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AN APPROXIMATIVE METHOD OF SIMULATING A DUEL

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ABSTRACT

We develop a dynamic Markovian method of simulating a battle between two infantry units. Its key feature is that the probabilities of the outcomes of the battle can be computed efficiently, without the joint distribution of the strengths of the units or their transition matrix, making the method feasible even with larger unit strengths. We find the probabilities of the outcomes to be close to the ones obtained from a more elaborate, but computationally more costly, joint Markov-chain model of strengths. Additionally, using our method we are able to compute the conditional distributions of the strength of a unit, given that it has, respectively, won the battle or been defeated by the enemy.

1 INTRODUCTION

The history of modern mathematical modeling of warfare essentially begins from the differential equations introduced by F. W. Lanchester (1868–1946). In particular, he attempted to model the sea battle of Trafalgar, which was fought between a British fleet and a combined Franco-Spanish fleet in 1805, by coupled, linear ordinary differential equations describing the number of operational ships in the opposing fleets (Lanchester 1916). Despite their simplicity and appealing mathematical elegance, a well-documented shortcoming of Lanchester's equations—already noted by Lanchester himself in the context of the battle of Trafalgar—is that they fail to capture the *randomness* and *uncertainty* that pervades virtually all theaters of war. In reality, say, drastic losses suffered by either of the fleets initially would very likely determine how the battle unfolds and might even help the, a priori, weaker fleet with fewer ships or less firepower to win the battle. Such phenomena are ruled out by Lanchester's equations, which imply gradual attrition of the fleets and a predetermined winner for the battle.

Motivated by the caveats of Lanchester's equations, various stochastic counterparts have been suggested in the literature. The canonical example of those is the Markov-process model commonly known as the *stochastic Lanchester model* that, according to Ancker and Gafarian (1988), first appears in the work of Snow (1948). Kingman (2002) gives a more recent description and analysis of this model, which represents the evolution of the strengths of opposing forces engaging each other in a duel as a bivariate continuous-time Markov process. Assuming that the strengths of the opposing forces are n and m, respectively, the Markov process assumes values in a state space with $(n+1)(m+1)$ elements. Alas, for large n and m, fast numerical computation of the probability distribution of the state of the stochastic Lanchester model becomes difficult. It has been proved that when n and m tend to infinity, with n/m fixed, the rescaled trajectories of the stochastic Lanchester model converge to the solution of Lanchester's classical differential equations—see Ancker and Gafarian (1988) for a comprehensive review of such results. This, of course, provides a way to analyze the stochastic Lanchester model with larger troop sizes, but as a first-order (i.e., in the *mean*) deterministic approximation, suffering from the shortcomings discussed above, it is clearly insufficient, e.g., for the purposes of risk analysis.

In this paper, we present a numerically efficient, approximative, Markovian method of modeling a duel, which is computationally feasible even with larger troop strengths. The method is geared towards providing the key probabilities that the battle is won by either of the troops involved in the duel or that both are defeated—which is typically unlikely, though. The novel feature of this method is that the computation of the probabilities of the outcomes of the battle can be done using a simple recursion, which does not require knowledge of the full joint distribution of the strengths of the troops nor the associated state transition matrix. One needs to keep track only of the conditional marginal distributions of strengths given that the battle continues. Furthermore, the conditional distributions of the strengths of a unit, given that it has won the battle or been defeated, respectively, can be computed in a similar recursive fashion rather efficiently.

The results of this paper are a part of the ongoing development of *Sandis* combat simulation software (Lappi 2008). Sandis is a land warfare simulator developed and used by Finnish Defence Forces to model combat scenarios involving military units ranging from platoons to brigades, with strengths ranging from (roughly) 20 to 3000 soldiers. It comprises detailed models of the armament and equipment of the troops, effects of direct and indirect fire (with fragmenting ammunition), medical evacuation in the battle field, and electronic warfare. Mathematically, Sandis is based on Markovian modeling of the probability distributions the strengths of the involved units. Currently, only marginal distributions of the strengths of the units in a scenario are computed and the probable outcome of the battle needs to be judged solely based on them, e.g., by comparing the expected values. While the actual probabilities of friendly or enemy forces winning are, thus, not readily available in the current version of Sandis, a *manual* method of deriving them has been recently described by Lappi (2012). The refined method described in this paper will be used as a foundation for a proper duel simulation feature that will be implemented in future versions of Sandis.

2 MATHEMATICAL FRAMEWORK

Consider a duel between Blue and Red infantry, fought with *direct-fire* weaponry (e.g., rifles or machine guns). Let us denote by B_t and R_t the strengths of the Blue and Red units, respectively, at time $t = 0, 1, \ldots, T$, where $T \in \mathbb{N}$ is the total number of time steps in the simulation. The initial strengths $B_0 \in \mathbb{N}$ and $R_0 \in \mathbb{N}$ are deterministic constants, whereas the subsequent strengths are random variables. Neither of the units receive any reinforcements during the battle, so we have $B_t \leqslant B_{t-1}$ and $R_t \leqslant R_{t-1}$. Both units have their respective minimal strengths $\underline{b} \in \{1, \ldots, B_0\}$ and $\underline{r} \in \{1, \ldots, R_0\}$ to be operational. If the strength of the unit falls below this threshold, the unit is unable to continue the battle and is considered defeated. In this case, the strengths of the units are frozen to signify the end of the battle. (What happens after that—are the remaining soldiers of the defeated unit captured as *prisoners of war*, or are they able to retreat—is beyond the scope of our modeling problem.) Formally, if we have $B_t < \underline{b}$ or $R_t < \underline{r}$ for some t, then $B_s = B_t$ and $R_s = R_t$ for all $s \geqslant t$. At any given time t, there are four distinct possibilities for the state of the battle:

- the battle continues, $C_t = \{B_t \geqslant \underline{b}, R_t \geqslant \underline{r}\}$,
- Blue troops have won, $W_{B,t} = \{B_t \geqslant \underline{b}, R_t < \underline{r}\}$,
- Red troops have won, $W_{R,t} = \{B_t < \underline{b}, R_t \geqslant \underline{r}\}$,
- both troops are defeated, $D_t = \{B_t < \underline{b}, R_t < \underline{r}\}$.

Since B_t and R_t take decreasing trajectories due to attrition, it follows that the probability of C_t will decrease and the probabilities of the terminal states, or *outcomes*, $W_{B,t}$, $W_{R,t}$, and D_t will increase over time.

It should be stressed that we have chosen to interpret Blue and Red here as *infantry* units and measure their strengths in terms of individual *soldiers* merely to provide an illustrative exposition. Indeed, our simulation method is not necessarily restricted to infantry—being equally applicable to the modeling of, e.g., *armored warfare* using the interpretation that the strengths are the numbers of operational *tanks* involved.

2.1 Reference Method

We first describe a reference method of defining a probability model for the evolution of the strengths (B_t, R_t) of the units over time. It is a generalization of the *point-fire* model employed by the current version of Sandis (Lappi and Pottonen 2006), and bears some similarity to the stochastic Lanchester model. We shall denote the associated probabilities by P. With the reference method, the t-th time step of the simulation is taken as follows.

1. In the beginning of the time step, the strengths of the units are B_{t-1} and R_{t-1}, respectively.
2. Each Blue (resp. Red) soldier fires independently $\lambda_B > 0$ (resp. $\lambda_R > 0$) rounds, on average, directed randomly and evenly at the R_{t-1} (resp. B_{t-1}) targets that consist of the enemy soldiers.
3. The probability that a round fired by a Blue (resp. Red) soldier produces a hit that incapacitates an enemy soldier is $p_B \in (0,1)$ (resp. $p_R \in (0,1)$).
4. In the end of the time step, soldiers that have suffered incapacitating hits are removed from the battle field and new strengths B_t and R_t are computed.

In accordance to this procedure, the strengths (B_t, R_t) follow a Markov chain with binomial transition probabilities

$$P[B_t = b, R_t = r | B_{t-1} = b_{-1}, R_{t-1} = r_{-1}] = \binom{b_{-1}}{b_{-1} - b} \pi_R(b_{-1}, r_{-1})^{b_{-1}-b} \big(1 - \pi_R(b_{-1}, r_{-1})\big)^b \\ \times \binom{r_{-1}}{r_{-1} - r} \pi_B(b_{-1}, r_{-1})^{r_{-1}-r} \big(1 - \pi_B(b_{-1}, r_{-1})\big)^r \quad (1)$$

if $b_{-1} \geqslant \underline{b}$, $b_{-1} \geqslant b$, $r_{-1} \geqslant \underline{r}$, and $r_{-1} \geqslant r$, where

$$\pi_B(b_{-1}, r_{-1}) = 1 - (1 - p_B)^{\frac{\lambda_B b_{-1}}{r_{-1}}} \quad \text{and} \quad \pi_R(b_{-1}, r_{-1}) = 1 - (1 - p_R)^{\frac{\lambda_R r_{-1}}{b_{-1}}}.$$

Otherwise, we set

$$P[B_t = b, R_t = r | B_{t-1} = b_{-1}, R_{t-1} = r_{-1}] = \begin{cases} 1, & \text{if } b = b_{-1} \text{ and } r = r_{-1}, \\ 0, & \text{if } b \neq b_{-1} \text{ or } r \neq r_{-1}. \end{cases} \quad (2)$$

Note that, mathematically, the strengths B_t and R_t are *conditionally* independent, given the previous strengths B_{t-1} and R_{t-1}, that is,

$$P[B_t = b, R_t = r \,|B_{t-1} = b_{-1}, R_{t-1} = r_{-1}] = \\ P[B_t = b \,|B_{t-1} = b_{-1}, R_{t-1} = r_{-1}] P[R_t = r \,|B_{t-1} = b_{-1}, R_{t-1} = r_{-1}]$$

for any $b, b_{-1} \in \{0, 1, \ldots, B_0\}$ and $r, r_{-1} \in \{0, 1, \ldots, R_0\}$. For an introduction to the conditional independence property, we refer to Pfeiffer (1978).

Equations (1) and (2) give us the entries of the state transition matrix of the Markov chain (B_t, R_t) under P. Thus, numerical evaluation of the joint distribution of (B_t, R_t) boils down to multiplying the initial (degenerate) distribution vector of (B_0, R_0) by the state transition matrix t times. Probabilities of the states C_t, $W_{B,t}$, $W_{R,t}$, and D_t can then be calculated from the joint distribution. The state transition matrix is $(B_0+1)(R_0+1) \times (B_0+1)(R_0+1)$-dimensional, but relatively sparse, so for small initial strengths B_0 and R_0, the construction of the matrix and the subsequent multiplications are manageable from the computational point of view. However, with larger initial strengths, the computational burden grows rapidly, which motivates a search for more efficient methods of simulating the battle, as discussed above.

2.2 Approximative Method

We now present the approximative method of defining a probability model for (B_t, R_t), which aims to be close to the reference model at least in terms of probabilities of the state of the battle. We shall denote by $\bar{P}$ the probabilities associated to this approximative method. The method relies on the mathematical assumption that for any $t = 1, 2, \ldots, T$ the strengths B_t and R_t of the units are conditionally independent given that *the battle has continued at time* $t-1$, that is,

$$\bar{P}[B_t = b, R_t = r \,|C_{t-1}] = \bar{P}[B_t = b \,|C_{t-1}]\bar{P}[R_t = r \,|C_{t-1}]. \tag{3}$$

We take the t-th time step as with the reference method, except that we draw the previous strengths of Blue and Red troops independently from the *marginal conditional distributions* $\bar{P}[B_{t-1} = b_{-1} \,|C_{t-1}]$ and $\bar{P}[R_{t-1} = r_{-1} \,|C_{t-1}]$, respectively. Thus, we set

$$\bar{P}[B_t = b \,|C_{t-1}] = \sum_{b_{-1}=\underline{b}}^{B_0} \sum_{r_{-1}=\underline{r}}^{R_0} \bar{P}[B_{t-1} = b_{-1}|C_{t-1}]\bar{P}[R_{t-1} = r_{-1}|C_{t-1}]$$
$$\times \binom{b_{-1}}{b_{-1} - b} \pi_R(b_{-1}, r_{-1})^{b_{-1}-b} \big(1 - \pi_R(b_{-1}, r_{-1})\big)^b, \tag{4}$$

where the $\pi_R(b_{-1}, r_{-1})$ and the associated parameters are as in the reference model. For $\bar{P}[R_t = r \,|C_{t-1}]$ we specify an analogous formula. As the battle continues only if the Blue strength is above $\underline{b}$, we have

$$\bar{P}[B_t = b|C_t] = \begin{cases} \dfrac{\bar{P}[B_t = b|C_{t-1}]}{\bar{P}[B_t \geqslant \underline{b}|C_{t-1}]}, & \text{if } b \geqslant \underline{b}, \\ 0, & \text{if } b < \underline{b}, \end{cases} \tag{5}$$

and an analogous expression for $\bar{P}[R_t = r|C_t]$.

Let us look into the key properties of the approximative method. Since $C_t \subset C_{t-1}$, by conditional independence (3), we may update the probability that the battle continues through

$$\bar{P}[C_t] = \bar{P}[C_t \cap C_{t-1}] = \bar{P}[B_t \geqslant \underline{b}|C_{t-1}]\bar{P}[R_t \geqslant \underline{r}|C_{t-1}]\bar{P}[C_{t-1}].$$

Moreover, we have

$$\bar{P}[B_t = b, R_t = r] = \bar{P}[B_t = b \,|C_{t-1}]\bar{P}[R_t = r \,|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[B_{t-1} = b, R_{t-1} = r] \tag{6}$$

provided that $b < \underline{b}$ or $r < \underline{r}$. This identity enables us to evaluate recursively the decisive probabilities $\bar{P}[B_t = b, R_t = r]$, where $b < \underline{b}$ or $r < \underline{r}$, *without* keeping track of the whole joint distribution of (B_t, R_t). This procedure requires only a record of the marginal conditional distributions $\bar{P}[B_t = b \,|C_{t-1}]$ and $\bar{P}[R_t = r \,|C_{t-1}]$ up to the t-th time step. From (6) we immediately obtain recursive formulae for the probabilities of the terminal states of the battle:

$$\bar{P}[W_{B,t}] = \bar{P}[B_t \geqslant \underline{b}|C_{t-1}]\bar{P}[R_t < \underline{r}|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[W_{B,t-1}], \tag{7}$$

$$\bar{P}[W_{R,t}] = \bar{P}[B_t < \underline{b}|C_{t-1}]\bar{P}[R_t \geqslant \underline{r}|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[W_{R,t-1}], \tag{8}$$

and

$$\bar{P}[D_t] = \bar{P}[B_t < \underline{b}|C_{t-1}]\bar{P}[R_t < \underline{r}|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[D_{t-1}]. \tag{9}$$

Recall that, initially, $\bar{P}[W_{B,0}] = \bar{P}[W_{R,0}] = \bar{P}[D_0] = 0$.

2.3 Conditional Distributions of Strengths

In addition to the probabilities of the states of the battle, we are interested in the *conditional distributions* of the strength of a unit *given that it has won or been defeated*, respectively. With the reference method, such distributions can be easily obtained from the joint distribution of (B_t, R_t), whereas with the approximative one, we can easily derive recursive formulae, akin to (7), (8), and (9), for them. The conditional distribution of the strength of Blue troops given that they have won satisfies

$$\bar{P}[B_t = b\,|W_{B,t}] = \frac{\bar{P}[B_t = b\,|C_{t-1}]\bar{P}[R_t < \underline{r}\,|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[B_{t-1} = b\,|W_{B,t-1}]\bar{P}[W_{B,t-1}]}{\bar{P}[W_{B,t}]} \tag{10}$$

for $b \geqslant \underline{b}$, whereas the distribution given that they are defeated satisfies

$$\bar{P}[B_t = b\,|D_t \cup W_{R,t}] = \frac{\bar{P}[B_t = b\,|C_{t-1}]\bar{P}[C_{t-1}] + \bar{P}[B_{t-1} = b\,|D_{t-1} \cup W_{R,t-1}](\bar{P}[D_{t-1}] + \bar{P}[W_{B,t-1}])}{\bar{P}[D_t] + \bar{P}[W_{B,t}]} \tag{11}$$

for $b < \underline{b}$. The initial distributions $\bar{P}[B_0 = b\,|W_{B,0}]$ and $\bar{P}[B_0 = b\,|D_0 \cup W_{R,0}]$ cannot be defined unambiguously, as $\bar{P}[W_{B,0}] = \bar{P}[W_{R,0}] = \bar{P}[D_0] = 0$, but this is not an issue because their contribution to (10) and (11) is multiplied by zero in any case. Again, analogous formulae hold for the strength of Red troops.

3 NUMERICAL RESULTS

We evaluated the accuracy of the approximative method relative to the reference method with numerical experiments. Specifically, we studied how closely the methods match when we compute the probabilities of the states of the battle or the conditional distributions of the strength of a unit, given that it has won or been defeated, respectively.

3.1 Example Scenario

To illustrate the output of the methods, we first made a simple experiment involving Blue and Red platoons, initially with $B_0 = 26$ and $R_0 = 30$ soldiers, respectively. To make the scenario more even, we compensated for the larger initial strength of Red by assuming that the marksmanship of Blue soldiers is superior and setting $p_B = 0.03$ and $p_R = 0.02$, respectively. These parameter values are comparable to typical hitting probabilities recorded in field experiments involving infantry, armed with assault rifles (Lappi and Pottonen 2006, Lappi and Vulli 2008). Moreover, we set the firing rates to be equal, $\lambda_B = \lambda_R = 2$. Finally, we used the typical criterion that a platoon is no longer operational if at least half of its soldiers have been hit in an incapacitative way, implying that $\underline{b} = 14$ and $\underline{r} = 16$, respectively.

Figure 1 displays the trajectories of the probabilities of the states and the conditional distributions, computed using both methods, in this scenario. We observe that with both methods, the battle ends and, thus, the outcome is clear with overwhelming probability after $T = 25$ time steps. The approximative method gives a slightly exaggerated picture of the outcome of the battle—in the sense that it overestimates the probability of Blue winning and, conversely, underestimates the probability of Red winning. Compared to the reference method, the approximative method gives more pessimistic conditional distributions of the strength, given that the platoon has won—in the sense that the probability mass is shifted towards lower strength. However, in terms of the conditional distributions, given that the platoon is defeated, the methods are in close agreement.

3.2 Experiment with Varied Parameter Values

To gain an understanding of how the approximation error varies across the parameter space, we made a more elaborate experiment using a two-dimensional gridded design, varying the parameters of Red troops, viz. R_0 (between 10 and 50) and p_R (between 0.002 and 0.032) over $41 \times 41 = 1681$ design points. For Blue

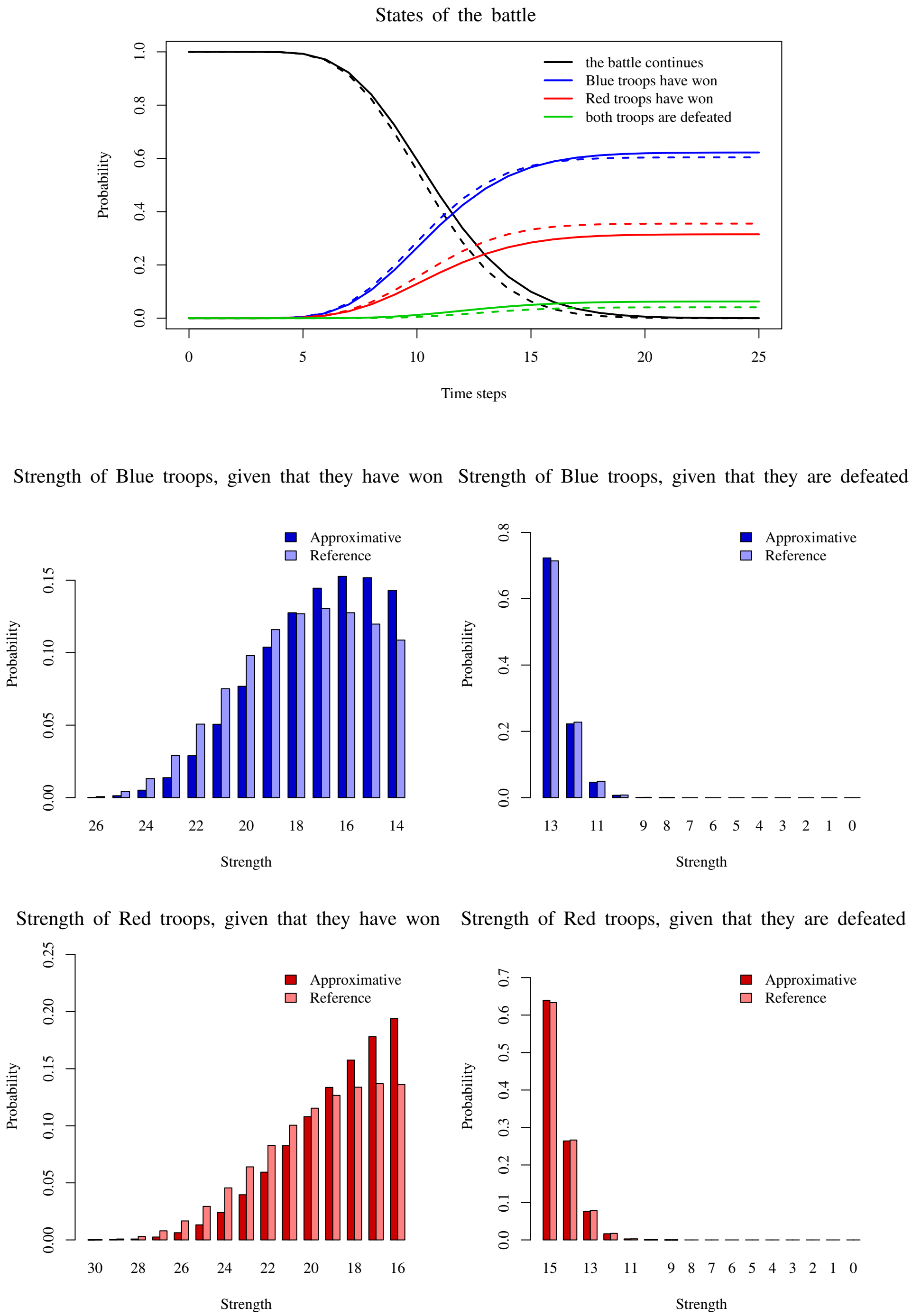

Figure 1: Top: evolution of the probabilities of the states of the battle according to the approximative method (solid lines) and to the reference method (dashed lines). Bottom: the conditional distributions of the strengths of the platoons. Values of the parameters: $T = 25$, $B_0 = 26$, $p_B = 0.03$, $\lambda_B = 2$, $\underline{b} = 14$, $R_0 = 30$, $p_R = 0.02$, $\lambda_R = 2$, and $\underline{r} = 16$.

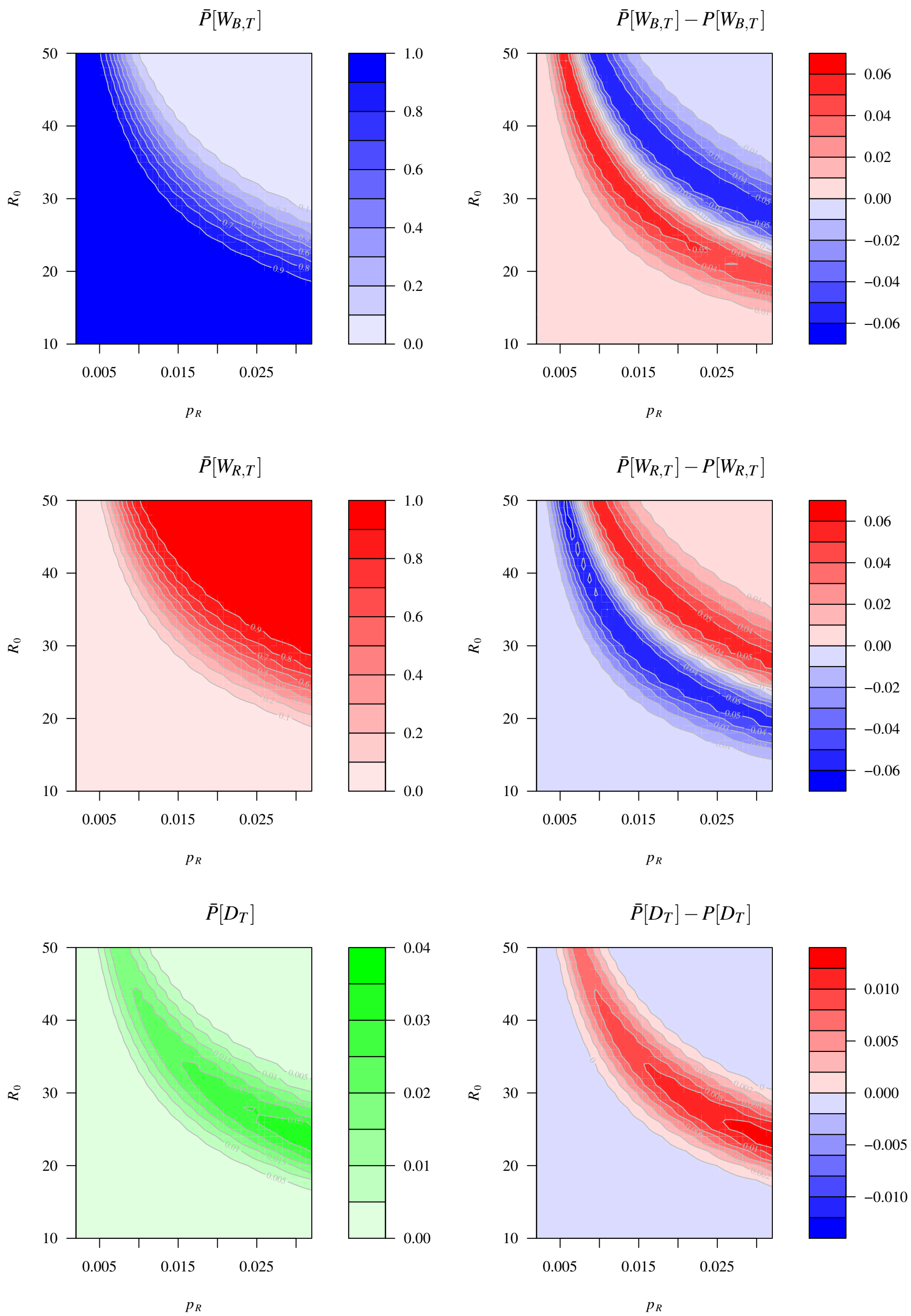

Figure 2: Probabilities of the states of the battle, as given by the approximative method (left column) and the corresponding error relative to the reference method (right column). Values of the other parameters: $T = 500$, $B_0 = 30$, $p_B = 0.02$, $\lambda_B = 1$, $\underline{b} = 16$, $\lambda_R = 1$, and $\underline{r} = \lfloor R_0/2 \rfloor + 1$, where $\lfloor \cdot \rfloor$ stands for the floor function.

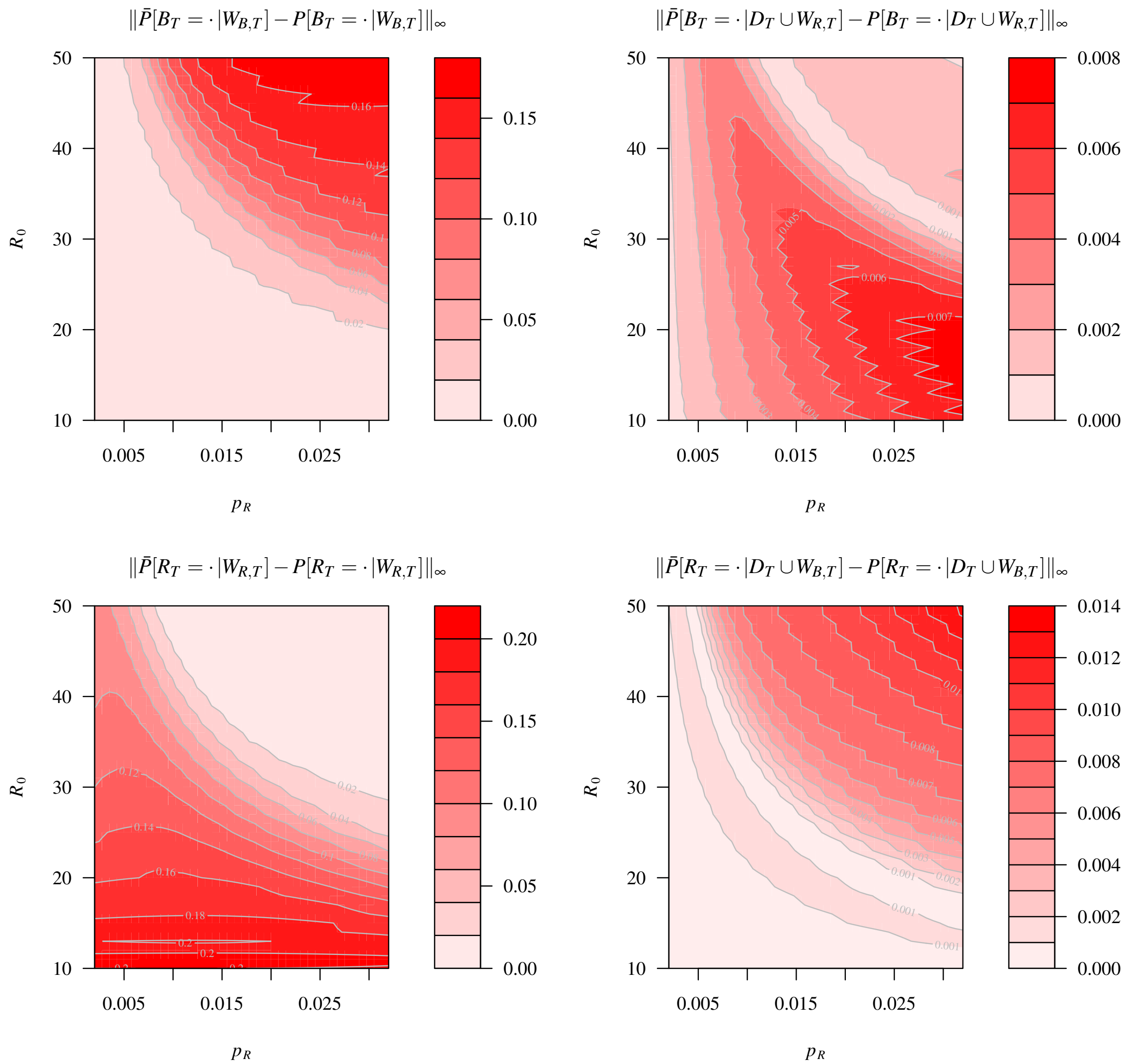

Figure 3: Error of the approximative method relative to the reference method with conditional distributions of the strengths. Values of the other parameters: $T = 500$, $B_0 = 30$, $p_B = 0.02$, $\lambda_B = 1$, $\underline{b} = 16$, $\lambda_R = 1$, and $\underline{r} = \lfloor R_0/2 \rfloor + 1$, where $\lfloor \cdot \rfloor$ stands for the floor function.

troops, we fixed $B_0 = 30$ and $p_B = 0.02$. As before, we used the criterion that a unit is no longer operational if at least half of its soldiers have been hit in an incapacitative way. Thus, $\underline{b} = 16$ and $\underline{r} = \lfloor R_0/2 \rfloor + 1$, where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leqslant x\}$, for $x \in \mathbb{R}$, stands for the floor function. Parameters λ_B and λ_R were set to unity. The number of time steps, $T = 500$, was chosen as to ensure that the probability of the battle continuing at T would be negligible. Indeed, we found that both $\bar{P}[C_T]$ and $P[C_T]$ were less than 10^{-70} across all design points.

Figure 2 displays contour plots of the probabilities of the terminal states of the battle obtained using the approximative method and the corresponding errors relative to the reference method. As expected, the probability of that Blue troops have won increases as the initial strength R_0 of Red troops and their hitting probability p_R decreases (and for Red troops, vice versa). The probability that both units end up being defeated is rather low, being less than 0.04 in all design points. The highest values are obtained when Blue and Red have roughly equal initial strengths or when Blue troops outnumber Red slightly, but the difference is offset by Red troops' higher hitting probability. In the plots of errors, we observe consistently the pattern that was already evident in Figure 1: the approximative method slightly exaggerates the probability that the more-likely winner has won. However, for practical purposes the magnitude of error is small, being less than 0.07 in all design points. When the winning probability is close to unity, the error essentially disappears.

Contour plots of the error of the approximative method for the conditional distributions of the strengths are displayed in Figure 3. To measure the magnitude of the error, we used the maximum norm of finite dimensional vectors,

$$\|x\|_\infty = \max_{0 \leqslant i \leqslant d} |x_i|, \quad \text{for } x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

that simply gives the maximal distance of the point probabilities in the compared distributions. The conditional distributions, given that the unit has won, exhibit errors of magnitude at most 0.07. However, a glance at Figure 2 reveals that the largest errors actually appear in the regions of the parameter space where the conditioning event occurs with very low probability. In such cases the conditional distribution would be of limited informational value, anyway. The errors of conditional distributions, given that the unit is defeated, are smaller. This is largely due to the fact that the probability mass of these distributions is tightly concentrated to values immediately below the thresholds $\underline{b}$ and $\underline{r}$ leaving less room for error.

The numerical experiments were made using prototype implementations of the methods, written in R programming language (R Development Core Team 2011). They were not particularly optimized in terms of performance, so only very tentative performance comparisons can be made based on the current experiments. With this caveat in mind, even for moderate initial strengths—e.g. with $B_0 = 26$ and $R_0 = 30$, as in the first illustrative experiment—the observed run times of the reference method were roughly 100-fold compared to the approximative one.

4 CONCLUSIONS

We have introduced an approximative Markovian method of modeling a duel between two infantry units. The method is based on the mathematical assumption that, in each time step of the simulation, the strengths of the units are conditionally independent, given that the battle continues after the preceding time step. This assumption allows us to compute the probabilities of the outcomes of the battle in an efficient recursive manner. The conditional distributions of the strength of a unit, given that it has won the battle or been defeated, respectively, can be computed in a similar way. We compared numerically the approximative method to a more elaborate reference method that entails modeling the strengths of the units jointly as Markov chain, requiring the joint distribution of the strengths to be evaluated in each time step. The key finding is that the approximative method performs efficiently and its output is numerically close and qualitatively parallel to the one obtained from the reference method.

Based on this study, we find the approximative method to be accurate enough to be used a foundation for a duel simulation feature, to be implemented in future versions of Sandis combat simulation software. As a further theoretical work, it would be of interest to study, whether there exist a simple analytical bound for the error of the approximative method. This could then be output by Sandis, as a quick measure of robustness, whenever the duel simulation is performed.

REFERENCES

Ancker, C. J., and A. V. Gafarian. 1988. "The Validity of Assumptions Underlying Current Uses of Lanchester Attrition Rates". Technical Report TRAC-WSMR-TD-7-88, US Army TRADOC Analysis Command-WSMR, White Sands Missile Range, New Mexico.

Kingman, J. F. C. 2002. "Stochastic Aspects of Lanchester's Theory of Warfare". *Journal of Applied Probability* 39:455–465.

Lanchester, F. W. 1916. *Aircraft in Warfare: the Dawn of the Fourth Arm*. London: Constable and Company, Ltd.

Lappi, E. 2008. "Sandis Military Operation Analysis Tool". 2nd Nordic Military Analysis Symposium, 17–18 November, Stockholm, Sweden.

Lappi, E. 2012. *Computational Methods for Tactical Simulations*. Doctor of military science thesis, National Defence University, Finland.

Lappi, E., and O. Pottonen. 2006. "Combat parameter estimation in Sandis OA software". In *Lanchester and Beyond—A Workshop on Operational Analysis Methodology*, edited by J. S. Hämäläinen, Volume 11 of *Publications of the Defence Forces Technical Research Centre*, 31–38. Riihimäki: Defence Forces Technical Research Centre.

Lappi, E. and Vulli, M. 2008. "Field Test for Parameter Estimation of Small Arms Fire". 2nd Nordic Military Analysis Symposium, 17–18 November, Stockholm, Sweden.

Pfeiffer, P. E. 1978. *Concepts of Probability Theory*. 2nd ed. New York: Dover.

R Development Core Team 2011. *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.

Snow, R. N. 1948. "Contributions to Lanchester Attrition Theory". Technical Report RA-15078, Project RAND, Douglas Aircraft Company, Inc., Santa Monica, California.

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