

An area law and sub-exponential algorithm for 1D systems

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Understanding the structure and complexity of ground states of gapped local Hamiltonians is the central problem in Condensed Matter Physics and Quantum Complexity Theory. A remarkably general conjecture about the structure of ground states, *The Area Law*, bounds the entanglement that such states can exhibit: specifically, for any subset S of particles, it bounds the entanglement entropy of ρ_S , the reduced density matrix of the ground state restricted to S , by the surface area of S . i.e. the number of local interactions between S and \bar{S} [1].

Although the general area law is still open, a lot of progress has been made on proving it for 1D systems. The breakthrough came with Hastings' result [2], which shows that the entanglement entropy across a cut for a 1D system is a constant independent of n , the number of particles in the system, and scales as $\exp(\frac{\log d}{\epsilon})$, where d is the dimension of each particle and ϵ is the spectral gap. This result implies that the ground state of a gapped 1D Hamiltonian can be approximated in the complexity class NP .

In this paper, we:

- give an exponential improvement to $\tilde{O}(\frac{\log^3 d}{\epsilon})$ in the bound of entanglement entropy for the general (frustrated) 1D Hamiltonians. The dependence on the gap even improves the previous best bound for frustration free 1D Hamiltonians and may possibly be tight to within log factors.
- prove the existence of sublinear bond dimension Matrix Product State approximations of ground states for general 1D Hamiltonians. This implies a subexponential time algorithm for finding such states thus providing evidence that this task is not NP-hard.

We also establish the following properties of local Hamiltonians which may be of independent interest:

- "Random walk like" behavior of entanglement: for a 1D Hamiltonian H , the Schmidt Rank (SR) of H^ℓ is bounded by $d^{O(\sqrt{\ell})}$.
- Let H' be the Hamiltonian consisting only of terms acting on a subset S of particles. Then the ground state of H has an exponentially small amount of norm in the "high" energy spectrum of H' : the total norm with energy above t is $2^{-\Omega(t-|\partial S|)}$ where $|\partial S|$ is the size of the boundary between S and \bar{S} .

The work here has its origins in the combinatorial approach of [3], which used the Detectability lemma, introduced earlier in [4], to give an alternate proof of Hastings' result for the special case of

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frustration-free Hamiltonians. The results there were greatly strengthened in [5] and [6], which introduced Chebyshev polynomials in conjunction with the detectability lemma to construct very strong AGSPs (approximate ground state projectors), leading to an exponential improvement of Hastings' bound in the frustration-free case to $O((\frac{\log d}{\epsilon})^3)$.

1.1 Background

The overall strategy is to start with a product state $|\psi\rangle$ and repeatedly apply some operator K such that $\frac{1}{\|K^j|\psi\rangle\|}K^j|\psi\rangle$ approximates the ground state and the SR of $K^j|\psi\rangle$ is not too large. This property of an operator K is captured in the following definition of an approximate ground state projection (AGSP):

Definition 1.1 *An Approximate Ground-Space Projection (AGSP)*

With respect to a ground state $|\Omega\rangle$ of a 1D Hamiltonian, an operator K is a (D, Δ) -Approximate Ground Space Projection (with respect to a cut) if the following three properties hold:

- $K|\Omega\rangle = |\Omega\rangle$.
- If $|\Omega^\perp\rangle$ is perpendicular to $|\Omega\rangle$, then $K|\Omega^\perp\rangle$ is also perpendicular to $|\Omega\rangle$ and $\|K|\Omega^\perp\rangle\|^2 \leq \Delta$.
- For any state $|\phi\rangle$, the SR of $K|\phi\rangle$ is at most D times that of $|\phi\rangle$.

The parameters Δ and D capture the tradeoff between the rate of movement towards the ground state and the amount of entanglement that applying the operator K incurs. In [5, 6], it was shown that a favorable tradeoff gives an area law:

Theorem 1.2 (Area Law) *If there exists a (D, Δ) -AGSP such that $D \cdot \Delta \leq \frac{1}{2}$, the ground state entropy is bounded by:*

$$S \leq \mathcal{O}(1) \cdot \log D . \tag{1}$$

1.2 Our Results

For our construction, the first critical step is to exchange local structure far from the cut for a valuable **reduction in the norm** of the Hamiltonian. To do this, we isolate a neighborhood of $s + 1$ particles around the cut in question, and then separately truncate the sum of the terms to the left and to the right of these $s + 1$ particles. Specifically, we define the truncation of an operator as follows:

Definition 1.3 (Truncation) *For any self-adjoint operator A , form $A^{\leq t}$, the truncation of A , by keeping the eigenvectors the same, keeping the eigenvalues below $\leq t$ the same, and replacing any eigenvalue $\geq t$ with t .*

We then define $H^{(t)} = (\sum_{i < 1} H_i)^{\leq t} + H_1 + \dots + H_s + (\sum_{i > s} H_i)^{\leq t}$, where the s middle terms act on the the isolated string of $s + 1$ particles around the cut. The result is a Hamiltonian H that is now

norm bounded by $u = s + 2t$ acting on n particles with the following structure:

$$H = H^{(t)} = H_L + H_1 + H_2 + \cdots + H_s + H_R, \quad (2)$$

where each H_i are norm bounded by 1 and acts locally on particles $m + i$ and $m + i + 1$, H_L acts on particles $1, \dots, m$ and H_R acts on particles $m + s + 1, \dots, n$. In the frustration free case, it is clear that the ground state of $H^{(t)}$ is the same as that of the original Hamiltonian and it can be shown that the spectral gap is preserved for some constant value of t . For the frustrated case, the ground state of $H^{(t)}$ is no longer that of the original Hamiltonian and a limiting argument (see below) will be needed to complete the proof.

Having cut down the problem to a Hamiltonian with bounded norm u of the form (2), we turn to the next critical step of constructing the AGSP, the **use of Chebyshev polynomials** to approximate the projection onto the ground state. We begin with a suitably shifted Chebyshev polynomial $C_\ell(x)$ of degree ℓ with the properties that $C_\ell(0) = 1$ and $|C_\ell(x)| \leq e^{-\sqrt{\epsilon} \frac{\ell}{\sqrt{u}}}$ for $\epsilon \leq x \leq u$. The AGSP is then $K = C_\ell(H)$ and it is clear that $\Delta = e^{-\sqrt{\epsilon} \frac{\ell}{\sqrt{u}}}$. Bounding the SR for K requires important new ideas. By the entanglement flows approach of [5, 6] the SR of each term in the expansion of H^ℓ is bounded by $d^{\ell/s+s}$. The difficulty is that the number of such terms, $(s+2)^\ell$, is much too large. To address this issue, we introduce formal commuting variables Z_i and consider the expression

$$P(Z) = (H_L Z_0 + H_1 Z_1 + \cdots + H_R Z_{s+1})^\ell = \sum_{a_0 + \cdots + a_{s+1} = \ell} f_{a_0, \dots, a_{s+1}} Z_0^{a_0} Z_1^{a_1} \cdots Z_{s+1}^{a_{s+1}}.$$

Now expand $P(Z) = (A + H_i Z_i + B)^\ell$ where A and B commute, and restrict attention to only those terms where Z_i appears at most ℓ/s times. The number of terms is bounded by $\binom{\ell + 2\frac{\ell}{s}}{2\frac{\ell}{s}}$, and the SR increase by $\left(\frac{\ell + 2\frac{\ell}{s}}{2\frac{\ell}{s}}\right) d^{\ell/s} d^s$ for all values of Z . We then use a **polynomial interpolation** argument to bound the SR of each $f_{a_0, \dots, a_{s+1}}$ and therefore of $H^\ell = \sum_{a_0 + \cdots + a_{s+1} = \ell} f_{a_0, \dots, a_{s+1}}$ (and hence for K as well) by $D = d^{\tilde{O}(\ell/s+s)}$.

Applying Theorem 1.2 to the above AGSP with $\ell = O(s^2)$, $s = \tilde{O}(\log^2(d)/\epsilon)$ yields our Area Law for frustration free Hamiltonians providing an entanglement entropy bound of the form $\tilde{O}(\log^3(d)/\epsilon)$.

To address the frustrated case, a third critical result is needed: that the ground states of $H^{(t)}$ are very good approximations of the ground state of the original Hamiltonian. Intuitively, the structure of the small eigenvectors and eigenvalues of $H^{(t)}$ should approach those of H as t grows and we show that to be the case, showing a **robustness theorem**: that the ground states of $H^{(t)}$ and H are exponentially close in t and the spectral gaps are of the same order.

We would like to apply Theorem 1.2 to an AGSP for $H^{(t)}$, for t sufficiently large, however, if we try to do this in one step, the SR cost becomes a large function of t . Instead we use a well chosen arithmetic sequence t_0, t_1, \dots and the associated AGSP's to $H^{(t_i)}$ to guide the movement towards the ground state. The robustness theorem allows for very rapid convergence, the result of which is the area law in the general (frustrated) case.

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