_i^2 as unknown coefficients; but they do not satisfy the boundary conditions.

In order to define the behavior of ω^2 (and consequently of u), these outer solutions must be matched with inner solutions which themselves satisfy the boundary conditions. This matching process will determine the behavior of u near r = 0 and r = 1, and it will also uniquely determine the unknown frequency terms ω_i^2. It should be noted that the nomenclature "outer" and "inner" is mathematical rather than physical. The "outer" solution is for the beam region away from the root and tip. The root and tip regions each have a distinctive "inner" solution.

Inner Expansion

Expansion near root. The yet unfulfilled boundary conditions at the beam root are u(0) = 0, u'(0) = 0. From an ordering analysis, the region in which these conditions are dominant must be of order $\eta^{1/2}$. To determine the behavior of the solution in this inner region (designated as $\bar{u}$ to distinguish it from the outer solution), Eq. (1a) must be rewritten in terms of an inner variable $\bar{r} = r/\eta^{1/2}$.

$$- 2 \frac{d^4\bar{u}}{d\bar{r}^4} + (1 - \eta\bar{r}^2) \frac{d^2\bar{u}}{d\bar{r}} - 2\eta\bar{r}\frac{d\bar{u}}{d\bar{r}} + 2\eta\omega^2\bar{u} = 0 \qquad (10)$$

Expanding $\bar{u}$ in powers of $\eta^{1/2}$,

$$\bar{u} = \eta^{1/2}\,\bar{u}_1 + \eta\,\bar{u}_2 + \ldots$$

substituting into Eq. (10), and collecting like powers of η, the equations for the $\bar{u}_i$ become

$$- 2 \frac{d^4\bar{u}_1}{d\bar{r}^4} + \frac{d^2\bar{u}_1}{d\bar{r}^2} = 0$$

$$- 2 \frac{d^4\bar{u}_2}{d\bar{r}^4} + \frac{d^2\bar{u}_2}{d\bar{r}^2} = 0 \qquad (11)$$

$$- 2 \frac{d^4\bar{u}_i}{d\bar{r}^4} + \frac{d^2\bar{u}_i}{d\bar{r}^2} = \bar{r}^2 \frac{d^2\bar{u}_{i-2}}{d\bar{r}^2} + 2\bar{r}\frac{d\bar{u}_{i-2}}{d\bar{r}} - 2\sum_{j=3}^{i} \omega_{j-3}^2\,\bar{u}_{i-j+1}$$

$$i = 3, 4, \ldots$$

The first two expansion variables $\bar{u}_1$ and $\bar{u}_2$ can be quickly found from Eq. (11).

$$\bar{u}_1 = a_1 + b_1\bar{r} + c_1\, e^{-\bar{r}/\sqrt{2}} + d_1 e^{+\bar{r}/\sqrt{2}}$$

$$\bar{u}_2 = a_2 + b_2\bar{r} + c_2\, e^{-\bar{r}/\sqrt{2}} + d_2 e^{+\bar{r}/\sqrt{2}}$$

The boundary conditions $\bar{u}(0) = \bar{u}'(0) = 0$ and $\bar{u}$ remaining finite in the far field immediately determine all but one of the constants for each variable.

$$\begin{aligned} \bar{u}_1 &= b_1\left[-\sqrt{2} + \bar{r} + \sqrt{2}\, e^{-\bar{r}/\sqrt{2}}\right] \\ \bar{u}_2 &= b_2\left[-\sqrt{2} + \bar{r} + \sqrt{2}\, e^{-\bar{r}/\sqrt{2}}\right] \end{aligned} \tag{12}$$

Thus, the unknown constants to be determined are the b_i in the inner expansion and the ω_i^2 in the outer expansion. These must be chosen so that the slopes and displacements of the inner and outer solutions match.

<u>Expansion near tip</u>. In addition to the geometric boundary conditions at the beam root, there are two natural boundary conditions at the beam tip, $u''(1) = u'''(1) = 0$, which must be satisfied. An ordering analysis reveals that the region in which these conditions are dominant is of order $\eta^{1/3}$. Rewriting Eq. (1a) in terms of an inner variable $\hat{r} = (1 - r)/\eta^{1/3}$ (and designating the solution as $\hat{u}$) yields

$$\frac{d^4\hat{u}}{d\hat{r}^4} - \hat{r}\left(1 - \frac{1}{2}\eta^{1/3}\hat{r}\right)\frac{d^2\hat{u}}{d\hat{r}^2} - \left(1 - \eta^{1/3}\hat{r}\right)\frac{d\hat{u}}{d\hat{r}} - \eta^{1/3}\omega^2\hat{u} = 0 \tag{13}$$

The variable $\hat{u}$ must be expanded in powers of $\eta^{1/6}$ because of the combination of $\eta^{1/3}$ with the $\eta^{1/2}\,\omega^2$ expansion.

$$\hat{u} = \hat{u}_o + \eta^{1/6}\,\hat{u}_1 + \eta^{2/6}\,\hat{u}_2 + \ldots$$

Substituting the above into Eq. (13), collecting like powers of η, and taking the first integral of the equations yields

$$\frac{d^3\hat{u}_o}{d\hat{r}^3} - \hat{r}\frac{d\hat{u}_o}{d\hat{r}} = a_o$$

$$\frac{d^3\hat{u}_1}{d\hat{r}^3} - \hat{r}\frac{d\hat{u}_1}{d\hat{r}} = a_1 \qquad (14)$$

$$\frac{d^3\hat{u}_i}{d\hat{r}^3} - \hat{r}\frac{d\hat{u}_i}{d\hat{r}} = a_i - \frac{1}{2}\hat{r}^2\hat{u}_{i-2} + \sum_{j=o}^{(i-2)/3} \omega_j^2 \int_o^{\hat{r}} \hat{u}_{i-2-3j}\, d\zeta \quad .$$

The boundary condition

$$\frac{d^3\hat{u}}{dr^3}(r = 1) = -\frac{1}{\eta}\frac{d^3\hat{u}}{d\hat{r}^3}(\hat{r} = 0) = 0$$

implies that the constants of integration a_i must vanish for all orders of η.

The general homogeneous solution to Eq. (14) can now be expressed in terms of Airy's functions.

$$\hat{u}_i = b_i\left[\frac{1}{3} - \int_o^{\hat{r}} Ai\, d\zeta\right] + c_i\int_o^{\hat{r}} Bi\, d\zeta + d_i$$

The constants c_i must be zero so that $\hat{u}_i$ remains finite in the outer field, and the constants b_i must be chosen so that the boundary condition

$$\frac{d^2\hat{u}}{d\hat{r}^2}(\hat{r} = 0) = 0$$

is fulfilled. The only available coefficients for matching with the outer solutions are simply the additive constants d_i. Therefore, only the tip displacement for u can be matched, and derivatives of u must match automatically from application of Eq. (14).

Matched Expansion

Matching near root. The matching condition near the beam root can be formally expressed as

$$\lim_{\overline{r} \to \infty} \overline{u}(\overline{r}) = \lim_{\overline{r} \to 0} u(\overline{r}) \quad .$$

In other words, if the outer expansion u is written terms of the inner variable $\bar{r}$, the expansions u and $\bar{u}$ must match term for term to each order of η. Now $u(\bar{r})$ can be expressed for small $\bar{r}$ as

$$u(\bar{r}) = \eta^{1/2}[u_1(0) + u_o'(0)\bar{r}] + \eta[u_2(0) + u_1'(0)\bar{r}] + \ldots$$

Comparing this with Eq. (12), as $\bar{r}$ becomes large, provides the matching relations

$$b_i = u_{i-1}'(0) \qquad \text{(matching slopes)}$$

$$u_i(0) = -\sqrt{2}\, b_i \qquad \text{(matching displacements)}$$

Using the relations in Table II and Appendix A for u(0) and u'(0), the unknown constants b_1, ω_1^2, and b_2 can be found for the nth mode.

$$b_1 = u_o'(0) = \frac{(-1)^{n-1}\ (2n)!}{(n-1)!\ (n)!\ 2^{2n-1}}$$

$$\omega_1^2 = \frac{(4n-1)}{\sqrt{2}}\left[\frac{(2n)!}{(n)!\ (n-1)!}\ \frac{1}{2^{2n-1}}\right]^2$$

$$b_2 = 0$$

The inner solution, written in terms of the outer variable r, then becomes

$$\bar{u} = \frac{(-1)^{n-1}(2n)!}{(n-1)!(n)!\ 2^{2n-1}}\left[r + \sqrt{2}\,\eta^{1/2}\left(e^{\frac{-r}{\sqrt{2}\,\eta^{1/2}}} - 1\right)\right] + \ldots$$

Since u_2 has been determined for the first mode, ω_2^2 for n = 1 may now also be found.

$$u_2(0) = -\frac{2}{3}\omega_2^2 + \frac{4}{3}\omega_1^4\left[\frac{7}{9} - \frac{2}{3}\ln(2)\right] = b_2 = 0$$

$$\omega_2^2 = 7 - 6\ln(2) = 2.84\,, \qquad (n = 1)$$

Matching the root solutions has therefore given the first mode and frequency to order η and higher modes and frequencies to order $\eta^{1/2}$.

Matching near tip. The matching conditions near the root have specified ω_1^2 and ω_2^2. The next step is to match the tip expansion to the outer expansion and thus determine the effect of tip boundary conditions on the frequency. Writing u in terms of $\tilde{r}$ yields for small $\tilde{r}$

$$u = u_o(1) + \eta^{1/2}[0] + \eta^{2/6}[-u_o'(1)\tilde{r}] + \eta^{3/6}[u_1(1)] + \eta^{4/6}[\tfrac{1}{2} u_o''(1)\tilde{r}^2]$$

$$+ \eta^{5/6}[- u_1'(1)\tilde{r}] + \eta^{6/6}[- \tfrac{1}{6} u_0'''(1)\tilde{r}^3 + u_2(1)] + \ldots$$

Repeated applications of Eq. (14) yield

$$\tilde{u}_o = d_o = u_o(1) = 1$$

$$\tilde{u}_1 = d_1 = 0$$

$$\tilde{u}_2 = d_2 - \omega_o^2 \tilde{r} = - \omega_o^2 \tilde{r}$$

$$\tilde{u}_3 = d_3 = u_1(1)$$

$$\tilde{u}_4 = \frac{1}{2} \omega_o^2(\omega_o^2 - 1) \left\{\frac{\tilde{r}^2}{2} + \frac{1}{A_i'(0)} \left[\frac{1}{3} - \int_o^{\tilde{r}} Ai d\zeta\right]\right\}, \; d_4 = 0$$

$$\tilde{u}_5 = - \tilde{r}[\omega_1^2 + \omega_o^2 u_1(1)], \; d_5 = 0$$

$$\tilde{u}_6 = d_6 - \frac{\omega_o^2(\omega_o^2 - 1)(\omega_o^2 - 3)}{6} \left[\frac{\tilde{r}^3}{6} + \pi\int_o^{\tilde{r}} Gi \; d\tilde{r}\right]$$

+ integral terms

These solutions automatically match the higher powers of η, for example

$$u_o'(1) = \omega_o^2$$

$$u_o''(1) = \frac{1}{2} \omega_o^2(\omega_o^2 - 1)$$

$$u_o'''(1) = \frac{1}{6} \omega_o^2(\omega_o^2 - 1)(\omega_o^2 - 3)$$

$$u_1'(1) = \omega_1^2 + \omega_o^2 u_1(1)$$

so that the only constants required for matching are the previously determined d_1.

The solutions provide some useful information concerning the tip region. First, while fulfilling $u'(0) = 0$ introduced exponentials, fulfilling $u''(1) = 0$ and $u'''(1) = 0$ has introduced Ai and Gi into the solution. Second, although Ai does decay exponentially (for large $\hat{r}$) Gi only decays as $1/\hat{r}$ (for large $\hat{r}$), so that the tip expansion will affect the root slope to an algebraic order $\eta^{5/3}$ for $n \geq 2$ and $\eta^{13/6}$ for $n = 1$ (because $\omega_0^2 - 1 = 0$ for $n = 1$). The effect of the tip conditions on frequency is therefore delayed into higher orders of η and can be neglected to the order considered here.

<u>Combined expressions.</u> Now that the inner expansions and outer expansions have been matched, they can be combined to form uniform expressions. By adding u, $\bar{u}$, and $\hat{u}$ and subtracting like terms, the modal deflections (and frequencies) may be expressed.

$n = 1$

$$\omega^2 = 1 + \frac{3}{\sqrt{2}}\eta^{1/2} + [7 - 6\,\ell n(2)]\eta + O(\eta^{3/2})$$

$n \geq 2$

$$\omega^2 = n(2n - 1) + \frac{(4n - 1)}{\sqrt{2}}\eta^{1/2}\left[\frac{(2n)!}{(n)!(n - 1)!}\ \frac{1}{2^{2n-1}}\right]^2 + O(\eta)$$

$n = 1$

$$u = r + \sqrt{2}\,\eta^{1/2}\left[-1 + r\,\ell n(1 + r) + \exp(-r/\sqrt{2}\,\eta^{1/2})\right]$$
$$- 2\eta\left[r\int_0^r \xi\,\frac{\ell n(1 + \xi)}{1 - \xi}\,d\xi + r\,\ell n(2)\ell n(1 - r) - \ell n(1 + r)\right.$$
$$\left. - (2 - \ell n(2))r\,\ell n(1 + r)\right] + O(\eta^{3/2}) \qquad (15)$$

$n = 2$

$$u = \frac{1}{2}(5r^3 - 3r) + \frac{65\sqrt{2}}{8}\eta^{1/2}\left[\frac{4}{21} - \frac{5}{7}r^2 + \frac{6}{35}r^3 - \frac{5}{7}(r - \frac{5}{3}r^3)\ell n(1 + r)\right.$$
$$\left. - \frac{4}{21}\exp(-r/\sqrt{2}\eta^{1/2})\right] + 15\eta^{2/3}\frac{\eta}{A_i'(0)}\left[G_i'(\hat{r})A_i(\hat{r}) - G_i(\hat{r})A_i'(\hat{r})\right] + O(\eta)$$

A comparison of the asymptotic frequency expression in Eq. (15) with Galerkin's method results is given in Fig. 2. This comparison reveals that below $\omega_{NR} = 0.15$ the $n = 10$ Galerkin results break down. The figure also reveals that the $O(\eta)$ expansion is within 10% of the actual value even up to $\omega_{NR} = 0.45$. It may be concluded that for the first mode the expansions show good convergence. In Fig. 3, a comparison of Galerkin and expansion mode shapes is given. For $\omega_{NR} = 0$, the Galerkin results naturally fail to predict the actual straight line mode shape. For $\omega_{NR} = 0.35$, in which the Galerkin results can be assumed to be nearly exact, the asymptotic expansion gives an excellent approximation even to order $\eta^{1/2}$. Therefore, the first mode shape also shows good convergence.

Two valuable pieces of information result from this asymptotic expansion. First of all, Eq. (15) gives the behavior of ω^2 and u for small η. This is necessary information for obtaining a simple frequency expression valid at all η. Second, the form of the inner expansion can be used to estimate the required number of orthogonal functions needed for convergence of the eigenvalue determinant. Since the expansion indicates an

$\exp\left(\frac{-r}{\sqrt{2}\eta^{1/2}}\right)$, it follows that

$$\beta_N > \frac{1}{\sqrt{2}\eta^{1/2}}$$

for convergence. Since $\beta_N \doteq \pi(N - 1/2)$, a simple expression for the required number of modal functions can be obtained.

$$N > \frac{1}{2} + \left[\pi\sqrt{2}\eta^{1/2}\right]^{-1} = \frac{1}{2} + \frac{0.8}{\omega_{NR}}$$

Figure 1 confirms the validity of this formula.

EMPIRICAL COMPOSITE EXPRESSIONS FOR BENDING

Two steps now remain in determining a simple frequency expression for flapping. First, the behavior of ω^2 for large η must be found. The method used here is to expand the eigenvalue determinant in powers of η. Second, a search must be made for a simple empirical function which can model both the low η and high η characteristics.

Large Stiffness Expansion

Eigenvalue determinant. The determinant of Eq. (3) is characterized by the fact that, for large η, it is dominated by its diagonal elements

$$D_{ij}(i = j) >> D_{ij}(i \neq j) \quad .$$

In this case, the eigenvalues of $D(\Lambda)$ may be written as

$$\Lambda_n = D_{nn} - \delta\Lambda_n \quad , \quad \delta\Lambda_n << \Lambda_n \quad .$$

The eigenvalue determinant, therefore, can be rewritten in terms of $\delta\Lambda_n$

$$\left| \left[D_{ij}\right] - \Lambda_n[I] \right| = \left| \left[\overline{D}_{ij}\right] + \delta\Lambda_n[I] \right| = 0$$

where

$$\overline{D}_{ij} = D_{ij} \qquad i \neq j$$

$$\overline{D}_{ij} = D_{ij} - D_{nn} \qquad i = j \qquad (\overline{D}_{nn} = 0) \quad .$$

Expanding the determinant, keeping only the highest power of $\overline{D}_{ij} (i = j)$ in each term, gives

$$(\delta\Lambda_n)^N + (\delta\Lambda_n)^{N-1} \sum_{\substack{i=1 \\ i\neq n}}^{N} \overline{D}_{ii} + \ldots$$

$$+ \delta\Lambda_n \prod_{\substack{i=1 \\ i\neq n}}^{N} \overline{D}_{ii} - \sum_{\substack{i=1 \\ i\neq n}}^{N} \left[(\overline{D}_{ni}\ \overline{D}_{in}) \prod_{\substack{j=1 \\ j\neq n \\ j\neq i}}^{N} \overline{D}_{jj} \right] = 0 \tag{16}$$

But $\delta\Lambda$ is of the order of the off diagonal terms, so that to first order only the last two terms in Eq. (16) need be retained. Discarding the higher order terms and using $\overline{D}_{ij} = \overline{D}_{ji}$ gives

$$\delta\Lambda_n = \sum_{\substack{i=1 \\ i\neq n}}^{N} \overline{D}^2_{ni}/\overline{D}_{ii} \quad , \quad \Lambda_n = D_{nn} - \sum_{\substack{i=1 \\ i\neq n}}^{N} \frac{D^2_{ni}}{D_{ii} - D_{nn}} \quad . \tag{17}$$

The eigenvalue, therefore, is given as a summation of combinations of the elements of D.

<u>Expansion for flapping</u>. When the ϕ_i are taken to be nonrotating modes, the frequency expansion for flapping becomes

$$\omega^2 \approx A_{nn} + \eta\beta_n^4 - \sum_{\substack{i=1 \\ i\neq n}}^{N} \frac{A^2_{ni}}{(A_{ii} - A_{nn}) + \eta(\beta_i^4 - \beta_n^4)} \quad . \tag{18}$$

There are two types of truncations associated with Eq. (18). The first type of truncation, from Eq. (16), is of order less than $(1/\eta)^2$. The second type is the truncation of the summation in Eq. (18). For example, the frequencies for n = 1,2 from Eq. (18) are

n = 1

$$\omega^2 = 1.19 + 12.36\eta - \frac{.0890}{1 + 89.3\eta} - \frac{.0376}{1 + 228.\eta} - \frac{.0085}{1 + 419.\eta} - \dots$$

n = 2 (19)

$$\omega^2 = 6.48 + 485.\eta + \frac{.0890}{1 + 89.3\eta} - \frac{.0025}{1 + 292.\eta} - \frac{.2860}{1 + 476.\eta} - \dots$$

Truncating the series is similar to truncating the number of orthogonal functions. The truncation error will clearly be much smaller than $1/\eta$, as it is bounded at a fixed percentage as η goes to zero. The formulas in Eq. (19) are compared, in Fig. (4) and Fig. (5), with the exact eigenvalues of D for several truncation values N. The results indicate that Eq. (19) is a good approximation for the eigenvalues of D even down to very small η.

Similarly, approximate eigenvectors can be determined by

$$\Big[[D] - \Lambda_n[I]\Big] \{a_n\} \simeq [\overline{D}]\{a_n\} = \{0\}$$

from which it immediately follows that for flapping

$$a_{in} \simeq \begin{cases} 1 & i = n \\ \dfrac{-A_{ni}}{A_{ii} - A_{nn} + \eta(\beta_i^4 - \beta_n)} & i \neq n \end{cases} \qquad (20)$$

Eq. (20) also agrees rather well with the Galerkin results. For example, comparing the expansion modes from Eq. (20)

$$a_1 = \begin{Bmatrix} 1.0 \\ .130/(1 + 89.3\eta) \\ .048/(1 + 228.\eta) \\ .002/(1 + 419.\eta) \end{Bmatrix}, \qquad a_2 = \begin{Bmatrix} -.130/(1 + 89.3\eta) \\ 1.0 \\ -.015/(1 + 292.\eta) \\ .099/(1 + 476.\eta) \end{Bmatrix} \qquad (21)$$

with the Galerkin modes for $\eta = 0$

$$a_1 = \begin{Bmatrix} 1.0 \\ .136 \\ .041 \\ .002 \end{Bmatrix} , \quad a_2 = \begin{Bmatrix} -.136 \\ 1.0 \\ -.050 \\ .099 \end{Bmatrix}$$

it is clear that the modal expansion converges very rapidly. Therefore, Eq. (18) and Eq. (20) are seen to be excellent approximations to ω^2 and a_{ij} for moderate to large η.

Approximate Frequencies

Choice of function. Now that the behavior of ω^2 is known for small and large values of η, it remains to find a function which will be valid at all values of η. When the order of the infinite series in Eq. (18) is estimated by using an integral rather than a summation, products of $\eta^{1/2}$ and $\tan^{-1}(\eta^{1/2})$ are introduced. These possess potential for use in a composite function for ω^2. Of particular interest is the combination

$$\omega^2 = a + b\eta + c\eta^{1/2} \tan^{-1} (d/\eta^{1/2}) \quad . \tag{22}$$

This function has the advantage of possessing an expansion in $\eta^{1/2}$ for small η and an expansion in η for large η. The next step, therefore, is to determine if Eq. (22) can be made to approximate the behavior of ω^2.

Choice of constants. For small η, Eq. (22) can be expanded as

$$\omega^2 = a + \frac{c\pi}{2}\eta^{1/2} + (b - c/d)\eta + . . . \tag{23a}$$

To fit the first two terms to the small η asymptotic expansion, Eq. (15), requires

$$a = n(2n - 1), \; c = \frac{\sqrt{2}}{\pi} (4n - 1) \left[\frac{(2n)!}{(n)!(n - 1)!} \frac{1}{2^{2n-1}}\right]^2$$

For large η, Eq. (22) can be expanded as

$$\omega^2 = b\eta + (a + cd) - \frac{cd^3}{3} \frac{1}{\eta} + . . . \tag{23b}$$

To fit the first two terms to the large η eigenvalue expansion, b and d must satisfy

$$b = \beta^4 \quad , \quad d = \frac{A_{nn} - n(2n - 1)}{c} \quad .$$

In terms of $\omega_{NR} = \beta_n^2 \eta^{1/2}$ Eq. (22) then becomes

$$\omega^2 = n(2n - 1) + \omega_{NR}^2 + f\omega_{NR} \tan^{-1}\left[\frac{A_{nn} - n(2n - 1)}{f\omega_{NR}}\right] \qquad (24)$$

$$f = c/\beta_n^2 \quad .$$

Thus, Eq. (24) represents an approximation valid for all frequencies. The terms in this approximation are simply ω_{NR} (the nonrotating frequency), $n(2n - 1)$ (the small η centrifugal stiffness), A_{nn} (the large η centrifugal stiffness, and f (the order $\eta^{1/2}$ frequency correction). The equation applies equally to lead-lag with $\omega^2{}_{FLAP}$ replaced by $\omega^2{}_{LEAD\text{-}LAG} + 1$.

Evaluation of results. Eq. (24) is a simple empirical expression representing ω^2 and is compared with Galerkin's results (for n = 1, 2) in Fig. (6) and Fig. (7). It is clearly seen that Eq. (24) is a good fit to ω^2 at all η values. This closeness of fit may also be viewed quantitatively by comparing the third terms in Eq. (23a) and Eq. (23b) with the corresponding terms in the asymptotic and eigenvalue expansions. For example, the order η term, for small values of η and n = 1, is found to be

$$b - c/d = 12.36 - 9.44 = 2.92 \qquad \text{(composite)}$$

$$7 - 6\ \ell n(2) = 2.84 \qquad \text{(asymptotic)}$$

Similarly, the order $1/\eta$ term for large values of η is found to be

$$-\ cd^3/3 = -.0013 \qquad \text{(composite)}$$

$$-\ \frac{.089}{89.3} - \frac{.0376}{228.} - \frac{.0085}{419.} = -.0012 \qquad \text{(eigenvalue)}$$

Eq. (24) is therefore an excellent fit to the exact frequency behavior.

Approximate Modes

Observations. The final step in solving the flap equation is a determination of approximate expressions for the mode shapes u(r). In Fig. 3, the low stiffness first mode (to order $\eta^{1/2}$) and the high stiffness first mode (to order $1/\eta$) are seen to be virtually identical. Similarly, Fig. 8 provides a comparison of the second mode. Except near the blade root, the high stiffness expansion (to order $1/\eta$) is very good even for $\eta = 0$. Furthermore, the mode experiences only minor changes as η is increased. Because of the η range where the two modes are in close proximity, it is possible to form approximate modes as linear combinations of the two expansions.

Normalization. Before combining the small and large η expressions, it is necessary to normalize the modes such that

$$\int_0^1 u^2\, dr = 1 \quad .$$

In Eq. (20) and Eq. (21), the high stiffness modes are seen to be completely dominated by the nth nonrotating mode (which is itself normalized) so that no normalization procedure is necessary. The small stiffness modes, however, are not normalized so that some additional work is required.

For all but the first mode, only the first order mode shape P_{2n-1} will be used. Table II provides the relation for normalizing these modes.

$n \geq 2$

$$u = \sqrt{4n-1}\, P_{2n-1}\, (-1)^{n+1} \tag{25a}$$

(The negative sign is added to give the small stiffness modes the same sign as the nonrotating modes.) For the first mode, the expansion will be taken to order $\eta^{1/2}$. The normalized version of this mode, neglec· ... higher order terms, can be calculated to be

$n = 1$

$$u = \sqrt{3}\left\{r + \sqrt{2}\,\eta^{1/2}\left[-1 + \left(\frac{7}{3} - 2\,\ell n\,(2)\right) r + r\ell n\,(1+r) + \exp(-r/\sqrt{2}\eta^{1/2})\right]\right\} \tag{25b}$$

Eq. (20), Eq. (25a), and Eq. (25b) are the normalized modes which will be combined to form the composite modes.

Modal composition. When two normalized modes are nearly the same, having two similar frequencies, the formation of a third normalized mode having some intermediate frequency can be accomplished by a simple linear composition. Designating $\omega_S(u_S)$ as the smaller frequency (with its mode) and $\omega_L(u_L)$ as the larger frequency (and its mode), the desired mode u having frequency ω is reasonably chosen as

$$u = u_S \frac{\omega_L^2 - \omega^2}{\omega_L^2 - \omega_S^2} + u_L \frac{\omega^2 - \omega_S^2}{\omega_L^2 - \omega_S^2} \tag{26}$$

As ω approaches either frequency, the mode corresponding to that frequency dominates.

In the context of this paper, u_S is taken to be the mode of Eq. (25), and ω_S is taken to be the asymptotic expansion frequency.

$n = 1,\qquad \omega_S^2 = 1 + \frac{3}{\sqrt{2}}\eta^{1/2} + [7 - 6\,\ell n\,(2)]\eta$

$n \geq 2,\qquad \omega_S^2 = n(2n-1) + \frac{(4n-1)}{\sqrt{2}}\eta^{1/2}\left[\frac{(2n)!}{(n)!\,(n-1)!}\;\frac{1}{2^{2n-1}}\right]^2$

Similarly ω_L and u_L are taken from Eq. (18) and Eq. (20). The choice of ω^2 as given by Eq. (24) insures that $\omega_S^2 < \omega^2 < \omega_L^2$, making the linear composition of Eq. (26) a smooth transition from Legendre polynomial ($\eta = 0$) to nonrotating mode ($\eta = \infty$). It should be pointed out that by replacing $\omega^2{}_{FLAP}$ with $\omega^2{}_{LEAD\text{-}LAG} + 1$, these modes also apply to the lead-lag case (see Eq. (1)).

ANALYTIC SOLUTION FOR TORSION

Now that the flapping and lead-lag solutions are in hand, the solution of the torsion equation remains. The torsion equation from Eq. (1c) is

$$-\gamma\theta'' - \frac{1}{2}[(1 - r^2)\theta']' - (\omega^2 - 1)\theta = 0 \quad .$$

The first term, the torsional rigidity term, is only of second order distinguishing this equation from the fourth order bending equations. The second term is the rigidity from centrifugal tension, and the -1 is the so-called "tennis racket" term resulting from gyroscopic forces in the rotating reference frame. Although the torsional frequency (and the stiffness γ) are often large enough that effects of rotation can be neglected, the lower limit of practical frequencies does include significant departures from the nonrotating case. The task at hand is to develop simple frequency and mode shape approximations which will: 1.) be valid for all values of torsional stiffness γ, and 2.) thereby determine the range of γ and ω for which rotational effects may be neglected.

Closed Form Modes

Limiting cases. Before preceeding to the direct solution of the torsion equation, it is interesting to study its behavior in the limits of $\gamma = 0$ and $\gamma = \infty$. When γ is identically equal to zero, the solutions to the torsion equation are Legendre polynomials.

$$\omega^2 = 1 + n(2n - 1)$$

$$\theta = P_{2n-1}(r) \qquad n = 1,2,3,\ldots$$

The lower limit, therefore, for the torsional frequency of uniform rotating beams is $\sqrt{2}$. In the limit as γ becomes very large the torsion solutions become the nonrotating modes and frequencies given in Table II.

$$\omega^2 = \frac{(2n - 1)^2\pi^2}{4}\gamma$$

$$\theta = \sqrt{2}\sin\left[\frac{(2n - 1)\pi}{2} r\right]$$

These nonrotating modes, unlike the $\gamma = 0$ modes, fulfill the boundary condition $\theta(1) = 0$.

Change of variable. In contrast to the flapping and lead-lag equations, the torsion equation admits a closed form solution for the mode shapes. By a simple change of independent variable

$$r = x\sqrt{1 + 2\gamma} \quad , \quad x = r/\sqrt{1 + 2\gamma}$$

the torsion equation and its boundary conditions can be brought into the form

$$-\frac{1}{2}\frac{d}{dx}\left[(1 - x^2)\frac{d\theta}{dx}\right] - (\omega^2 - 1) = 0 \tag{27a}$$

$$\theta(0) = 0 \quad , \quad \frac{d\theta}{dx}\left(\frac{1}{\sqrt{1 + 2\gamma}}\right) = 0 \quad . \tag{27b}$$

The torsion equation has, therefore, been transformed into Legendre's equation. The beam tip, however, is at $x = \frac{1}{\sqrt{1 + 2\gamma}}$ rather than $x = 1$. The general solution to Eq. (27a) is, from Eq. (7),

$$\theta = aP_\nu + bP_\nu \int^x \frac{1}{P_\nu^2(1 - x^2)} dx = aP_\nu + bQ_\nu \tag{28a}$$

$$2\omega^2 = \nu(\nu + 1) + 2 \quad , \quad \nu > 0 \quad . \tag{28b}$$

From the properties of Legendre functions, the boundary condition $\theta(0) = 0$ implies that

$$a/b = \frac{\pi}{2}\tan\left(\frac{\nu\pi}{2}\right)$$

yielding a series representation for θ.[5]

$$\theta = C\left[x - \frac{(\nu - 1)\,(\nu + 2)}{3!}x^3 + \frac{(\nu - 1)\,(\nu - 3)(\nu + 2)(\nu + 4)}{5!}x^5 - \ldots\right] \tag{29}$$

The remaining boundary condition, $\frac{d\theta}{dx}(1/\sqrt{1 + 2\gamma}) = 0$ is fulfilled only for certain discrete values of ν. These eigenvalues for ν then imply the frequencies and modes through Eq. (28b) and Eq. (29).

The modes of Eq. (29) are valid for all values of γ, recovering both limiting cases ($\gamma = 0, \infty$). When $\gamma = 0$, Eq. (27a) becomes singular and the θ' boundary condition is relaxed. The series expansion of Eq. (29), however, diverges unless ν takes on one of the eigenvalues

$$\nu = 2n - 1 \qquad n = 1,2,3 \quad .$$

This results in a truncation of Eq. (29) yielding the Legendre polynomials. For large values of γ, Eq. (29) reduces to

$$\theta = \frac{C}{\nu} \sin\left[\nu r/\sqrt{2}\, \gamma^{1/2}\right] \quad .$$

The boundary condition $\theta'(1) = 0$ then implies

$$\frac{\nu}{\sqrt{2}\gamma^{1/2}} = \frac{(2n - 1)\pi}{2} \qquad , \qquad \omega^2 = \frac{\nu^2}{2} = \frac{(2n - 1)^2\pi^2}{4}\gamma$$

recovering the nonrotating solution.

Approximate Frequencies

Small γ expansion. Eq. (28) and Eq. (29) now describe the torsion mode in closed form, but the solution of a transcendental equation

$$\frac{d\theta}{dx}\left(\frac{1}{\sqrt{1 + 2\gamma}}\right) = 0$$

is required to determine the natural frequencies and undefined constants of the modes. In order to obtain a simple frequency expression, this transcendental equation needs to be approximated in closed form. To begin, the behavior of Eq. (27b) is studied for small values of γ.

In the limit of small γ, the boundary condition of Eq. (27b) must be satisfied for x very nearly 1.

$$x_{TIP} = \frac{1}{\sqrt{1 + 2\gamma}} = 1 - \gamma + \frac{3}{2}\gamma^2 - \ldots \tag{30a}$$

Correspondingly, ν is expected to be very near to one of its odd integer values

$$\nu = (2n - 1) + \varepsilon \qquad , \qquad \varepsilon << 1 \quad . \tag{30b}$$

The tip boundary condition, therefore, may be written to first order in γ, using Eq. (28) and Eq. (30), as

$$\frac{2}{\pi}\tan\left(\frac{\pi\varepsilon}{2}\right) = \left[\frac{dP_\nu/dx}{dQ_\nu/dx}\right]_{x = 1 - \gamma} .$$

Using the properties of Legendre functions, this boundary condition becomes

$$\varepsilon = \frac{2}{\pi}\tan^{-1}\left[\frac{\pi}{2}\,\frac{\left.\frac{dP_{2n-1}}{dx}\right|_{x=1} + O(\varepsilon) + O(\gamma)}{\frac{1}{2}\gamma - \frac{1}{2}n(2n-1)\ln\gamma + O(1)}\right]$$

$$= \frac{2}{\pi}\tan^{-1}[\gamma\pi\{n(2n - 1)\ [1 + n(2n - 1)\gamma\ln\gamma + O(\gamma)] + O(\varepsilon)\}]$$

$$= 2n(2n - 1)\ [\gamma + n(2n - 1)\gamma^2\ln\gamma] + O(\gamma^2) .$$

Thus, the behavior of ν and ω^2 for small γ is found as an expansion in powers of γ and $\gamma\ln\gamma$.

$$\omega^2 = 1 + \frac{\nu(\nu + 1)}{2} = 1 + n(2n - 1) + n(2n - 1)\ (4n - 1)\gamma$$

$$+ n^2(2n - 1)^2(4n - 1)\gamma^2\ln\gamma + O(\gamma^2) \qquad (31)$$

<u>Empirical frequency expression</u>. The behavior of ω^2 for large γ can be found by applying Eq. (17) to the torsional determinant, and using Table I for determining A_{ij}.

$$\omega^2 = \frac{5}{4} + \frac{\pi^2}{12}(2n - 1)^2(1 + 3\gamma)$$

$$- \frac{192(2n - 1)^4}{\pi^2(1 + 3\gamma)}\sum_{\substack{i=1\\ i\neq n}}^{N}\frac{(2i - 1)^4}{[(2i - 1)^2 - (2n - 1)^2]^5} \qquad (32)$$

As before the summation converges rapidly giving a very accurate approximation for ω^2. The form of the asymptotic expansion and of the eigenvalue expansion provide motivation for attempting to fit the frequency behavior with a function of the form

$$\omega^2 = a + b\gamma + c\gamma^2\ \ln\left|\frac{\gamma^2}{d + \gamma^2}\right| - e\,\frac{\gamma^2}{1 + 3\gamma} . \qquad (33)$$

For small γ, this formula reduces to

$$\omega^2 = a + b\gamma + 2c\gamma^2\ln\gamma - [e + c\ \ln\ (d)]\gamma^2 + \ldots$$

The constants a, b, and c can be immediately determined by comparing with Eq. (31).

$$a = 1 + n(2n - 1)$$
$$b = n(2n - 1)(4n - 1) \qquad (34c\text{-}c)$$
$$c = n^2(2n - 1)^2(4n - 1)/2$$

For large γ, Eq. (33) becomes

$$\omega^2 = a + b\gamma - c\gamma^2\left(\frac{d}{\gamma^2} - \frac{1}{2}\frac{d^2}{\gamma^4}\right) - \frac{e\gamma}{3}\left(1 - \frac{1}{3\gamma} + \frac{1}{9\gamma^2}\right)$$
$$= \left(b - \frac{e}{3}\right)\gamma + \left(a - cd + \frac{e}{9}\right) - \frac{e}{27}\frac{1}{\gamma} \quad .$$

From Eq. (32), this gives the values of e and d

$$e = 3n(2n - 1)(4n - 1) - \frac{3\pi^2}{4}(2n - 1)^2 \qquad (34e,d)$$

$$d = \frac{8n(2n - 1)(2n + 1) - 3 - 2\pi^2(2n - 1)^2}{6n^2(2n - 1)^2(4n - 1)} \quad .$$

The final formula for torsion is therefore,

$$\omega^2 = 1 + n(2n - 1) + n(2n - 1)(4n - 1)\gamma$$
$$+ \frac{1}{2}n^2(2n - 1)^2(4n - 1)\gamma^2 \ln\left|\frac{\gamma^2}{d + \gamma^2}\right| \qquad (35)$$
$$+ \frac{\frac{3\pi^2}{4}(2n - 1)^2 - 3n(2n - 1)(4n - 1)}{1 + 3\gamma}\gamma^2 \quad .$$

To complete the solution, the mode shape is then given by Eq. (29) with

$$x = \frac{r}{\sqrt{1 + 2\gamma}} \quad , \qquad \nu = \frac{\sqrt{8\omega^2 - 7} - 1}{2} \quad .$$

Practical Considerations

Comparison of methods. A comparison of the various frequency approximations reveals three facts. First, the Galerkin expression, Eq. (32), with only one mode gives frequencies to within 5% for γ as low as 0.8 (first torsion frequency 2.0) for $n = 1, 2$. Second, the frequency expansion, Eq. (31), gives frequencies to within 5% for γ as high as 0.8 for $n = 1,2$. And third, a frequency expression neglecting the $[(1 - r^2)\theta']'$ term

$$\omega^2 = 1 + \frac{\pi^2}{4}(2n - 1)^2\gamma$$

will introduce errors of the order of 15% at $\gamma = .8$. The impact of these three observations is double-edged. On the one hand, the low γ expansion (in terms of logarithms) is seen to be an excellent approximation even as γ approaches unity. On the other hand, it appears that for γ in the practical range of interest the Galerkin solution is perfectly adequate so that the low γ expansion is actually not required. This should not, however, be construed to imply that the effects of rotation are small. The importance of the tension term $\frac{1}{2}[(1 - r^2)\theta']'$ reveals that rotation is important, but may be accounted for by a single function Galerkin analysis. Similarly, Fig. (9) shows that for $\gamma > 0.8$ the modes have also converged to the nonrotating modes so that the mode shape, Eq. (29) is also not necessary.

Special cases. There may be, however, special cases for which Eq. (29) and Eq. (35) are necessary. That is, in Eq. (1c) the area and mass radii of gyration (k_A, k_M) were assumed equal. In the event that they are unequal, the tension term in Eq. (1c) must be multiplied by k_A^2/k_M^2. Therefore, the region of validity of Eq. (32) will become

$$\omega > \sqrt{1 + 3\, k_A^2/k_M^2}$$

rather than $\omega > 2$. For example, if $k_A^2/k_M^2 = 2$, then the region of validity of Eq. (32) is reduced to $\omega > 3.6$. It follows that for beams with certain types of cross-sections, Eq. (29) and Eq. (35) may be required if

$$\omega^2 < 1 + 3\, k_A^2/k_M^2 \quad .$$

CONCLUSIONS

By combining a low stiffness expansion and a high stiffness expansion, a simple empirical closed form expression has been obtained for the flapping and lead-lag bending frequencies of rotating uniform beams.

$$\omega^2_{FLAP} = 1 + \omega^2_{LEAD\text{-}LAG} =$$

$$n(2n - 1) + \omega^2_{NR} + f\omega_{NR} \tan^{-1}\left[\frac{A_{nn} - n(2n - 1)}{f\omega_{NR}}\right]$$

where

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n = mode number

ω_{NR} = nonrotating frequency (flap or lead-lag)

A_{nn} = Galerkin integral (Table I)

$$f = \frac{\sqrt{2}(4n - 1)}{\pi\beta_n^2}\left[\frac{(2n)!}{(n)!(n - 1)!}\,\frac{1}{2^{2n-1}}\right]^2$$

β_n = Galerkin constant (Table I) .

This formula is uniformly valid for all values of ω_{NR} from zero to infinity. A similar expression has been obtained for torsion, although for most practical examples the frequencies are adequately estimated by the Galerkin formula

$$\omega^2_{TORSION} = \omega^2_{NR} + \frac{5}{4} + \frac{\pi^2}{12}(2n - 1)^2 \quad .$$

These frequency expressions, along with the modal equations which have been developed, provide a practical alternative to energy methods for natural frequency determination. They provide assured convergence throughout the frequency range, require only minimal numerical calculations, and allow rapid determination of the mode shapes and beam properties for any desired rotating frequencies.

APPENDIX

BEHAVIOR OF u_1 NEAR BEAM ROOT

From Eq. (8), the formula for u_1 (of the nth mode) can be written as

$$u_1 = 2\omega_1^2 \, P_{2n-1} \int^r \frac{\int_r^1 P_{2n-1}^2 \, d\zeta}{P_{2n-1}^2 (1 - r^2)} \, dr \quad .$$

From Table II, P_{2n-1} can be expressed for small r as

$$P_{2n-1} = a_1 r + a_3 r^3$$

where

$$a_1 = \frac{1}{2^{2n-1}} \, \frac{(-1)^{n-1}(2n)!}{(n)!(n-1)!} \quad , \quad a_3 = \frac{1}{2^{2n}} \, \frac{(-1)^{n-2}(2n+2)!}{(n)!(n+1)! \; 3} \quad .$$

Therefore, near the root section u_1 behaves as

$$u_1 = 2\omega_1^2 \, (a_1 r + a_3 r^3) \int^r \frac{\frac{1}{4n-1} - \frac{a_1 r^3}{3}}{(a_1^2 r^2 + 2a_1 a_3 r^4)(1 - r^2)} \, d\zeta$$

$$= 2\omega_1^2 \, (a_1 r + a_3 r^3) \, \frac{1}{(4n-1)a_1^2} \left[\frac{1}{r^2} + 1 - \frac{2a_3}{a_1}\right] d\zeta$$

$$= \frac{2\omega_1^2}{(4n-1)a_1^2} \, (a_1 r + a_3 r^3) \left[-\frac{1}{r} + \left(1 - \frac{2a_3}{a_1}\right) r\right]$$

$$= \frac{2\omega_1^2}{(4n-1)a_1} \left[-1 + \left(1 - \frac{3a_3}{a_1}\right) r^2\right]$$

$$u_1(0) = \frac{\omega_1^2 (n-1)!(n)! \; 2^{2n}}{(-1)^n (2n)!(4n-1)} \quad , \quad u_1'(0) = 0 \quad .$$

REFERENCES

1. Hodges, D. H and Ormiston, R. A., "Nonlinear Equations for Bending of Rotating Beams with Application to Linear Flap-Lag Stability of Hingeless Rotors," TM X-2770, November 1972.

2. Houbolt, J. C. and Brooks, G. W., "Differential Equations of Motion for Combined Flapwise Bending, Chordwise Bending, and Torsion of Twisted Nonuniform Rotor Blades," NACA Report 1346, October 1956.

3. Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., Aeroelasticity, Addison-Wesley Publishing Company, Cambridge, Mass., 1955.

4. Cole, J. D. Perturbation Methods in Applied Mechanics, Blaisdell Company, 1958.

5. Hildebrand, F. B., Advanced Calculus for Applications, Prentice Hall, Inc., Englewood, New Jersey, 1965.

TABLE I.

A. FLAPPING AND LEAD-LAG $\omega_{NR} = \beta_n^2 \eta^{1/2}$

n	α_n	β_n	β_n^2	β_n^4
1	0.73410	1.87510	3.51600	12.3623
2	1.01847	4.69409	22.0345	485.518
3	0.99922	7.85476	61.697[illegible]	3806.55
4	1.00003	10.9955	120.901	14617.1
>4	1.00000	$\frac{\pi}{2}(2n-1)$	$\frac{\pi^2}{4}(2n-1)^2$	$\frac{\pi^4}{16}(2n-1)^4$

$$\varphi_n = \cosh(\beta_n r) - \cos(\beta_n r) - \alpha_n \left[\sinh(\beta_n r) - \sin(\beta_n r)\right]$$

(NONROTATING MODE)

i	A_{i1}	A_{i2}	A_{i3}	A_{i4}
1	1.19334	-0.685855	-0.792379	-0.546413
2	–	6.47823	0.169408	-2.91185
3	–	–	17.8603	3.27427
4	–	–	–	36.0553

B. TORSION $\omega_{NR} = \frac{(2n-1)\pi}{2} \gamma^{1/2}$

$$\varphi_n = \sqrt{2} \sin\left[\frac{(2n-1)\pi}{2} r\right]$$

(NONROTATING MODE)

$$A_{ij} = \begin{cases} \left[1/4 + \frac{\pi^2}{12}(2i-1)^2\right] & i = j \\ \frac{-4(2i-1)^2(2j-1)^2}{\left[(2i-1)^2 - (2j-1)^2\right]^2} & i \neq j \end{cases}$$

TABLE II.

n	P_{2n-1}
1	r
2	$1/2\,(5r^3 - 3r)$
3	$1/8\,(63r^5 - 70r^3 + 15r)$
4	$1/16\,(429r^7 - 693r^5 + 315r^3 - 35r)$
GENERAL CASE	$\frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k\,[2\,(2n-1-k)]!}{(k)!\,(2n-1-k)!\,(2n-1-2k)!}\, r^{2n-1-2k}$

$P_{2n-1}(0) = 0$ $\qquad$ $P_{2n-1}(1) = 1$ $\qquad$ $P'_{2n-1}(1) = n(2n-1)$

n	1	2	3	4	GENERAL CASE
$P'_{2n-1}(0)$	1	-3/2	15/8	-35/16	$\frac{(-1)^{n-1}\,(2n)!}{(n-1)!\,(n)!\,2^{2n-1}}$
$\int_0^1 P_{2n-1}\,dr$	1/2	-1/8	1/16	-5/128	$\frac{(-1)^{n-1}\,(2n-2)!}{4^{n-1}\,(2n)\,[(n-1)!]^2}$
$\int_0^1 P^2_{2n-1}\,dr$	1/3	1/7	1/11	1/15	$\frac{1}{4n-1}$

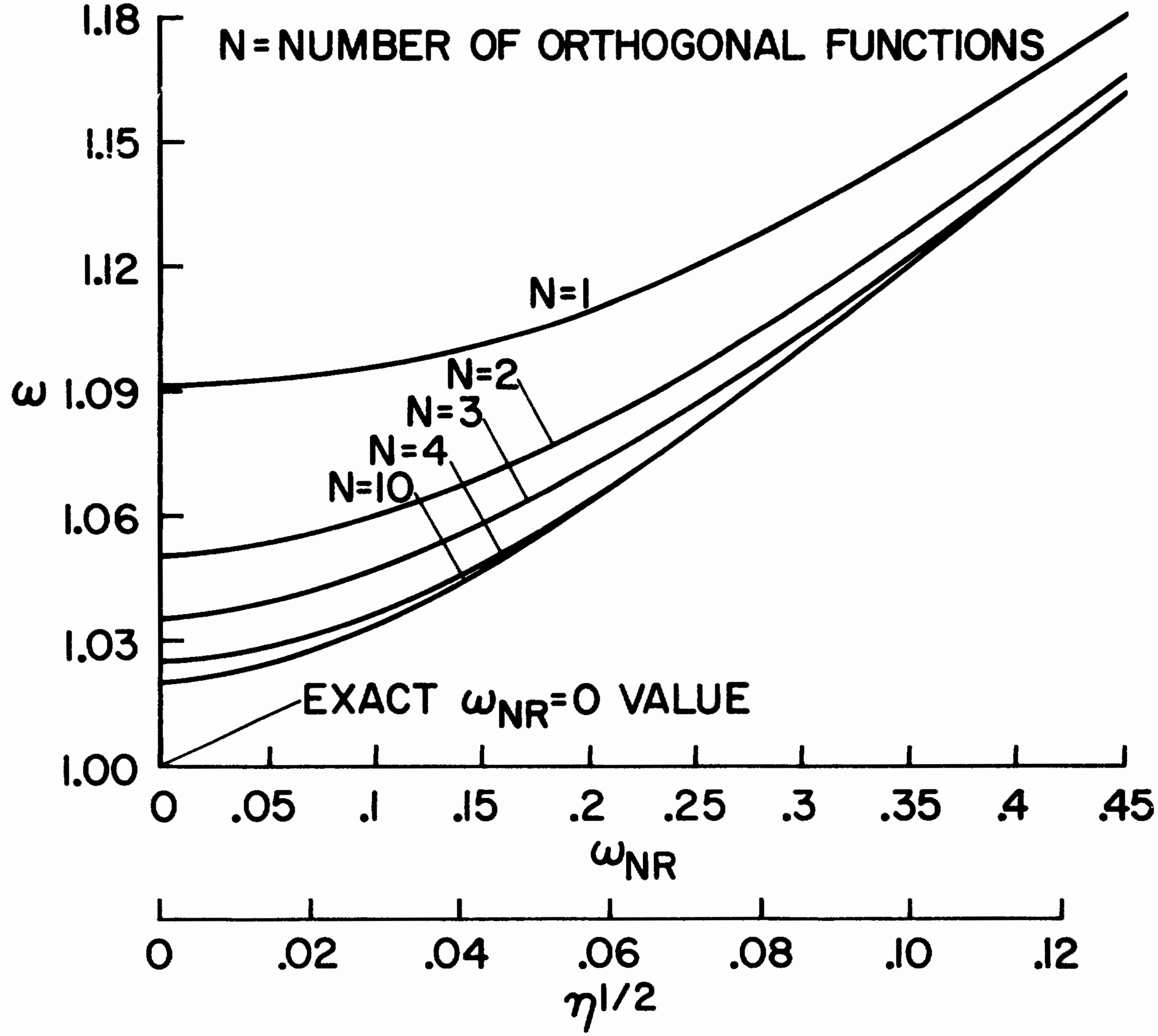

Figure 1. First frequency from Galerkin's method.

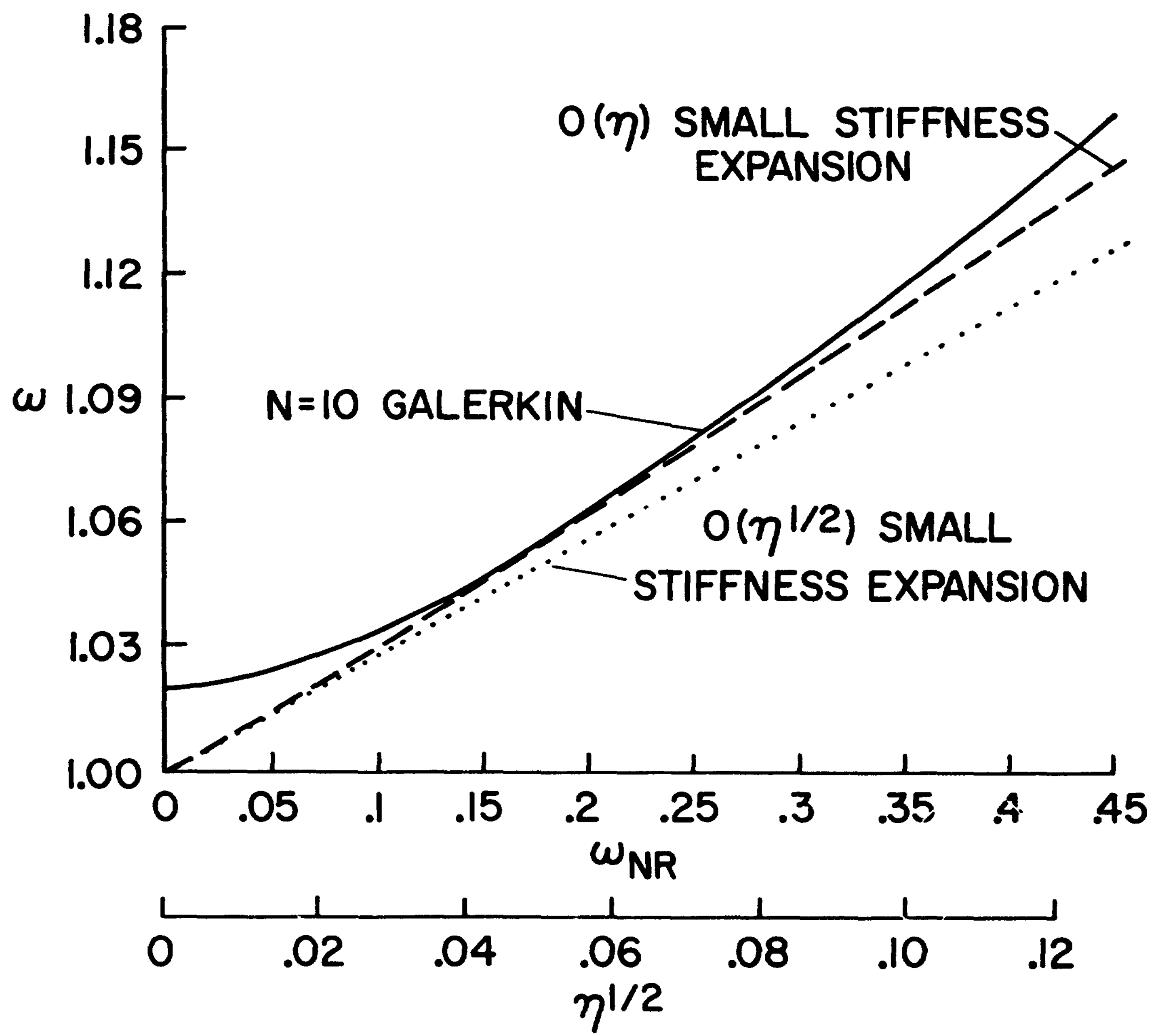

Figure 2. First frequency from small stiffness expansion.

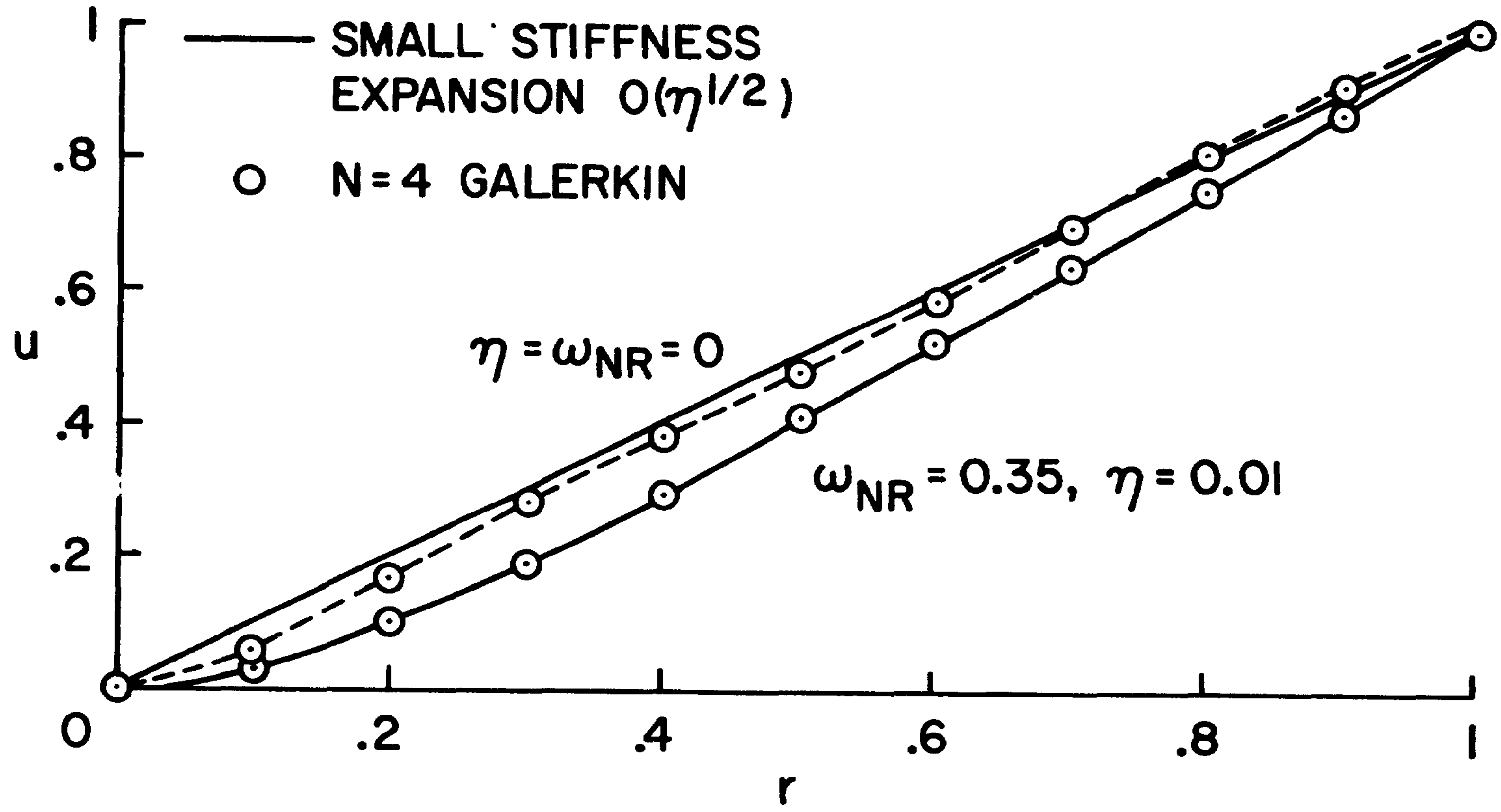

Figure 3. First flap mode shape.

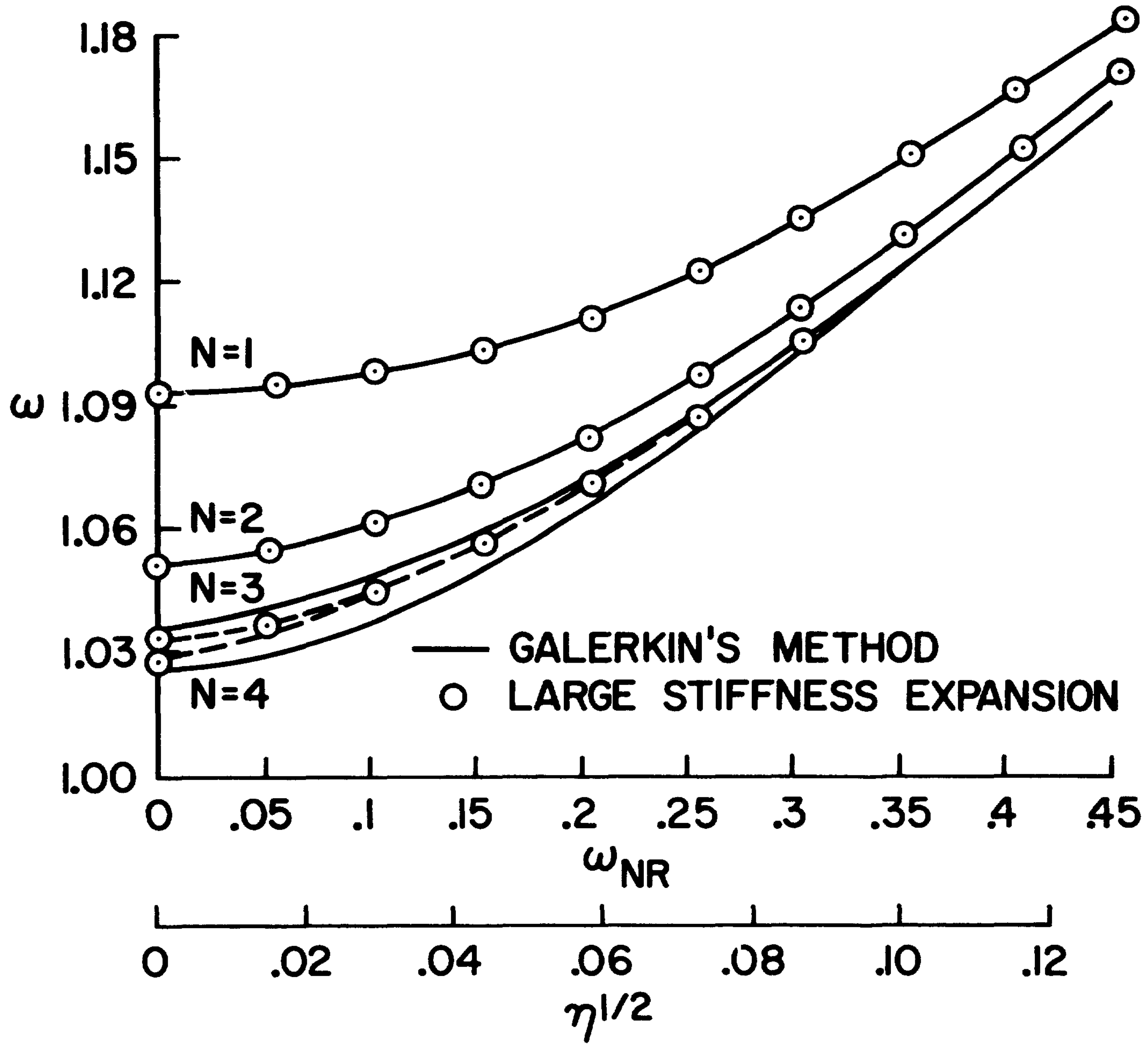

Figure 4. First frequency from large stiffness expansion.

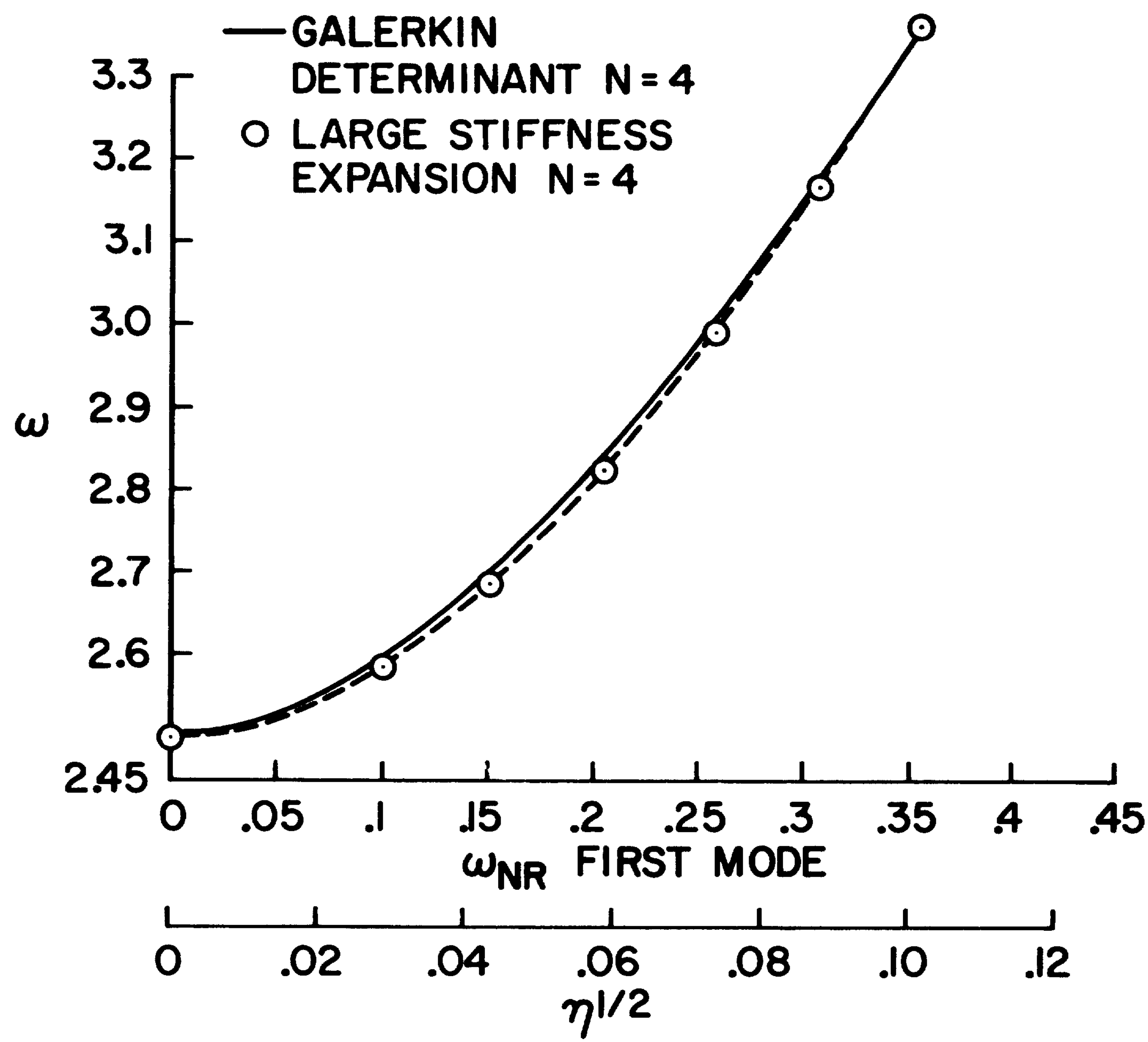

Figure 5. Second frequency from large stiffness expansion.

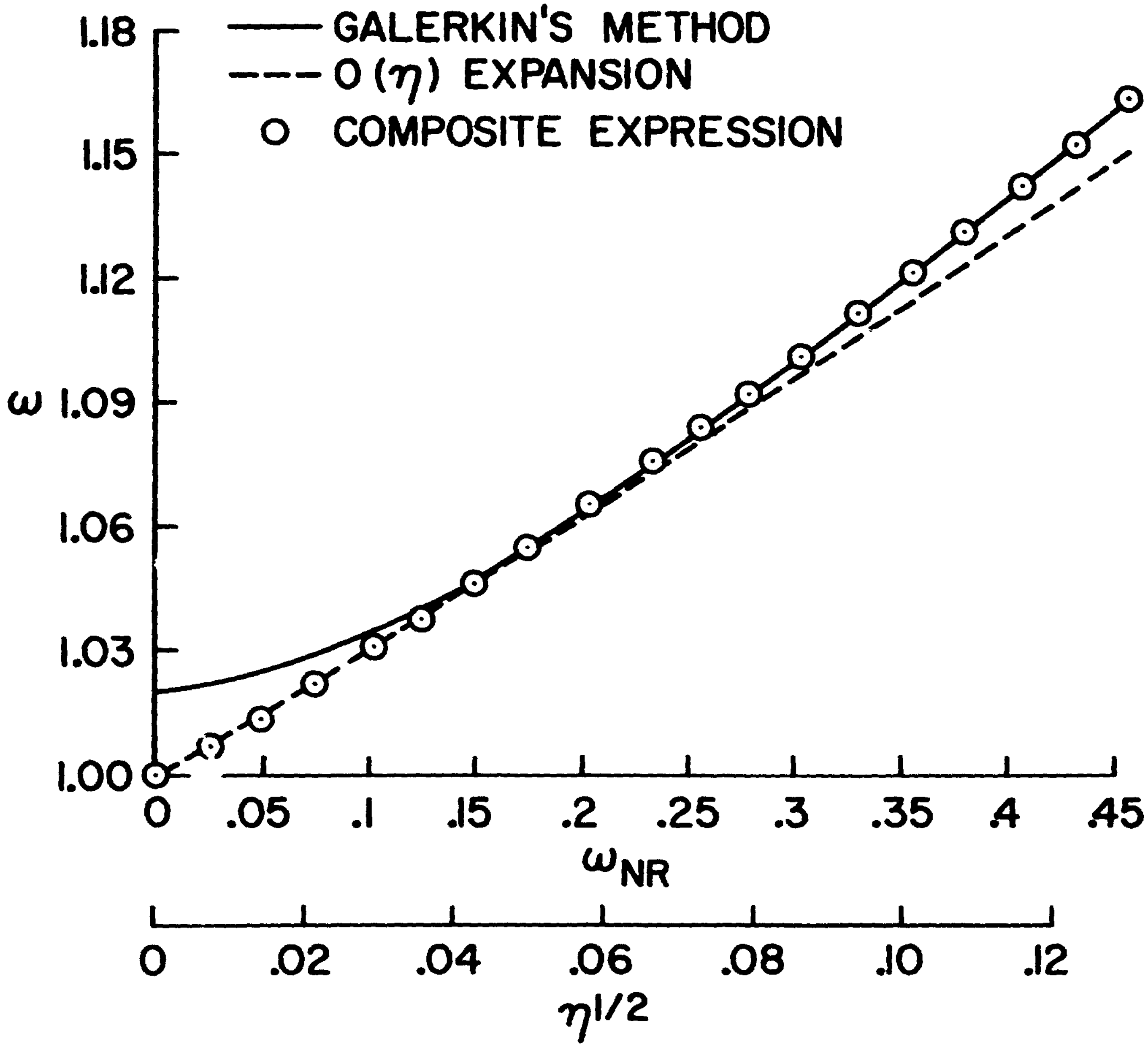

Figure 6. First frequency from composite expansion.

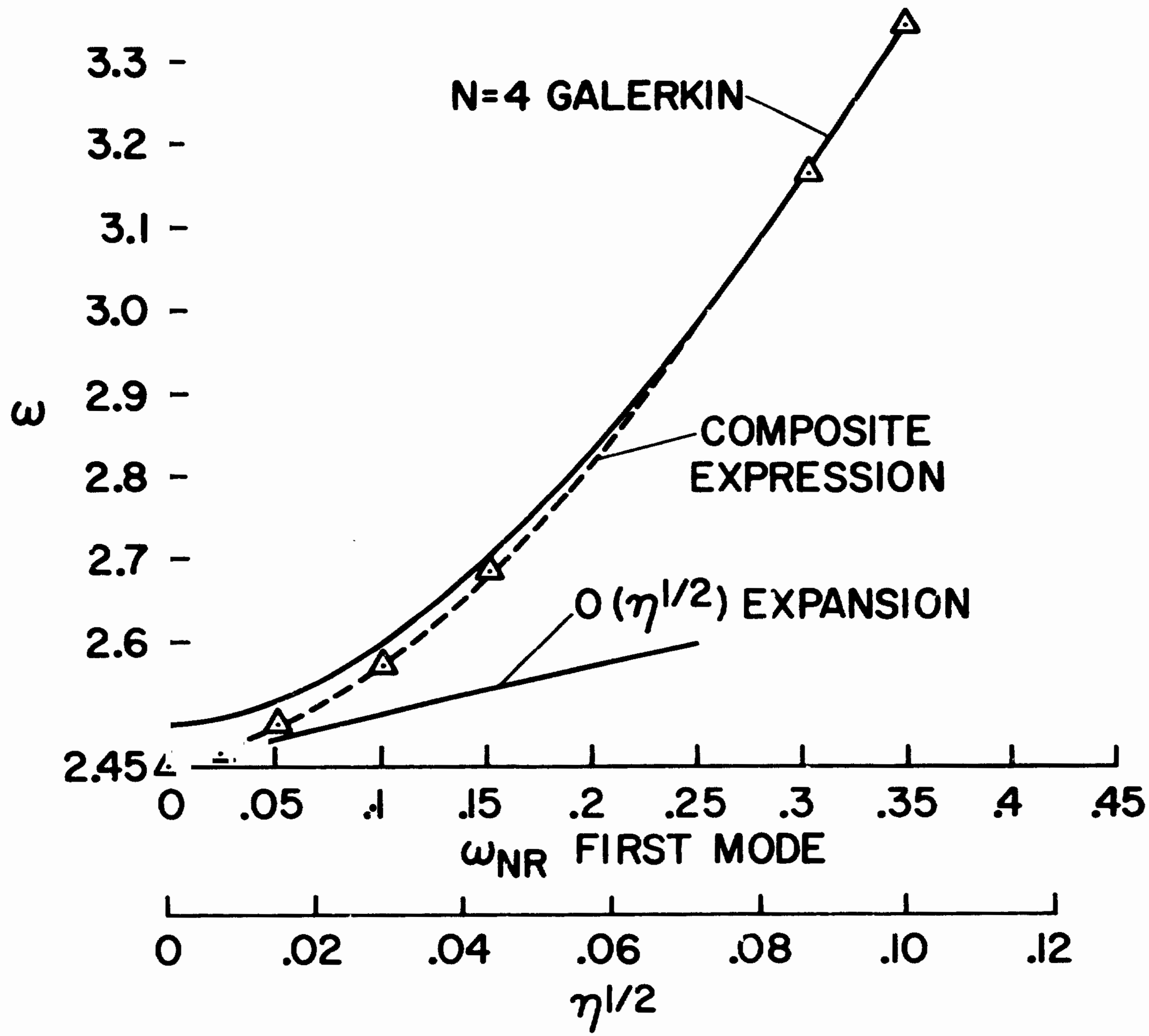

Figure 7. Second frequency from composite expansion.

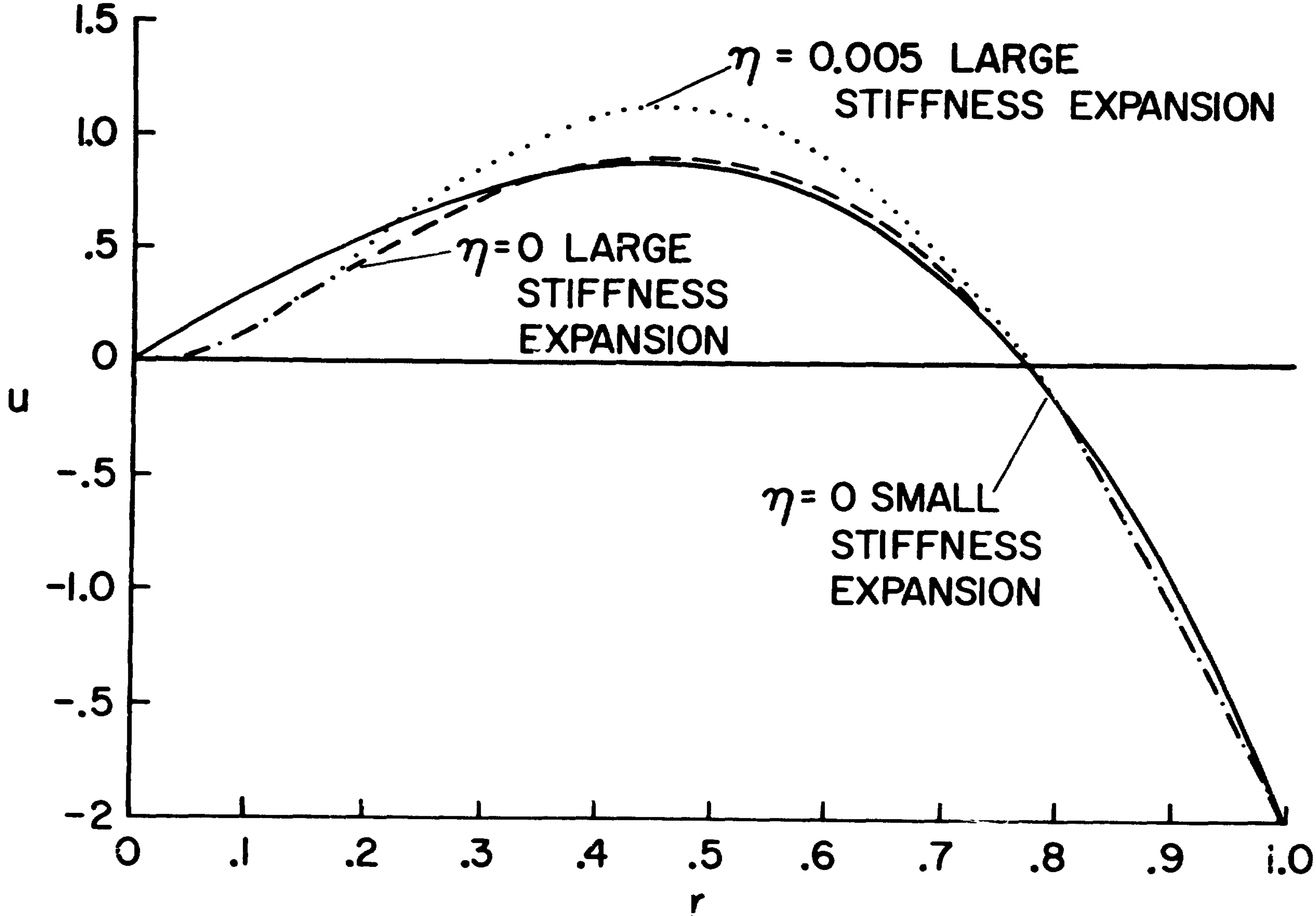

Figure 8. Second flap mode shape.

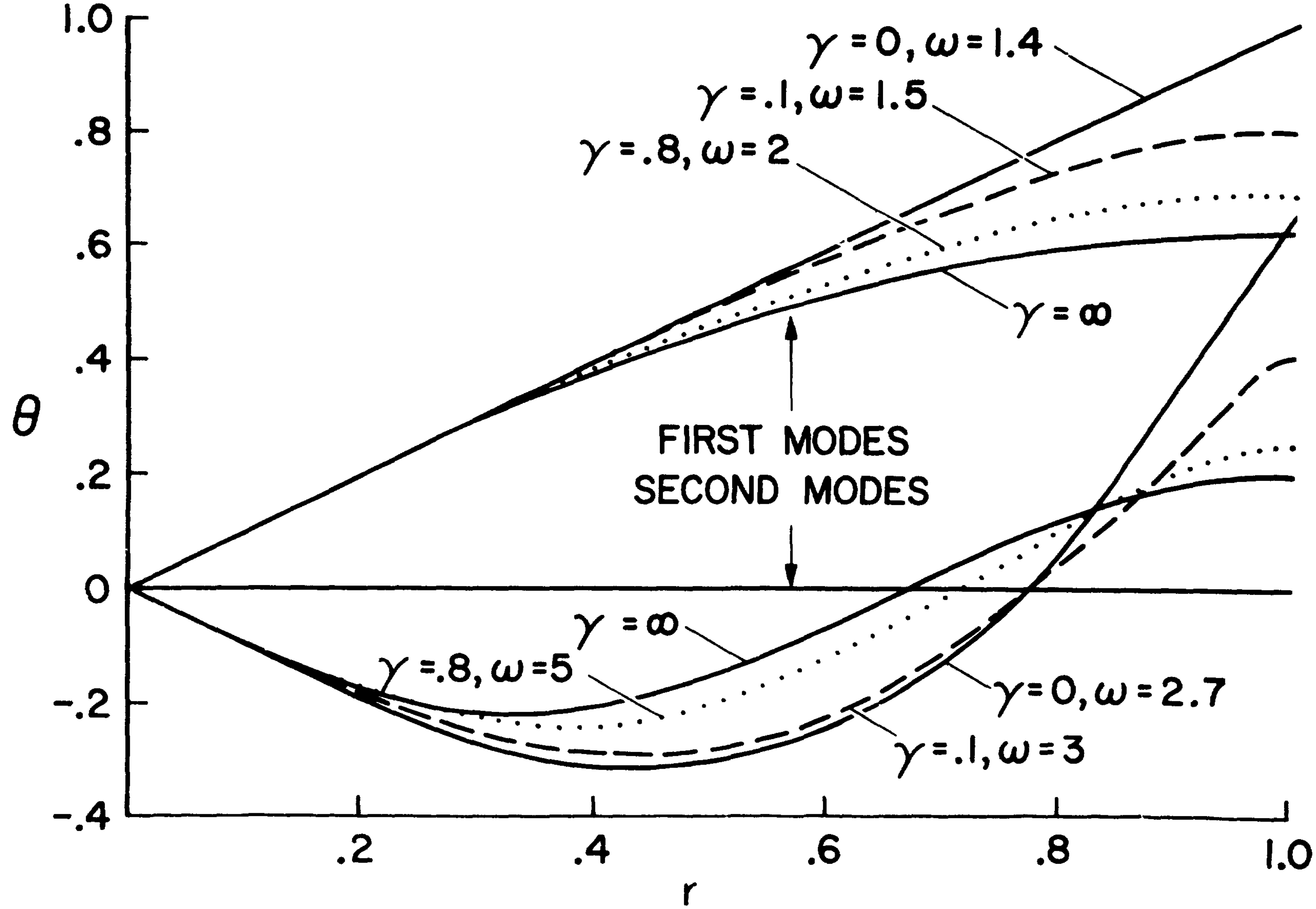

Figure 9. Torsion mode shapes.