An approximation result for special functions with bounded deformation

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Abstract

We show in this paper that in a domain $\Omega \subset \mathbb{R}^2$ with some regularity, a function $u \in SBD(\Omega)$ with $u, e(u) \in L^2$ and $\mathcal{H}^1(J_u) < +\infty$ can be approximated with a sequence u_n with relatively closed jump set J_{u_n} in Ω, such that u_n and $e(u_n)$ respectively converge to u and $e(u)$ in L^2 (strong) while $\lim_{n\to\infty} \mathcal{H}^1(\overline{J}_{u_n}) = \mathcal{H}^1(J_u)$.

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1 Introduction

Special Bounded Deformation displacements have been introduced by Ambrosio, Bellettini, Dal Maso, Coscia [4, 8] to represent displacements in linearized elasticity problems with discontinuities (that may model cracks in the material). Given $u \in \Omega$, where Ω is an open subset of $\mathbb{R}^N$, one says that a displacement $u : \Omega \to \mathbb{R}^N$ has bounded deformation whenever the symmetric part of the distributional derivative $\mathcal{E}(u) = (Du + Du^T)/2$ is a bounded Radon measure. In this case, it is proven in [4] that the measure $\mathcal{E}(u)$ can be decomposed into three parts, one absolutely continuous with respect to the Lebesgue measure dx, denoted by $e(u)\,dx$, and two other that are singular: a jump part, carried by the rectifiable $(N-1)$–dimensional set J_u of points where the function u as two different approximate limits u_+ and u_-, together with a normal vector ν_u, and a "Cantor part", which vanishes on Borel sets of finite $\mathcal{H}^{N-1}$ measure.

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The space $SBD(\Omega)$ is defined as the space of the bounded deformation functions u such that the Cantor part of $\mathcal{E}(u)$ vanishes, so that this measure can be written

$$\mathcal{E}(u) \;=\; e(u)(x)\,dx \;+\; (u_+(x) - u_-(x)) \odot \nu_u(x)\mathcal{H}^{N-1} \llcorner J_u(x) \tag{1}$$

where $\mathcal{H}^{N-1} \llcorner J_u$ is the $(N-1)$–dimensional Hausdorff measure restricted to J_u and $a \odot b$ denotes the symmetrized tensor product $(a \otimes b + b \otimes a)/2$.

These functions are useful in the theory of brittle crack evolution, following a model proposed by Francfort and Marigo [22, 23]. One can define a "Mumford-Shah"–like potential energy of the form $E(u) = \int_\Omega W(e(u))\,dx + \mathcal{H}^{N-1}(J_u)$, with W some linearized elasticity bulk energy, and roughly define a discrete evolution with timestep $\delta t > 0$ by letting, for every $n \in \mathbb{N}$, u_n be a minimizer of $E(u)$ among all u with $u = g(n\delta t)$ and $J_{u_n} \supset J_{u_{n-1}}$, where $g(t)$ is a given boundary displacement and the second condition expresses the fact that the fracture is irreversible and can only grow. At this point, several problems arise. Does each minimization problem have a solution? Does there exist some limit evolution as $\delta t \downarrow 0$? Some of these issues are addressed in [3, 18, 16, 21, 17], for variants of this problem (scalar versions, topological restrictions on the cracks, nonlinear elasticity). However, in the case of linearized elasticity, a study of this problem is still out of reach for many technical reasons. Interesting also would be to find a way to numerically minimize energy E, in order to simulate crack growth. In [11], such experiments have been conducted, that are based on a Ambrosio and Tortorelli [6, 7] approximation of energy E, in the case where W is a positive definite quadratic form of the deformation $e(u)$. But the Γ–convergence of this approximation to E is not known. A major issue is in the proof of the Γ–limsup: in Ambrosio and Tortorelli's works, it relies strongly on the fact that any function in $SBV(\Omega)$ with finite Mumford-Shah energy $\int |\nabla u|^2 + \mathcal{H}^{N-1}(S_u)$ can be approximated by functions u_n such that the jump set S_{u_n} is closed. No such result exists up to now for SBD functions.

In this paper we propose an approach to prove such a property, and show, only in dimension $N = 2$ and for W with quadratic growth, that provided Ω is bounded and $\partial\Omega$ is locally a subgraph, any $u \in SBD(\Omega) \cap L^2(\Omega;\mathbb{R}^2)$ with $E(u) < +\infty$ can be approximated (in L^2) by a sequence u_n such that $\limsup_{n\to\infty} \int_\Omega W(e(u_n))\,dx + \mathcal{H}^1(\overline{J}_{u_n}) \le \int_\Omega W(e(u))\,dx + \mathcal{H}^1(J_u)$. It turns out that the jump set J_{u_n} that we build is included in a finite union of closed connected C^1 curves, whose total length goes to $\mathcal{H}^1(J_u)$ as $n \to \infty$. The proof we give is probably valid in any dimension, up to a few modifications, however, one step requires an inequality that depends strongly on the dimension, and that we only have proven in dimension 2 (see Appendix A).

Using a SBD semicontinuity result proven in [8], we deduce the convergence of $e(u_n)$ to $e(u)$ in L^2–strong, and the convergence of $\mathcal{H}^1(\overline{J}_{u_n})$ to $\mathcal{H}^1(J_u)$. On the other hand, we do not know whether the sequence $(u_n)_{n\ge1}$ we build can be uniformly bounded in BD.

As a consequence we deduce the Γ–convergence of an Ambrosio and Tortorelli [6, 7] approximation of the elasticity Mumford Shah functional (in 2D), with an L^∞ constraint. This justifies in part the numerical computations presented in [11].

2 Mathematical preliminaries

In this section we recall some of the results of [4] and [8] on BD and SBD functions that will be useful for our analysis. We assume that the corresponding properties for BV and SBV functions are known to the reader, we refer to [5] for a good monograph on the topic.

2.1 Main notations.

In this paper, we will denote by dx the Lebesgue measure in $\mathbb{R}^N$, $N \geq 1$ (we will sometimes also denote $|E| = \int_E dx$ the measure of the set E), while $\mathcal{H}^n$, $n \leq N$, is the n–dimensional Hausdorff measure (see for instance [20]). Given $E, F \in \mathbb{R}^N$, we denote by $E \triangle F = (E \setminus F) \cup (F \setminus E)$ their symmetric difference. In $\mathbb{R}^N$, $a \cdot b = \sum_{i=1}^N a_i b_i$ is the Euclidean scalar product, and we denote the norm by $|a| = \sqrt{a \cdot a}$. For any $a \in \mathbb{R}^N$, $a^\perp = \{x \in \mathbb{R}^N : a \cdot x = 0\}$ is the hyperplane (if $a \neq 0$) orthogonal to a. $B(x, r) = \{y \in \mathbb{R}^N : |x - y| < r\}$ is the (open) ball of center x and radius r, and $\overline{B}(x, r) = \{|y - x| \leq r\}$ is its closure. The notation ω_N stands for the volume of the unit ball in $\mathbb{R}^N$, $|B(0, 1)|$, and one has $N\omega_N = \mathcal{H}^{N-1}(\mathbb{S}^{N-1})$, where $\mathbb{S}^{N-1} = \partial B(0, 1)$.

We will also let $\mathcal{S}^{N\times N}$ be the $(N(N+1)/2)$–dimensional vector space of the symmetric $N \times N$ matrices. For A a matrix of size $N \times N$, we let $|A| = \sqrt{\operatorname{Tr}(AA^T)}$ (A^T is the transpose of A and $\operatorname{Tr} A$ its trace)—this defines the standard Euclidean norm in the space of all $N \times N$ matrices. If $a, b \in \mathbb{R}^N$, the tensor product $a \otimes b$ is the matrix $(a_i b_j)_{i,j=1}^N$ while $a \odot b \in \mathcal{S}^{N\times N}$, the symmetrized tensor product, is $(a \otimes b + b \otimes a)/2$. Notice that $|a||b|/\sqrt{2} \leq |a \odot b| \leq |a||b|$.

2.2 BD functions.

As mentionned in the introduction, the space $BD(\Omega)$ of displacements with bounded deformation in $\Omega \subset \mathbb{R}^N$ is the set of $u \in L^1(\Omega; \mathbb{R}^N)$ such that the symmetrized distributional gradient

$$\mathcal{E}(u)_{i,j} = \frac{1}{2}(D_i u_j + D_j u_i)$$

($i, j = 1, \ldots, N$) is a bounded Radon measure in Ω (a matrix–valued measure with finite total variation). We refer to [4] and the references herein for more details on this space, which has been introduced in order to describe plastic deformations in a solid.

Given u in $BD(\Omega)$, one says that $x \in \Omega$ has one–sided limits $u_-(x)$ and $u_+(x)$ at x, with respect to the direction $\nu_u(x) \in \mathbb{S}^{N-1}$, if the rescaled functions $u_\rho(y) := u(x + \rho y)$, $y \in B(0, 1)$, converge in $L^1(B(0, 1); \mathbb{R}^N)$ to

$$u_0(y) = \begin{cases} u_+(x) & \text{if } y \cdot \nu_u(x) > 0\,, \\ u_-(x) & \text{if } y \cdot \nu_u(x) < 0\,, \end{cases}$$

as $\rho \to 0$. If $u_+(x) \neq u_-(x)$, then the triplet $(u_+(x), u_-(x), \nu_u(x))$ is unique up to a change of sign of $\nu_u(x)$ together with a permutation of $u_+(x)$ and $u_-(x)$. In this case, we say that $x \in J_u$, the jump set of u. (If $u_+(x) = u_-(x)$ then x is a Lebesgue point of u, with Lebesgue limit $u_+ = u_-$, and $\nu_u(x)$ is arbitrary.)

It is shown in [4, Prop. 3.5] that J_u is a countably $(\mathcal{H}^{N-1}, N-1)$–rectifiable Borel set: there exists $(\Gamma_i)_{i=1}^\infty$ a sequence of C^1 hypersurfaces covering almost all of J_u, that is, $\mathcal{H}^{N-1}(J_u \setminus (\cup_{i=1}^\infty \Gamma_i)) = 0$.

At $\mathcal{H}^{N-1}$–almost all $x \in J_u$, $\nu_u(x)$ is an approximate normal to J_u, characterized by

$$\nu_u(x) \;=\; \pm\nu_{\Gamma_i}(x) \;\text{ at } \mathcal{H}^{N-1}\text{–a.e. } x \in J_u \cap \Gamma_i.$$

2.3 Structure of $\mathcal{E}(u)$. SBD functions.

The structure of the distributional deformation $\mathcal{E}(u)$ of u is described in Section 4 of [4]: one has (see Def. 4.1, Thm. 4.3, Prop. 4.4)

$$\mathcal{E}(u) \;=\; e(u)\,dx \;+\; (u_+ - u_-) \odot \nu_u \mathcal{H}^{N-1} \llcorner J_u \;+\; \mathcal{E}^c(u)$$

where:

- $e(u) \in L^1(\Omega; \mathcal{S}^{N\times N})$ is the Radon-Nikodym derivative of $\mathcal{E}(u)$ with respect to the Lebesgue measure dx. It is called the approximate symmetric differential of u, and is characterized (Lebesgue–) almost everywhere in Ω by
$$\lim_{\rho\to 0} \frac{1}{\rho^N} \int_{B(x,\rho)} \frac{|u(y) - u(x) - (e(u)(y-x))\cdot(y-x)|}{|y-x|^2}\, dy \;=\; 0.$$
- The measure $\mathcal{E}^c(u)$ (the "Cantor part") vanishes on any Borel set $B \subset \Omega$ which is σ–finite with respect to $\mathcal{H}^{N-1}$.

The space $SBD(\Omega)$ is defined as the set of all displacements $u \in BD(\Omega)$ such that $\mathcal{E}^c(u) = 0$. It means that the singular part (with respect to the Lebesgue measure) of the derivative of u is entirely carried by the jump set J_u. An important compactness result is given by [8, Thm. 1.1]: it is shown that if a sequence $(u_n)_{n\geq 1}$ in $SBD(\Omega)$ is such that

$$\sup_{n\geq 1} \int_\Omega |u_n|\,dx + \int_{J_{u_n}} |(u_n)_+ - (u_n)_-|\,d\mathcal{H}^{N-1}(x) + \int_\Omega W(e(u_n))\,dx + \mathcal{H}^{N-1}(J_{u_n}) \;<\; +\infty$$

for some nonnegative bulk energy W with $\lim_{|A|\to\infty} W(A)/|A| = +\infty$, then, up to a subsequence, there exists $u \in SBD(\Omega)$ such that $u_n \to u$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $e(u_n) \rightharpoonup e(u)$ weakly in $L^1(\Omega;\mathcal{S}^{N\times N})$, $\mathcal{E}(u_n) \overset{*}{\rightharpoonup} \mathcal{E}(u)$ weakly$-*$ as a bounded measure and $\mathcal{H}^{N-1}(J_u) \leq \liminf_{n\to\infty} \mathcal{H}^{N-1}(J_{u_n})$. We will need a variant of this result, where the a priori bound on the measures $|\mathcal{E}(u_n)|$ is replaced by the knowledge that u_n converges to $u \in SBD(\Omega)$. Since we will need the result only when W is a particular quadratic form of $e(u)$, a case in which the proof is quite simpler than in [8], in order to make this paper more self-contained we will give a short proof of our variant (Lemma 5.1 in Section 5).

2.4 Slicing properties.

Essential to the proofs in this paper are the slicing properties of SBD functions, that allow to characterize them by means of SBV functions on lines. If $u \in SBD(\Omega)$, $e \in$

$\mathbb{S}^{N-1}$ and $z \in e^\perp$, we denote by $u^e_z(s)$ the function $u(z+se)\cdot e$, that is defined on $\Omega^e_z = \{s \in \mathbb{R} : z+se \in \Omega\}$. We also let $J^e_u = \{x \in J_u : [u(x)]\cdot e \neq 0\}$ (where $[u(x)]$ denotes the jump $u_+(x) - u_-(x)$). Then, from the Structure Theorem [4, Thm. 4.5], we have that for $\mathcal{H}^{N-1}$–a.e. $z \in e^\perp$, the function u^e_z is in $SBV(\Omega^e_z)$ (unless Ω^e_z is empty), $(e(u)(z+se)e)\cdot e = (u^e_z)'(s)$ a.e. in Ω^e_z, $S_{u^e_z} = \{s \in \mathbb{R} : z+se \in J^e_u\}$, and for all $s \in S_{u^e_z}$,

$$\left\{u^+(z+se)\cdot e, u^-(z+se)\cdot e\right\} \;=\; \left\{(u^e_z)^+(s), (u^e_z)^-(s)\right\}\,.$$

One has that

$$\int_{e^\perp} \mathcal{H}^0(S_{u^e_z})\, d\mathcal{H}^{N-1}(z) \;=\; \int_{J^e_u} |\nu_u(x)\cdot e|\, d\mathcal{H}^{N-1}(x),$$

while

$$\frac{1}{2\omega_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(e) \int_{e^\perp} \mathcal{H}^0(S_{u^e_z})\, d\mathcal{H}^{N-1}(z) \;=\; \mathcal{H}^{N-1}(J_u).$$

Notice that $\mathcal{H}^{N-1}(J_u \setminus J^e_u) = 0$ for $\mathcal{H}^{N-1}$–a.e. $e \in \mathbb{S}^{N-1}$ (see [4], eqn. (4.5)).

3 Some technical lemmas

Here we will show some technical results that will be useful in the rest of the paper.

Throughout the whole paper Ω will be an open subset of $\mathbb{R}^N$, usually bounded and with some regularity. Given A an open subset of $\mathbb{R}^N$, $c > 0$ and $u \in SBD(A)$, we let

$$E_c(u,A) \;=\; \int_A W(e(u)(x))\,dx \;+\; c\mathcal{H}^{N-1}(J_u)$$

while

$$\overline{E}_c(u,A) \;=\; \int_A W(e(u)(x))\,dx \;+\; c\mathcal{H}^{N-1}(\overline{J_u}).$$

Here the closure $\overline{J_u}$ is intended as the essential closure in $\mathbb{R}^2$ (not A) of the set J_u, that is, the smallest closed set in $\mathbb{R}^2$ that contains J_u up to a $\mathcal{H}^1$–negligible set. (J_u is supposed to be a subset of A, if u is the restriction to A of a SBD function defined in a larger set, it has to be replaced with $J_u \cap A$.) When $c = 1$, we denote E_c by simply E, and $\overline{E}_c$ by $\overline{E}$. The function $W : \mathbb{R}^{N\times N} \to \mathbb{R}$ is a quadratic form, which is positive definite on the subspace $\mathcal{S}^{N\times N}$ of symmetric matrices.

The next (obvious) lemma allows us to approximate an SBD function locally on a finite open covering of a set and then glue together the approximations.

Lemma 3.1 *Let Ω, $(A_i)_{i=1}^k$ be open subsets of $\mathbb{R}^N$ such that $\overline{\Omega} \subset \cup_{i=1}^k A_i$. Let $u \in SBD(\Omega)$, and assume that for each $i = 1,\dots,k$, there exists a sequence $(u^i_n)_{n\geq 1}$ in $SBD(A_i \cap \Omega)$ such that $\lim_{n\to\infty} \|u - u^i_n\|_{L^2(A_i\cap\Omega;\mathbb{R}^N)} \to 0$. Let $\ell_i = \limsup_{n\to\infty} \overline{E}(u_i, A_i \cap \Omega)$. Then there exists $(u_n)_{n\geq 1}$ a sequence in $SBD(\Omega)$ with $\|u - u_n\|_{L^2(\Omega;\mathbb{R}^N)} \to 0$ and such that $\limsup_{n\to\infty} \overline{E}(u_n,\Omega) \leq \sum_{i=1}^k \ell_i$.*

Proof. The idea is to consider a partition of unity $(\varphi_i)_{i=1}^k$ on Ω subject to the $(A_i)_{i=1}^k$: each φ_i is C^∞, nonnegative, compactly supported in A_i and $\sum_{i=1}^k \varphi_i(x) = 1$ for all $x \in \Omega$.

Then, we let $u_n = \sum_{i=1}^k \varphi_i u_n^i$. Clearly, $\|u_n - u\|^2_{L^2(\Omega;\mathbb{R}^N)} \le \sum_{i=1}^k \int_{A_i\cap\Omega} \varphi_i |u_n^i - u|^2 \to 0$ as $n\to\infty$. Let us explain why $\limsup_{n\to\infty} \overline{E}(u_n,\Omega) \le \sum_{i=1}^k \ell_i$. One has

$$\begin{aligned} e(u_n) \;&=\; \sum_{i=1}^k u_n^i \odot \nabla\varphi_i \,+\, \varphi_i e(u_n^i)\,, \\ J_{u_n} \;&\subset\; \bigcup_{i=1}^k J_{u_n^i}\,. \end{aligned}$$

We first deduce that $\mathcal{H}^{N-1}(\overline{J}_{u_n}) \le \sum_{i=1}^k \mathcal{H}^{N-1}(\overline{J}_{u_n^i})$. Then, since $\sum_{i=1}^k \nabla\varphi_i = \nabla 1 = 0$, we can rewrite the first equation

$$e(u_n) \;=\; \sum_{i=1}^k (u_n^i - u)\odot\nabla\varphi_i \,+\, \varphi_i e(u_n^i).$$

W is a nonnegative quadratic form, so that for any $\varepsilon > 0$ and $A, B \in \mathcal{S}^{N\times N}$, $W(A+B) \le (1+\varepsilon)W(A) + (1+1/\varepsilon)W(B)$. Using also the convexity of W, we find that

$$\int_\Omega W(e(u_n)) \;\le\; \sum_{i=1}^k k\frac{1+\varepsilon}{\varepsilon}\int_\Omega W\left((u_n^i - u)\odot\nabla\varphi_i\right)\,dx \,+\, (1+\varepsilon)\int_\Omega \varphi_i W(e(u_n^i))\,dx.$$

We deduce

$$\overline{E}(u_n,\Omega) \;\le\; (1+\varepsilon)\sum_{i=1}^k \overline{E}(u_n^i, A_i\cap\Omega) \,+\, c\sum_{i=1}^k \int_{A_i\cap\Omega} |u_n^i - u|^2\,dx$$

where c is some constant depending on ε, k and $\sup_{i,x} |\nabla\varphi_i(x)|$. Letting $n\to\infty$ we get $\limsup_{n\to\infty} E(u_n,\Omega) \le (1+\varepsilon)\sum_{i=1}^k \ell_i$, and since ε is arbitrary we get the thesis. □

We will say that Ω, a bounded open set of $\mathbb{R}^N$, satisfies "assumption (H)" if

$$\text{(H)}\quad \begin{cases} \text{At every boundary point } x\in\partial\Omega, \text{ there exist coordinates} \\ (\xi_1,\dots,\xi_N) \text{ and a continuous function } f:\mathbb{R}^{N-1}\to\mathbb{R} \text{ such that} \\ \text{near } x,\ \Omega \text{ coincides with the subgraph } \{\xi_N < f(\xi_1,\dots,\xi_{N-1})\}. \end{cases}$$

We now show the following approximation lemma, that allows to extend sligthly out of an open set Ω satisfying (H) a function in $SBD(\Omega)$, without perturbing much its energy.

Lemma 3.2 *Assume Ω satisfies (H) and $u \in SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^N)$, with $E(u,\Omega) < +\infty$. Then, for any $\varepsilon > 0$, there exists Ω' with $\Omega \subset\subset \Omega'$ and u' with $\|u'-u\|_{L^2(\Omega;\mathbb{R}^N)} \le \varepsilon$, such that*

$$\int_{\Omega'} W(e(u'))\,dx \;\le\; \int_\Omega W(e(u))\,dx + \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}(J_{u'}) \le \mathcal{H}^{N-1}(J_u) + \varepsilon. \qquad (2)$$

In order to prove this result we first need the following lemma.

Lemma 3.3 *Let Ω, $(A_i)_{i=1}^k$ be open subsets of $\mathbb{R}^N$ such that $\overline{\Omega} \subset \cup_{i=1}^k A_i$. Let μ be positive, finite Borel measure on $\mathbb{R}^N$. Then for each $\varepsilon > 0$, there exists a partition of unity in Ω subject to the $(A_i)_{i=1}^k$, that is, functions $(\varphi_i)_{i=1}^k$ with each $\varphi_i \in C^\infty(A_i)$, nonnegative, compactly supported in A_i and that satisfy $\sum_{i=1}^k \varphi_i(x) = 1$ for all $x\in\Omega$, such that $\mu\left(\cup_{i=1}^k \mathrm{supp}\,\overline{\{0<\varphi_i<1\}}\right) \le \varepsilon$.*

Proof. For any open set $A \subset \mathbb{R}^N$ let us denote $A_s = \{x \in A : \text{dist}(x, \mathbb{R}^N \setminus A) > s\}$. One first finds positive numbers $(s_i)_{i=1}^k$ such that $\overline{\Omega} \subset \cup_{i=1}^k (A_i)_{s_i}$ and $\mu(\cup_{i=1}^k A_i \setminus (A_i)_{s_i}) \leq \varepsilon$. Let $i_0 \in \{1, \ldots, k\}$, and assume we have found the s_i for $i < i_0$ with $\overline{\Omega} \subset (\cup_{i<i_0}(A_i)_{s_i}) \cup (\cup_{i \geq i_0} A_i))$ and $\mu(A_i \setminus (A_i)_{s_i}) \leq \varepsilon/k$ for each $i < i_0$. Let $\delta > 0$ be the distance between the disjoint compact sets $\overline{\Omega} \setminus A_{i_0}$ and $\overline{\Omega} \setminus [(\cup_{i<i_0}(A_i)_{s_i}) \cup (\cup_{i>i_0} A_i)]$. Since $\bigcap_{s>0} A_{i_0} \setminus (A_{i_0})_s = \emptyset$, $\lim_{s \to 0} \mu(A_{i_0} \setminus (A_{i_0})_s) = 0$. One can therefore choose $s_{i_0} \in (0, \delta)$ such that $\mu(A_{i_0} \setminus (A_{i_0})_{s_{i_0}}) \leq \varepsilon/k$. The fact that $s_{i_0} < \delta$ yields that $\overline{\Omega} \setminus (A_{i_0})_{s_{i_0}}$ is still disjoint from $\overline{\Omega} \setminus [(\cup_{i<i_0}(A_i)_{s_i}) \cup (\cup_{i>i_0} A_i)]$, in other words $\overline{\Omega} \subset (\cup_{i \leq i_0}(A_i)_{s_i}) \cup (\cup_{i>i_0} A_i))$.

Now, for each $i = 1, \ldots, k-1$, one easily finds a C^∞ function ψ_i with $0 \leq \psi_i \leq 1$, $\text{supp}\, \psi_i \subset\subset A_i$ and $\psi_i = 1$ in a neighborhood of $\overline{(A_i)_{s_i}}$ (for instance, by mollifying the characteristic function of $(A_i)_{s_i/2}$). We have $\overline{\{0 < \psi_i < 1\}} \subset\subset A_i \setminus \overline{(A_i)_{s_i}}$. We let $\varphi_1 = \psi_1$, $\varphi_i = \psi_i(1 - \sum_{j<i} \varphi_j)$ for $i = 2, \ldots, k$. These functions are clearly C^∞.

It is clear that $\text{supp}\, \varphi_i \subset\subset A_i$ for every i. Let us show by induction that $\sum_{j \leq i} \varphi_j \in [0,1]$, and is 1 on $\cup_{j \leq i}(A_i)_{s_i}$. It will yield in particular that $\varphi_i = \psi_{i+1}(1 - \sum_{j<i} \varphi_j) \in [0,1]$. If $i = 1$, these properties are clear by construction of $\varphi_1 = \psi_1$. If $i \geq 2$ and these properties are true for $i - 1$, then $\sum_{j \leq i} \varphi_j = \sum_{j<i} \varphi_j + \psi_i(1 - \sum_{j<i} \varphi_j)$ is a convex combination of 1 and $\psi_i \in [0,1]$. Hence it is in $[0,1]$. Moreover, it takes the value 1 whenever either $\sum_{j<i} \varphi_j = 1$, or $\psi_i = 1$, so that it is 1 on $\cup_{j \leq i}(A_i)_{s_i}$. If $i = k$, since $\overline{\Omega} \subset \cup_{i=1}^k (A_i)_{s_i}$, we get that $\sum_{i=1}^k \varphi_i(x) = 1$ for all $x \in \Omega$.

We have shown that $(\varphi_i)_{i=1}^k$ is a partition of unity on Ω subject to the covering $(A_i)_{i=1}^k$, now, it is easy to show that $\cup_{i=1}^k \text{supp}\, \overline{\{0 < \varphi_i < 1\}} \subseteq \cup_{i=1}^k A_i \setminus \overline{(A_i)_{s_i}}$, so that $\mu\left(\cup_{i=1}^k \text{supp}\, \overline{\{0 < \varphi_i < 1\}}\right) \leq \varepsilon$. □

Proof of Lemma 3.2. To prove the lemma we first consider a finite covering $A_1, \ldots, A_k$ of $\partial\Omega$ with open sets such that in each A_i, there is a direction $e^i \in \mathbb{S}^{N-1}$ and a continuous function $f : (e^i)^\perp \to \mathbb{R}$ such that $A_i \cap \Omega$ is represented by the subgraph $\{x \cdot e^i < f(x - (x \cdot e^i)e^i)\}$. In such a A_i we will define the function u_t^i, for $t > 0$, as $u_t^i(x) = u(x - te^i)$, which is defined slightly outside of Ω (in A_i), more precisely, on $A_i \cap (\Omega + [0,t)e^i)$, for t small enough. (By convention we extend it with the value zero in the rest of A_i.) It is standard that $u_t^i \to u$ in $L^2(A_i; \mathbb{R}^N)$ as $t \to 0$, where u is extended with the value 0 outside of Ω. Let us observe that, also, $e(u_t^i) \to e(u)$ in $L^2(A_i; \mathcal{S}^{N \times N})$ as $t \to 0$, extending again $e(u_t^i)$ (respectively, $e(u)$) with 0 out of $\Omega + [0,t)e^i$ (respectively, Ω).

We choose $A_0 \subset\subset \Omega$ such that $\overline{\Omega} \subset \cup_{i=0}^k A_i$, and for conveniency we let for any $t > 0$, $u_t^0 = u$ in A_0. Then we fix $\varepsilon > 0$ and invoke Lemma 3.3, with the measure $\mathcal{H}^{N-1} \llcorner J_u$ (which is a bounded Borel measure on $\mathbb{R}^N$), to find a partition of unity $\varphi_0, \ldots, \varphi_k$ subject to the $(A_i)_{i=0}^k$, with $\mathcal{H}^{N-1}\left((J_u \cap \left(\cup_{i=0}^k \text{supp}\, \overline{\{0 < \varphi_i < 1\}}\right)\right) \leq \varepsilon/(2(k+1))$.

Given $\bar{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ with each $t_i > 0$, small, we let $u_{\bar{t}} = u\varphi_0 + \sum_{i=1}^k u_{t_i}^i \varphi_i$, it is a function in $SBD(\Omega_{\bar{t}})$ where $\Omega_{\bar{t}} = A_0 \cup (\cup_{i=1}^k (A_i \cap (\Omega + [0, t_i)e^i))$ strictly contains Ω. It is easy to check that $u_{\bar{t}} \to u$ in $L^2(\Omega; \mathbb{R}^N)$, as $\bar{t} \to 0$. Let us estimate $\int_{\Omega_{\bar{t}}} W(e(u_{\bar{t}}))\, dx$ and $\mathcal{H}^{N-1}(J_{u_{\bar{t}}})$.

One has, using the fact that $\sum_{i=0}^k \nabla\varphi_i = 0$ inside Ω, whereas (by convention) $u = 0$

outside Ω,

$$e(u_{\bar t}) \;=\; \sum_{i=1}^k (u^i_{t_i}-u)\odot\nabla\varphi_i \,+\, \sum_{i=0}^k \varphi_i e(u^i_{t_i})$$

(letting for instance $t_0=0$, remember that $u^0_t=u$ for all t). The first part, $\sum_{i=1}^k(u^i_{t_i}-u)\odot\nabla\varphi_i$, converges to 0 in $L^2(\cup_{i=1}^k A_i;\mathcal{S}^{N\times N})$ as $\bar t$ goes to 0. The second part, $\sum_{i=0}^k\varphi_i e(u^i_{t_i})$, converges strongly to $e(u)$ in $L^2(\cup_{i=0}^k A_i;\mathcal{S}^{N\times N})$. Hence $e(u_{\bar t})\to e(u)$ as $\bar t\to 0$. We deduce that if $\bar t$ is small enough,

$$\int_{\Omega_{\bar t}} W(e(u_{\bar t}))\,dx \;\le\; \int_\Omega W(e(u))\,dx \,+\,\varepsilon.$$

Now, $J_{u_{\bar t}}\subset\cup_{i=0}^k(J_{u^i_{t_i}}\cap\operatorname{supp}\varphi_i)$. Since the measure $\mathcal{H}^{N-1}\llcorner J_{u^i_{t_i}}$ obviously converges to $\mathcal{H}^{N-1}\llcorner J_u$ as $t_i\to 0$, and since each $\operatorname{supp}\varphi_i$ is closed, one has

$$\limsup_{\bar t\to 0}\mathcal{H}^{N-1}(J_{u_{\bar t}}) \;\le\; \sum_{i=0}^k \limsup_{t_i\to 0}\mathcal{H}^{N-1}\llcorner J_{u^i_{t_i}}(\operatorname{supp}\varphi_i) \;\le\; \sum_{i=0}^k \mathcal{H}^{N-1}\llcorner J_u(\operatorname{supp}\varphi_i).$$

But $\sum_{i=0}^k \mathcal{H}^{N-1}\llcorner J_u(\operatorname{supp}\varphi_i) \;\le$

$$\mathcal{H}^{N-1}(J_u) \;+\; (k+1)\mathcal{H}^{N-1}\left(J_u\cap\left(\cup_{i=0}^k \operatorname{supp}\overline{\{0<\varphi_i<1\}}\right)\right),$$

so that it is less than $\mathcal{H}^{N-1}(J_u)+\varepsilon/2$. Hence if $\bar t$ is small enough, one has

$$\mathcal{H}^{N-1}(J_{u_{\bar t}}) \;\le\; \mathcal{H}^{N-1}(J_u) \,+\,\varepsilon.$$

Choosing $\Omega'=\Omega_{\bar t}$, $u'=u_{\bar t}$ for a very small $\bar t$ hence shows the thesis of Lemma 3.2. $\square$

4 A first result with a bad constant

In this section, the dimension of the space is fixed to $N=2$, and we will consider only the following bulk energy:

$$W(A) \;=\; \operatorname{Tr}(AA^T) \,+\, \frac{1}{2}(\operatorname{Tr}(A))^2\,, \tag{3}$$

defined for any 2×2 matrix A.

We prove the following theorem.

Theorem 1 *Assume Ω satisfies (H) and let $u\in SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^2)$, such that $E(u,\Omega)<+\infty$. Then, there exists a sequence (u_n) of displacements in $SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^2)$, with $\|u_n-u\|_{L^2(\Omega;\mathbb{R}^2)}\to 0$, such that each J_{u_n} is essentially closed in Ω (that is, $\mathcal{H}^1(\overline{J}_u\cap\Omega\setminus J_u)=0$), while each u_n is in $H^1(\Omega\setminus J_{u_n};\mathbb{R}^2)$, with the estimate*

$$\limsup_{n\to\infty}\overline{E}(u_n,\Omega) \;\le\; E_{c_0}(u,\Omega) \tag{4}$$

where c_0 is a universal constant ($c_0=8\sqrt{4+2\sqrt{2}}$). For each n, the set J_{u_n} is included in a finite union of closed segments. If $\|u\|_{L^\infty}<+\infty$, one can ensure that $\|u_n\|_{L^\infty}\le\|u\|_{L^\infty}$ for all n.

Proof. The proof is based on a discretization argument, similar to what is used in [14, Sec. 3.3] (see also [24]), together with an interpolation argument that is inspired from [13]. Let $u \in SBD(\Omega) \cap L^2(\Omega;\mathbb{R}^2)$. We fix $\varepsilon > 0$ and consider Ω' and u' given by Lemma 3.2. (Observe that if u is bounded, then the u' built in Lemma 3.2 is also clearly bounded by $\|u\|_{L^\infty}$.)

We consider a system of coordinates (e_1, e_2) such that for all $e \in \{e_1, e_2, e_1-e_2, e_1+e_2\}$, $\mathcal{H}^1(\{x \in J_{u'} : [u'(x)]\cdot e = 0\}) = 0$ (almost any $e_1 \in \mathbb{S}^1$ suits), and a small discretization step $h > 0$ (in practice, less than $\operatorname{dist}(\partial\Omega, \partial\Omega')/2\sqrt{2}$). Given $y \in [0,1)^2$, we will denote by $u_h^y(\xi)$ the discretization of u' given by $u_h^y(\xi) = u'(hy+\xi)$ for any $\xi \in h\mathbb{Z}^2 \cap (\Omega' - hy)$. For any $\tau \in \mathbb{R}^2$, we also denote, by J^τ the set $\cup_{x\in J_u}[x, x-\tau]$ (the union of the translates of $-s\tau$ of J_u, for $s \in [0,1]$). We let $D = \{e_1, e_2, e_1 - e_2, e_1 + e_2\}$ be a set of directions of interactions, and for each $e \in D$ and $\xi \in h\mathbb{Z}^2$ we set $l_{e,h}^y(\xi) = \chi_{J^{he}}(hy+\xi) \in \{0,1\}$, where $\chi_{J^{he}}$ is the characteristic function of J^{he}.

Given u_h^y, $l_h^y = (l_{e,h}^y)_{e\in D}$, we define a discrete energy

$$E_h^y(u_h^y, l_h^y) \;=\; h^2 \sum_{e\in D}\sum_{\xi} \frac{\left((u_h^y(\xi+he) - u_h^y(\xi))\cdot e\right)^2}{|e|^4h^2}(1 - l_{e,h}^y(\xi)) \;+\; \beta\frac{l_{e,h}^y(\xi)}{|e|h} \tag{5}$$

where the sum on the ξ runs on all the points $\xi \in h\mathbb{Z}^2$ such that both $hy+\xi$ and $hy+\xi+he$ are in Ω'. Here the parameter $\beta > 0$ will be fixed later on.

Let us compute the average of $E_h^y(u_h^y, l_h^y)$ over $y \in [0,1)^2$:

$$\int_{[0,1)^2} E_h^y(u_h^y, l_h^y)\, dy \;=\; \sum_{e\in D}\int_{[0,h)^2} dy \sum_{\xi}\frac{\left((u'(y+\xi+he) - u'(y+\xi))\cdot e\right)^2}{|e|^4h^2}(1 - \chi_{J^{he}}(y+\xi)) \;+\; \beta\frac{\chi_{J^{he}}(y+\xi)}{|e|h}$$

This is less than (letting $x = \xi + y$)

$$\sum_{e\in D}\int_{\Omega'\cap\Omega'-he}\frac{\left((u'(x+he) - u'(x))\cdot e\right)^2}{|e|^4h^2}(1 - \chi_{J^{he}}(x)) \;+\; \beta\frac{\chi_{J^{he}}(x)}{|e|h}\, dx\,. \tag{6}$$

For each $e \in D$, we will make a change of variable $x = z + se'$ where $e' = e/|e|$. The integral above becomes (to simplify we denote $d\mathcal{H}^{N-1}(z)$ by dz)

$$\int_{z\in e^\perp} dz\int_{I_{z,h}^e}\frac{\left((u'(z+(s+h|e|)e') - u'(z+se'))\cdot e'\right)^2}{|e|^2h^2}(1 - \chi_{J^{he}}(z+se')) \;+\; \beta\frac{\chi_{J^{he}}(z+se')}{|e|h}\, ds$$

where $I_{z,h}^e = \{s \in \mathbb{R} : z + se', z + (s+h|e|)e' \in \Omega'\}$ (we also denote $I_z^e = I_{z,0}^e$).

As mentionned in section 2.4, for almost all z the function $u_z^e : s \mapsto u'(z+se')\cdot e'$ is in $SBV(I_z^e)$, and its jump set $S_{u_z^e}$ is given by $\{s \in I_z^e : z + se' \in J_{u'} \text{ and } [u'(z+se)]\cdot e' \neq 0\}$. Moreover, since $\int_{\Omega'} W(e(u'))\, dx + \mathcal{H}^1(J_u') < +\infty$, one checks easily that this jump set is finite for almost any z, and that u_z^e has regularity H^1 in the complement of its jump set (this will be justified in the sequel). In particular, if $\chi_{J^{he}}(z+se')=0$, then $S_{u_z^e}\cap[s,s+h|e|]\neq 0$ and

$$((u'(z+(s+h|e|)e')-u'(z+se'))\cdot e')^2 \\ = (u_z^e(s+h|e|)-u_z^e(s))^2 \;\le\; h|e|\int_s^{s+h|e|}\left(\frac{\partial u_z^e}{\partial s}(t)\right)^2 dt\,.$$

We deduce that

$$\int_{I_{z,h}^e}\frac{((u'(z+(s+h|e|)e')-u'(z+se'))\cdot e')^2}{|e|^2h^2}(1-\chi_{J^{he}}(z+se'))\,ds \\ \le \int_{I_z^e}\left(\frac{\partial u_z^e}{\partial s}(t)\right)^2 dt\,.$$

On the other hand,

$$\int_{I_{z,h}^e}\frac{\chi_{J^{he}}(z+se')}{|e|h}\,ds \;\le\; \frac{1}{|e|h}\left|\left\{s\in I_z^e\,:\,[s-h|e|,s]\cap S_{u_z^e}\neq\emptyset\right\}\right|$$

which is less than $\mathcal{H}^0(S_{u_z^e})$. We find that the integral in (6) is dominated by

$$\int_{z\in e^\perp}dz\left(\int_{I_z^e}\left(\frac{\partial u_z^e}{\partial s}(t)\right)^2 dt+\beta\mathcal{H}^0(S_{u_z^e})\right) = \int_{\Omega'}((e(u')e')\cdot e')\,dx+\beta\int_{J_{u'}}|\nu_{u'}\cdot e'|\,d\mathcal{H}^1.$$

It turns out that our choice of W satisfies $W(A)=\sum_{e\in D}((Ae')\cdot e')^2$ for any $A\in\mathcal{S}^{2\times 2}$, hence the sum of these integrals over all $e\in D$ is

$$\int_{\Omega'}W(e(u')(x))\,dx\;+\;\beta\int_{J_{u'}}h(\nu_{u'}(x))\,d\mathcal{H}^1(x)$$

which thus provides a bound for $\int_{[0,1)^2}E_h^y(u_h^y,l_h^y)\,dy$. Here $h(p)=|p\cdot e_1|+|p\cdot e_2|+(|p\cdot(e_1+e_2)|+|p\cdot(e_1-e_2)|)/\sqrt{2}$. We notice that $(1+\sqrt{2})|p|\le h(p)\le\sqrt{4+2\sqrt{2}}|p|$ for all $p\in\mathbb{R}^2$, in particular, we have, letting $\beta'=\sqrt{4+2\sqrt{2}}\beta$,

$$\int_{[0,1)^2}E_h^y(u_h^y,l_h^y)\,dy\;\le\;\int_{\Omega'}W(e(u'))\,dx\;+\;\beta'\mathcal{H}^1(J_{u'}) \tag{7}$$

This inequality guarantees that, for y in a subset of positive measure of $(0,1)^2$, the discrete energy $E_h^y(u_h^y,l_h^y)$ is less than $\int_{\Omega'}W(e(u'))+\beta'\mathcal{H}^1(J_{u'})$. The idea, at this point, is to interpolate the discrete data u_h^y,l_h^y in order to find a displacement with energy close to $E_h^y(u_h^y,l_h^y)$. But in doing so, we also need to ensure that the interpolates will converge to u' in $L^2(\Omega;\mathbb{R}^2)$ as $h\to 0$. In order to achieve this property, we introduce the function $\Delta(x)=(1-|x\cdot e_1|)^+(1-|x\cdot e_2|)^+$ (here $t^+=\max(t,0)$) and to any discretization (u_h^y) of u' we associate the displacement

$$w_h^y(x)\;=\;\sum_{\xi\in h\mathbb{Z}^2\cap\Omega'}u_h^y(\xi)\Delta\left(\frac{x-\xi}{h}-y\right).$$

Notice that since $\Omega \subset\subset \Omega'$, it is well defined for $x \in \Omega$ as soon as h is small enough. We have (using $\sum_\xi \Delta((x-\xi)/h - y) = 1$ at every x)

$$\begin{aligned}\int_{[0,1)^2} dy \int_\Omega |u'(x) - w_h^y(x)|^2\,dx \\ = \int_{[0,1)^2} dy \int_\Omega \left[\sum_{\xi\in h\mathbb{Z}^2\cap\Omega'} \Delta\left(\frac{x-\xi}{h} - y\right)(u'(x) - u'(hy+\xi))\right]^2 dx \\ \le \int_{[0,1)^2} dy \int_\Omega \sum_{\xi\in h\mathbb{Z}^2\cap\Omega'} \Delta\left(\frac{x-\xi}{h} - y\right)|u'(x) - u'(hy+\xi)|^2\,dx\,,\end{aligned}$$

and, letting $z = (x-\xi)/h - y$, we get

$$\int_{[0,1)^2} dy \int_\Omega |u'(x) - w_h^y(x)|^2\,dx \le \int_{(-1,1)^2} \Delta(z)\,dz \int_\Omega |u'(x) - u'(x-hz)|^2\,dx\,.$$

Since for all z, $\int_\Omega |u'(x) - u'(x-hz)|^2\,dx \to 0$ as $h \to 0$ (and is uniformly bounded by $2\|u'\|_{L^2}^2$), we deduce that $\lim_{h\to 0} \int_{[0,1)^2} dy \int_\Omega |u' - w_h^y|^2\,dx = 0$. Hence, there is a subsequence $(h_k)_{k\ge 1}$ of h (with $h_k \downarrow 0$ as $k\to\infty$), and a measurable set $A \subset [0,1)^2$ with Lebesgue measure 1, such that for each $y \in A$, $\lim_{k\to\infty} \|u' - w_{h_k}^y\|_{L^2(\Omega;\mathbb{R}^2)} = 0$. Now, we observe that (7) yields (using Fatou's lemma)

$$\int_{[0,1)^2} \liminf_{k\to\infty} E_{h_k}^y(u_{h_k}^y, l_{h_k}^y)\,dy \;\le\; \int_{\Omega'} W(e(u'))\,dx \;+\; \beta'\mathcal{H}^1(J_{u'})\,,$$

so that we can find $y \in A$ with the additional property

$$\liminf_{k\to\infty} E_{h_k}^y(u_{h_k}^y, l_{h_k}^y) \;\le\; \int_{\Omega'} W(e(u'))\,dx \;+\; \beta'\mathcal{H}^1(J_{u'})\,.$$

Hence, extracting another subsequence $(h_{k_l})_{l\ge 1}$ from $(h_k)_{k\ge 1}$, we find a sequence of discretizations $(u_{h_{k_l}}^y, l_{h_{k_l}}^y)_{l\ge 1}$ with both

$$\begin{cases} \lim_{l\to\infty} \|u' - w_{h_{k_l}}^y\|_{L^2(\Omega;\mathbb{R}^2)} = 0 \quad\text{and} \\ \lim_{l\to\infty} E_{h_{k_l}}^y(u_{h_{k_l}}^y, l_{h_{k_l}}^y) \le \int_{\Omega'} W(e(u'))\,dx + \beta'\mathcal{H}^1(J_{u'})\,. \end{cases} \tag{8}$$

In the sequel, we will fix y to this particular value (and consequently drop the corresponding superscripts), and simply denote by $(h)_{h>0}$ the subsequence $(h_{k_l})_{l\ge 1}$.

We now are able to achieve the proof of Theorem 1. We say that the square $\xi + hy + [0,h)^2$, $\xi \in h\mathbb{Z}^2$, is a "jump square" at scale h if any of the "line processes" $l_{e_1,h}(\xi)$, $l_{e_2,h}(\xi)$, $l_{e_1+e_2,h}(\xi)$, $l_{e_1,h}(\xi+he_2)$, $l_{e_1-e_2,h}(\xi+he_2)$, $l_{e_2,h}(\xi+he_1)$ is equal to 1. Then, we define the displacement $v_h : \Omega \to \mathbb{R}^2$ by letting $v_h(x) = w_h(x)$ whenever x does not belong to a jump square, and 0 otherwise. Such a v_h is clearly in $SBD(\Omega)$. Its jump set J_{v_h} is contained in the union of the boundaries of the jump squares, which is a closed set.

Le us estimate the energy of v_h. First, the length $\mathcal{H}^1(\overline{J}_{v_h})$ is bounded by $4h \times K_h$ where K_h is the total number of jump squares at scale h. But for any of these squares

$C = \xi + hy + [0,h)^2$, one has

$$h\beta\left(\frac{l_{e_1,h}(\xi)+l_{e_2,h}(\xi)+l_{e_1,h}(\xi+he_2)+l_{e_2,h}(\xi+he_1)}{2}\right.$$
$$\left. + \quad \frac{l_{e_1-e_2,h}(\xi+he_2)+l_{e_1+e_2,h}(\xi)}{\sqrt{2}}\right) \;\geq\; h\frac{\beta}{2},$$

(since at least one of all these $l_{e,h}$'s is 1). The left-hand side expression is the contribution of the square C to the second part $h^2\sum_{e\in D}\sum_{\xi}\beta\frac{l_{e,h}(\xi)}{|e|h}$ of the energy $E_h(u_h,l_h)$ defined in (5). Hence if we choose $\beta = 8$, summing on all the jump squares we find that

$$\mathcal{H}^1(\overline{J}_{v_h}) \;\leq\; h^2\sum_{e\in D}\sum_{\xi}\beta\frac{l_{e,h}(\xi)}{|e|h}.$$

Let us observe that the total area of the jump squares is h^2K_h, and repeating the same arguments we find that, thanks to (8), it is $O(h)$.

On the other hand, if $C = \xi + hy + [0,h)^2$ is not a "jump square", then Lemma A.1 in Appendix A shows that $\int_C W(e(v_h))\,dx$ is less than

$$h^2\left(\frac{((u_h(\xi+he_1)-u_h(\xi))\cdot e_1)^2}{2h^2}+\frac{((u_h(\xi+h(e_1+e_2))-u_h(\xi+he_2))\cdot e_1)^2}{2h^2}\right.$$
$$+\frac{((u_h(\xi+he_2)-u_h(\xi))\cdot e_2)^2}{2h^2}+\frac{((u_h(\xi+h(e_1+e_2))-u_h(\xi+he_1))\cdot e_2)^2}{2h^2}$$
$$+\frac{((u_h(\xi+h(e_1+e_2))-u_h(\xi))\cdot(e_1+e_2))^2}{4h^2}$$
$$\left.+\frac{((u_h(\xi+he_2)-u_h(\xi+he_1))\cdot(e_2-e_1))^2}{4h^2}\right)$$

which is exactly the contribution of the square C to the first part (the "bulk part") of energy (5). On a jump square C, $\int_C W(e(v_h))\,dx = 0$. We find therefore that, having chosen $\beta = 8$,

$$\int_\Omega W(e(v_h))\,dx + \mathcal{H}^1(\overline{J}_{v_h}) \;\leq\; E_h(u_h,l_h) \;\leq\; \int_{\Omega'} W(e(u'))\,dx + \beta'\mathcal{H}^1(J_{u'}).$$

Here, $\beta' = 8\sqrt{4+2\sqrt{2}}$. Now, we observe that $\|v_h - w_h\|^2_{L^2(\Omega;\mathbb{R}^2)}$ is less than the integral $\int_{J_h\cap\Omega} w_h^2\,dx$, where J_h is the union of the jump squares at scale h, and since $|J_h| = O(h)$ and w_h converges strongly in $L^2(\Omega;\mathbb{R}^2)$, we find that $\|v_h - w_h\|_{L^2(\Omega;\mathbb{R}^2)} \to 0$ as $h \to 0$, so that v_h also goes to u' in $L^2(\Omega;\mathbb{R}^2)$ as $h \to 0$.

Therefore, if h is small enough, the displacement v_h will satisfy

$$\|v_h - u\|_{L^2(\Omega;\mathbb{R}^2)} \;\leq\; \|v_h - u'\|_{L^2(\Omega;\mathbb{R}^2)} + \|u' - u\|_{L^2(\Omega;\mathbb{R}^2)} \;\leq\; 2\varepsilon,$$

$$\overline{E}(v_h,\Omega) \;\leq\; \int_{\Omega'} W(e(u'))\,dx + \beta'\mathcal{H}^1(J_{u'}) \;\leq\; E_{c_0}(u,\Omega) + 2\varepsilon$$

with $c_0 = \beta'$. This proves Theorem 1 (the final assertion is clear from the construction).$\square$

5 The main result

Now, using Theorem 1, a localization argument, and Lemma 3.1, we will deduce the following Theorems 2 and 3. The first one shows that any $u \in SBD(\Omega)$ can be approximated in $L^2(\Omega;\mathbb{R}^2)$ with displacements u_n, such that $\limsup_{n\to\infty} \overline{E}(u_n,\Omega) \le E(u,\Omega)$, for our particular choice of the quadratic form W. The second one is a corollary of the first and of a variant of [8, Thm. 1.1] (Lemma 5.1 below), that ensures that there is in fact strong convergence in $L^2(\Omega;\mathcal{S}^{2\times 2})$ of the approximate deformations $e(u_n)$ to $e(u)$, hence convergence of the energies $E(u_n,\Omega)$ to $E(u,\Omega)$ for any other choice of the positive-definite quadratic form W.

Theorem 2 *Assume Ω satisfies (H) and let $u \in SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^2)$, such that $E(u,\Omega) < +\infty$. Then, there exists a sequence (u_n) of displacements in $SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^2)$, with $\|u_n - u\|_{L^2(\Omega;\mathbb{R}^2)} \to 0$, such that each J_{u_n} is closed in Ω, contained in a finite union of closed connected pieces of C^1 curves, $u_n \in H^1(\Omega\setminus J_{u_n};\mathbb{R}^2)$, and*

$$\limsup_{n\to\infty} \overline{E}(u_n,\Omega) \;\le\; E(u,\Omega)\,. \tag{9}$$

Moreover, if $\|u\|_{L^\infty} < +\infty$, one can ensure that $\|u_n\|_{L^\infty} \le \|u\|_{L^\infty}$ for all n.

Proof. We first recall that J_u is $(\mathcal{H}^1,1)$–rectifiable in the sense of Federer [20] (see [4]), which means that there exists a countable union of C^1 curves $(\Gamma_i)_{i=1}^\infty$ such that $\mathcal{H}^1(J_u \setminus \cup_{i=1}^\infty \Gamma_i) = 0$. For each $i \ge 1$, we can define a set

$$S_i \;=\; \left\{ x \in J_u\cap\Gamma_i \setminus \cup_{j<i} S_j \;:\; \lim_{\rho\to 0}\frac{\mathcal{H}^1(J_u\cap \overline{B}(x,\rho))}{2\rho} = 1 \text{ and } \lim_{\rho\to 0}\frac{\mathcal{H}^1(J_u\cap\Gamma_i\cap \overline{B}(x,\rho))}{2\rho} = 1 \right\},$$

that is, the set of points where J_u has $\mathcal{H}^1$–density 1, as well as density 1 along the smooth curve Γ_i (and i is the first index such that it happens). We have that $\mathcal{H}^1(J_u\setminus\cup_{i=1}^\infty S_i) = 0$ (since $\mathcal{H}^1$–almost all points in J_u have $\mathcal{H}^1$–density 1, and $\mathcal{H}^1$–almost all points in $J_u\cap\Gamma_i$ have density 1 along Γ_i). Observe that if $x \in S_i$, then $\lim_{\rho\to 0}\mathcal{H}^1(J_u\cap\overline{B}(x,\rho)\setminus\Gamma_i)/(2\rho) = 0$.

If we fix $\varepsilon > 0$, then for every i, at each $x \in S_i$, for almost all ρ that is small enough, we have that $\overline{B}(x,\rho)\subset\Omega$, $\mathcal{H}^1(J_u\triangle\Gamma_i\cap\overline{B}(x,\rho)) \le 2\varepsilon\rho$, $\mathcal{H}^1(J_u\cap\overline{B}(x,\rho)) \ge 2(1-\varepsilon)\rho$, $\mathcal{H}^1(J_u\cap\partial B(x,\rho)) = 0$, and, as well, that Γ_i separates $B(x,\rho)$ in exactly two connected components, each one being a domain satisfying the property (H) (this is true simply because Γ_i is C^1, so that it is almost a diameter of $B(x,\rho)$ as ρ goes to zero).

Now, if we invoke Besicovitch's covering theorem (with the measure $\mathcal{H}^1 \llcorner \cup_{i=1}^\infty S_i$, *cf* [19, Cor. 1 p. 35]), then we find a covering $(\overline{B}_j)_{j=1}^\infty$ of $\mathcal{H}^1$–almost all of $\cup_{i=1}^\infty S_i$, of such closed balls (we denote by x_j the center of B_j and ρ_j its radius). Since $\sum_{j=1}^\infty \mathcal{H}^1(J_u\cap B_j) = \mathcal{H}^1(J_u) < +\infty$[1] there exists k with $\sum_{j>k}\mathcal{H}^1(J_u\cap B_j) < \varepsilon$. For each B_j, $j = 1,\dots,k$, there is an index i such that $\mathcal{H}^1(J_u\triangle\Gamma_i\cap\overline{B}_j) \le 2\varepsilon\rho_j \le \varepsilon/(1-\varepsilon)\mathcal{H}^1(J_u\cap\overline{B}_j)$. We can

[1] Remember $\mathcal{H}^1(J_u\cap\partial B_j) = 0$, so that $\mathcal{H}^1(J_u\cap B_j) = \mathcal{H}^1(J_u\cap\overline{B}_j)$ for all j.

invoke Theorem 1 in each of the two components of $B_j \setminus \Gamma_i$, to find a sequence $(u_n^j)_{n\geq 1}$ converging to u in $L^2(B_j;\mathbb{R}^2)$, such that

$$\limsup_{n\to\infty} \int_{B_j} W(e(u_n^j))\,dx \,+\, \mathcal{H}^1(\overline{J_{u_n^j} \cap B_j} \setminus \Gamma_i) \;\leq\; \int_{B_j} W(e(u))\,dx \,+\, c_0\mathcal{H}^1(J_u \cap B_j \setminus \Gamma_i)\,.$$

This yields

$$\begin{aligned}\limsup_{n\to\infty} \int_{B_j} W(e(u_n^j))\,dx \,+\, \mathcal{H}^1(\overline{J_{u_n^j} \cap B_j}) \\ \leq \int_{B_j} W(e(u))\,dx \,+\, \mathcal{H}^1(J_u \cap B_j) \,+\, c_0\frac{\varepsilon}{1-\varepsilon}\mathcal{H}^1(J_u \cap B_j)\,.\end{aligned}$$

On the other hand, for $t > 0$, let $A_t = \{x \in \mathbb{R}^2 \;:\; \mathrm{dist}\,(x, \Omega \setminus \cup_{j=1}^k \overline{B}_j) < t\}$. Since $\mathcal{H}^1(J_u \cap (\cap_{t>0} A_t)) = \mathcal{H}^1(J_u \setminus \cup_{j=1}^k \overline{B}_j) \leq \varepsilon$, if t is small enough, we have that $\mathcal{H}^1(J_u \cap A_t) \leq 2\varepsilon$. Also $A_t \cap \Omega$ satisfies (H). Hence there exists $(u_n^0)_{n\geq 1}$ in $SBD(A_t \cap \Omega)$, converging to u in $L^2(A_t \cap \Omega;\mathbb{R}^2)$ with

$$\limsup_{n\to\infty} \int_{A_t\cap\Omega} W(e(u_n^0))\,dx \,+\, \mathcal{H}^1(\overline{J_{u_n^0}}) \;\leq\; \int_{A_t\cap\Omega} W(e(u))\,dx \,+\, 2c_0\varepsilon\,.$$

Invoking Lemma 3.1 with the covering A_t, $(B_j)_{j=1}^k$ of $\overline{\Omega}$ and the sequences $(u_n^j)_{n\geq 1}$, $j = 0, \ldots, k$, we find a sequence $(u_n)_{n\geq 1}$ that converges to u in $L^2(\Omega;\mathbb{R}^2)$ such that

$$\limsup_{n\to\infty} \overline{E}(u_n, \Omega) \;\leq\; E(u, \Omega) \,+\, 2c_0\varepsilon \,+\, c_0\frac{\varepsilon}{1-\varepsilon}\mathcal{H}^1(J_u)\,.$$

Since ε is arbitrary, a standard diagonalization argument shows Theorem 2. Notice that here again, if u is bounded, then u_n is bounded with same bound. □

Theorem 3 *Assume Ω satisfies (H) and let $u \in SBD(\Omega) \cap L^2(\Omega;\mathbb{R}^2)$, such that $E(u,\Omega) < +\infty$. Then, there exists a sequence (u_n) of displacements in $SBD(\Omega) \cap L^2(\Omega;\mathbb{R}^2)$, with $\|u_n - u\|_{L^2(\Omega;\mathbb{R}^2)} \to 0$, such that each J_{u_n} is closed in Ω, contained in a finite union of closed connected pieces of C^1 curves, $u_n \in H^1(\Omega \setminus J_{u_n};\mathbb{R}^2)$, and*

- *(i)* $e(u_n) \to e(u)$ *strongly in* $L^2(\Omega;\mathcal{S}^{2\times 2})$,
- *(ii)* $\lim_{n\to\infty} \mathcal{H}^1(\overline{J}_{u_n}) = \lim_{n\to\infty} \mathcal{H}^1(J_{u_n}) = \mathcal{H}^1(J_u)$.

Again, if $\|u\|_{L^\infty} < +\infty$, one can ensure that $\|u_n\|_{L^\infty} \leq \|u\|_{L^\infty}$ for all n.

Proof. We will show in fact that the sequence given by Theorem 2 enjoys the desired properties. For this we need the following (simpler) variant of the semicontinuity result of Theorem 1.1 in [8], where no assumption is made on $\sup_n \|u_n\|_{BD}$, but we assume instead that $u_n \to u$ in $L^2(\Omega;\mathbb{R}^N)$, and consider only completely isotropic quadratic forms of $e(u)$. We state the lemma in any dimension N, replacing W with

$$W(A) = \frac{1}{N\omega_N}\int_{\mathbb{S}^{N-1}} ((A\xi)\cdot\xi)^2\,d\mathcal{H}^{N-1}(\xi)$$

which defines a quadratic form of $A \in \mathcal{S}^{N\times N}$, that is positive definite. This extends to any dimension the definition (3) (up to a factor 4).

Lemma 5.1 (*cf* [8, Thm. 1.1]) *Let Ω be an open subset of $\mathbb{R}^N$. Assume $(u_n)_{n\geq 1}$ is a sequence in $SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^N)$ such that $\sup_{n\geq 1}\int_\Omega W(e(u_n))\,dx+\mathcal{H}^{N-1}(J_{u_n})<+\infty$ and u_n converges strongly in $L^2(\Omega;\mathbb{R}^N)$ to some $u\in SBD(\Omega)$. Then*

(i) $e(u_n)\rightharpoonup e(u)$ weakly in $L^2(\Omega;\mathcal{S}^{N\times N})$,

(ii) $\mathcal{H}^{N-1}(J_u)\leq\liminf_{n\to\infty}\mathcal{H}^1(J_{u_n})$.

Proof. The proof reproduces essentially the proof of [8] in a simpler situation (see also [2]), and we will sketch it briefly.

We will show that for any smooth function $\varphi\in C_c^\infty(\Omega;\mathcal{S}^{N\times N})$ and any $\lambda>0$, one has

$$\int_\Omega W(e(u)+\varphi)\,dx\;+\;\lambda\mathcal{H}^{N-1}(J_u)\leq\;\liminf_{n\to\infty}\int_\Omega W(e(u_n)+\varphi)\,dx+\lambda\mathcal{H}^{N-1}(J_{u_n}).\quad(10)$$

The lemma will follow. Indeed, if (10) holds, we have

$$\begin{aligned}\mathcal{H}^{N-1}(J_u)\;&\leq\;\liminf_{n\to\infty}\mathcal{H}^{N-1}(J_{u_n})+\frac{1}{\lambda}\int_\Omega W(e(u_n)+\varphi)\,dx\\ &\leq\;\liminf_{n\to\infty}\mathcal{H}^{N-1}(J_{u_n})+\frac{1}{\lambda}\limsup_{n\to\infty}\int_\Omega W(e(u_n+\varphi))\,dx\,.\end{aligned}$$

Sending λ to $+\infty$ we get point (ii) of the lemma.

The same argument, sending this time λ to 0, shows that

$$\int_\Omega W(e(u+\varphi))\,dx\;\leq\;\liminf_{n\to\infty}W(e(u_n+\varphi))\,dx\,.\quad(11)$$

Upon extracting a subsequence, we can assume that $e(u_n)\rightharpoonup\sigma$ in $L^2(\Omega;\mathcal{S}^{N\times N})$. But (11) yields, if we denote by $B(\,\cdot\,,\,\cdot\,)$ the symmetric quadratic form associated to W (such that $W(\varepsilon)=B(\varepsilon,\varepsilon)$),

$$\int_\Omega B(e(u),\varphi)\,dx\;\leq\;\int_\Omega B(\sigma,\varphi)\,dx\;+\;\frac{1}{2}\left(\liminf_{n\to\infty}\int_\Omega W(e(u_n))\,dx-\int_\Omega W(e(u))\,dx\right).$$

Since φ is arbitrary, we easily deduce $\int_\Omega B(e(u),\varphi)\,dx=\int_\Omega B(\sigma,\varphi)\,dx$ for all smooth φ, which implies $\sigma=e(u)$, and shows point (i) of the lemma.

It remains to show (10). Given $v\in SBD(\Omega)$, $\xi\in\mathbb{S}^{N-1}$ and $z\in\xi^\perp$, we denote by $v_z^\xi(s)$ the function $v(z+s\xi)\cdot\xi$, defined on the open (possibly empty) set $\Omega_z^\xi=\{s\,:\,z+s\xi\in\Omega\}$. For all ξ and almost all $z\in\xi^\perp$, the function $v_z^\xi(s)$ is in $SBV(\Omega_z^\xi)$. Moreover, we can write (to simplify we denote $d\mathcal{H}^{N-1}(\xi)$ by $d\xi$ and $d\mathcal{H}^{N-1}(z)$ by dz, and denote by $\varphi_z^\xi(s)$ the function $s\mapsto(\varphi(z+s\xi)\xi)\cdot\xi$)

$$\int_\Omega W(e(v)+\varphi)\,dx\;=\;\frac{1}{N\omega_N}\int_{\mathbb{S}^{N-1}}d\xi\int_{\xi^\perp}dz\int_{\Omega_z^\xi}\left((v_z^\xi)'(s)+\varphi_z^\xi(s)\right)^2\,ds$$

whereas (see [4, 8])

$$\mathcal{H}^{N-1}(J_v)\;=\;\frac{1}{2\omega_{N-1}}\int_{\mathbb{S}^{N-1}}d\xi\int_{\xi^\perp}\mathcal{H}^0(S_{v_z^\xi})\,dz\,.$$

Since

$$\int_\Omega|u_n-u|^2\,dx\;=\;\frac{1}{N\omega_N}\int_{\mathbb{S}^{N-1}}d\xi\int_{\xi^\perp}dz\left(\int_{\Omega_z^\xi}|(u_n)_z^\xi-u_z^\xi|^2\,ds\right),$$

upon extracting a subsequence (still denoted by (u_n)), one can assume that $(u_n)^\xi_z \to u^\xi_z$ strongly in $L^2(\Omega^\xi_z)$ for a.e. $\xi \in \mathbb{S}^{N-1}$ and for a.e. $z \in \xi^\perp$ (we first identify $\xi^\perp$ to $\mathbb{R}^{N-1}$ to get the convergence for a.e. $(z,\xi) \in \mathbb{S}^{N-1} \times \mathbb{R}^{N-1}$).

Using Fatou's lemma, one sees that for every $\lambda > 0$ (denoting $\kappa = N\omega_N/(2\omega_{N-1})$),

$$\int_{\mathbb{S}^{N-1}} d\xi \int_{\xi^\perp} dz \liminf_{n\to\infty} \left(\int_{\Omega^\xi_z} \left(((u_n)^\xi_z)'(s) + \varphi^\xi_z(s)\right)^2 ds + \frac{\lambda\kappa}{2}\mathcal{H}^0\left(S_{(u_n)^\xi_z}\right)\right) \\ \leq (N\omega_N) \liminf_{n\to\infty} \int_\Omega W(e(u_n)+\varphi)\, dx + \lambda\mathcal{H}^{N-1}(J_{u_n}) < +\infty\,.$$

For almost every ξ and z, hence, one sees that

$$\liminf_{n\to\infty} \int_{\Omega^\xi_z} \left(((u_n)^\xi_z)'(s) + \varphi^\xi_z(s)\right)^2 ds + \frac{\lambda\kappa}{2}\mathcal{H}^0\left(S_{(u_n)^\xi_z}\right) < +\infty,$$

and we can apply Ambrosio's Theorem [2, Thm 2.1] of (compactness and) semicontinuity in $GSBV(\Omega^\xi_z)$ to deduce that

$$\int_{\Omega^\xi_z} \left((u^\xi_z)'(s) + \varphi^\xi_z(s)\right)^2 ds + \frac{\lambda\kappa}{2}\mathcal{H}^0\left(S_{u^\xi_z}\right) \\ \leq \liminf_{n\to\infty} \int_{\Omega^\xi_z} \left(((u_n)^\xi_z)'(s) + \varphi^\xi_z(s)\right)^2 ds + \frac{\lambda\kappa}{2}\mathcal{H}^0\left(S_{(u_n)^\xi_z}\right).$$

Integrating again over ξ and z, we find (10). Lemma 5.1 is proven. □

Proof of Theorem 3. Consider the sequence given by Theorem 2. By Lemma 5.1, one has

- $e(u_n) \rightharpoonup e(u)$ in $L^2(\Omega; \mathcal{S}^{2\times 2})$,
- $\int_\Omega W(e(u))\, dx \leq \liminf_{n\to\infty} \int_\Omega W(e(u_n))\, dx$,
- $\mathcal{H}^{N-1}(J_u) \leq \liminf_{n\to\infty} \mathcal{H}^1(J_{u_n})$.

Thanks to (9), we deduce that point (ii) of the thesis of the theorem holds, as well as $\lim_{n\to\infty} \int_\Omega W(e(u_n))\, dx = \int_\Omega W(e(u))$. This yields also the strong convergence of $e(u_n)$ to $e(u)$, that is, point (i) of the thesis. This shows Theorem 3. □

Remark 5.2 *As mentioned before, the main drawback of our proof is that it does not provide any global bound in $BD(\Omega)$ of the approximating sequence $(u_n)_{n\geq 1}$. On the other hand, one sees from the construction that each u_n is Lipschitz continuous on $\Omega \setminus J_{u_n}$, with continuous limits on $\partial\Omega$ on both sides of the jump J_{u_n}.*

Remark 5.3 *Strictly speaking, we have not shown that each $u \in SBD(\Omega)$ can be approximated by a u_n such that J_{u_n} is closed in Ω, but more precisely, by a u_n such that there exists a closed set J_n, finite union of closed, connected pieces of C^1 curves, with $J_{u_n} \subset J_n \cap \Omega$ and $\mathcal{H}^1(J_n) \to \mathcal{H}^1(J_u)$. However, if really needed, an infinitesimal perturbation of each u_n could be made in order to ensure $J_{u_n} = J_n \cap \Omega$ (again, up to a negligible set), yielding $\mathcal{H}^1(\overline{J}_{u_n} \cap \Omega \setminus J_{u_n}) = 0$.*

Remark 5.4 *If the boundary of Ω is oscillating rapidly it might happen that, in our construction, $\mathcal{H}^1(\overline{J}_{u_n}) > \mathcal{H}^1(J_{u_n})$ (although one always have $\mathcal{H}^1(\overline{J}_{u_n} \cap \Omega \setminus J_{u_n}) = 0$). The essential point is that $\mathcal{H}^1(\overline{J}_{u_n})$ converges to $\mathcal{H}^1(J_u)$.*

6 An application

Here, in order to illustrate the interest of Theorem 3, we show how it yields the extension to the SBD case of a now "classical" Γ–convergence result in SBV, proven by Ambrosio and Tortorelli [6, 7, 5].

We show the following result (here W is any positive-definite quadratic form on $\mathcal{S}^{2\times 2}$):

Theorem 4 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-regular open set. Let $M > 0$. For $\varepsilon > 0$ let us define the functional, for $(u, v) \in L^2(\Omega; \mathbb{R}^2) \times L^2(\Omega)$*

$$E_\varepsilon(u, v) \;=\; \begin{cases} \displaystyle\int_\Omega (v^2 + \eta_\varepsilon) W(e(u))\, dx \;+\; \int_\Omega \varepsilon |\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\, dx & \\ \qquad\qquad \text{if } (u,v) \in H^1(\Omega;\mathbb{R}^2) \times H^1(\Omega) \text{ and } \|u\|_{L^\infty} \le M\,; & \\ +\infty \qquad\qquad \text{otherwise,} & \end{cases} \tag{12}$$

with $\eta_\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0$. Then, as $\varepsilon \to 0$, E_ε Γ–converges (in $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega)$) to

$$E(u, v) \;=\; \begin{cases} \displaystyle\int_\Omega W(e(u))\, dx \;+\; \mathcal{H}^1(J_u) \quad \text{if } u \in SBD(\Omega), \quad v = 0, & \\ \qquad\qquad\qquad\qquad \text{and } \|u\|_{L^\infty} \le M\,; & \\ +\infty \qquad\qquad \text{otherwise.} & \end{cases} \tag{13}$$

Proof. The proof of most of this result is now standard [6, 7, 1, 15]. We just sketch the proof of the Γ–$\liminf$ inequality, following an approach of Braides and Solci [12] (*cf* also [9]). We choose u_j, v_j that converge to some u, v in L^2, and such that $\sup_{j\ge 1} E_{\varepsilon_j}(u_j, v_j) < +\infty$, where (ε_j) is a sequence that goes to 0. First, we notice that we must have $v = 1$ (since $\int_\Omega (1-v_j)^2\, dx \le c\varepsilon_j$). We write that $\int_\Omega \varepsilon_j |\nabla v_j|^2 + (1-v_j)^2/(4\varepsilon_j)\, dx \ge \int_\Omega |1-v_j||\nabla v_j|\, dx$, so that, using the coarea formula,

$$E_{\varepsilon_j}(u_j, v_j) \;\ge\; \int_0^1 ds \left(\int_{\{v_j > s\}} 2sW(e(u_j))\, dx \;+\; (1-s)\mathcal{H}^1(\partial_*\{v_j > s\}) \right).$$

($\partial_*\{v_j > s\}$ denotes the reduced boundary of the finite perimeter set $\{x : v_j(x) > s\}$, see [19, 20].) Then, we need to adapt [10, Lemma 2] to the SBD case (with uniform L^∞ bound M), with the assumption that $u_j \to u$ in $L^2(\Omega;\mathbb{R}^2)$, following essentially the lines of the proof we gave of Lemma 5.1. We will deduce that for almost each $s \in (0,1)$,

$$\int_\Omega 2sW(e(u))\, dx + 2(1-s)\mathcal{H}^1(J_u) \;\le\; \liminf_{j\to\infty} \int_{\{v_j > s\}} 2sW(e(u_j))\, dx \;+\; (1-s)\mathcal{H}^1(\partial_*\{v_j > s\})\,.$$

Integrating over $s \in (0,1)$ and using Fatou's lemma, we get the inequality $E(u,v) \le \liminf_{j\to\infty} E_{\varepsilon_j}(u_j, v_j)$.

To prove the Γ–$\limsup$ inequality, we first notice that because of Theorem 3, we just need to prove it for a (u,v) with $v = 0$ and $u \in SBD(\Omega)$ with $\mathcal{H}^1(\overline{J}_u) < +\infty$, replacing $\mathcal{H}^1(J_u)$ by $\mathcal{H}^1(\overline{J}_u)$ in the energy (assuming also $\overline{J}_u$ is rectifiable). Then, a standard diagonalization argument will yield the result. We follow the approach in [9]. We notice that

$$\lim_{s\to 0} \frac{|\{x \in \mathbb{R}^2 \,:\, \mathrm{dist}\,(x, \overline{J}_u) < s\}|}{2s} \;=\; \mathcal{H}^1(\overline{J}_u).$$

Indeed, the left–hand side of this equality is the Minkowsky contents of the set $\overline{J}_u$, which is known to coincide with the 1–dimensional Hausdorff measure for closed and rectifiable subsets of $\mathbb{R}^2$ [5, 20].

We let $d(x) = \mathrm{dist}\,(x, \overline{J}_u)$, and $f(s) = |\{x \in \Omega : d(x) < s\}|$ for all $s > 0$. We have $\limsup_{s\to 0} f(s)/(2s) \le \mathcal{H}^1(\overline{J}_u)$. Let $\alpha_\varepsilon < \varepsilon$ be a small parameter (that goes to 0 and will be precised later on). We let, for every $\varepsilon > 0$, $v_\varepsilon(x) = 1 - \exp(-(d(x) - \alpha_\varepsilon)/2\varepsilon)$ if $d(x) > \alpha_\varepsilon$, $v_\varepsilon(x) = 0$ otherwise, while $u_\varepsilon(x) = u(x)$ if $d(x) \ge \alpha_\varepsilon$, $u_\varepsilon(x) = (2d(x)/\alpha_\varepsilon - 1)u(x)$ if $\alpha_\varepsilon > d(x) \ge \alpha_\varepsilon/2$, and $u_\varepsilon(x) = 0$ if $d(x) < \alpha_\varepsilon/2$. This u_ε is in $H^1(\Omega)$. It is clear that $v_\varepsilon \to 0$ as $\varepsilon \to 0$, while $u_\varepsilon \to u$ (in L^2). On the other hand,

$$\begin{aligned}\int_\Omega (v_\varepsilon^2 + \eta_\varepsilon) W(e(u_\varepsilon))\,dx \;&\le\; (1+\eta_\varepsilon)\int_\Omega W(e(u))\,dx \,+\, c\left|\left\{\frac{\alpha_\varepsilon}{2} < d < \alpha_\varepsilon\right\}\right| \frac{\eta_\varepsilon M^2}{\alpha_\varepsilon^2} \\ &=\; (1+\eta_\varepsilon)\int_\Omega W(e(u))\,dx \,+\, O\left(\frac{\eta_\varepsilon}{\alpha_\varepsilon}\right).\end{aligned}$$

We see that if $\eta_\varepsilon = o(\alpha_\varepsilon)$, then the limit of the right-hand side is $\int_\Omega W(e(u))\,dx$. Let us estimate the other term of $E_\varepsilon(u_\varepsilon, v_\varepsilon)$. One has

$$\int_\Omega \varepsilon|\nabla v_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \;=\; \frac{1}{2\varepsilon}\int_{\{d>\alpha_\varepsilon\}} e^{-\frac{d-\alpha_\varepsilon}{\varepsilon}}\,dx \;=\; \frac{1}{2\varepsilon}\int_{\alpha_\varepsilon}^{+\infty} e^{-\frac{s-\alpha_\varepsilon}{\varepsilon}} \mathcal{H}^1(\partial\{d > s\})\,ds.$$

Since $f(s) = \int_0^s \mathcal{H}^1(\partial\{d > t\})\,dt$, integrating by parts we get that this integral is

$$-\frac{1}{2\varepsilon} f(\alpha_\varepsilon) \,+\, \frac{1}{2\varepsilon^2}\int_{\alpha_\varepsilon}^\infty f(s) e^{-\frac{s-\alpha_\varepsilon}{\varepsilon}}\,ds \;=\; -\frac{\alpha_\varepsilon}{\varepsilon}\frac{f(\alpha_\varepsilon)}{2\alpha_\varepsilon} \,+\, \int_0^\infty \left(\frac{\alpha_\varepsilon}{\varepsilon} + t\right) \frac{f(\alpha_\varepsilon + \varepsilon t)}{2(\alpha_\varepsilon + \varepsilon t)} e^{-t}\,dt\,.$$

Since $\int_0^\infty t e^{-t}\,dt = 1$ and $\limsup_{\varepsilon\to 0} f(\alpha_\varepsilon + \varepsilon t)/(2(\alpha_\varepsilon + \varepsilon t)) \le \mathcal{H}^1(\overline{J}_u)$, the lim sup of the above expression is not greater than $\mathcal{H}^1(\overline{J}_u)$ as soon as $\alpha_\varepsilon = o(\varepsilon)$. Hence, choosing $\alpha_\varepsilon = \sqrt{\varepsilon\eta_\varepsilon}$, we have both $\eta_\varepsilon = o(\alpha_\varepsilon)$ and $\alpha_\varepsilon = o(\varepsilon)$, and we deduce $\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \le \int_\Omega W(e(u))\,dx \,+\, \mathcal{H}^1(\overline{J}_u)$. □

Remark 6.1 *A more carefully written proof would show that it is possible to take* $\alpha_\varepsilon = O(\varepsilon)$*, which is interesting from a numerical analysis point of view.*

Remark 6.2 *One shows also easily that if* $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ *is such that* $\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$*, then some subsequence* $(u_{\varepsilon_j}, v_{\varepsilon_j})_{j\ge 1}$ *will converge in* L^2*. To do so, one notices that one can select for each* ε *a level* $s_\varepsilon \simeq 1/2$ *such that* $\sup_{\varepsilon>0} \mathcal{H}^1(\partial_*\{v_\varepsilon > s_\varepsilon\}) < +\infty$*. Then, we apply the compactness result in [8, Thm. 1.1] to the functions* $u'_\varepsilon = u_\varepsilon\,\chi_{\{v_\varepsilon > s\}}$*, which are uniformly bounded in* $BD(\Omega)$ *thanks to the* L^∞ *bound in the definition (12) of* E_ε*.*

A A simple inequality

The following lemma is essential in the proof of Theorem 1. Given $U = (u^\alpha_{i,j})_{\substack{\alpha=1,2 \\ i,j=0,1}} \in \mathbb{R}^8$, we associate a displacement $u(x_1, x_2)$ by letting

$$u(x_1, x_2) \;=\; \left(\sum_{i,j=0,1} u^\alpha_{i,j} \Delta(x_1 - i, x_2 - j)\right)_{\alpha=1,2}$$

where $\Delta(x_1,x_2) = (1-|x_1|)^+(1-|x_2|)^+$. We can define a positive quadratic form of U by letting $Q_1(U) = \int_{(0,1)^2} W(e(u))\,dx_1dx_2$ where W is given by (3). Another quadratic form is given by the formula

$$Q_2(U) = \frac{1}{2}\left((u^1_{1,0}-u^1_{0,0})^2+(u^1_{1,1}-u^1_{0,1})^2+(u^2_{0,1}-u^2_{0,0})^2+(u^2_{1,1}-u^2_{1,0})^2\right)$$
$$+\frac{1}{4}\left((u^1_{1,1}+u^2_{1,1}-u^1_{0,0}-u^2_{0,0})^2+(u^1_{0,1}-u^2_{0,1}-u^1_{1,0}+u^2_{1,0})^2\right).$$

We show the following result.

Lemma A.1 $Q_1 \le Q_2$

Proof. There are several ways to show this inequality, however, we did not find any that is really satisfactory. Indeed, this lemma is the only point in the proof of Theorem 1 that is not straightforward to extend in higher dimension (Theorems 2 and 3 would then also easily follow in any dimension). In fact, given a fixed dimension N, it is possible to show the N–dimensional version of this result, by a "straightforward" matrix calculation (that we will perform here in dimension 2). However, the matrices that are involved are of dimension $(N2^N)\times(N2^N)$, and it would be much nicer to find some general and systematic proof of the result not depending on the dimension. A possible approach would be to consider a general discrete energy $Q\left((u(\xi))_{\xi\in\{0,1\}^N}\right)$ defined on the values $u(\xi)$ at the vertices ξ of the unit cube (with some reasonable properties, nonnegative, invariant by addition of a constant, maybe quadratic, maybe with other symmetries, etc...), scale it appropriately to define a discrete energy at scale $h>0$ in the unit cube, consider its Γ–limit (for instance in H^1-weak), which should be of the form $\int_{(0,1)^N} W(\nabla u)\,dx$, and then show that if $u(x)$ is the function $\sum_{\xi\in\{0,1\}^N} u(\xi)\Pi_{i=1}^N(1-|x_i-\xi_i|)^+$ then $\int_{(0,1)^N} W(\nabla u)\,dx \le Q\left((u(\xi))_{\xi\in\{0,1\}^N}\right)$. We believe that such a result should hold, for a reasonably large class of functions Q.

Since we do not know how to prove such a result, let us just compute the matrices A_1 and A_2 of Q_1 and Q_2 and compare them. In order to do so we will use the following ordering of the 8 coefficients of U:

$$U = (u^1_{0,0},u^1_{1,0},u^1_{0,1},u^1_{1,1},u^2_{0,0},u^2_{0,1},u^2_{1,0},u^2_{1,1}).$$

Then, because of the symmetries, we see that for $i=1,2$,

$$A_i = \begin{pmatrix} B_i & C_i \\ C_i^T & B_i \end{pmatrix}$$

where B_i, C_i are 4×4 matrices (B_i is symmetric). A_2 is easy to compute:

$$B_2 = \frac{1}{4}\begin{pmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \end{pmatrix} \quad\text{and}\quad C_2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

To compute A_1, we need to compute the scalar products $\int_{(0,1)^2} \mathbf{A}e(w_i):e(w_j)$, where $(w_i)_{i=1}^8$ are the basis functions defining u ($u(x)=\sum_{i=1}^8 u_iw_i(x)$), and $\mathbf{A}$ is the tensor associated to the quadratic form W, that is, such that $\mathbf{A}\sigma = \sigma + (1/2)(\mathrm{Tr}\,\sigma)I$ for any $\sigma \in \mathcal{S}^{2\times 2}$. The Table 1 gives the 8 functions w_i, their symmetrized gradients $e(w_i)$ and the corresponding $\mathbf{A}e(w_i)$.

i	w_i	$e(w_i)$	$\mathbf{A}e(w_i)$
1	$\begin{pmatrix}(1-x_1)(1-x_2)\\0\end{pmatrix}$	$\begin{pmatrix}-(1-x_2) & -\frac{1-x_1}{2}\\ -\frac{1-x_1}{2} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-3(1-x_2) & -(1-x_1)\\ -(1-x_1) & -(1-x_2)\end{pmatrix}$
2	$\begin{pmatrix}x_1(1-x_2)\\0\end{pmatrix}$	$\begin{pmatrix}1-x_2 & -\frac{x_1}{2}\\ -\frac{x_1}{2} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}3(1-x_2) & -x_1\\ -x_1 & 1-x_2\end{pmatrix}$
3	$\begin{pmatrix}(1-x_1)x_2\\0\end{pmatrix}$	$\begin{pmatrix}-x_2 & \frac{1-x_1}{2}\\ \frac{1-x_1}{2} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-3x_2 & 1-x_1\\ 1-x_1 & -x_2\end{pmatrix}$
4	$\begin{pmatrix}x_1x_2\\0\end{pmatrix}$	$\begin{pmatrix}x_2 & \frac{x_1}{2}\\ \frac{x_1}{2} & 0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}3x_2 & x_1\\ x_1 & x_2\end{pmatrix}$
5	$\begin{pmatrix}0\\(1-x_1)(1-x_2)\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{1-x_2}{2}\\ -\frac{1-x_2}{2} & -(1-x_1)\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-(1-x_1) & -(1-x_2)\\ -(1-x_2) & -3(1-x_1)\end{pmatrix}$
6	$\begin{pmatrix}0\\(1-x_1)x_2\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{x_2}{2}\\ -\frac{x_2}{2} & 1-x_1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1-x_1 & -x_2\\ -x_2 & 3(1-x_1)\end{pmatrix}$
7	$\begin{pmatrix}0\\x_1(1-x_2)\end{pmatrix}$	$\begin{pmatrix}0 & \frac{1-x_2}{2}\\ \frac{1-x_2}{2} & -x_1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}-x_1 & 1-x_2\\ 1-x_2 & -3x_1\end{pmatrix}$
8	$\begin{pmatrix}0\\x_1x_2\end{pmatrix}$	$\begin{pmatrix}0 & \frac{x_2}{2}\\ \frac{x_2}{2} & x_1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}x_1 & x_2\\ x_2 & 3x_1\end{pmatrix}$

Table 1: The basis $(w_i)_{i=1}^{8}$ and its derivatives

Using $\int_0^1 x(1-x)\,dx = 1/6$, $\int_0^1 x^2\,dx = \int_0^1 (1-x)^2\,dx = 1/3$, and $\int_{(0,1)^2} x_1x_2\,dx = 1/4$, we deduce easily that

$$B_1 \;=\; \frac{1}{12}\begin{pmatrix} 8 & -5 & 1 & -4\\ -5 & 8 & -4 & 1\\ 1 & -4 & 8 & -5\\ -4 & 1 & -5 & 8\end{pmatrix} \qquad \text{and} \qquad C_1 \;=\; C_2\,.$$

Hence $A_2 - A_1$ has the form

$$A_2 - A_1 \;=\; \begin{pmatrix} B_2 - B_1 & 0\\ 0 & B_2 - B_1\end{pmatrix}.$$

The matrix $B_2 - B_1$, given by

$$B_2 - B_1 = \frac{1}{12}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

has eigenvalues 0 (with multiplicity 3) and 1/3: it is nonnegative, which shows the lemma. □

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