An Asymmetric Version of the Two Car Pursuit-Evasion Game

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***Abstract*— In this paper we consider a differential game of pursuit and evasion involving two players with constant, but different, speeds, and different maneuverability constraints. Specifically, the evader has limited maneuverability, while the pursuer is completely agile. This problem is an asymmetric version of the well-known Game of Two Cars. The aim of this paper is to derive the optimal strategies of the two players and characterize areas of initial conditions that lead to capture if the pursuer acts optimally, and areas that guarantee evasion regardless of the pursuer's strategy. It is shown that the problem reduces to a special version of Zermelo's Navigation Problem (ZNP) for the pursuer. Therefore, the well-known ZNP solution can be used to validate the results obtained through the differential game framework as well as to characterize the time-optimal trajectories. The results are directly applicable to collision avoidance problems.**

I. INTRODUCTION

The topic of pursuit and evasion has received particular attention in the theory of differential/dynamic games. The pioneering work of Isaacs [1] on the extension of game theory to the framework of differential games, includes a plethora of examples of pursuit and evasion. Classical results also include [2], wherein the authors examined conditions under which capture is possible in a two-player linear-quadratic pursuit-evasion game: if both players are subject to single integrator dynamics and have no control constraints, then the necessary and sufficient condition for interception is that the speed of the pursuer is higher than that of the evader.

When curvature constrains are introduced, two-player pursuit-evasion games exhibit significantly more complicated solutions. Games as such include Isaacs' Homicidal Chauffeur problem [1], [3] and the Game of Two Cars [4]. Necessary and sufficient conditions for capture, regardless of the initial conditions of the players were derived in [5]. Reference [5] states that a pursuer is guaranteed to capture the evader regardless of initial conditions only if she is faster than the evader, and does not have a major maneuverability disadvantage against the evader.

One of the most extensively studied games in this setting is the Game of Two Cars [4]. The Game of Two Cars entails two players having the same speed and the same maneuverability restrictions, i.e., they are identical, and capture occurs when the distance between the players becomes less than a constant, which is known as the "kill zone." As with any pursuit-evasion game in which different initial conditions lead to different game outcomes, an essential part of the solution of the game is the determination of the *barrier* [1]. Simply put, the barrier is the surface that separates initial states of the game that lead to capture under optimal play, from states in which capture is impossible, as long as the evader plays optimally, and evasion is guaranteed.

Several extensions and generalizations of the Game of Two Cars appear in the literature under the name *maritime collision avoidance* [6], [7], [8], [9], [10], [11], [12]. In this game setting, the two agents have different constant speeds and different minimum turning radii. Analytic expressions for the barriers do exist [10], but practical implementation is problematic in several cases because of the underlying assumptions: the exact maneuvering capability of the opponent is assumed to be known a priori, and a continuous measurement of her instantaneous orientation is also necessary. Motivated by these technical difficulties, in this paper we propose to investigate the following variation of the Game of Two Cars: the players have different constant speeds, and the pursuer is assumed to be completely agile. This leads to a game of reduced dimensionality that utilizes the least amount of information on the characteristics of the pursuing agent. Because of this reduced dimensionality, the game exhibits an inherently different solution, which cannot be obtained as a special case from similar games of higher dimensionality. Relevant work on this problem can be found in [13], [14]. The first one treats this problem as a special case of the Game of Two Cars and presents a purely geometric procedure to obtain a solution, without offering an analytic expression for the barrier, while the second one focuses on the investigation of feedback control laws for the pursuer.

II. PROBLEM STATEMENT

Consider two players, a pursuer and an evader, moving on a plane. The subscripts p and e will be reserved for the "pursuer" (P) and the "evader" (E), respectively. The pursuer's objective is *capture*, that is, interception of the evader in finite time, whereas the evader's objective is *evasion*, a state in which she avoids interception indefinitely. The agents have the different constant speeds v_e and v_p. The pursuer is assumed to be agile, in the sense that she can change the orientation of her velocity vector instantaneously. On the other hand, the evader is less agile and cannot take turns that have a radius smaller than her minimum turning radius R. In this setting, it is a well known fact [5] that if the pursuer is also faster, then capture is guaranteed regardless of initial conditions. We will therefore limit our analysis to the more interesting case when $v_p \leq v_e$. We wish to investigate the conditions under which capture is possible, and extract the corresponding optimal strategies for both P and E.

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A. A Sufficient Condition for Evasion

The equations of motion for the pursuer and the evader, written in an inertial frame of reference with coordinates x and y are given by

$$\dot{x}_p = v_p \cos\phi_p, \tag{1}$$
$$\dot{y}_p = v_p \sin\phi_p, \tag{2}$$
$$\dot{x}_e = v_e \cos\phi_e, \tag{3}$$
$$\dot{y}_e = v_e \sin\phi_e, \tag{4}$$
$$\dot{\phi}_e = -\frac{v_e}{R}u, \qquad u \in [-1, 1]. \tag{5}$$

Let $\boldsymbol{z} \triangleq (x_p - x_e, y_p - y_e)$ be the vector pointing from the evader's instantaneous position to the pursuer's instantaneous position. Furthermore, let $\mathbf{v}_e(t)$ and $\mathbf{v}_p(t)$ be the velocity vectors, at time t, of the evader and the pursuer, respectively, such that $\|\mathbf{v}_e(t)\| = v_e \geq \|\mathbf{v}_p(t)\| = v_p$.

Theorem 1: If, at any time t_0, the inner product between the velocity vector of the evader, $\mathbf{v}_e(t_0)$, and the relative position vector $\boldsymbol{z}(t_0)$ is non-positive, that is, if

$$\langle \boldsymbol{z}(t_0), \mathbf{v}_e(t_0)\rangle \leq 0, \tag{6}$$

then capture is not possible for all $t \geq t_0$, and the evader escapes.

Proof: Without loss of generality, we will assume that $t_0 = 0$. The proof is similar to the proof of Theorem 1.a of [15], for the limiting case of one pursuer. To prove the theorem, it suffices to find a strategy for the evader that leads to escape if (6) holds. To this end, assume that the evader will not change the orientation of her velocity vector after $t = 0$, that is, $\mathbf{v}_e(t) = \mathbf{v}_e(0) = \mathbf{v}_e$ for all $t \geq 0$. The relative position vector $\boldsymbol{z}$ then satisfies

$$\boldsymbol{z}(t) = \boldsymbol{z}_0 + \int_0^t \mathbf{v}_p(\tau)\,\mathrm{d}\tau - t\mathbf{v}_e. \tag{7}$$

where $\boldsymbol{z}_0 = \boldsymbol{z}(0)$. We may define $\hat{\mathbf{v}}_p(t) = \frac{1}{t}\int_0^t \mathbf{v}_p(\tau)\,\mathrm{d}\tau$, which satisfies

$$\|\hat{\mathbf{v}}_p(t)\| = \|\frac{1}{t}\int_0^t \mathbf{v}_p(\tau)\,\mathrm{d}\tau\| \leq \frac{1}{t}\int_0^t \|\mathbf{v}_p(\tau)\|\,\mathrm{d}\tau = v_p, \tag{8}$$

for all $t \geq 0$, and thus equation (7) becomes $\boldsymbol{z}(t) = \boldsymbol{z}_0 + t\hat{\mathbf{v}}_p(t) - t\mathbf{v}_e$. Since $\|\mathbf{v}_e\| = v_e$, and $v_e \geq v_p$, it follows from (6) that $\|\boldsymbol{z}(t)\| \geq \|\boldsymbol{z}_0 - t\mathbf{v}_e\| - \|t\hat{\mathbf{v}}_p(t)\| \geq \|\boldsymbol{z}_0 - t\mathbf{v}_e\| - tv_p = \sqrt{\|\boldsymbol{z}_0\|^2 - 2t\langle \mathbf{v}_e, \boldsymbol{z}_0\rangle + v_e^2 t^2} - v_p t > 0$, for all $t \geq 0$, and the proof is complete. ■

The proof of Theorem 1 states that, given E's maneuverability restriction, capture is possible only if E's velocity at some time $t \geq 0$ has a component pointing towards P. In such a case, E's best strategy is clearly to eliminate this velocity component as fast as possible. If E cannot eliminate her velocity component pointing towards P fast enough, capture will occur. On the other hand, as stated in Theorem 1, if at no point in time E's velocity has a component pointing towards P, then the maneuverability superiority of P is inconsequential and P will not be able to intercept E. Next, we investigate the more interesting case when E's initial velocity vector has a component pointing towards P. In this case, as evidenced by Theorem 1, capture may be possible.

III. Differential Game Formulation and Solution

We seek to answer the following problem: given that the condition of Theorem 1 is *not* satisfied, determine the initial positions of the two agents that lead to capture and the initial positions that lead to evasion under optimal play by both agents. The answer to this problem is obtained through the solution of a *game of kind*. In a game of kind, we are interested in an outcome which is essentially an event (in our case, capture or evasion), as opposed to a *game of degree*, in which the outcome is the value of a certain variable (e.g., how much time did P need to intercept E). However, instead of solving the game of kind, we will consider a corresponding game of degree, the solution of which will illuminate the solution of the game of kind as well.

A. Differential Game Setup

We transform the problem from the fifth-dimensional *realistic game space* (1)-(5) to a two-dimensional *reduced game space*, by fixing the origin of a coordinate system on E's current position and by aligning the y-axis with E's velocity vector ([1], see also Figure 1). The evader action then consists of choosing her center of curvature at a point $C = (R/u, 0)$ on the x-axis as shown in Figure 1. Consequently, the reduced game space has only two coordinates, namely the (x, y) coordinates of P relative to E in the evader's fixed, velocity-aligned frame. The equations of motion of P in this frame are given by

$$\dot{x} = -\frac{v_e}{R}yu + v_p\cos\phi, \tag{9}$$
$$\dot{y} = \frac{v_e}{R}xu + v_p\sin\phi - v_e, \qquad u \in [-1, 1], \tag{10}$$

where ϕ is P's control in this new reference frame, given by $\phi \triangleq \phi_p + \pi/2 - \phi_e$.

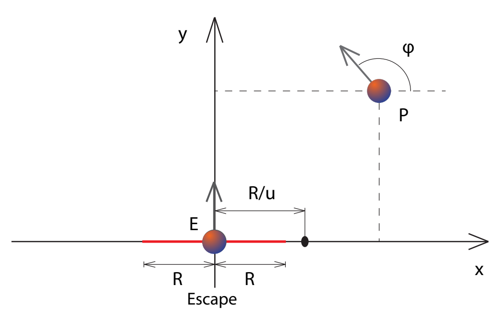


Fig. 1. Reference frame for the reduced state space.

With the assistance of Theorem 1, one can reduce the state space under investigation by formulating a simplified differential game of degree. To this end, note that Theorem 1 implies that if, at any instant of time, the game state (x, y)

reaches the closed lower half plane of the reduced game space (i.e., $y \leq 0$) while $x \neq 0$, evasion is guaranteed. Therefore, we only need to consider the upper half plane as the game space, and introduce as terminal surface the entire x-axis, thus eliminating the problem of dealing with a low-dimensional terminal surface. Furthermore, instead of considering time until interception as the payoff (which may be infinite if no interception occurs), one can equivalently consider as payoff the distance traveled from the origin when the game state vector penetrates the x-axis. The pursuer's objective is then to minimize this distance, with interception occurring if the zero value is attained (the minimum possible value), while the evader's objective is to maximize this distance. Finally, one can easily observe that the problem is symmetric; analysis of the upper right quadrant is enough to extend the results to the case of the upper left quadrant.

To proceed, we define the cost

$$J(\mathbf{x}, \phi, u) = \frac{1}{2} x^2(t_f), \tag{11}$$

where $\mathbf{x} = [x, y]^T \in \mathcal{E} \triangleq \{\mathbf{x} \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is the state, and t_f denotes the time at game termination. We seek to solve the problem of conflicting actions represented by u (maximizing control) and ϕ (minimizing control) that maximize/minimize the terminal cost (11) with state space $\mathcal{E}$ and terminal surface $\mathcal{C} \triangleq \{\mathbf{x} \in \mathcal{E} : y = 0\}$ under the dynamic equations (9) and (10).

B. Solution of the Game

In order to solve the game defined above, we apply the framework developed in [1]. Specifically, the *Hamilton Jacobi Isaacs* (HJI) equation for this problem is:

$$\min_{\phi} \max_{u} \left\{ \frac{\partial V(\mathbf{x})}{\partial x} \dot{x} + \frac{\partial V(\mathbf{x})}{\partial y} \dot{y} \right\} = 0, \tag{12}$$

or, alternatively,

$$\begin{aligned} \min_{\phi} \max_{|u| \leq 1} \Big\{ & \frac{v_e}{R} \big(V_y(\mathbf{x}) x - V_x(\mathbf{x}) y\big) u \\ & + v_p \big(V_x(\mathbf{x}) \cos\phi + V_y(\mathbf{x}) \sin\phi\big) - v_e V_y(\mathbf{x}) \Big\} = 0, \end{aligned} \tag{13}$$

where V_x and V_y are the partial derivatives with respect to x and y of the Value $V \in C^1$ of the game, which is defined as

$$V(\mathbf{x}) \triangleq \min_{\phi} \max_{u} J(\mathbf{x}, \phi, u) = \max_{u} \min_{\phi} J(\mathbf{x}, \phi, u),$$

with boundary condition $V(\mathbf{x})|_{\mathbf{x} \in \mathcal{C}} = \frac{1}{2} x^2$. Notice that the *Minimax Assumption* [1], which states that the order of minimization and maximization is inconsequential, holds in our case because the dynamic equations (9), (10) are separable in terms of the control inputs, and the cost is terminal.

In general, the game space may be divided into regions separated by *singular surfaces* [1]. Within each such region of the game space, $V(\mathbf{x})$, if it exists, will be a C^1 class function and satisfies the HJI equation. Similarly, in each region, except possibly on the singular surfaces, the optimal controls will be continuous functions of the state.

We may now proceed to the calculation of the optimal controls from (13). Since $u \in [-1, 1]$, it follows from (13) that

$$u^*(\mathbf{x}) = \operatorname{sign}\big(V_y(\mathbf{x}) x - V_x(\mathbf{x}) y\big) \triangleq \sigma(\mathbf{x}), \tag{14}$$

which implies that E's optimal control is *bang-bang*. Furthermore, applying the Lemma on Circular Vectograms [1] on (13) for the minimization of the term $V_x(\mathbf{x}) \cos\phi + V_y(\mathbf{x}) \sin\phi$ in terms of ϕ, yields the optimal action for the pursuer, as follows:

$$\begin{gathered} \cos\phi^*(\mathbf{x}) = -\frac{V_x(\mathbf{x})}{\rho(\mathbf{x})}, \quad \sin\phi^*(\mathbf{x}) = -\frac{V_y(\mathbf{x})}{\rho(\mathbf{x})}, \\ \rho(\mathbf{x}) = \sqrt{V_x^2(\mathbf{x}) + V_y^2(\mathbf{x})} > 0. \end{gathered} \tag{15}$$

Note that the corresponding minimum value of the term $V_x(\mathbf{x}) \cos\phi^* + V_y(\mathbf{x}) \sin\phi^*$ in equation (13) is $-\rho(\mathbf{x})$.

The next step is to derive the *Retrogressive Path Equations* [1]. These are the equations arising when one solves the game backwards in time, starting from the terminal surface $\mathcal{C}$. For this reason, we introduce the reverse time variable $\tau = t_f - t$. Denoting with $(\circ)$ the partial derivative with respect to τ, the (inverse) evolution of the partial derivatives of the value V are given by

$$\overset{\circ}{V}_{\mathbf{x}_k} = \sum_{i=1}^{2} V_{\mathbf{x}_i}(\mathbf{x}) \frac{\partial f_i(\mathbf{x}, \phi^*, u^*)}{\partial \mathbf{x}_k}, \qquad k = 1, 2, \tag{16}$$

where f_i $(i = 1, 2)$ denotes the right-hand-side of the differential equations (9) and (10) respectively (for our problem, $x_1 = x$ and $x_2 = y$). One readily computes from (16) and (9), (10) that

$$\overset{\circ}{V}_x = \sigma(\mathbf{x}) \frac{v}{R} V_y(\mathbf{x}), \tag{17}$$

$$\overset{\circ}{V}_y = -\sigma(\mathbf{x}) \frac{v}{R} V_x(\mathbf{x}). \tag{18}$$

To obtain the boundary conditions for (17) and (18), we parameterize the terminal surface $\mathcal{C}$ with $n - 1$ variables, n being the dimension of the game space. In our case, only a single variable (s) is needed, and the parameterization has the form $\big(h_1(s), h_2(s)\big) = (s, 0)$, $s \geq 0$, i.e., $x = s$ on the x-axis. The value V on $\mathcal{C}$ should be equal with the terminal cost J, i.e. $V(\mathbf{x})|_{\mathbf{x} \in \mathcal{C}} = J(s) = \frac{1}{2} s^2$. Then, the boundary conditions for the Value partial derivatives can be acquired from the relation [1]

$$\frac{\partial J}{\partial s} = \sum_{i=1}^{2} V_{\mathbf{x}_i}(\mathbf{x}) \frac{\partial h_i}{\partial s}, \qquad \mathbf{x} \in \mathcal{C}, \tag{19}$$

which yields,

$$s = V_x(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{C}, \; s \geq 0. \tag{20}$$

The second boundary condition is obtained by enforcing the HJI equation (13) on the terminal surface under the parameterization $x = s$. Noting that on $\mathcal{C}$, $y = 0$ and $V_x = s$, and by virtue of the optimal controls (14) and (15), we obtain

$$\frac{v_e}{R} |V_y(\mathbf{x}) s| - v_p \sqrt{V_y^2(\mathbf{x}) + s^2} - v_e V_y(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{C}. \tag{21}$$

Solving for $V_y(\mathbf{x})$ yields two solutions, depending on its sign. Keeping in mind that $s \geq 0$, equation (21) yields, for $V_y(\mathbf{x}) > 0$,

$$V_y(\mathbf{x}) = \frac{Rsv_p}{\sqrt{v_e^2(R-s)^2 - R^2v_p^2}}, \tag{22}$$

while for $V_y(\mathbf{x}) \leq 0$ one readily computes

$$V_y(\mathbf{x}) = -\frac{Rsv_p}{\sqrt{v_e^2(R+s)^2 - R^2v_p^2}}, \qquad s \geq 0,\ \mathbf{x} \in \mathcal{C}. \tag{23}$$

The corresponding signs of u^* are obtained immediately with the help of equation (14), which on $\mathcal{C}$ becomes

$$u^*(\mathbf{x}) = \operatorname{sign}(V_y(\mathbf{x})s), \qquad s \geq 0,\ \mathbf{x} \in \mathcal{C}. \tag{24}$$

Thus, (22) corresponds to $u^* = 1$ on $\mathcal{C}$ (game ends while the evader is steering towards the pursuer), while (23) corresponds to $u^* = -1$ on $\mathcal{C}$ (game ends while the evader is steering away from the pursuer).

It is claimed that only the latter is a valid boundary condition for our game. The argument to support this claim emerges through the analysis of the *usable part* [1] of the terminal surface for both cases of boundary conditions provided by (22) and (23). It is not uncommon for a terminal surface of a game to be divided into two regions: the *usable part* and the *nonuseable part*, which are separated by what is known in the literature as the *boundary of the usable part (BUP)*. The usable part is the subset of the terminal surface in which the game can end under optimal play, that is, if both players act optimally. The nonusable part, on the contrary, is the rest of the terminal surface in which the game would end only if at least one of the players does not play optimally. Essentially, no retrograde optimal paths exist emanating from the nonusable part (see [1] for further details). To identify the usable part, let $\nu \triangleq [\nu_1\ \nu_2]^T$ be the vector normal to $\mathcal{C}$ from $\mathbf{x}$ on $\mathcal{C}$ and extending into $\mathcal{E}$. Then, the usable part of $\mathcal{C}$ is the region in which the following inequality holds:

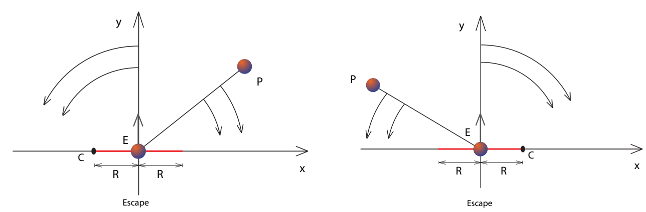
$$\min_{\phi} \max_{u} \sum_{i=1}^{2} \nu_i f_i(\mathbf{x}, u, \phi) < 0, \qquad \mathbf{x} \in \mathcal{C}, \tag{25}$$

while the nonusable part has the inequality sign reversed. Since the terminal surface in the game we consider is the (nonnegative) x-axis, the vector normal to the terminal surface extending into the game space at each point of the terminal axis is $\nu = [0\ 1]^T$. Recall that on $\mathcal{C}$ we have $x = s$, $y = 0$, and $V_x = s$. Thus, condition (25) can be rewritten, by virtue of the dynamics (9), (10) and the optimal controls (14) and (15), as

$$\frac{v_e s}{R}\operatorname{sign}(V_y(\mathbf{x})) - v_p \frac{V_y(\mathbf{x})}{\sqrt{V_y^2(\mathbf{x}) + s^2}} - v_e < 0, \quad \mathbf{x} \in \mathcal{C},\ s \geq 0. \tag{26}$$

Evaluating the right-hand-side of the above expression for the two different boundary values of V_y given by equations (22) and (23) leads to the following two observations: for the negative boundary value of V_y given by (23), the inequality (26) is satisfied for any $s > 0$. This implies that the entire terminal surface is usable, i.e., the game can terminate optimally anywhere on $\mathcal{C}$. However, for the positive boundary value of V_y given by (22), the right-hand-side is positive for any $s > 0$, rendering the entire terminal surface nonusable.

The evader's best response is therefore established: Regardless of P's strategy, E will try to steer away from P with her maximum turning capability ($u^* = -1$), in an attempt to eliminate the velocity vector component pointing towards P as fast as possible. The evader's strategy is depicted in Figure 2.



(a) $u^* = -1$ (b) $u^* = 1$

Fig. 2. Optimal evader strategies: (a) for $x > 0$ and (b) for $x < 0$, by virtue of the symmetry of our problem. The arrows indicate the corresponding rotation of the LOS and the reference frame. If the game state reaches the closed lower half plane, except the origin, escape is guaranteed.

It remains to integrate the system of ordinary differential equations subject to the appropriate boundary conditions. Since we have limited our analysis to the upper right quadrant, we apply $u = -1$ for the rest of this section. The corresponding retrogressive path equations for the game states can be established if one applies the optimal controls $u^* = -1$ and ϕ^* in equations (9) and (10) and switches the sign to reverse the time flow. Thus, setting $c \triangleq v_e/R$ we are left with the system:

$$\overset{\circ}{V}_x(\tau) = -cV_y(\tau), \tag{27}$$

$$\overset{\circ}{V}_y(\tau) = cV_x(\tau), \tag{28}$$

$$\overset{\circ}{x}(\tau) = -cy(\tau) + v_p\frac{V_x(\tau)}{\rho(\tau)}, \tag{29}$$

$$\overset{\circ}{y}(\tau) = cx(\tau) + v_p\frac{V_y(\tau)}{\rho(\tau)} + v_e, \tag{30}$$

subject to the boundary conditions (20) and (23) for V_x and V_y respectively, and $x(0) = s$ and $y(0) = 0$ for the state. The solution to this system can be obtained analytically, and since we are primarily interested in paths that lead to capture, we may take the limit as s tends to zero. After extensive algebraic manipulation involving trigonometric formulas, we finally obtain

$$x(\tau) = -R + R\cos(c\tau) + v_p\tau\sin(\gamma - c\tau), \tag{31}$$

$$y(\tau) = R\sin(c\tau) + v_p\tau\cos(\gamma - c\tau), \quad \tau \in [0, \tau_{\max}], \tag{32}$$

where $\gamma = \arccos(-v_p/v_e)$. Equations (31) and (32) define the *barrier* of the game that separates the game space into two regions; a region in which optimal play of both agents leads to capture and a region in which optimal play leads to evasion. To obtain $\tau_{\max}$, it is important to note that the barrier expression is invalidated as soon as two barrier branches intersect – the part of the barrier arc beyond the point of intersection is then no longer valid and is therefore discarded. In our case, the two branches of the barrier intersect on the y-axis, because of the inherent symmetry of the problem at hand. Thus, we may obtain $\tau_{\max}$ as the root of $x(\tau) = 0$, i.e., $\tau_{\max}$ is the solution of the transcendental equation

$$v_p \tau_{\max} \sin(\gamma - c\tau_{\max}) = R - R\cos(c\tau_{\max}). \tag{33}$$

Figure 3 depicts the barrier for $v_p = v_e = 1$, $R = 0.7$.

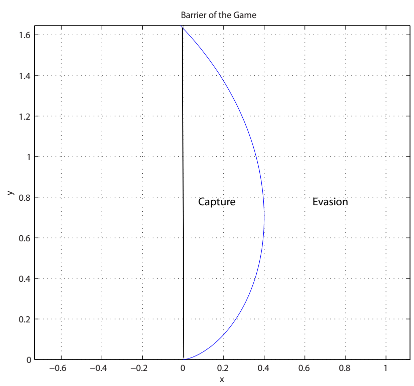


Fig. 3. The barrier, given by equations (31) and (31) for $v_e = v_p = 1$ and $R = 0.7$, for the case in which the pursuer appears in the upper right quadrant. By virtue of symmetry, a reflection on the y-axis will provide the barrier for the case in which the pursuer is located at the upper left quadrant.

So far, we have characterized which states lead to capture and which states lead to evasion. We will now turn our attention to the time-optimal problem when the outcome is capture, that is, we shall consider the states that lead to capture under optimal play, and examine the characteristics of the time-optimal capture trajectories.

IV. Time-Optimal Characteristics and Equivalence to ZNP

Zermelo's Navigation Problem (ZNP) is a well-known result in optimal navigation which has received lots of attention in the literature (see for example [16], [17], [18]). Initially stated by the German mathematician E. Zermelo in 1931, the problem formally reads: *"In a given vector field of currents, which is a function of position (and possibly time), a vehicle moves with constant speed relative to the currents. How should the vehicle be navigated in order to reach a given destination in minimum time?"* [19], [20]. In ZNP, the equations of motion for the vehicle are

$$\dot{x} = v\cos\phi + U(x, y), \tag{34}$$

$$\dot{y} = v\sin\phi + Q(x, y), \tag{35}$$

where U, Q are known functions that correspond to the components of the vectorfield along the x and y direction, and ϕ is the control input. The goal is to minimize time until the vehicle reaches a target location. Returning to our problem, it is easy to observe that if we apply E's optimal strategy ($u = -1$ for the upper right quadrant), equations (9) and (10) assume the form

$$\dot{x} = v_p \cos\phi + \frac{v_e}{R} y, \tag{36}$$

$$\dot{y} = v_p \sin\phi - \frac{v_e}{R} x - v_e, \qquad x, y \geq 0, \tag{37}$$

and the target location which P intends to reach in minimum time is the origin $(0, 0)$. By comparing equations (34)-(35)

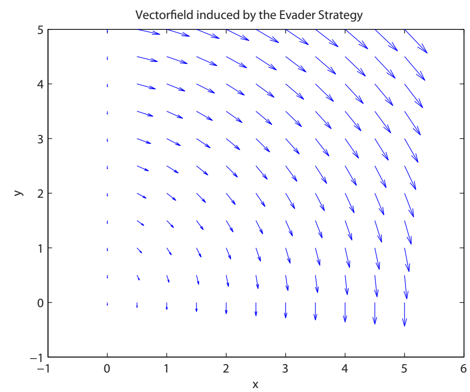


Fig. 4. Applying the evader's strategy induces a vectorfield that resembles a current, which P needs to overcome in order to intercept E in minimum time. Plotted for $v_e = v_p = 1$, $R = 0.7$.

with (36)-(37) it is evident that E's optimal control results in an induced vectorfield that resembles a current, which P needs to overcome in order to intercept E in minimum time. This vectorfield is shown in Figure 4. This interesting fact allows us to use the well-known *Zermelo's Navigation Formula* [20] which states that the optimal control ϕ^* obeys

$$\begin{aligned}\dot{\phi}^* &= \sin^2\phi^* \frac{\partial Q(x,y)}{\partial x} - \cos^2\phi^* \frac{\partial U(x,y)}{\partial y} \\ &\quad + \sin\phi^* \cos\phi^* \left(\frac{\partial U(x,y)}{\partial x} - \frac{\partial Q(x,y)}{\partial y}\right),\end{aligned} \tag{38}$$

which, for $U(x, y) = v_e y/R$ and $Q(x, y) = -v_e x/R - v_e$, yields

$$\dot{\phi}^* = -\frac{v_e}{R}. \tag{39}$$

The problem therefore reduces to a two-point boundary value problem consisting of integrating equations (36), (37) and (39) subject to initial conditions (x, y) and $\phi^*(0)$ that will lead to a trajectory passing through the origin $(0, 0)$. Alternatively, one can consider integrating this system of ODEs backwards in time, i.e., by flipping the sign of the right-hand sides of (36), (37) and (39) and using the variable τ, subject to the retrograde boundary conditions $(x, y) = (0, 0)$ and a variable retrograde boundary condition $\phi_f^* \in [0,\ 2\pi]$ for (39). This will yield a parametric family of curves, and it remains to locate the one that passes through the original point (x, y) of interest. In fact, this integration can be performed analytically to obtain the following parametric family of curves

$$x(\phi_f^*;\tau) = -R + R\cos(c\tau) - v_p\tau\cos(\phi_f^* + c\tau), \tag{40}$$
$$y(\phi_f^*;\tau) = R\sin(c\tau) - v_p\tau\sin(\phi_f^* + c\tau), \quad \tau \in [0, \tau_{\max}], \tag{41}$$

where ϕ_f^* is the free parameter and $\tau_{\max}$ is the solution to the transcendental equation

$$v_p\tau_{\max}\cos(\phi_f^* + c\tau_{\max}) = R - R\cos(c\tau_{\max}). \tag{42}$$

Figure 5 illustrates several time optimal trajectories, members of the parametric family of curves given by (40) and (41), corresponding to different values of ϕ_f^*. The barrier, i.e., the rightmost time optimal trajectory, is obtained for $\phi_f^* = 3\pi/2 - \gamma$, wherein the expressions (40) and (41) become identical to the barrier given by (31) and (32).

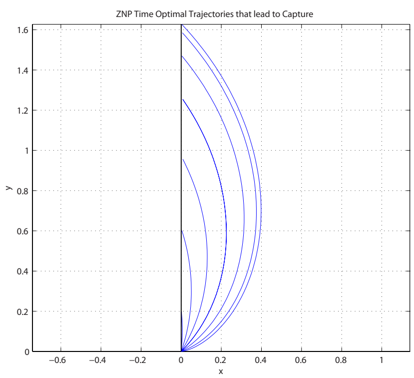


Fig. 5. Members of the parametric family of curves given by (40) and (41), corresponding to different values of ϕ_f^*, $v_e = v_p = 1$, $R = 0.7$.

Remark 1 Note that, although the pursuer control action leads to a curved path of the game state in the evader fixed reference frame, as seen in Figure 5, its trajectory in the inertial reference frame is a straight line. This can be easily seen from the fact that $\phi = \phi_p + \pi/2 - \phi_e$, thus $\dot{\phi}_p = \dot{\phi} + \dot{\phi}_e$ which, by virtue of the ZNP solution of equation (39) and the evader dynamics given by equation (5) for $u = -1$, results in $\dot{\phi}_p^* = -v/R + v/R = 0$.

V. Conclusions

In this paper we have investigated the pursuit and evasion differential game between an agile pursuer and an evader having maneuverability restrictions, both of different constant speeds. It was shown that if the initial velocity vector orientation of the evader does not have a component pointing towards the pursuer, then capture is not possible and the evader escapes without having to alter her velocity vector orientation. If this condition is not satisfied, capture may be possible. For this case, the solution of the game admits a characterization of the barrier that separates states that lead to capture under optimal play, and states that lead to evasion regardless of the pursuer's actions. Time-optimal trajectories were obtained by recognizing the equivalence of this problem to the well-known Zermelo navigation problem in optimal control. The results have immediate application in collision avoidance problems. Specifically, the barrier delineates a region of non-capturability outside of which a collision is not possible even against a malicious – and more agile – pursuer.

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