Pergamon

Int. J. Non-Linear Mechanics, Vol. 30, No. 3, pp. 371–380, 1995

0020-7462/95 $9.50 + 0.00

0020-7462(94)00054-9

AN APPROXIMATE SOLUTION TECHNIQUE NOT DEPENDING ON SMALL PARAMETERS: A SPECIAL EXAMPLE

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(*Received* 7 *January* 1993; *accepted in final form* 21 *November* 1994)

Abstract—One simple, typical non-linear equation is used in this paper to describe a kind of analytical technique for non-linear problems. This technique is based on both homotopy in topology and the Maclaurin series. In contrast to perturbation techniques, the proposed method *does not* require small or large parameters. The example shows that the proposed method can give much better approximations than those given by perturbation techniques. In addition the proposed method can be used to obtain formulae *uniformly* valid for both small and large parameters in non-linear problems.

1. INTRODUCTION

After the appearance of supercomputers, it is not difficult for us to find the solutions of linear problems. It is however still difficult to solve non-linear problems, especially by means of analytical methods. Although the non-linear analytical techniques are fast developing, they still do not completely satisfy mathematicians and engineers.

The non-linear analytical methods most widely applied by engineers are perturbation techniques. Using pertubation methods, engineers have obtained many interesting and important results. These successes lead up to wider applications of perturbation methods in engineering. But, like other non-linear analytical techniques, perturbation methods have their own limitations. At first, all perturbation techniques are based on small or large parameters so that at least one unknown must be expressed in a series of small parameters. But unfortunately, not every non-linear equation has such a small parameter. Secondly, even if there exists such a parameter, the results given by perturbation methods are valid, in most cases, only for the small values of the parameter. Mostly, the simplified linear equations have different properties from the original non-linear equation, and sometimes some initial or boundary conditions are superfluous for the simplified linear equations. As a result, the corresponding initial approximations are perhaps far from exact. Clearly, these limitations of perturbation techniques come from the small parameter assumption. So, it seems necessary to develop a kind of new non-linear analytical method which *does not* require small parameters at all.

Liao ([7–10]) has described a non-linear analytical technique which does not require small parameters and thus can be applied to solve non-linear problems without small or large parameters. This technique is based on homotopy [15], which is an important part of topology ([1, 2, 12–14]). Using one interesting property of homotopy, one can transform any non-linear problem into an infinite number of linear problems, no matter whether or not there exists a small or large parameter. (Note that, in perturbation methods, a non-linear equation is transformed into an infinite number of linear equations by means of the small parameter assumption.) In [8], the proposed method is applied to solve two-dimensional non-linear progressive gravity waves and we obtained better analytical approximations at the fourth order than those given by perturbation methods. In [9], this method is used to give a high order streamfunction-vorticity formulation of the non-linear Navier–Stokes equations in fluid mechanics, which is suited to the boundary element

Contributed by P. B. Hagedorn.

method and has a better property of convergence so that results at high R_e have been obtained. (At present, the author has proved that this kind of BEM formulations are still stable for the shear-driven viscous flow in square cavity in the case $R_e = 10^4$.) Both gravity wave equations and Navier–Stoke's equations are difficult non-linear problems in fluid mechanics. The success of the proposed method in solving these non-linear equations show its effectiveness, although modifications and improvements may be possible.

Many researchers are unfamiliar with the non-linear gravity wave equations and Navier–Stoke's equations in fluid mechanics. So, in this paper, one simple and typical non-linear equation is used to describe the basic ideas of the proposed method. The exact solution of the selected example can be expressed by an elliptic integral of the second kind so that we can compare the approximate results to the exact solution. We will show that the proposed method can give much better approximations than perturbation techniques. In the last part of this paper, we discuss the limitations and possible modifications of the proposed method.

2. BASIC IDEAS OF HOMOTOPY METHOD

At first, let us simply introduce the basic ideas of homotopy. Homotopy is an important part of differential topology. Homotopy techniques ([3–6]) are widely applied to give all zeros of non-linear algebraic equations, for example eigenvalues.

The basic idea of the homotopy technique is simple. For example, in the case where we need to solve a non-linear algebraic equation $f(x) = 0$, $(x \in \mathbf{R})$, we can at first introduce a variable $p \in [0, 1]$, called an imbedding parameter, and select a much simpler algebraic equation $x - a = 0$. Then, we can construct a homotopy $[0, 1] \times \mathbf{R} \to \mathbf{R}$:

$$\mathscr{H}(x; p) = pf(x) + (1 - p)(x - a) \quad x \in \mathbf{R}, p \in [0, 1].$$

Clearly, we have $\mathscr{H}(x, 0) = x - a$ and $\mathscr{H}(x, 1) = f(x)$: the changing process of p from zero to unity is just that of $\mathscr{H}(x, p)$ from $(x - a)$ to $f(x)$. In topology, this is called *deformation*, and $(x - a), f(x)$ are called homotopic. Now, the solution x of the algebraic equation

$$\mathscr{H}(x, p) = pf(x) + (1 - p)(x - a) = 0 \quad p \in [0, 1], x \in \mathbf{R}$$

is a function of imbedding parameter p. Thus, differentiating the above equation with respect to p, we obtain an initial problem about $x(p)$ as follows:

$$\begin{cases} \mathrm{d}x/\mathrm{d}p = -\left(\dfrac{\partial \mathscr{H}}{\partial p} \Big/ \dfrac{\partial \mathscr{H}}{\partial x}\right), & p \in [0, 1], x \in \mathbf{R}, \\ x = a, & p = 0. \end{cases}$$

The above initial problem can be easily solved by numerical techniques, such as the Runge–Kutta method [14]. The value of x at $p = 1$ is just the solution of the original algebraic equation $f(x) = 0$. The advantage of the above technique is that, even if the initial approximation $x = a$, which can be *freely* selected, is far from the solution of the equation $f(x) = 0$, we can generally obtain one of the solutions of the equation $f(x) = 0$.

3. BASIC IDEAS OF THE PROPOSED METHOD

Now, we introduce the basic ideas of the proposed method by considering the following non-linear differential equation:

$$(1 + 4q^2x^2)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \Lambda x + 4q^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 x = 0 \tag{1}$$

with two boundary conditions

$$x(t) = A \quad \text{when } t = 0, \tag{2}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \quad \text{when } t = 0, \tag{3}$$

where q and Λ are known constants.

When $\Lambda > 0$, the above problem has a periodic solution. The exact period T can be expressed by an elliptic integral of the second kind

$$\frac{T}{T_0} = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{1 + 4q^2A^2\cos^2\phi}\,\mathrm{d}\phi, \tag{4}$$

where $T_0 = 2\pi/\sqrt{\Lambda}$, as mentioned by Nayfeh and Mook in [11].

In the case where either A or q is small, one can obtain the perturbation solution of equation (1) at the first and second order of approximation by means of perturbation techniques such as the Lindstedt–Poincaré technique or the method of multiple scales ([11]), respectively, as follows:

$$\text{first order: } \frac{\bar{T}_1}{T_0} = 1 + q^2A^2, \tag{5}$$

$$\text{second order: } \frac{\bar{T}_2}{T_0} = 1 + q^2A^2 - q^4A^4. \tag{6}$$

The comparison of the exact period (4) with the perturbation approximations (5), (6) is shown in Fig. 1. Clearly, only in the case $(qA)^2 \ll 1$, the perturbation formulas (5), (6) give good approximations of the period T. And only in the case $(qA)^2 \ll 1$, the perturbation approximations at the second order are better than those at the first-order. The convergence radius of the corresponding series given by perturbation methods may be very small. But even for a very large value of $(qA)^2$, for example, $(qA)^2 = 100000$, the motion described by equation (1) is still periodic and thus there exists certainly a value of the period T. So, it seems necessary to give a formula of T which is valid for both small and large values of $(qA)^2$.

Let $\tau = \omega t$ and $\lambda = \sqrt{\Lambda}/\omega = T/T_0$, then equation (1) becomes

$$(1 + 4q^2x^2)\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \lambda^2 x + 4q^2\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 x = 0. \tag{7}$$

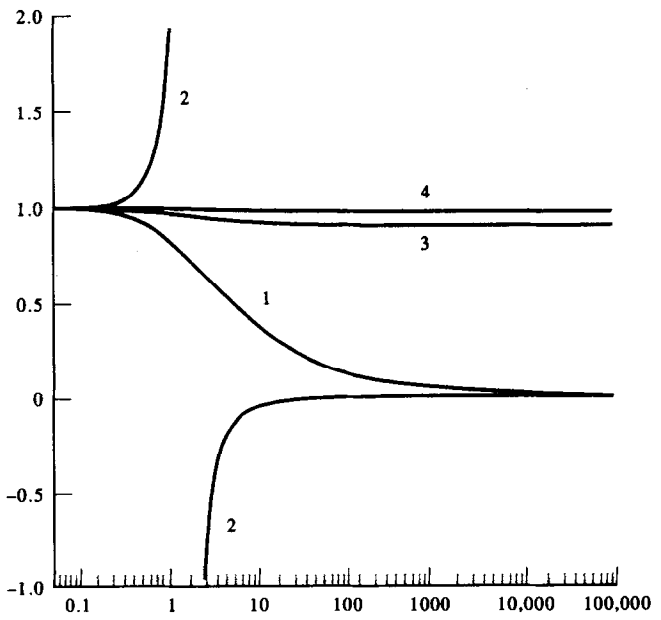

Fig. 1. The comparisons of the exact period with the approximate period. Horizontal axis: $(qA)^2$; Vertical axis: $T_{exact}/T_{approximate}$; Curve 1: first-order perturbation solution; Curve 2: second-order perturbation solution; Curve 3: solutions given by (46) [$M = 1$]; Curve 4: solutions given by (47) [$M = 2$].

We construct a transformation $x(\tau) \to X(\tau; p)$ and $\lambda \to \mu(p)$, which satisfies the following equation

$$(1 + 4q^2X^2)\frac{d^2X}{d\tau^2} + \mu^2 X + 4q^2\left(\frac{dX}{d\tau}\right)^2 X$$

$$= (1 - p)\left\{(1 + 4q^2x_0^2)\frac{d^2x_0}{d\tau^2} + \lambda_0^2 x_0 + 4q^2\left(\frac{dx_0}{d\tau}\right)^2 x_0\right\} \quad \tau \in \mathbf{R}, p \in [0, 1] \tag{8}$$

with the two boundary conditions

$$X(\tau; p) = A \quad p \in [0, 1], \tau = 0, \tag{9}$$

$$\frac{dX(\tau; p)}{d\tau} = 0 \quad p \in [0, 1], \tau = 0, \tag{10}$$

where $p \in [0, 1]$ is the imbedding parameter. Note that $\mu(p)$ is now also a function of p, because the frequency ω of the solution $X(\omega t; p)$ is unknown and is also a function of p.

When $p = 0$, we obtain, from equations (8)–(10), the initial equations ε_0, which have the solutions

$$X(\tau; 0) = x_0(\tau), \tag{11}$$

$$\mu(0) = \lambda_0, \tag{12}$$

where $x_0(\tau)$ is a *freely* selected periodic function which satisfies the two boundary conditions (2) and (3), and λ_0 is a *freely* selected constant. Therefore, $x_0(\tau)$ and λ_0 are the values of $X(\tau; p)$ and $\mu(p)$ at $p = 0$, respectively.

When $p = 1$, equation (8) is the same as the original equation (7) so that its solutions are just $x(\tau)$ and λ, that is

$$X(\tau; 1) = x(\tau), \tag{13}$$

$$\mu(1) = \lambda. \tag{14}$$

Let ε_0 and ε_f denote the equations (8)–(10) at $p = 0$ and $p = 1$, respectively. Clearly, different values of p in equations (8)–(10) correspond to different equations $\varepsilon(p)$ and to different solutions $X(\tau; p)$ and $\mu(p)$. The process of the change of p from zero to unity is the process of the change of equations $\varepsilon(p)$ from ε_0 to ε_f, and is also the process of the continuous change of the solution $X(\tau; p)$ from $x_0(\tau)$ to $x(\tau)$ and the solution $\mu(p)$ from λ_0 to λ, respectively. So, $X(\tau; p): x_0(\tau) \simeq x(\tau)$ and $\mu(p): \lambda_0 \simeq \lambda$ are homotopies. Here, $x_0(\tau)$ and $x(\tau)$ are homotopic, as are λ_0 and λ. (In fact, the equations (8)–(10) construct a homotopy operator.) For convenience, we call the equations (8)–(10) *zero order deformation equations* or *zero order process equations*, and call the proposed method the *Homotopy Analysis Method* or the *Process Analysis Method*. The reason for these terminologies will be made clear in the following development.

Let us first assume that the equations (8)–(10) have solutions $X(\tau; p)$ and $\mu(p)$ in the whole domain $p \in [0, 1]$, which are smooth enough so that $X(\tau; p)$ and $\mu(p)$ have the kth-order partial derivatives with respect to p at $p = 0$, denoted respectively as

$$x_0^{[k]}(\tau) = \left.\frac{\partial^k X(\tau; p)}{\partial p^k}\right|_{p=0} \quad (k = 1, 2, 3, \ldots), \tag{15}$$

$$\lambda_0^{[k]} = \left.\frac{\partial^k \mu(p)}{\partial p^k}\right|_{p=0} \quad (k = 1, 2, 3, \ldots). \tag{16}$$

For convenience, call $x_0^{[k]}(\tau)$ and $\lambda_0^{[k]}$ *kth-order deformation derivatives* of $X(\tau; p)$ and $\mu(p)$ at $p = 0$.

Then, one can obtain the Maclaurin series of $X(\tau; p)$ and $\mu(p)$, respectively, as follows:

$$X(\tau; p) = x_0(\tau) + \sum_{k=1}^{\infty}\left\{\frac{x_0^{[k]}(\tau)}{k!}\right\}p^k, \tag{17}$$

$$\mu(p) = \lambda_0 + \sum_{k=1}^{\infty}\left\{\frac{\lambda_0^{[k]}}{k!}\right\}p^k. \tag{18}$$

Generally, the Maclaurin series have a finite radius ρ of convergence. Clearly, in the case $\|x(\tau) - x_0(\tau)\| = 0$ and $\|\lambda - \lambda_0\| = 0$, one can easily obtain

$$x_0^{[k]}(\tau) = 0$$

and

$$\lambda_0^{[k]} = 0$$

from equations (8)–(10), so that the Maclaurin series (17) and (18) hold for any value of p. It seems that, when $\|x(\tau) - x_0(\tau)\|$ and $\|\lambda - \lambda_0\|$ increase, the radius ρ of convergence of the Maclaurin series (17) and (18) will decrease. In addition to the assumption formulated above, let us now also assume that $\|x(\tau) - x_0(\tau)\|$ and $\|\lambda - \lambda_0\|$ are small enough so that the radius ρ of convergence of the Maclaurin series (17) and (18) would be equal to or greater than one, then

$$x(\tau) = X(\tau; 1) = x_0(\tau) + \sum_{k=1}^{\infty} \frac{x_0^{[k]}(\tau)}{k!}, \tag{19}$$

$$\lambda = \mu(1) = \lambda_0 + \sum_{k=1}^{\infty} \frac{\lambda_0^{[k]}}{k!}, \tag{20}$$

which gives relations between the solutions $x(\tau)$, λ and the free selected initial approximations $x_0(\tau)$, λ_0, respectively.

At the first order of approximation, we have from (19) and (20) that

$$x(\tau) \approx x_0(\tau) + x_0^{[1]}(\tau), \tag{21}$$

$$\lambda \approx \lambda_0 + \lambda_0^{[1]}. \tag{22}$$

The $x_0^{[k]}(\tau)$ and $\lambda_0^{[k]}$ $(k = 1, 2, 3, \ldots)$ can be obtained in the similar way as follows.

Differentiating equations (8)–(10) with respect to p and then setting $p = 0$, we obtain the *first order deformation equation*

$$(1 + 4q^2x_0^2)\frac{d^2x_0^{[1]}}{d\tau^2} + \left(8q^2x_0\frac{dx_0}{d\tau}\right)\frac{dx_0^{[1]}}{d\tau} + \left\{8q^2x_0\frac{d^2x_0}{d\tau^2} + 4q^2\left(\frac{dx_0}{d\tau}\right)^2 + \lambda_0^2\right\}x_0^{[1]}$$
$$= -\left\{(1 + 4q^2x_0^2)\frac{d^2x_0}{d\tau^2} + \sigma x_0 + 4q^2\left(\frac{dx_0}{d\tau}\right)^2 x_0\right\} \tag{23}$$

with boundary conditions

$$x_0^{[1]}(0) = 0, \tag{24}$$

$$\frac{dx_0^{[1]}(0)}{d\tau} = 0, \tag{25}$$

where

$$\sigma = \lambda_0^2 + 2\lambda_0\lambda_0^{[1]}. \tag{26}$$

Select the periodic function $x_0(\tau) = A\cos(\tau)$ as the initial approximation and let

$$x_0^{[1]}(\tau) = \sum_{k=0}^{\infty} B_{2k+1}\cos(2k + 1)\tau \tag{27}$$

denote the solution of the first order deformation equations (23)–(25), where B_{2k+1} $(k \geq 0)$ are unknowns. Then we have

$$\frac{dx_0^{[1]}}{d\tau} = -\sum_{k=0}^{\infty} (2k + 1)B_{2k+1}\sin(2k + 1)\tau, \tag{28}$$

$$\frac{d^2x_0^{[1]}}{d\tau^2} = -\sum_{k=0}^{\infty} (2k + 1)^2 B_{2k+1}\cos(2k + 1)\tau, \tag{29}$$

and the boundary condition (25) can be satisfied automatically.

Substituting $x_0(\tau) = A\cos(\tau)$ into the right side of equation (23), we have

$$-\left\{(1+4q^2x_0^2)\frac{d^2x_0}{d\tau^2}+\sigma x_0+4q^2\left(\frac{dx_0}{d\tau}\right)^2 x_0\right\}$$
$$= A\{[(1+2q^2A^2)-\sigma]\cos\tau+2q^2A^2\cos 3\tau\}. \tag{30}$$

Substituting equations (27)–(29) and $x_0(\tau) = A\cos(\tau)$ into the left side of equation (23), we obtain

$$(1+4q^2x_0^2)\frac{d^2x_0^{[1]}}{d\tau^2}+\left(8q^2x_0\frac{dx_0}{d\tau}\right)\frac{dx_0^{[1]}}{d\tau}+\left\{8q^2x_0\frac{d^2x_0}{d\tau^2}+4q^2\left(\frac{dx_0}{d\tau}\right)^2+\lambda_0^2\right\}x_0^{[1]}$$
$$=\sum_{k=0}^{\infty}\{(\lambda_0^2-2q^2A^2)-(1+2q^2A^2)(2k+1)^2\}B_{2k+1}\cos(2k+1)\tau$$
$$-q^2A^2\sum_{k=0}^{\infty}\{(2k+1)^2+2(2k+1)+3\}B_{2k+1}\cos(2k+3)\tau$$
$$-q^2A^2\sum_{k=0}^{\infty}\{(2k+1)^2-2(2k+1)+3\}B_{2k+1}\cos(2k-1)\tau$$
$$=\{[(\lambda_0^2-1)-6q^2A^2]B_1-6q^2A^2B_3+\cdots\}\cos\tau$$
$$+\{-6q^2A^2B_1+[(\lambda_0^2-9)-20q^2A^2]B_3+\cdots\}\cos 3\tau+\cdots. \tag{31}$$

In expression (27), we only use a finite number of terms M. We give the solutions at the first order of approximation in the case of one term only ($M = 1$) or two terms ($M = 2$), respectively.

3.1. *Approximation of T/T_0 in the case $M = 1$*

In the case of only one term in (27), comparing the coefficients of equations (30) and (31), which are respectively the two sides of equation (23), we have

$$B_1 = \left[\frac{(1+2q^2A^2)-\sigma}{\lambda_0^2-1-6q^2A^2}\right]A. \tag{32}$$

Owing to (24), we get $B_1 = 0$. Therefore, we have

$$\sigma = 1+2q^2A^2. \tag{33}$$

Due to (26), we also obtain

$$\lambda_0^{[1]} = \frac{\sigma-\lambda_0^2}{2\lambda_0}. \tag{34}$$

Thus, we obtain from (22) that

$$\lambda \approx \lambda_0 + \frac{\sigma-\lambda_0^2}{2\lambda_0}. \tag{35}$$

which would be a better approximation than λ_0. Note that λ_0 can be freely selected, so that the equation (35) gives an iterative procedure:

$$\lambda_{k+1} = \lambda_k + \frac{\sigma-\lambda_k^2}{2\lambda_k}. \tag{36}$$

It should be emphasized that, as mentioned before, λ_0 can be now *freely* selected. Clearly, when $(qA)^2$ is very small, $T_0 = 2\pi/\sqrt{\Lambda}$ is a good approximation of the period T. Therefore, we can at first select $\lambda_0 = 1$ as a rough approximation of T. Thus, according to (36) and (33), we have

$$\lambda_1 = 1+q^2A^2,$$

which is the same as the perturbation result (5) at first-order of approximation.

But, different from perturbation methods, we can now select different initial values of λ. Clearly, $(1 + q^2A^2)$ is a better initial value of λ than 1.0, which is used in the perturbation methods. Thus, substituting $\lambda_1 = 1 + q^2A^2$ into (36), we have

$$\lambda_2 = 1 + q^2A^2\left(\frac{2 + q^2A^2}{2 + 2q^2A^2}\right),$$

which is a better approximation of $\lambda = T/T_0$ than the perturbation approximation (5). This gives a series of the approximate values of λ. For example, in the case $(qA)^2 = 1.0$, we have

$$\lambda_0 = 1.000,$$

$$\lambda_1 = 2.000,$$

$$\lambda_2 = 1.750,$$

$$\lambda_3 = 1.732,$$

$$\lambda_4 = 1.732,$$

$$\ldots\ .$$

Note that, according to (4), the exact value of T/T_0 in the case $(qA)^2 = 1.0$ is 1.678. Thus, using the iterative formulas (36), we can obtain a series $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots$ of the approximate values of T/T_0, which has the property that λ_{k+1} is a better approximation than λ_k. Therefore, in the limit

$$\lambda = \lim_{k \to +\infty} \lambda_k,$$

one obtains

$$\lambda = \lambda + \frac{\sigma - \lambda^2}{2\lambda}. \tag{37}$$

Therefore we have

$$\lambda = \sqrt{\sigma} \tag{38}$$

as the 'best' approximation at the first order. Setting $\lambda_0^{[1]} = 0$ in expression (26) gives the same result as (38). This means the selected value of λ_0 is so good that no modification of λ_0 is needed.

Substituting (33) into (38) gives

$$\lambda = \sqrt{1 + 2q^2A^2}. \tag{39}$$

3.2. *Approximation of T/T_0 in the case $M = 2$*

If two terms are used in (27), comparing the coefficients of equations (30) and (31), which are respectively the two sides of the equation (23), we obtain two linear algebraic equations for the two unknowns B_1 and B_3:

$$[(\lambda_0^2 - 1) - 6q^2A^2]B_1 - 6q^2A^2B_3 = A[(1 + 2q^2A^2) - \sigma], \tag{40}$$

$$-6q^2A^2B_1 + [(\lambda_0^2 - 9) - 20q^2A^2]B_3 = 2q^2A^2. \tag{41}$$

From the above set of equations, we can obtain the expressions of the unknowns B_1 and B_3, which are functions of λ_0 and σ.

Owing to the boundary condition (24), one has $B_1 + B_3 = 0$ and from this follows

$$\sigma = \frac{(9 + 34q^2A^2 + 28q^4A^4) - (1 + 4q^2A^2)\lambda_0^2}{14q^2A^2 + (9 - \lambda_0^2)}. \tag{42}$$

According to (38)

$$\lambda^2 = \frac{(9 + 34q^2A^2 + 28q^2A^4) - (1 + 4q^2A^2)\lambda^2}{14q^2A^2 + (9 - \lambda^2)} \tag{43}$$

also holds. This is an algebraic equation in λ^2, which has the solution

$$\lambda = \sqrt{5 + 9q^2A^2 \pm \sqrt{16 + 56q^2A^2 + 53q^4A^4}}. \tag{44}$$

Clearly

$$\lim_{(qA)^2 \to 0} \lambda = \sqrt{5 \pm 4}$$

must be equal to one, so we select the ' $-$ ' sign in expression (44). Therefore, we have

$$\lambda = \sqrt{5 + 9q^2A^2 - \sqrt{16 + 56q^2A^2 + 53q^4A^4}}. \tag{45}$$

Thus, we obtain two approximate expressions for the non-dimensional period λ at the first order of approximation as follows:

$$\text{one term used } (M = 1): \quad \frac{T_1}{T_0} = \sqrt{1 + 2q^2A^2}, \tag{46}$$

$$\text{two terms used } (M = 2): \quad \frac{T_2}{T_0} = \sqrt{5 + 9q^2A^2 - \sqrt{16 + 56q^2A^2 + 53q^4A^4}}, \tag{47}$$

where $T_0 = 2\pi/\sqrt{\Lambda}$.

The comparison of the above approximate formulas with the perturbation formulas (5) and (6) is shown in Fig. 1.

As mentioned before, the perturbation formulas (5) and (6) can give good approximations only in the case $(qA)^2 \ll 1$. Even at $(qA)^2 = 1$, these perturbation approximations have errors larger than 16% and 67%, respectively. On the other hand, in the case $(qA)^2 = 1.0$, the expressions (46) and (47) give respectively only 3.1% and 0.1% errors. It is interesting that even at $(qA)^2 = 10^5$, the formulas (46) and (47) lead to errors smaller than 10% and 3%, respectively, whereas the perturbation formula (5) gives errors larger than 99%. Note that, in the case $(qA)^2 > (1 + \sqrt{5})/2$, the perturbation formula (6) at the second order of approximation gives negative values of λ which have certainly no meaning. But, even in case $(qA)^2$ tending to infinity, we have

$$\lim_{|qA| \to \infty} \frac{T}{T_1} = \lim_{|qA| \to \infty} \frac{\frac{2}{\pi}\int_0^{\pi/2} \sqrt{1 + 4q^2A^2\cos^2\phi}\,\mathrm{d}\phi}{\sqrt{1 + 2q^2A^2}} = \frac{2\sqrt{2}}{\pi} \approx 0.900 \tag{48}$$

and

$$\lim_{|qA| \to \infty} \frac{T}{T_2} = \lim_{|qA| \to \infty} \frac{\frac{2}{\pi}\int_0^{\pi/2} \sqrt{1 + 4q^2A^2\cos^2\phi}\,\mathrm{d}\phi}{\sqrt{5 + 9q^2A^2 - \sqrt{16 + 56q^2A^2 + 53q^2A^4}}}$$

$$= \left(\frac{4}{\pi}\right)\frac{1}{\sqrt{9 - \sqrt{53}}} \approx 0.971. \tag{49}$$

Therefore, for *any* value of $(qA)^2$, it can be easily proved that

$$0 \le \frac{|T - T_1|}{T_1} \le 10\%, \tag{50}$$

$$0 \le \frac{|T - T_2|}{T_0} \le 3\%. \tag{51}$$

So, the formulas (46) and (47) are *uniformly valid* for *any* values of $(qA)^2$. (Note that the perturbation formulas (5) and (6) are valid only in the case $(qA)^2 \ll 1$.) It should be pointed out that the formulas (46) and (47) are much simpler and cheaper to evaluate than the exact expression (4). Note also that, formulas (46) and (47) are only first order approximations.

4. LIMITATIONS AND NEEDED IMPROVEMENTS

In this paper, we have used a simple example to describe the basic ideas of a non-linear analytical technique, showing its reasonableness and effectiveness. Based on this simple example, we make some additional remarks about this method.

At first, the proposed method can give much better analytical approximations than perturbation methods. Secondly, the results obtained by the proposed method are uniformly valid for both small and large parameters. This is so mainly because the proposed method is based on the simple property of homotopy in topology, that is, any kth-order deformation equations are linear about the kth-order deformation derivatives. By means of this property of homotopy, any non-linear problem can be transformed into infinite number of linear problems, no matter whether or not there exists a small or large parameter. This is the principal difference between the proposed method and the perturbation techniques. (Note that all perturbation methods need one or more small parameters to transform a non-linear equation into a series of linear equations, therefore, it is certainly reasonable that the perturbation results are usually valid only for the small values of the parameters. It seems that this limitation of perturbation methods can be overcome by the proposed method.)

In opposition to perturbation techniques, the initial approximations can be now *freely* selected. So, in this paper, we can use the methods of iteration and limit to find the 'best' approximation of the non-dimensional period λ. This means that the initial approximations are *not* simply *freely* selected, although we *can* do so. The example shows that, this kind of iterative technique is reasonable and very effective, and can give much better approximations than perturbation techniques. It is interesting that improvements are obtained if we consider the selection of the initial approximations from a totally new view point.

Without the limitation of the small parameter assumption, the proposed method gives much greater freedom to select the initial approximations. Thus, we can use many other techniques to find some simple and reasonable initial approximations. This is perhaps why the proposed method can give much better approximations, which are valid for both small and large parameters, than perturbation techniques.

Although the simple example given in this paper is a non-linear differential equation, the proposed method can be also applied to solve non-linear algebraic equations and partial differential equations, such as the full Navier–Stokes equations, the full equations of gravity waves, the KdV equation, the Boussinesq's equation, the Sine–Gorden equation and so on.

However, the proposed method is still unsatisfactory and certainly many improvements are needed. Although the proposed method has shown its effectiveness and reasonableness in many examples ([7–10]), the theoretical research is urgently needed, because the examples do not constitute mathematical proof. One problem which needs to be answered is: under which conditions is the radius of convergence of the Maclaurin series [in form similar as expression (17)] equal to or greater than one? Note that, when $p = 0$, the right side of equation (8) expresses the errors of the original equation. Clearly, if the selected initial solutions are just the exact ones, then there exist no errors at all and the right side of expression (8) is zero. In this case, it can be easily proved that any kth-order deformation derivatives are zero, so that the corresponding Maclaurin series converges for any values of p. So, if the selected initial approximations are good enough, the radius of convergence ρ should be greater than unity. The Maclaurin series given by the proposed method seems to be somewhat special: its radius of convergence is dependent upon the selection of the initial approximations. Thus, if the initial solutions are suitably selected, the radius of convergence of the Maclaurin series given by the proposed method should be greater than one, as shown in this paper. This is however presently an unproved assumption. Although most Maclaurin series have finite radius of convergence greater than zero, unfortunately, there exist indeed some Maclaurin series having radius of convergence equal to zero, and in these cases, the proposed method will fail. So, in practice, we should examine the results obtained by the proposed method. (Note that any perturbation solutions must be examined also.)

The proposed method can overcome some limitations of perturbation techniques: it does not require small or large parameters, and may give better approximations which are uniformly valid for both small and large parameters or variables. However, as the perturbation techniques, the proposed method is not perfect and has its own limitations: it must be

assumed that, the solutions of the zero-order deformation equations, similar to equations (8)–(10), exist and are analytic on $p \in [0, 1]$, and the radius of convergence of the corresponding Maclaurin series, similar to (17) and (18) as mentioned in this paper, should be greater than or equal to one. Each of these may fail to happen. Even then, however, the proposed technique has been successfully applied in some considered non-linear problems to give satisfactory results. So, as one of the attempts to develop a satisfactory non-linear analytical technique which does not require small parameters and can give uniformly valid approximations for both small and large parameters or variables, the proposed method seem to be promising, although more applications and especially the theoretical research are needed to improve it.

Acknowledgements—This work is supported by the Chinese State Education Committee. The author wishes to thank the reviewers of the *International Journal of Non-Linear Mechanics* for their suggestions. In particular, I would like to express my sincere thanks to Professor Peter Hagedorn (Institut für Mechanik, Technische Hochschule Darmstadt) for his helpful suggestions and also for improving the wording of this paper.

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