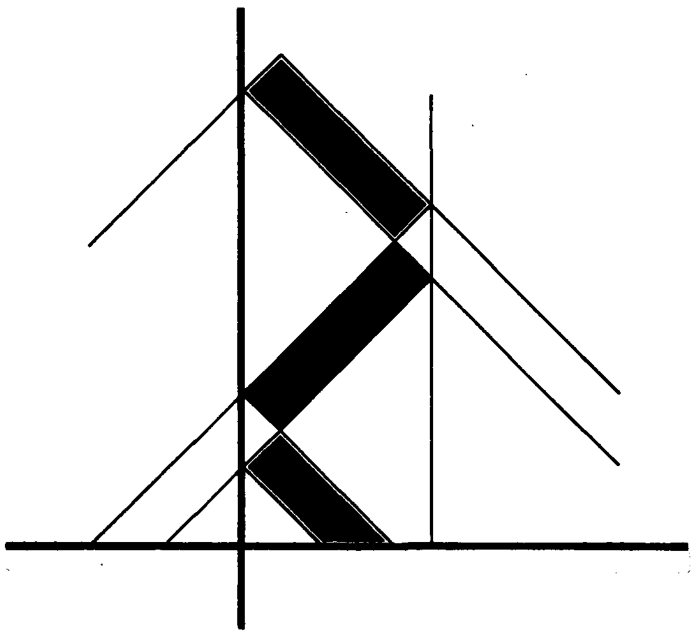
An Asymptotic Analysis of a Class of Nonlinear Hyperbolic Equations



W.T. van Horssen

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van

W.T. van Horssen.

Stelling 10.

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Het vrijwel niet verbaliseren van fietsendieven door de Nederlandse overheid kan tot gevolg hebben dat indien over duizend jaar een wetenschappelijk onderzoek wordt verricht naar de criminaliteit in de Primitief–Industriële Eeuwen een conclusie kan luiden: in de tweede helft van de twintigste eeuw was fietsendiefstal een niet voorkomend verschijnsel in Nederland.

Een asymptotische analyse van een klasse van niet-lineaire hyperbolische vergelijkingen

Aan mijn ouders

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INTRODUCTION

In a number of physical systems oscillations can be described by one or more weakly non-linear second order partial differential equations of the hyperbolic type (see for instance [4,6,11-15,17,18,20,22,23]). In this thesis the following initial-boundary value problem will be considered for a real-valued function $u(x,t;\epsilon)$, which is either scalar-valued or vector-valued:

$$u_{tt} - c^2 u_{xx} + du + \epsilon F(u;\epsilon) = 0, \qquad 0 < x < \pi,\ t > 0, \tag{1}$$

$$u(x,0;\epsilon) = u_0(x;\epsilon), \qquad 0 < x < \pi, \tag{2}$$

$$u_t(x,0;\epsilon) = u_1(x;\epsilon), \qquad 0 < x < \pi, \tag{3}$$

$$u(0,t;\epsilon) = u(\pi,t;\epsilon) = 0, \qquad t \geq 0, \tag{4}$$

where ϵ is small, c independent of ϵ, and $d = 0$ or $d = 1$. The operator F is defined to be

$$F(u;\epsilon)(x,t) \equiv f(x,t,u(x,t;\epsilon),\ u_t(x,t;\epsilon),\ u_x(x,t;\epsilon);\epsilon).$$

Furthermore, the real-valued functions u_0, u_1 and f have to satisfy certain smoothness conditions, which will be mentioned in the following chapters.

For scalar-valued functions the initial-boundary value problem (1)-(4) for the perturbed wave equation ($d = 0$) is considered in chapter 1 and for the perturbed telegraph equation ($c = d = 1$) in chapter 2. Moreover, in chapter 2 the restriction has been made that f solely depends on x, t and $u(x,t;\epsilon)$. In chapter 3 the initial-boundary value problem (1)-(4) for a system of perturbed wave equations ($d = 0$) is considered for vector-valued functions. In that case c is a diagonal matrix with positive and ϵ-independent diagonal elements.

In [4,5,18,20] several initial-boundary value problems and initial value problems for second order, weakly nonlinear hyperbolic equations involving a small parameter ϵ have been considered and for these problems several methods have been developed to construct formal asymptotic approximations of the solutions. As usual formal asymptotic approximations are defined to be functions satisfying the differential equation(s) and the initial conditions up to some order depending on the small parameter ϵ. In a number of papers [4,5, 18,20] it is suggested or assumed that a theory for the asymptotic validity of formal approximations of the solutions of initial-boundary value problems like (1)-(4) is available. However, this is incorrect. In this thesis an asymptotic theory for a class of initial-boundary value problems for (systems of) weakly nonlinear hyperbolic equations of order two will be presented. In fact, this asymptotic theory can be regarded as an extension of the asymptotic theory for ordinary differential equations as for instance described in [2,8,25]. The asymptotic theory presented in this thesis implies the well-posedness (in the classical sense) of the initial-boundary value problem (1)-(4) and the asymptotic validity (as ϵ tends to zero) of a class of formal approximations on long and ϵ-dependent time-scales.

The asymptotic theory is applied to several initial-boundary value problems for (systems of) weakly nonlinear hyperbolic equations of order two. In chapter 1 an initial-boundary value problem for the Rayleigh wave equation $u_{tt} - u_{xx} = \epsilon\left(u_t - \frac{1}{3}u_t^3\right)$ is studied. In the early seventies an initial-boundary value problem for the Rayleigh wave equation has been postulated in [22,23] to describe full span galloping oscillations of overhead transmission lines. From an aero-elastic analysis it is shown in chapter 1 that this initial-boundary value problem indeed may be regarded as a simple model describing the galloping oscillations (in the vertical direction) of overhead transmission lines. In chapter 2 an initial-boundary value problem for a weakly nonlinear telegraph equation $u_{tt} - u_{xx} +$

$+ u + \epsilon u^3 = 0$ is studied. Finally in chapter 3 an initial-boundary value problem for the following system of weakly nonlinear wave equations is studied:

$$\begin{aligned} v_{tt} - v_{xx} &= \epsilon\left(a_{10}v_t + a_{01}w_t + a_{20}v_t^2 + a_{11}v_tw_t + a_{02}w_t^2 + a_{03}w_t^3\right), \\ w_{tt} - w_{xx} &= \epsilon\left(b_{01}w_t + b_{11}v_tw_t + b_{02}w_t^2 + b_{03}w_t^3\right), \end{aligned}$$

where $a_{10}, a_{01}, \ldots, b_{03}$ are ϵ-independent constants. It is also shown in chapter 3 that this initial-boundary value problem may be regarded as a model describing the galloping oscillations (in the vertical and in the horizontal direction) of overhead transmission lines. For the aforementioned initial-boundary value problems asymptotic approximations (as ϵ tends to zero) of the solutions will be constructed using a two-timescales perturbation method. In the chapters 1 and 3 the initial-boundary value problems for the (systems of) weakly nonlinear wave equations are studied by rewriting these problems in the characteristic coordinates $\sigma = x - t$ and $\xi = x + t$. Although it seems natural to investigate the initial-boundary value problems for the (systems of) weakly nonlinear wave equations by means of a Fourier series expansion of the solution, it turns out that this approach leads to computational difficulties. In fact, in this approach a system of infinitely many, coupled, nonlinear, ordinary differential equations is obtained, which in general is hard to solve. To approximate the solution of this system of differential equations the truncation method of Galerkin may be used. However, for the so-obtained approximation asymptotic validity can often only be proved on a time-scale which in general is smaller than the time-scale for which the original initial-boundary value problem has been proved to be well-posed. In chapter 2, however, it turns out that the method of Fourier series expansion of the solution is applicable to the initial-boundary value problem for the weakly semi-linear telegraph equation. From [4,6,17,18,20] and from this thesis it may be concluded that the method of characteristic coordinates is applicable to a special class of nonlinear partial differential equations, which are non-dispersive in the unperturbed case (that is $\epsilon = 0$) and that the method of Fourier series expansion of the solution is applicable to a class of nonlinear partial differential equations, which are dispersive in the unperturbed case.

CHAPTER 1

AN ASYMPTOTIC THEORY FOR A CLASS OF INITIAL-BOUNDARY VALUE PROBLEMS FOR WEAKLY NONLINEAR WAVE EQUATIONS WITH AN APPLICATION TO A MODEL OF THE GALLOPING OSCILLATIONS OF OVERHEAD TRANSMISSION LINES

Abstract

This chapter aims to contribute to the foundation of the asymptotic methods for initial-boundary value problems and initial value problems for weakly nonlinear hyperbolic partial differential equations of order two. In this chapter an asymptotic theory for a class of initial-boundary value problems for weakly nonlinear wave equations is presented. The theory implies the well-posedness of the problem in the classical sense and the validity of formal approximations on long time-scales.

As an application of the theory an initial-boundary value problem for a Rayleigh wave equation is studied in detail using a two-timescales perturbation method. From an aero-elastic analysis it is shown that this initial-boundary value problem may be regarded as a model describing the growth of wind-induced oscillations of overhead transmission lines.

1.1. Introduction

In this chapter an asymptotic theory is presented for the following initial-boundary value problem for a nonlinearly perturbed wave equation

$$u_{tt} - u_{xx} + \epsilon f(x,t,u,u_t,u_x;\epsilon) = 0, \qquad 0 < x < \pi,\ t > 0, \tag{1.1.1}$$

$$u(x,0;\epsilon) = u_0(x;\epsilon) \text{ and } u_t(x,0;\epsilon) = u_1(x;\epsilon), \qquad 0 < x < \pi, \tag{1.1.2}$$

$$u(0,t;\epsilon) = u(\pi,t;\epsilon) = 0, \qquad t \geq 0, \tag{1.1.3}$$

with $0 < |\epsilon| \leq \epsilon_0 << 1$ and where the nonlinearity f and the initial values u_0 and u_1 have to satisfy certain smoothness properties, which are mentioned in section 1.2. The asymptotic theory implies the well-posedness (in the classical sense) of the initial-boundary value problem (1.1.1)-(1.1.3) and the asymptotic validity of formal approximations. In this chapter formal approximations are defined to be functions that satisfy the differential equation and the initial values up to some order depending on the small parameter ϵ. In [11] a similar asymptotic theory has been developed for an initial-boundary value problem for the weakly semi-linear telegraph equation

$$u_{tt} - u_{xx} + u + \epsilon f(x,t,u;\epsilon) = 0, \qquad 0 < x < \pi,\ t > 0,$$

subject to the inital and boundary conditions (1.1.2) and (1.1.3). The well-posedness of that problem and the asymptotic validity of formal approximations could be established on a time-scale of order $|\epsilon|^{-1/2}$. For the initial-boundary value problem (1.1.1)-(1.1.3) it will be shown that a time-scale of order $|\epsilon|^{-1}$ can be obtained.

The asymptotic theory in [11] and the asymptotic theory presented in this chapter can be regarded as an extension of the asymptotic theory for ordinary differential equations as for instance described in [1,2,8,25]. In a number of papers for instance in [5,6,18,22,23], it is suggested or assumed that an asymptotic theory for the validity of formal approximations of the solutions of initial-boundary value problems like (1.1.1)-(1.1.3) is available. In [5,20] it is taken for granted that in [8] a justification is given of a perturbation method introduced in [4]. An important part of the justification, namely an estimate of the difference between the exact solution and the formal approximation is not given in [8]. Furthermore, the time-scale on which the results might be valid, is not specified in [8]. Some authors, as for instance [3,8,20], have noticed that these validity proofs were absent or far from complete. In the literature only recently some asymptotic validity proofs have been given. For instance in [3] a rather successful approach has been introduced to justify a number of formal perturbation methods. However, this approach is incomplete because in [3] the presumption is made that on sufficiently large time-scales the initial value problems under consideration are well-posed in some (not specified) sense. Some other asymptotic results have been obtained in [6,19,27] by rewriting (1.1.1)-(1.1.3) as an initial value problem for a system of infinitely many ordinary differential equations in a Hilbert or Sobolev space.

This chapter, being an attempt to contribute to the foundations of the asymptotic methods for weakly nonlinear hyperbolic partial differential equations, is organized as follows. In section 1.2 the well-posedness of the problem is investigated and established on a time-scale of order $|\epsilon|^{-1}$ and in section 1.3 the asymptotic validity of formal approximations is studied. The asymptotic theory is applied in section 1.5 to the initial-boundary value problem (1.1.1)-(1.1.3) with $f(x,t,u,u_t,u_x;\epsilon) \equiv -u_t + \frac{1}{3} u_t^3$. In the early seventies this initial-boundary value problem for the Rayleigh wave equation has been postulated in [22] to describe full span galloping oscillations of overhead transmission lines. In section 1.4 it follows from an aero-elastic analysis that this initial-boundary value problem may indeed be regarded as a model which describes the growth of wind-induced oscillations of overhead transmission lines. Using a two-timescales perturbation method, as for instance successfully used in [4,6,11,17,18], an asymptotic approximation of the solution of the aforementioned initial-boundary value problem will be constructed. Finally in section 1.6 some remarks are made on the asymptotic theory applied to initial and initial-boundary value problems for the weakly nonlinear wave equations. Furthermore, some of the results obtained in the literature are discussed.

1.2. The well-posedness of the problem

In this chapter the following weakly nonlinear initial-boundary value problem for a (with respect to x and t) twice continuously differentiable function $u(x,t;\epsilon)$ is considered.

$$u_{tt} - u_{xx} + \epsilon F(u;\epsilon) = 0, \qquad t > 0,\ 0 < x < \pi, \tag{1.2.1}$$

$$u(x,0;\epsilon) = u_0(x;\epsilon), \qquad 0 < x < \pi, \tag{1.2.2}$$

$$u_t(x,0;\epsilon) = u_1(x;\epsilon), \qquad 0 < x < \pi, \tag{1.2.3}$$

$$u(0,t;\epsilon) = u(\pi,t;\epsilon) = 0, \qquad t \geq 0, \tag{1.2.4}$$

where

$$F(u;\epsilon)(x,t) \equiv f(x,t,u(x,t;\epsilon),u_t(x,t;\epsilon),u_x(x,t;\epsilon);\epsilon), \tag{1.2.5}$$

$0 < |\epsilon| \leq \epsilon_0 << 1$, and where $f(x,t,u,p,q;\epsilon)$, $u_0(x;\epsilon)$ and $u_1(x;\epsilon)$ satisfy

$$f,\ \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial u},\ \frac{\partial f}{\partial p},\ \frac{\partial f}{\partial q} \in C([0,\pi] \times [0,\infty> \times \mathbb{R}^3 \times [-\epsilon_0,\epsilon_0],\mathbb{R})$$

with $F(u;\epsilon)(0,t) = F(u;\epsilon)(\pi,t) = 0$ for $t \geq 0$, (1.2.6)

$$u_0,\ \frac{\partial u_0}{\partial x},\ \frac{\partial^2 u_0}{\partial x^2} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0],\mathbb{R})$$

with $u_0(0;\epsilon) = u_0(\pi;\epsilon) = u_0''(0;\epsilon) = u_0''(\pi;\epsilon) = 0$, and (1.2.7)

$$u_1,\ \frac{\partial u_1}{\partial x} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0],\mathbb{R}) \quad \text{with} \quad u_1(0;\epsilon) = u_1(\pi;\epsilon) = 0. \tag{1.2.8}$$

Furthermore, $f(x,t,u,p,q;\epsilon)$ and its partial derivatives with respect to x,u,p and q are assumed to be uniformly bounded for those values of t under consideration.

To prove existence and uniqueness in the classical sense of the solution of the initial-boundary value problem (1.2.1)-(1.2.4) an equivalent integral equation will be used. In order to derive this integral equation the initial-boundary value problem is transformed into an initial value problem by extending the functions f, u_0 and u_1 in x to odd and 2π-periodic functions (see for instance [7, chapter 5] or [28, chapter 2]). The extensions of u, f, u_0 and u_1 are denoted by u^*, f^*, u_0^* and u_1^* respectively. Then, assuming that the solution u^* of the initial value problem is twice continuously differentiable, an integral equation for the solution of the initial value problem is given by

$$u^*(x,t;\epsilon) = -\frac{\epsilon}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f^*(\xi,\tau,u^*(\xi,\tau;\epsilon),u^*_\tau(\xi,\tau;\epsilon),u^*_\xi(\xi,\tau;\epsilon);\epsilon)\, d\xi d\tau + $$

$$+ \frac{1}{2} u_0^*(x+t;\epsilon) + \frac{1}{2} u_0^*(x-t;\epsilon) + \frac{1}{2}\int_{x-t}^{x+t} u_1^*(\xi;\epsilon)\, d\xi. \tag{1.2.9}$$

Using reflection principles (1.2.9) can be rewritten as an integral equation on the semi-infinite strip $0 \le x \le \pi$, $0 \le t < \infty$, yielding

$$u(x,t;\epsilon) = \frac{\epsilon}{2}\int_0^t \int_0^\pi G(\xi,\tau;x,t)F(u;\epsilon)(\xi,\tau)d\xi d\tau + u_\ell(x,t;\epsilon), \tag{1.2.10}$$

where G and u_ℓ are given by

$$G(\xi,\tau;x,t) = \sum_{k\in\mathbb{Z}} \{ H(t-\tau-\xi+2k\pi-x)H(t-\tau+\xi-2k\pi+x) +$$
$$-H(t-\tau+\xi+2k\pi-x)H(t-\tau-\xi-2k\pi+x) \} \tag{1.2.11}$$

and

$$u_\ell(x,t;\epsilon) = \frac{1}{2}\int_0^\pi \left\{ u_0(\xi;\epsilon)\frac{\partial G}{\partial \tau}(\xi,0;x,t) - u_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi, \tag{1.2.12}$$

in which H(a) is a function on $\mathbb{R}$ which is equal to 1 for $a > 0$, $\frac{1}{2}$ for $a = 0$ and zero otherwise. In (1.2.12) it is assumed that G is differentiated according to the rule $\frac{d}{d\tau}\{ H(f(\tau))H(g(\tau)) \} = \delta_0(f(\tau)) \frac{df(\tau)}{d\tau} H(g(\tau)) + H(f(\tau))\delta_0(g(\tau)) \frac{dg(\tau)}{d\tau}$, where δ_0 is the Dirac delta function. In fact, G as defined by (1.2.11) is the Green's function for the differential operator $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and the boundary conditions (1.2.4). It is worth noticing that the solution of the linear initial-boundary value problem (1.2.1)-(1.2.4) (that is with $F \equiv 0$) is given by $u_\ell(x,t;\epsilon)$.

Some elementary calculations show that if $v(x,t;\epsilon)$ is a twice continuously differentiable solution of the initial-boundary value problem (1.2.1)-(1.2.4) then $v(x,t;\epsilon)$ is a solution of the integral equation (1.2.10). And if $w(x,t;\epsilon)$ is a twice continuously differentiable solution of the integral equation (1.2.10) then it can easily be shown that $w(x,t;\epsilon)$ is a solution of the initial-boundary value problem (1.2.1)-(1.2.4). Hence, the integral equation (1.2.10) and the initial-boundary value problem (1.2.1)-(1.2.4) are equivalent if twice continuously differentiable solutions exist. Now it will be proved that a unique, twice continuously differentiable solution of the integral equation (1.2.10) exists on a region J_L of the (x,t)-plane. And so, a unique and twice continuously differentiable solution exists for the initial-boundary value problem (1.2.1)-(1.2.4) on J_L.

In order to prove existence and uniqueness in the classical sense of the solution of the nonlinear integral equation (1.2.10) a fixed point theorem will be used. Let J_L be given by

$$J_L = \left\{(x,t) \mid 0 \le x \le \pi,\ 0 \le t \le L\,|\epsilon|^{-1}\right\} \tag{1.2.13}$$

in which L is a sufficiently small, positive constant independent of ϵ. Let $C^2_M(J_L)$ be the space of all real-valued and twice continuously differentiable functions w on J_L with norm $\|\cdot\|_{J_L}$ defined by

$$\| w \|_{J_L} = \sum_{\substack{i,j=0 \\ i+j\le 2}}^{2} \max_{(x,t)\in J_L} \left| \frac{\partial^{i+j} w(x,t)}{\partial x^i \partial t^j} \right| \le M.$$

From the smoothness properties of u_0 and u_1 it follows that (for fixed u_0 and u_1) there exists a positive constant M_1 independent of ϵ such that,

$$\| u_\ell \|_{J_L} \le \frac{1}{2} M_1, \tag{1.2.14}$$

and from the smoothness properties of $F(u;\epsilon)(x,t)$ (as defined by (1.2.5) and (1.2.6)) it follows that there exist ϵ-independent constants M_2 and M_3 such that,

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} F(v;\epsilon)(x,t) \right| \le M_2, \tag{1.2.15}$$

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} \Big(F(v;\epsilon)(x,t) - F(w;\epsilon)(x,t)\Big) \right| \le M_3 \; \| v-w \|_{J_L}, \tag{1.2.16}$$

for all $(x,t) \in J_L$, $\epsilon \in [-\epsilon_0,\epsilon_0]$ and $v,w \in C^2_{M_1}(J_L)$. Now let the integral operator T: $C^2(J_L) \to C^2(J_L)$, which is related to the integral equation (1.2.10), be defined by

$$(Tw)(x,t) \equiv \frac{\epsilon}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t) F(w;\epsilon)(\xi,\tau) \, d\xi d\tau + u_\ell(x,t;\epsilon), \tag{1.2.17}$$

where G, F and u_ℓ are given by (1.2.11), (1.2.5) and (1.2.12) respectively. According to Banach's fixed point theorem the integral operator T has a unique fixed point in $C^2_{M_1}(J_L)$ if the operator T satisfies

(i) T: $C^2_{M_1}(J_L) \to C^2_{M_1}(J_L)$, and

(ii) $\| Tv-Tw \|_{J_L} \le k \; \| v-w \|_{J_L}$ with $0 < k < 1$, for all $v,w \in C^2_{M_1}(J_L)$.

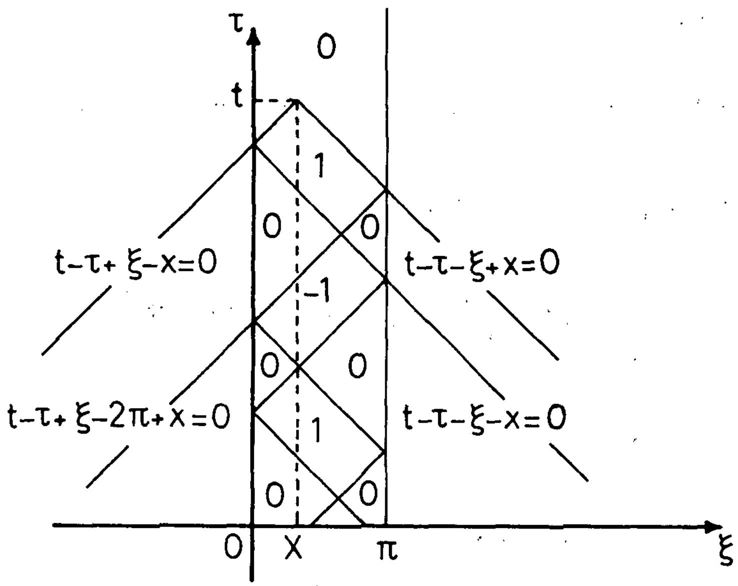

Figure 1.2.1: Subregions in V with the corresponding values of the Green's function $G(\xi,\tau;x,t)$.

Now, it will be proved that the integral operator T satisfies these two conditions. It is not difficult to show that T maps $C^2_{M_1}(J_L)$ into the space of twice continuously differentiable functions on J_L. In order to prove that T maps $C^2_{M_1}(J_L)$ into itself an estimate of the Green's function $G(\xi,\tau;x,t)$ should be obtained for $0 \leq \xi \leq \pi$, $0 \leq \tau \leq t$ and fixed x and t. In figure 1.2.1 the characteristics from the point (x,t) and the reflected characteristics at the boundaries $\xi = 0$ and $\xi = \pi$ are drawn in the (ξ,τ)-plane. These (reflected) characteristics divide the region $V = \{(\xi,\tau) \,|\, 0 \leq \xi \leq \pi,\ \tau \geq 0\}$ into a finite number of subregions. In each subregion $G(\xi,\tau;x,t)$ can be determined by evaluating (1.2.11). These values are given in figure 1.2.1. The following estimate of $G(\xi,\tau;x,t)$ can now be made for $0 \leq \xi \leq \pi$, $\tau \geq 0$ and fixed x and t:

$$|G(\xi,\tau;x,t)| \leq 1. \tag{1.2.18}$$

Using (1.2.13)-(1.2.15), (1.2.17)-(1.2.19) the following estimate can be made

$$\|Tv\|_{J_L} \leq \|Tv-u_\ell\|_{J_L} + \|u_\ell\|_{J_L} \leq$$

$$\leq \sum_{\substack{i,j=0 \\ i+j\leq 2}}^{2} \max_{(x,t)\in J_L} \left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} \Big((Tv)x,t) - u_\ell(x,t;\epsilon)\Big) \right| + \frac{1}{2} M_1 \leq$$

$$\leq \left(\frac{\pi}{2} + 5\right) M_2 L + \epsilon_0 M_2 + \frac{1}{2} M_1$$

for all $v \in C^2_{M_1}(J_L)$. Now ϵ_0 has been assumed to be sufficiently small and so, there exists an ϵ-independent constant L such that $\left(\frac{\pi}{2} + 5\right) M_2 L + \epsilon_0 M_2 \leq \frac{1}{2} M_1$. Hence, $\|Tv\|_{J_L} \leq M_1$ for all $v \in C^2_{M_1}(J_L)$. So, T maps $C^2_{M_1}$ into itself. Using (1.2.13), (1.2.16)-(1.2.19) it will be shown that T is a contraction on $C^2_{M_1}(J_L)$. Let v and $w \in C^2_{M_1}(J_L)$, then the following estimate can be obtained

$$\|Tv-Tw\|_{J_L} \leq \left(\left(\frac{\pi}{2} + 5\right) M_3 L + \epsilon_0 M_3 \right) \|v-w\|_{J_L}.$$

It is obvious that there exists an ϵ-independent constant L such that $\left(\frac{\pi}{2} + 5\right) M_3 L + \epsilon_0 M_3 \leq k < 1$. Since there always exists a constant L independent of ϵ such that $\left(\frac{\pi}{2} + 5\right) M_2 L + \epsilon_0 M_2 < \frac{1}{2} M_1$ and $\left(\frac{\pi}{2} + 5\right) M_3 L + \epsilon_0 M_3 \leq k < 1$, it follows that T maps $C^2_{M_1}(J_L)$ into itself and that T is a contraction on $C^2_{M_1}(J_L)$. Banach's fixed point theorem then implies that T has a unique fixed point in $C^2_{M_1}(J_L)$, that is, a unique and twice continuously differentiable function on J_L. Hence, the solution of the integral equation (1.2.10) is unique and twice continuously differentiable on J_L. And so, on J_L a unique and twice continuously differentiable solution exists for the initial-boundary value problem (1.2.1)-(1.2.4).

Next it will be shown that the solution of the initial-boundary value problem (1.2.1)-(1.2.4) depends continuously on the initial values. Let $u(x,t;\epsilon)$ satisfy (1.2.1)-(1.2.4) and let $\bar{u}(x,t;\epsilon)$ satisfy (1.2.1), (1.2.4), $\bar{u}(x,0;\epsilon) = \bar{u}_0(x;\epsilon)$ and $\bar{u}_t(x,0;\epsilon) = \bar{u}_1(x;\epsilon)$, where $\bar{u}_0$ and $\bar{u}_1$ satisfy (1.2.7) and (1.2.8). Let $\bar{u}_\ell$ be given by

$$\bar{u}_\ell(x,t;\epsilon) = \frac{1}{2}\int_0^\pi \left\{\bar{u}_0(\xi;\epsilon)\frac{\partial G}{\partial \tau}(\xi,0;x,t) - \bar{u}_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi.$$

After subtracting the integral equations for u and $\bar{u}$, using (1.2.10), (1.2.13), (1.2.16) and (1.2.18), assuming u and $\bar{u} \in C^2_{M_1}(J_L)$, one obtains the estimate

$$\begin{aligned}\| u-\bar{u} \|_{J_L} &\le \left(\left(\frac{\pi}{2} + 5\right) M_3 L + \epsilon_0 M_3 \right) \| u-\bar{u} \|_{J_L} + \| u_\ell-\bar{u}_\ell \|_{J_L} \le \\ &\le k \, \| u-\bar{u} \|_{J_L} + \| u_\ell-\bar{u}_\ell \|_{J_L} \qquad \text{with } 0 \le k < 1.\end{aligned}$$

This inequality implies $\| u-\bar{u} \|_{J_L} \le \frac{1}{1-k} \| u_\ell-\bar{u}_\ell \|_{J_L}$ with $0 \le k < 1$.
So, small differences between the initial values generate small differences between the solutions u and $\bar{u}$ on J_L. In other words the solution of the initial-boundary value problem depends continuously on the initial values. The following theorem on the well-posedness of the problem can now be formulated.

Theorem 1.2.1

Suppose that F, u_0 and u_1 satisfy the assumptions (1.2.6)-(1.2.8). Then for any ϵ satisfying $0 < |\epsilon| \le \epsilon_0 << 1$, the nonlinear initial-boundary value problem (1.2.1)-(1.2.4) and the equivalent nonlinear integral equation (1.2.10) have the same, unique and twice continuously differentiable solution for $0 \le x \le \pi$ and $0 \le t \le L|\epsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ϵ. Furthermore, this unique solution depends continuously on the initial values.

1.3. On the validity of formal approximations

Since the initial-boundary value problem (1.1.1)-(1.1.3) contains a small parameter ϵ perturbation methods may be applied for the construction of approximations to the solution. In most perturbation methods for weakly nonlinear problems a function is constructed that satisfies the differential equation and the initial conditions up to some order depending on the small parameter ϵ. Such a function is usually called a formal approximation. To show that this formal approximation is an asymptotic approximation (as $\epsilon \to 0$) requires an additional analysis. Therefore suppose that on J_L (given by (1.2.13)) a twice continuously differentiable function $v(x,t;\epsilon)$ is constructed satisfying

$$v_{tt} - v_{xx} + \epsilon F(v;\epsilon) = |\epsilon|^m c_1(x,t;\epsilon), \qquad m > 1, \tag{1.3.1}$$

$$v(x,0;\epsilon) = u_0(x;\epsilon) + |\epsilon|^{m-1} c_2(x;\epsilon) \equiv v_0(x;\epsilon), \qquad 0 < x < \pi, \tag{1.3.2}$$

$$v_t(x,0;\epsilon) = u_1(x;\epsilon) + |\epsilon|^{m-1} c_3(x;\epsilon) \equiv v_1(x;\epsilon), \qquad 0 < x < \pi, \tag{1.3.3}$$

$$v(0,t;\epsilon) = v(\pi,t;\epsilon) = 0, \qquad 0 \le t \le L|\epsilon|^{-1}, \tag{1.3.4}$$

where ϵ, F, u_0 and u_1 satisfy (1.2.5)-(1.2.8) and where c_1, c_2 and c_3 satisfy

$$c_1, \frac{\partial c_1}{\partial x} \in C([0,\pi] \times [0, L|\epsilon|^{-1}] \times [-\epsilon_0,\epsilon_0], \mathbb{R})$$

$$\text{with } c_1(0,t;\epsilon) = c_1(\pi,t;\epsilon) = 0, \qquad \text{for } 0 \le t \le L|\epsilon|^{-1}, \tag{1.3.5}$$

$$c_2, \frac{\partial c_2}{\partial x}, \frac{\partial^2 c_2}{\partial x^2} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0], \mathbb{R})$$

$$\text{with } c_2(0;\epsilon) = c_2(\pi;\epsilon) = c_2''(0;\epsilon) = c_2''(\pi;\epsilon) = 0, \qquad \text{and} \tag{1.3.6}$$

$$c_3, \frac{\partial c_3}{\partial x} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0], \mathbb{R}) \qquad \text{with } c_3(0;\epsilon) = c_3(\pi;\epsilon) = 0. \tag{1.3.7}$$

Furthermore, $c_1(x,t;\epsilon)$ and its derivative with respect to x are supposed to be uniformly bounded for those values of t and ϵ under consideration. From theorem 1.2.1 it follows that the initial-boundary value problem (1.3.1)-(1.3.4) has a unique, twice continuously differentiable solution on a time-scale of $O(|\epsilon|^{-1})$. This initial-boundary value problem can then be transformed into the equivalent integral equation

$$v(x,t;\epsilon) = \frac{\epsilon}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t)\tilde{F}(v;\epsilon)(\xi,\tau)\, d\xi d\tau + v_\ell(x,t;\epsilon), \tag{1.3.8}$$

where G is given by (1.2.11) and where $\tilde{F}$ and v_ℓ are given by

$$\tilde{F}(v;\epsilon)(x,t) \equiv F(v;\epsilon)(x,t) - |\epsilon|^{m-1} c_1(x,t;\epsilon) \qquad \text{and}$$

$$v_\ell(x,t;\epsilon) = \frac{1}{2} \int_0^\pi \left\{ v_0(\xi;\epsilon) \frac{\partial G}{\partial \tau}(\xi,0;x,t) - v_1(\xi;\epsilon)G(\xi,0;x,t) \right\} d\xi.$$

Now, it will be shown that the formal approximation v is an asymptotic approximation (as $\epsilon \to 0$) of the solution of the initial-boundary value problem (1.2.1)-(1.2.4) if m > 1, that is, it will be proved that

$$\| u-v \|_{J_L} = O(\delta(\epsilon)), \qquad \text{where } \lim_{\epsilon \to 0} \delta(\epsilon) = 0.$$

Moreover $\delta(\epsilon)$ will be derived explicitly. This result implies that

$$\lim_{\epsilon \to 0} | u(x,t;\epsilon) - v(x,t;\epsilon) | = 0 \qquad \text{for } (x,t) \in J_L.$$

Subtracting the integral equation (1.3.8) from the integral equation (1.2.10), supposing that v_ℓ satisfies (1.2.14) and that $\tilde{F}$ satisfies (1.2.15) and (1.2.16), using (1.2.13), (1.2.16), (1.2.18) and the fact that $u,v \in C^2_{M_1}(J_L)$, the following estimate is obtained

$$\|u-v\|_{J_L} \le \left(\left(\frac{\pi}{2}+5\right) M_3L + \epsilon_0 M_3\right) \|u-v\|_{J_L} + \|c\|_{J_L} + \|u_\ell - v_\ell\|_{J_L} \le$$
$$\le k\|u-v\|_{J_L} + \|c\|_{J_L} + \|u_\ell - v_\ell\|_{J_L},$$

with $0 \le k < 1$ and where c is given by

$$c(x,t;\epsilon) = \frac{|\epsilon|^m}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t)c_1(\xi,\tau;\epsilon)\, d\xi d\tau,$$

and where $u_\ell - v_\ell$ is given by

$$u_\ell(x,t;\epsilon) - v_\ell(x,t;\epsilon) = -\frac{|\epsilon|^{m-1}}{2} \int_0^\pi \left\{c_2(\xi;\epsilon)\frac{\partial G}{\partial \tau}(\xi,0;x,t) - c_3(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi.$$

Hence,

$$\|u-v\|_{J_L} \le \frac{1}{1-k}\left\{\|c\|_{J_L} + \|u_\ell - v_\ell\|_{J_L}\right\} \quad \text{with } 0 \le k < 1.$$

From the smoothness properties of c_1, c_2 and c_3 it follows that there exists a constant K independent of ϵ, such that

$$\|c\|_{J_L} \le \left(\left(\frac{\pi}{2}+5\right) KL + |\epsilon| K\right) |\epsilon|^{m-1} \quad \text{and}$$
$$\|u_\ell - v_\ell\|_{J_L} \le \left(\frac{\pi}{2}+11\right) K|\epsilon|^{m-1}.$$

So, $$\|u-v\|_{J_L} \le \frac{|\epsilon|^{m-1}K}{1-k}\left\{\left(\frac{\pi}{2}+5\right)L + |\epsilon| + \frac{\pi}{2} + 11\right\}.$$

For $m > 1$ this inequality implies the asymptotic validity (as $\epsilon \to 0$) of the formal approximation v. The following theorem has now been established.

Theorem 1.3.1

Let the formal approximation v satisfy (1.3.1)-(1.3.4), where ϵ, F, u_0 and u_1 are given by (1.2.5)-(1.2.8) and where c_1, c_2 and c_3 satisfy (1.3.5)-(1.3.7). Then for $m > 1$, the formal approximation v is an asymptotic approximation (as $\epsilon \to 0$) of the solution u of the nonlinear initial-boundary value problem (1.2.1)-(1.2.4). The asymptotic approximation v is valid for those values of the independent variables x and t for which problem (1.2.1)-(1.2.4) has been proved well-posed. That is,

$$\| u-v \|_{J_L} = O(|\epsilon|^{m-1}), \text{ implying } |u(x,t;\epsilon) - v(x,t;\epsilon)| = O(|\epsilon|^{m-1})$$

for $0 \leq x \leq \pi$ and $0 \leq t \leq L|\epsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ϵ.

1.4. A simple model of the galloping oscillations of overhead transmission lines

In this section a simple model describing the galloping oscillations of overhead transmission lines will be derived. Galloping can be described as a low frequency, large amplitude phenomenon involving an almost purely vertical oscillation of single-conductor lines on which for instance ice has accreted. The frequencies involved are so low that the assumption can be made that the aerodynamic forces are as in steady flow. Another consequence of these low frequencies is that structural damping may be neglected. In severe cases galloping may give rise to conductor damage due to impact of conductor lines and due to flashover as a result of a phase-difference between conductor lines, for which the mutual distance has become too small. The usual conditions (see [26]) causing galloping are those of incipient icing in a stable atmospheric environment implying uniform (but not necessarily high velocity) airflows.

A symmetric circular conductor in a horizontal airflow cannot exhibit galloping because it cannot generate a force that lifts the conductor against gravity. On the other hand, a conductor on which ice has accreted may gallop if it adopts a suitable attitude to the wind. To describe this phenomenon a right-handed coordinate system is set up where one of the endpoints of the conductor is the origin. Through this point three mutually perpendicular axes (the x-, y- and z-axis) are drawn, where the z-axis coincides with the direction of gravity. The three coordinate axes span the three coordinate planes in space, the (x,y)-, (x,z)- and (y,z)-planes. On each coordinate axis a unit vector is fixed: on the x-axis the vector $\underline{e}_x$, on the y-axis the vector $\underline{e}_y$ and on the z-axis the vector $\underline{e}_z$, which has a direction opposite to gravity. The coordinate axes are directed by these vectors, such that a right-handed coordinate system is obtained. The coordinates of the endpoints of the conductor are supposed to be (0,0,0) and (ℓ,0,0), where ℓ is the distance between the endpoints. To model galloping a cross-section (perpendicular to the x-axis) of the conductor with ice ridge is considered. Assume that all cross-sectional shapes are identical and symmetric. Along the axis of symmetry of a cross-section a vector $\underline{e}_s$ is defined to be directing away from the ice ridge and starting in the centre of the cross-section. In figure 1.4.1 the centre of the cross-section is considered to be at $x = x_0$, $y = y_0$ and $z = z_0$ with $0 < x_0 < \ell \leq \ell_c$, where ℓ_c is the length of the conductor.

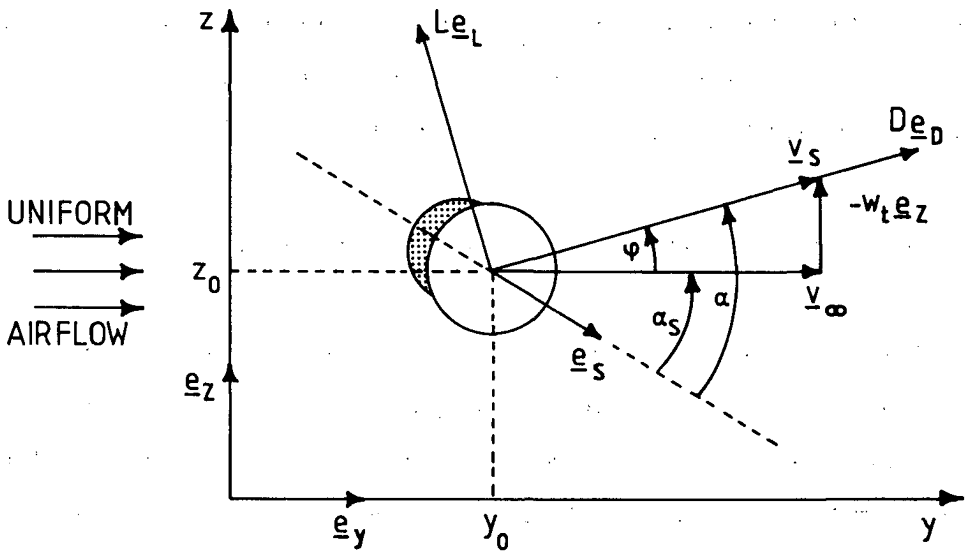

Figure 1.4.1. Cross-section of the circular conductor with ice ridge.

Let $w(x_0,t)$ denote the z-coordinate of the centre of the cross-section at $x = x_0$ and time t. Assume that every cross-section perpendicular to the x-axis oscillates in the (y,z)-plane. Furthermore, assume that torsion of the conductor may be neglected. Let the static angle of attack α_s (assumed to be constant and identical for all cross-sections) be the angle between $\underline{e}_s$ and the uniform airflow $\underline{v}_\infty$, that is, $\alpha_s := \angle(\underline{e}_s,\underline{v}_\infty)$ with $|\alpha_s| \leq \pi$. In this uniform airflow with flow velocity $\underline{v}_\infty = v_\infty\underline{e}_y$ $(v_\infty > 0)$ the conductor may oscillate due to the lift force $L\underline{e}_L$ and the drag force $D\underline{e}_D$. It should be noted that the drag force $D\underline{e}_D$

has the direction of the virtual windvelocity $\underline{v}_s \equiv \underline{v}_\infty - \frac{\partial w}{\partial t}\, \underline{e}_z$ and that the lift force $L\underline{e}_L$ has a direction perpendicular to the virtual windvelocity $\underline{v}_s$ ($\underline{e}_L$ is chosen perpendicular and anti-clockwise to $\underline{e}_D$). In figure 1.4.1 the forces $L\underline{e}_L$ and $D\underline{e}_D$ acting on the cross-section are given. Since galloping is an almost purely vertical oscillation only vertical displacements of the conductor will be considered. Furthermore, the conductor is considered to be an one-dimensional continuum in which the only interaction between different parts is a tension T, which is assumed to be constant in space and time. The validity of the assumption will be discussed in chapter 3, section 3.6. The equation describing the vertical motion of the conductor is given by

$$\rho_c A w_{tt} - TA(1 + w_x^2)^{-3/2}\, w_{xx} = -\rho_c A g + D \sin\phi + L \cos\phi, \tag{1.4.1}$$

where the magnitudes of the drag and lift force acting on the conductor per unit length of the conductor are D and L respectively, ρ_c the mass-density of the conductor (including the small ice ridge), A the constant cross-sectional area of the conductor (including the small ice ridge), ϕ the angle between $\underline{v}_\infty$ and $\underline{v}_s$ (that is, $\phi := \angle(\underline{v}_\infty, \underline{v}_s)$ with $|\phi| \leq \pi$) and g the gravitational acceleration. The magnitudes D and L of the aerodynamic forces may be given by

$$D = \frac{1}{2}\, \rho_a d c_D(\alpha)\, v_s^2, \tag{1.4.2}$$

$$L = \frac{1}{2}\, \rho_a d c_L(\alpha)\, v_s^2, \tag{1.4.3}$$

where ρ_a is the density of the air, d the diameter of the cross-section of the circular part of the conductor, $v_s = |\underline{v}_s|$, α the angle between $\underline{e}_s$ and $\underline{v}_s$ (that is, $\alpha := \angle(\underline{e}_s, \underline{v}_s)$ with $|\alpha| \leq \pi$), and $c_D(\alpha)$ and $c_L(\alpha)$ the quasi-steady drag- and lift-coefficients, which may be obtained from wind-tunnel measurements. For a certain range of values of v_∞ some

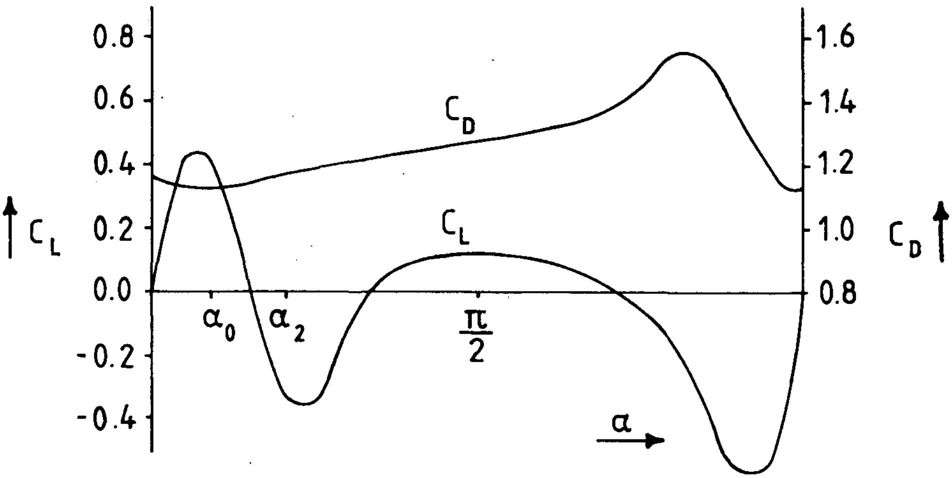

Figure 1.4.2. Typical variation of the drag and lift-coefficients c_D and c_L with angle of attack for a symmetric profile with small icy nose.

characteristic results from wind-tunnel experiments are given in figure 1.4.2 (see also [1,24,26]).

According to the Den Hartog criterion [10] a two-dimensional section is aerodynamically unstable if

$$c_D(\alpha) + \frac{dc_L(\alpha)}{d\alpha} < 0.$$

From figure 1.4.2 it follows that this condition is likely to be satisfied for some interval in α with $\alpha_0 < \alpha < \alpha_2$, where α_0 and α_2 are determined by $c_D(\alpha) + \frac{dc_L(\alpha)}{d\alpha} = 0$. For these values of α the drag- and lift-coefficients are approximated by (see also [1])

$$c_D(\alpha) = c_{D0} \quad \text{and} \quad c_L(\alpha) = c_{L1}(\alpha - \alpha_1) + c_{L3}(\alpha - \alpha_1)^3, \tag{1.4.4}$$

with $c_{D0} > 0$, $c_{L1} < 0$, $c_{L3} > 0$, $\alpha_0 < \alpha_1 < \alpha_2$ and $c_L(\alpha_1) = 0$. Since galloping is a low frequency oscillation it is assumed that $|w_t| << v_\infty$ (so $|\phi| << 1$). The right-hand side of equation (1.4.1) can now be expanded near $\phi = 0$. Also it is assumed that $|w_x| << 1$ and so, the left-hand side of equation (1.4.1) can be expanded near $w_x = 0$. Using the fact that $\phi = \arctan\left(\frac{-w_t}{v_\infty}\right)$ and neglecting terms of degree four and higher one obtains after some elementary calculations

$$w_{tt} - c^2\left(1 - \frac{3}{2}w_x^2\right)w_{xx} = -g + \frac{\rho_a d v_\infty^2}{2\rho_c A}\left\{a_0 + \frac{a_1}{v_\infty}w_t + \frac{a_2}{v_\infty^2}w_t^2 + \frac{a_3}{v_\infty^3}w_t^3\right\}, \quad (1.4.5)$$

where

$$\begin{aligned}
c &= \left(T\rho_c^{-1}\right)^{1/2},\\
a_0 &= c_{L1}(\alpha_s - \alpha_1) + c_{L3}(\alpha_s - \alpha_1)^3,\\
a_1 &= -c_{D0} - c_{L1} - 3c_{L3}(\alpha_s - \alpha_1)^2, \qquad (1.4.6)\\
a_2 &= \left(\frac{1}{2}c_{L1} + c_{L3}\right)(\alpha_s - \alpha_1) + \frac{1}{2}c_{L3}(\alpha_s - \alpha_1)^3, \quad \text{and}\\
a_3 &= -\frac{1}{2}c_{D0} - \frac{1}{6}c_{L1} - c_{L3}\left(1 + (\alpha_s - \alpha_1)^2\right).
\end{aligned}$$

Applying the transformation $w(x,t) = \tilde{w}(x,t) + \frac{\rho_c g}{2T}x(x - \ell)$ and using the dimensionless variables

$$\bar{w} = \frac{\pi c}{\ell v_\infty}\tilde{w}, \qquad \bar{x} = \frac{\pi}{\ell}x \qquad \text{and} \qquad \bar{t} = \frac{\pi c}{\ell}t$$

equation (1.4.5) becomes

$$\bar{w}_{\bar{t}\bar{t}} - \left\{1 - \frac{3}{2}\left(\frac{v_\infty}{c}\right)^2\left(\bar{w}_{\bar{x}} + \frac{g\ell}{2\pi c v_\infty}(2\bar{x} - \pi)\right)^2\right\}\bar{w}_{\bar{x}\bar{x}} +$$

$$+ \frac{3}{2}\left(\frac{v_\infty}{c}\right)^2\frac{g\ell}{2\pi c v_\infty}\left(\bar{w}_{\bar{x}} + \frac{g\ell}{2\pi c v_\infty}(2\bar{x} - \pi)\right)^2 =$$

$$= \frac{\rho_a d\ell}{2\pi\rho_c A}\left(\frac{v_\infty}{c}\right)\left\{a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right\}, \qquad (1.4.7)$$

where the dimensionless constants a_0, a_1, a_2 and a_3 are given by (1.4.6).

Typical values of the physical quantities in a practical application are: ℓ = 400 m, d = 0.04 m, $A = \pi\left(\frac{d}{2}\right)^2 = 4\pi.10^{-4}$ m^2, ρ_c = 4000 kg/m^3, ρ_a = 1.25 kg/m^3, g = 10 m/s^2 and v_∞ = 10 m/s. The tension T in the conductor is estimated by $\frac{1}{2}\rho_c g\left(\frac{\ell}{2}\right)^2 s_0^{-1}$, where s_0 (usually 2 or 3 per cent of ℓ) is the sag of the conductor. Let s_0 be 10 m, then T = 8.10^7 kg/ms^2 and consequently c = 140 m/s (c may be identified with the speed of propagation of transversal waves in the conductor). Then, it follows that

$$\frac{\rho_a d\ell}{2\pi\rho_c A} \approx \frac{5}{8}, \quad \frac{g\ell}{2\pi c v_\infty} \approx \frac{1}{2} \quad \text{and} \quad \frac{v_\infty}{c} = \frac{1}{14}.$$

Putting $\bar{\epsilon} = \frac{v_\infty}{c}$ and assuming that the static angle of attack α_s is such that galloping may set in according to the instability criterion of Den Hartog [10], that is, assuming that $\alpha_s = \alpha_1 + O(\bar{\epsilon})$, equation (1.4.7) becomes up to order $\bar{\epsilon}$

$$\bar{w}_{\bar{t}\bar{t}} - \bar{w}_{\bar{x}\bar{x}} = \bar{\epsilon}\,\frac{\rho_a d\ell}{2\pi\rho_c A}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right), \tag{1.4.8}$$

where $a = -c_{D0} - c_{L1}$ and $b = \frac{1}{2}c_{D0} + \frac{1}{6}c_{L1} + c_{L3}$. For the cross-sectional shape of the conductor with small ice ridge under consideration the aerodynamic coefficients c_{D0}, c_{L1} and c_{L3} may be determined from wind-tunnel measurements (as for instance given in figure 1.4.2). From figure 1.4.2 it follows that $c_{D0} > 0$, $c_{L1} < 0$, $|c_{L1}| > c_{D0}$, $c_{L3} > 0$, $a > 0$ and $b > 0$. If one considers a conductor with fixed endpoints the boundary conditions $\bar{w}(0,\bar{t}) = \bar{w}(\pi,\bar{t}) = 0$ are obtained. By a simple change of scale $u(\bar{x},\bar{t}) = \left(\frac{3b}{a}\right)^{1/2}\bar{w}(\bar{x},\bar{t})$ the model equation (1.4.8) can be simplified to a Rayleigh wave equation

$$u_{\bar{t}\bar{t}} - u_{\bar{x}\bar{x}} = \epsilon\left(u_{\bar{t}} - \frac{1}{3}u_{\bar{t}}^3\right), \tag{1.4.9}$$

where $\epsilon = \tilde{\epsilon}a \dfrac{\rho_a d\ell}{2\pi\rho_c A} = -\dfrac{v_\infty}{c}\dfrac{c_{D0} + c_{L1}}{2\pi}\dfrac{\rho_a d\ell}{\rho_c A}$ is a small, positive parameter. In the next section equation (1.4.9) subject to the boundary values $u(0,\bar{t}) = u(\pi,\bar{t}) = 0$ and the initial values $u(\bar{x},0) = w_0(\bar{x})$ and $u_{\bar{t}}(\bar{x},0) = w_1(\bar{x})$ will be studied, where $w_0(\bar{x})$ and $w_1(\bar{x})$ can be regarded as the initial displacement and the initial velocity of the conductor in vertical direction respectively.

It is worth noticing that in the early seventies ([22]) an equation similar to equation (1.4.9) has been postulated to describe the galloping oscillations of overhead transmission lines. In that paper it has been assumed that $\epsilon u_{\bar{t}}$ and $-\frac{\epsilon}{3}u_{\bar{t}}^3$ represent forces tending to increase and decrease respectively the magnitude of the oscillation-amplitudes. In this section it has been shown that this simple model can be derived using aerodynamical arguments.

1.5. An asymptotic approximation of the solution of a Rayleigh wave equation

In this section the following initial-boundary value problem for a twice continuously differentiable function u(x,t) will be considered

$$u_{tt} - u_{xx} + \epsilon\left(-u_t + \frac{1}{3}u_t^3\right) = 0, \qquad 0 < x < \pi,\ t > 0, \qquad (1.5.1)$$

$$u(x,0) = u_0(x) \equiv a_n \sin nx, \qquad 0 < x < \pi, \qquad (1.5.2)$$

$$u_t(x,0) = u_1(x) \equiv b_n \sin nx, \qquad 0 < x < \pi, \qquad (1.5.3)$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \geq 0, \qquad (1.5.4)$$

where a_n and b_n are constants, n an integer and $0 < \epsilon << 1$. From theorem 1.2.1 it follows that this initial-boundary value problem is well-posed on J_L (given by (1.2.13)). In [4] a similar initial-boundary value problem has been considered with $n = 1$, $a_n = 2$ and $b_n = 0$. However, in that paper the asymptotic validity of the formal approximation has not been given. In this section for arbitrary n, a_n and b_n an asymptotic approximation (as $\epsilon \to 0$) of the solution of (1.5.1)-(1.5.4) will be constructed. In view of computational difficulties (as has been noticed in [18]) whenever one assumes an infinite series representation for the solution of the nonlinear initial-boundary value problem, one may alternatively investigate the problem in the characteristic coordinates $\sigma = x - t$ and $\xi = x + t$. In this approach the initial-boundary value problem (1.5.1)-(1.5.4) is replaced by an initial value problem. This replacement requires to extend the dependent variable u(x,t) as well as the initial values $u_0(x)$ and $u_1(x)$ in x to odd and 2π-periodic functions. For simplicity the extended functions will be denoted by the same symbols. In constructing an approximation of the solution $u(x,t) = \tilde{u}(\sigma,\xi)$ of this initial value problem a two-timescales perturbation method will be used, since the straightforward perturbation expansion $\tilde{u}_0(\sigma,\xi) + \epsilon\tilde{u}_1(\sigma,\xi) + \ldots$ causes secular terms. Applying the two-timescales perturbation method $u(x,t)$ is supposed to be a function of $\sigma = x - t$, $\xi = x + t$ and $\tau = \epsilon t$. By putting $u(x,t) \equiv v(\sigma,\xi,\tau)$ the following initial value problem for v is obtained

$$-4v_{\sigma\xi} + 2\epsilon(v_{\xi\tau} - v_{\sigma\tau}) + \epsilon^2 v_{\tau\tau} + \epsilon\left(v_\sigma - v_\xi - \epsilon v_\tau + \frac{1}{3}(-v_\sigma + v_\xi + \epsilon v_\tau)^3\right) = 0,$$

$$\text{for } -\infty < \sigma < \xi < \infty,\ \tau > 0, \quad (1.5.5)$$

$$v(\sigma,\xi,\tau) = u_0(\sigma) = a_n \sin n\sigma, \qquad \text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0, \quad (1.5.6)$$

$$-v_\sigma(\sigma,\xi,\tau) + v_\xi(\sigma,\xi,\tau) + \epsilon v_\tau(\sigma,\xi,\tau) = u_1(\sigma) = b_n \sin n\sigma,$$

$$\text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0. \quad (1.5.7)$$

Furthermore, it is assumed that v may be approximated by the formal perturbation expansion $v_0(\sigma,\xi,\tau) + \epsilon v_1(\sigma,\xi,\tau) + \epsilon^2 v_2(\sigma,\xi,\tau) + \ldots$. By substituting this approximation into (1.5.5)-(1.5.7), and after equating the coefficients of like powers in ϵ, it follows from the powers 0 and 1 of ϵ that v_0 should satisfy

$$-4v_{0_{\sigma\xi}} = 0, \qquad -\infty < \sigma < \xi < \infty,\ \tau > 0, \quad (1.5.8)$$

$$v_0(\sigma,\xi,\tau) = u_0(\sigma) = a_n \sin n\sigma, \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \quad (1.5.9)$$

$$-v_{0_\sigma}(\sigma,\xi,\tau) + v_{0_\xi}(\sigma,\xi,\tau) = u_1(\sigma) = b_n \sin n\sigma, \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \quad (1.5.10)$$

and that v_1 should satisfy

$$-4v_{1_{\sigma\xi}} = 2v_{0_{\sigma\tau}} - 2v_{0_{\xi\tau}} - \left(v_{0_\sigma} - v_{0_\xi} + \frac{1}{3}(-v_{0_\sigma} + v_{0_\xi})^3\right)$$

$$\text{for } -\infty < \sigma < \xi < \infty,\ \tau > 0, \quad (1.5.11)$$

$$v_1(\sigma,\xi,\tau) = 0, \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \quad (1.5.12)$$

$$-v_{1_\sigma}(\sigma,\xi,\tau) + v_{1_\xi}(\sigma,\xi,\tau) = -v_{0_\tau}(\sigma,\xi,\tau), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0. \quad (1.5.13)$$

In the further analysis v_0 and v_1 will be determined, and it will be shown that on J_L

$\bar{u}(x,t) \equiv v_0(x-t,x+t,\epsilon t) + \epsilon v_1(x-t,x+t,\epsilon t)$ is an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $u(x,t)$ of the initial-boundary value problem (1.5.1)-(1.5.4).

The general solution of the partial differential equation (1.5.8) is given by $v_0(\sigma,\xi,\tau) = f_0(\sigma,\tau) + g_0(\xi,\tau)$. The initial values (1.5.9) and (1.5.10) imply that f_0 and g_0 have to satisfy $f_0(\sigma,0) + g_0(\sigma,0) = u_0(\sigma)$ and $-f_0'(\sigma,0) + g_0'(\sigma,0) = u_1(\sigma)$, where the prime denotes differentiation with respect to the first argument. From the odd and 2π-periodic extension of the dependent variable of problem (1.5.1)-(1.5.4) it follows that f_0 and g_0 also have to satisfy $g_0(\sigma,\tau) = -f_0(-\sigma,\tau)$ and $f_0(\sigma,\tau) = f_0(\sigma+2\pi,\tau)$ for $-\infty < \sigma < \infty$ and $\tau \geq 0$. The undetermined behaviour of f_0 with respect to τ will be used to avoid secular terms in v_1. From the well-posedness theorem it followed that u, u_t and u_x are $O(1)$ on J_L. So, v and its first derivatives have to remain $O(1)$ on $-\infty < x < \infty$ and $0 \leq t \leq L|\epsilon|^{-1}$. Furthermore, it should be noticed that the equations for v_0 and v_1 have been derived under the assumption that v_0, v_1 and their derivatives up to order two are $O(1)$. These boundedness conditions on v_0 and v_1 determine the behaviour of f_0 with respect to τ. From (1.5.11)-(1.5.13) v_{1_σ} and v_{1_ξ} may be obtained easily. For instance,

$$\begin{aligned}-4v_{1_\sigma}(\sigma,\xi,\tau) &= -4v_{1_\sigma}(\sigma,\sigma,\tau) + (\xi - \sigma)\left(2f_{0_{\sigma\tau}}(\sigma,\tau) - f_{0_\sigma}(\sigma,\tau) + \frac{1}{3} f_{0_\sigma}^3(\sigma,\tau)\right) + \\ &+ f_{0_\sigma}(\sigma,\tau) \int_\sigma^\xi g_{0_\theta}^2(\theta,\tau)d\theta + \int_\sigma^\xi \Big\{-2g_{0_{\theta\tau}}(\theta,\tau) + g_{0_\theta}(\theta,\tau) - f_{0_\sigma}^2(\sigma,\tau)g_{0_\theta}(\theta,\tau) + \\ &- \frac{1}{3} g_{0_\theta}^3(\theta,\tau)\Big\} d\theta + h(\sigma,\tau),\end{aligned} \tag{1.5.14}$$

where h will be determined later on. Since the first integral in (1.5.14) contains a non-negative and 2π-periodic integrand it follows that this integral will grow with the length $\xi - \sigma$ of the integration interval. It turns out that this integral can be written in a part which is $O(1)$ for all values of σ and ξ and in a part which is linear in $\xi - \sigma$.

$$\int_\sigma^\xi g_{0_\theta}^2(\theta,\tau)\,d\theta = \int_\sigma^\xi \left\{ g_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi} \int_0^{2\pi} g_{0_\psi}^2(\psi,\tau)\,d\psi \right\} d\theta \; +$$

$$+ \; \frac{\xi - \sigma}{2\pi} \int_0^{2\pi} g_{0_\psi}^2(\psi,\tau)\,d\psi.$$

Noticing that $\xi - \sigma = 2t$ it follows that $\xi - \sigma$ is of $O(|\epsilon|^{-1})$ on a time-scale of $O(|\epsilon|^{-1})$. So, v_{1_σ} will be of $O(|\epsilon|^{-1})$ unless f_0 and g_0 are such that in (1.5.14) the terms of $O(|\epsilon|^{-1})$ (that is, terms linear in $\xi - \sigma$) disappear. It turns out that both v_{1_σ} and v_{1_ξ} are $O(1)$ on a timescale of $O(|\epsilon|^{-1})$ if f_0 and g_0 satisfy the following two conditions

$$2f_{0_{\sigma\tau}} - f_{0_\sigma} + \frac{1}{3} f_{0_\sigma}^3 + f_{0_\sigma} \frac{1}{2\pi} \int_0^{2\pi} g_{0_\theta}^2(\theta,\tau)\,d\theta = 0, \text{ and}$$

$$-2g_{0_{\xi\tau}} + g_{0_\xi} - \frac{1}{3} g_{0_\xi}^3 - g_{0_\xi} \frac{1}{2\pi} \int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)\,d\theta = 0.$$

From $g_0(\theta,\tau) = -f_0(-\theta,\tau)$ it follows that these two conditions are equivalent. So, v_{1_σ} and v_{1_ξ} are both $O(1)$ on a time-scale of $O(|\epsilon|^{-1})$ if f_0 satisfies

$$2f_{0_{\sigma\tau}} - f_{0_\sigma} + \frac{1}{3} f_{0_\sigma}^3 + f_{0_\sigma} \frac{1}{2\pi} \int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)\,d\theta = 0. \tag{1.5.15}$$

In [4] an equation similar to equation (1.5.15) has been solved. If the method introduced in [4] is applied to equation (1.5.15) one obtains after some calculations $f_0(\sigma,\tau)$, and so $v_0(\sigma,\xi,\tau) = f_0(\sigma,\tau) - f_0(-\xi,\tau)$. It turns out that f_0 and v_0 are given by

$$f_0(\sigma,\tau) = \frac{\lambda(\tau)}{n\phi^{1/2}(\tau)} \arcsin\left[\left[\frac{c_n\phi(\tau)}{1 + c_n\phi(\tau)}\right]^{1/2} \sin(\alpha + n\sigma) \right] + k(\tau), \tag{1.5.16}$$

$$v_0(\sigma,\xi,\tau) = \frac{\lambda(\tau)}{n\phi^{1/2}(\tau)} \left\{ \arcsin\left[\left[\frac{c_n\phi(\tau)}{1 + c_n\phi(\tau)}\right]^{1/2} \sin(\alpha + n\sigma) \right] + \right.$$

$$\left. - \arcsin\left[\left[\frac{c_n\phi(\tau)}{1 + c_n\phi(\tau)}\right]^{1/2} \sin(\alpha - n\xi) \right] \right\}, \tag{1.5.17}$$

where $k(\tau)$ is an arbitrary function in τ with $k(0) = 0$, $\sigma = x - t$, $\xi = x + t$, $\tau = \epsilon t$, $c_n = n^2a_n^2 + b_n^2$, α given by $\cos\alpha = na_nc_n^{-1/2}$ and $\sin\alpha = b_nc_n^{-1/2}$, and $\lambda(\tau)$ and $\phi(\tau)$ are implicitly given by $\lambda(\tau) = 4e^{\tau/2}m^{-3}(\tau)$ and $\phi(\tau) = \frac{m(\tau)}{c_n}(m(\tau) - 2)$ with $m(\tau)$ determined by $m^8(\tau) - \frac{8}{7}m^7(\tau) = \frac{2^6c_n}{3}(e^\tau - 1) + \frac{3.2^8}{7}$.

Now the linear initial value problem (1.5.11)-(1.5.13) can be solved, and it turns out that v_1 is given by

$$v_1(\sigma,\xi,\tau) = \frac{1}{4}\left(f_0(\sigma,\tau) - f_0(-\xi,\tau)\right) \int_\sigma^\xi \left\{ f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)\,d\psi \right\} d\theta +$$

$$- \frac{1}{4}\int_\sigma^\xi \left\{ f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)\,d\psi \right\} \left(f_0(\theta,0) - f_0(-\theta,0)\right) d\theta +$$

$$+ f_1(\sigma,\tau) + g_1(\xi,\tau), \tag{1.5.18}$$

where f_0 is given by (1.5.16) and where (for $\sigma = \xi$ and $\tau = 0$) $f_1 + g_1$ is determined by the initial values (1.5.12) and (1.5.13). The undetermined behaviour of f_1 and g_1 with respect to τ can be used to avoid secular terms in v_2. However, in this analysis v_2 will not be determined. For that reason it may be assumed that $f_1 = f_1(\sigma)$ and $g_1 = g_1(\xi)$, and then

$$f_1(\sigma) + g_1(\xi) = -\frac{1}{2}\int_\sigma^\xi (f_{0_\tau}(\theta,0) + g_{0_\tau}(\theta,0))d\theta =$$

$$= \frac{b_n}{n^2}\left\{\frac{(3c_n - 2^4)}{2^5}(\sin(n\xi) - \sin(n\sigma)) + \frac{(3n^2a_n^2 - b_n^2)}{3^3.2^5}(\sin(3n\xi) - \sin(3n\sigma))\right\}.$$

It can be shown from (1.5.17) and (1.5.18) that v_0, v_1 and their derivatives up to order two are of $O(1)$ on J_L. So, the assumptions under which the equations for v_0 and v_1 have been derived, are justified. So far a function $v_0(\sigma,\xi,\tau) + \epsilon v_1(\sigma,\xi,\tau) \equiv \bar{v}(\sigma,\xi,\tau) =$ $= \bar{v}(x-t,x+t,\epsilon t) \equiv \bar{u}(x,t)$ has been constructed. It can easily be seen that $\bar{u}(x,t)$ satisfies (1.5.2) and (1.5.4) exactly, and (1.5.3) up to order ϵ^2 in the sense of theorem 1.3.1. After rather lengthy, but elementary calculations it can also be shown that $\bar{u}(x,t)$ satisfies (1.5.1) up to $\epsilon^2 c_1(x,t;\epsilon)$, where c_1, $\frac{\partial c_1}{\partial x} \in C([0,\pi] \times [0,L|\epsilon|^{-1}] \times [-\epsilon_0,\epsilon_0], \mathbb{R})$ with $c_1(0,t;\epsilon) =$ $= c_1(\pi,t;\epsilon) = 0$ for $0 \le t \le L|\epsilon|^{-1}$. Furthermore, c_1 and $\frac{\partial c_1}{\partial x}$ are uniformly bounded in ϵ. Then it follows from theorem 1.3.1 that $\bar{u}(x,t)$ is an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution of the initial-boundary value problem (1.5.1)-(1.5.4) for $(x,t) \in J_L$, that is $\|u-\bar{u}\|_{J_L} = O(\epsilon)$. From this estimate the following estimate can be obtained

$$\|u-v_0\|_{J_L} = \|u-\bar{u}+\bar{u}-v_0\|_{J_L} \le \|u-\bar{u}\|_{J_L} + \|\epsilon v_1\|_{J_L} = O(\epsilon).$$

Hence, $v_0(x-t,x+t,\epsilon t)$ given by (1.5.17) is also an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $u(x,t)$ of problem (1.5.1)-(1.5.4) for $0 \le x \le \pi$ and $0 \le t \le L\epsilon^{-1}$, in which L is an ϵ-independent, positive constant.

1.6. Some general remarks

The asymptotic theory presented in this chapter may directly be applied to initial value problems for weakly nonlinear wave equations. The well-posedness of these problems on the infinite domain $-\infty < x < \infty$ and the asymptotic validity of formal approximations may be established on a time-scale of order $|\epsilon|^{-1/2}$. This time-scale follows from the integration over the characteristic triangle (with an area of $O(t^2)$) in the integral equation, which is equivalent to the initial value problem. In some special cases (for instance, if (1.5.1)-(1.5.3) is considered as an initial value problem on $-\infty < x < \infty$) a time-scale of $O(|\epsilon|^{-1})$ can be obtained. However, the question remains open if for general initial value problems (that is, problems like (1.2.1)-(1.2.3) on $-\infty < x < \infty$) the well-posedness in the classical sense can be established on a time-scale of $O(|\epsilon|^{-1})$. To obtain such a time-scale one has most likely to use a different function space, perhaps a suitable Sobolev space.

In [4,18] formal approximations of the solutions of a number of initial value and initial-boundary value problems for weakly nonlinear wave equations have been constructed. In those references the asymptotic validity of the formal approximations has not been investigated. However, the asymptotic theory presented in this paper can be used successfully to justify those results, that is, estimates of the differences between the exact solutions and the formal approximations can be given on ϵ-dependent time-scales. It is also interesting to mention that only smoothness conditions are required (see (1.2.6)-(1.2.8)) and that no other assumptions are made about the nonlinear perturbation term F. Thus, the asymptotic theory presented in this chapter is applicable to those initial-boundary value problems whose solutions, while being bounded at times of $O(|\epsilon|^{-1})$, could eventually become unbounded. Such, for example, is the case for the initial-boundary value problem (1.2.1)-(1.2.4) with $F \equiv -u_t^3$ and $0 < \epsilon << 1$.

In section 1.3 the condition $m > 1$ is introduced. It should be noted that this condition for the asymptotic validity of formal approximations on a time-scale of $O(|\epsilon|^{-1})$ is a sufficient, but not a necessary one as can be seen from section 1.5. The asymptotic approximation v_0 (which is valid on a time-scale of $O(|\epsilon|^{-1})$) satisfies the partial differential equation and the initial values up to order ϵ, that is $m = 1$. It may be remarked that $u_\ell(x,t) = \left(a_n \cos nt + \frac{b_n}{n} \sin nt\right) \sin nx$, which is the solution of the linear initial-boundary value problem (1.5.1)-(1.5.4) (that is, (1.5.1) with $\epsilon = 0$), also satisfies the weakly nonlinear partial differential equation and the initial values up to order ϵ. In general u_ℓ will not approximate the exact solution of the nonlinear initial-boundary value problem on a time-scale of $O(|\epsilon|^{-1})$. However, on a smaller time-scale the asymptotic validity of u_ℓ can easily be established, that is, it can be shown using the methods discussed in sections 1.2 and 1.3 that $|u(x,t) - u_\ell(x,t)| \leq |\epsilon| Mt$, where M is a constant independent of ϵ. This inequality implies $u(x,t) = u_\ell(x,t) + O(|\epsilon|^{1-\alpha})$ for $0 \leq x \leq \pi$ and $0 \leq t \leq L|\epsilon|^{-\alpha}$ with $0 \leq \alpha < 1$. From the asymptotic validity of u_ℓ on a time-scale of order $|\epsilon|^{-\alpha}$ with $0 \leq \alpha < 1$ it follows that whenever one wants to study the effect of the small (ϵ-dependent) and nonlinear terms in the partial differential equation, one has to construct approximations with a validity on a time-scale of order $|\epsilon|^{-1}$.

In a number of papers [4,20,22,23] initial value and initial-boundary value problems for the Rayleigh wave equation have been studied by constructing formal approximations of the solutions or by deriving some properties of the approximations for large times. An interesting result (without an asymptotic justification) has been found in [20]. For a rather general class of initial values it has been shown in [20] that the first order approximation tends to a superposition of standing triangular waves as $\epsilon t \to \infty$. How the solution tends to these standing triangular waves can be determined by solving a nonlinear integro-differential equation. It is not made clear in [20] how to solve the integro-differential equation, but it is the author's opinion based upon the results in this chapter and in [4] that only for a restricted class of initial values, such as (1.5.2) and (1.5.3), this equation may be solved analytically.

As can be seen from (1.5.17) v_0 also tends to a standing triangular wave (with amplitude $\frac{\pi}{2n}\sqrt{3}$ and period $\frac{2\pi}{n}$) as $\epsilon t \to \infty$. However, it should be emphasized that nothing can be said about the asymptotic validity of v_0 as $\epsilon t \to \infty$, since only for finite ϵt (that is, $0 \leq |\epsilon t| \leq L < \infty$) the asymptotic validity of v_0 could be established.

In [9] it was concluded from the behaviour of the first order approximation as $\epsilon t \to \infty$ that the Rayleigh wave equation (postulated in [22]) is not a good model for galloping oscillations, since the approximation allows at least one and possibly infinitely many sharp bends. From a mathematical point of view the validity of this conclusion is rather doubtful since only for finite ϵt, that is for $|\epsilon t| \leq L < \infty$, the asymptotic validity of the results has been obtained so far. And on this finite time-scale the solution and the asymptotic approximations are at least two times continuously differentiable with respect to the independent variables if the initial values are sufficiently smooth.

In section 1.5 monochromatic initial values have been considered which applies to the description of galloping oscillations, because these oscillations often affect only a single mode of vibration. To obtain some information about the maximum oscillation-amplitudes, the following formula may be used

$$w(x,t) = \frac{\rho_c g}{2T}\, x(x - \ell) + \left(\frac{a}{3b}\right)^{1/2} \frac{\ell v_\infty}{\pi c}\, u\left(\frac{\pi}{\ell}\, x, \frac{\pi c}{\ell}\, t\right),$$

where $w(x,t)$, ρ_c, g, T, a, b, ℓ, v_∞, c and $u\left(\frac{\pi}{\ell}\, x, \frac{\pi c}{\ell}\, t\right)$ are defined as in section 1.4. The first term in this formula may be considered as the position of the conductor in rest, whereas the second term represents the change of the position of the conductor due to galloping. From (1.5.39) it follows that the maximum amplitude of $u\left(\frac{\pi}{\ell}\, x, \frac{\pi c}{\ell}\, t\right)$ for $\epsilon t \to \infty$ is $\frac{\pi}{2n}\sqrt{3}$. So, the maximum oscillation-amplitude of $w(x,t)$ may be approximated by

$$\left[\frac{-c_{D0} - c_{L1}}{\frac{1}{2} c_{D0} + \frac{1}{6} c_{L1} + c_{L3}}\right]^{1/2} \frac{\pi v_\infty}{2c} \frac{\ell}{n\pi},$$

where $\frac{\ell}{n\pi}$ is the frequency of the monochromatic initial values and where c_{D0}, c_{L1} and c_{L3} are the aerodynamic coefficients, which may be obtained from wind-tunnel measurements.

Finally, it should be noted that the two-timescales perturbation method is applicable to perturbations not solely depending on derivatives of the dependent variable, but also applicable to perturbations depending in a special way on the dependent variable and its derivatives. Consider for instance the initial-boundary value problem (1.2.1)-(1.2.4) with $f(x,t,u,u_t,u_x;\epsilon) \equiv (-1 + u^2)u_t$. The partial differential equation (1.2.1) can then be considered as a generalized Van der Pol equation. As is well-known this equation is related to the Rayleigh wave equation, which has been introduced in section 1.4 and treated in section 1.5. Again a two-timescales perturbation method can be used to construct an asymptotic approximation of the solution. The equation for $f_0(\sigma,\tau)$ now becomes

$$2f_{0\sigma\tau} - f_{0\sigma} + \frac{1}{3} f_0^2 f_{0\sigma} + f_{0\sigma} \frac{1}{2\pi} \int_0^{2\pi} f_0^2(\theta,\tau)\, d\theta = 0,$$

which can be integrated with respect to σ. As in section 1.5 an order ϵ asymptotic approximation can be constructed on a time-scale of order $|\epsilon|^{-1}$.

CHAPTER 2

ON INITIAL-BOUNDARY VALUE PROBLEMS FOR WEAKLY SEMI-LINEAR TELEGRAPH EQUATIONS. ASYMPTOTIC THEORY AND APPLICATION*)

Abstract

In this chapter an asymptotic theory for a class of initial-boundary value problems for weakly semi-linear telegraph equations is presented.
The theory implies the well-posedness of the problem and the validity of formal approximations on long time-scales. As an application of the theory an initial-boundary value problem for the equation $u_{tt} - u_{xx} + u + \epsilon u^3 = 0$ is considered. To construct an $O(\epsilon)$ approximation of the solution of this problem a two-timescales perturbation method is applied.

*) This chapter is a revised version of a paper [11] by the author of this thesis and A.H.P. van der Burgh.

2.1. Introduction

In this chapter an asymptotic theory is presented for the following initial-boundary value problem for a nonlinearly perturbed telegraph equation:

$$u_{tt} - u_{xx} + u + \epsilon F(x,t,u;\epsilon) = 0, \qquad 0 < x < \pi,\ t > 0, \tag{2.1.1}$$

$$u(x,0) = u_0(x;\epsilon) \quad \text{and} \quad u_t(x,0) = u_1(x;\epsilon), \qquad 0 < x < \pi, \tag{2.1.2}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \geq 0, \tag{2.1.3}$$

where $0 < |\epsilon| \leq \epsilon_0 << 1$ and where the functions F, u_0 and u_1 have to satisfy certain smoothness properties, which are mentioned in section 2.2.

The main problems which are studied in this asymptotic theory are the well-posedness of the problem and the asymptotic validity of formal approximations. The classical question of the well-posedness of a problem involving a small parameter has from asymptotical point of view an interesting aspect, which has been studied only in recent years.

In order to make clear what is meant by an interesting aspect, consider problem (2.1.1)-(2.1.3) with $\epsilon = 0$ and $\epsilon = 1$. For $\epsilon = 0$ it is easy to prove existence and uniqueness of the classical solution, that is a solution which is two times continuously differentiable with respect to x and t, on the semi-infinite strip $0 \leq x \leq \pi$ and $t \geq 0$. However, when $\epsilon = 1$ only a local theory may be given which states that a unique solution exists for $0 \leq x \leq \pi$ and $0 \leq t \leq T = O(1)$. It can be shown that when $\epsilon \in [-\epsilon_0,\epsilon_0]$ $T = T(\epsilon)$ where $T(\epsilon) \to \infty$ for $\epsilon \to 0$. Now the conjecture in the literature [6,20,27] is that $T = O(|\epsilon|^{-1})$. The conjecture is based on the assumption that u(x,t) may be expanded in eigenfunctions: $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$ where $u_n(t)$ $(n = 1,2,...)$ are the solutions of an initial value problem for a system of an infinite number of ordinary differential equations. As is well-known an initial-value problem for an analogous, finite-dimensional system of ordinary differential equations has a unique solution on a time-scale of $O(|\epsilon|^{-1})$. However, it is by no means clear that from this fact the existence of a unique classical solution for problem (2.1.1)-(2.1.3) follows on the time-scale of $O(|\epsilon|^{-1})$. Another crucial step in this approach, which is not mentioned in the literature, is to show that the infinite series, which is supposed to represent the solution of problem (2.1.1)-(2.1.3), converges uniformly on a time-scale of $O(|\epsilon|^{-1})$. From this it may be concluded that the proof of the conjecture still has to be given. In this chapter a weaker result namely that $T = O(|\epsilon|^{-1/2})$ will be proved. Extending the initial-boundary value problem to an initial value problem, this proof may be given by applying Banach's fixed point theorem to the (with the initial value problem) equivalent integral equation, which involves as kernel a uniformly bounded Bessel function of the first kind and order zero. The $O(|\epsilon|^{-1/2})$ time-scale is a consequence of the integration over a triangle-shaped region with an area of $O(t^2)$. In this chapter reflection principles are applied such that the integration over the aforementioned triangle can be reduced to an integration over a strip with an area of O(t). However, the integral equation obtained in this way involves an O(t) kernel, which may be identified as a Green's function. It may be concluded that if one wishes to obtain an $O(|\epsilon|^{-1})$ estimate for the time-scale one has probably to use a different technique and one has most likely to introduce supplementary conditions on the nonlinear term F. It looks like that this has been done recently in [19]. In this preprint some results are given on the existence and uniqueness of solutions for initial-boundary value problems, related to the type of problems discussed in this chapter. If the nonlinear term F satisfies some supplementary conditions the author of the preprint claims that in a suitable Sobolev space existence and uniqueness can be established on an $O(|\epsilon|^{-1})$ time-scale.

The remarks made above on the problem of the estimate of the time-scale also apply to the domain where continuous dependence of the solution on the initial values has to be established. One may roughly say that in proving continuous dependence of the solution on the initial values one has to show that small μ perturbations of the initial values correspond to small δ perturbations in the solution on the $O(|\epsilon|^{-1/2})$ time-scale where $\delta \to 0$ when $\mu \to 0$. This problem is closely related to the problem of showing that formal approximations $\bar{u}(x,t)$, which are functions which satisfy the partial differential equation and the initial-boundary conditions up to some order depending on the small parameter ϵ, are indeed asymptotic approximations of the solution of problem (2.1.1)-(2.1.3), that is, on the $O(|\epsilon|^{-1/2})$ time-scale

$$|u(x,t) - \bar{u}(x,t)| = O(\delta), \quad \text{where } \delta \to 0 \text{ for } \epsilon \to 0.$$

An interesting problem to study in asymptotics is the determination of the so-called order function $\delta = \delta(\epsilon)$. In the theory of ordinary differential equations these problems have been studied extensively (for instance in [2], one of the first papers dealing with these problems, and in [25] where a fairly complete review including references is given). However, in the theory of partial differential equation of the evolution type only little is known. In [3,8] some results are given for initial value problems and in [4,17,18,20] the problems outlined above are mentioned but not solved.

This chapter being an attempt to contribute to the questions outlined above, is organized as follows. In section 2.2 the well-posedness of the problem is investigated and established on the $O(|\epsilon|^{-1/2})$ time-scale. In section 2.3 the asymptotic validity of formal approximations is studied. It is remarkable that an estimate is obtained by a technique based on the use of an auxiliary function and an integral inequality. This technique was introduced in [2] for ordinary differential equations and applied to initial value problems for evolution equations in [3].

In section 2.4 the asymptotic theory is applied to the special example $F(x,t,u;\epsilon) \equiv u^3$. As a method to construct a formal approximation, which is also an asymptotic approximation, a two-timescales perturbation method is used. Application of this method yields an initial value problem for a system of infinitely many nonlinear first order ordinary differential equations. In [17] for the case that $F \equiv u_t - \frac{1}{3} u_t^3$ a similar infinite system has been solved exactly, which is judged to be rare in [18]. In this chapter the obtained infinite system for the case $F \equiv u^3$ is solved exactly, which also requires rather tedious calculations. On the basis of this result and the results obtained in [6] and [17] one may conclude that the applicability of the method is not restricted to some special cases, but applies to a general class of perturbations F with a polynomial structure in the dependent variable. Finally in section 2.5 some concluding remarks are made on the results obtained in this chapter.

Preliminary studies show that the asymptotic theory so far established can be extended to perturbations F which depend on derivatives of the dependent variable. This extension includes initial value and initial-boundary value problems for the weakly nonlinear telegraph and wave equations.

2.2. The well-posedness of the problem

In this chapter the following weakly semi-linear initial-boundary value problem for a twice continuously differentiable function u(x,t) is considered.

$$u_{tt} - u_{xx} + u + \epsilon F(x,t,u;\epsilon) = 0, \qquad 0 < x < \pi,\ t > 0, \tag{2.2.1}$$

$$u(x,0) = u_0(x;\epsilon), \qquad 0 < x < \pi, \tag{2.2.2}$$

$$u_t(x,0) = u_1(x;\epsilon), \qquad 0 < x < \pi, \tag{2.2.3}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \geq 0, \tag{2.2.4}$$

with $0 < |\epsilon| \leq \epsilon_0 << 1$, and (2.2.5)

where F, u_0 and u_1 satisfy:

$$F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial u} \in C([0,\pi]\times[0,\infty)\times\mathbb{R}\times[-\epsilon_0,\epsilon_0],\mathbb{R})$$

with $F(0,t,0;\epsilon) = F(\pi,t,0;\epsilon) = 0 \quad$ for $t \geq 0$, (2.2.6)

$$u_0, \frac{\partial u_0}{\partial x}, \frac{\partial^2 u_0}{\partial x^2} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R})$$

with $u_0(0;\epsilon) = u_0(\pi;\epsilon) = u_0''(0;\epsilon) = u_0''(\pi;\epsilon) = 0$, and

$$u_1, \frac{\partial u_1}{\partial x} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0], \mathbb{R}) \text{ with } u_1(0;\epsilon) = u_1(\pi;\epsilon) = 0. \tag{2.2.8}$$

Furthermore, F is assumed to be uniformly bounded for those values of t under consideration. (2.2.9)

In order to prove existence and uniqueness in the classical sense of the solution of the initial-boundary value problem (2.2.1)-(2.2.4) an equivalent integral equation will be used, which has been derived in appendix 2A.

This integral equation is given by

$$u(x,t) = \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)F(\xi,\tau,u(\xi,\tau);\epsilon)d\xi d\tau +$$

$$+ \int_0^\pi \left\{u_0(\xi;\epsilon)G_\tau(\xi,0;x,t) - u_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi \equiv (Tu)(x,t), \qquad (2.2.10)$$

where the Green's function G is given by:

$$G(\xi,\tau;x,t) = \frac{1}{2} \sum_{k\in\mathbb{Z}} \left\{H(t-\tau-\xi+2k\pi-x)H(t-\tau+\xi-2k\pi+x)J_0([(t-\tau)^2 - (\xi-2k\pi+x)^2]^{1/2}) + \right.$$

$$\left. - H(t-\tau+\xi+2k\pi-x)H(t-\tau-\xi-2k\pi+x)J_0([(t-\tau)^2 - (\xi+2k\pi-x)^2]^{1/2})\right\}, \qquad (2.2.11)$$

in which J_0 is the Bessel function of the first kind of order zero and in which H(a) is a step function which is equal to 1 for a > 0, $\frac{1}{2}$ for a = 0 and zero otherwise. From the integral equation (2.2.10) it follows that the solution u_ℓ of the linear initial-boundary value problem (2.2.1)-(2.2.4) (that is with $F \equiv 0$) is given by

$$u_\ell(x,t) = \int_0^\pi \left\{u_0(\xi;\epsilon)G_\tau(\xi,0;x,t) - u_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi. \qquad (2.2.12)$$

In the further analysis the abbreviation $u_\ell(x,t)$ will be used.

In proving existence and uniqueness of the solution of the nonlinear integral equation (2.2.10) a fixed point theorem due to Banach-Caccioppoli will be used. Let

$$D_L = \left\{(x,t) | 0 \le x \le \pi,\ 0 \le t \le L|\epsilon|^{-1/2}\right\}, \qquad (2.2.13)$$

in which L is a sufficiently small, positive constant independent of ϵ.

Let $C_M(D_L)$ be the space of all real-valued continuous functions w on D_L with norm $\| \cdot \|_{D_L}$ defined by

$$\| w \|_{D_L} = \max_{(x,t)\in D_L} |w(x,t)| \le M.$$

It is not difficult to show that T maps $C_M(D_L)$ into the space of continuous functions on D_L. In order to prove that T maps $C_M(D_L)$ into itself an estimate of the Green's function $G(\xi,\tau;x,t)$ should be obtained for $0 \le \xi \le \pi$, $0 \le \tau \le t$ and fixed x and t. In figure 2.2.1 some properties of the Green's function $G(\xi,\tau;x,t)$ are given in the (ξ,τ)-plane. In this figure the characteristics from the point (x,t) and the reflected characteristics at the boundaries $\xi = 0$ and $\xi = \pi$ are drawn. These (reflected) characteristics divide the region $V = \left\{(\xi,\tau) | 0 \le \xi \le \pi,\ \tau \ge 0\right\}$ into a finite number of subregions. In each subregion only a finite number of Bessel functions is defined by definition (2.2.11) of $G(\xi,\tau;x,t)$. From this definition it follows that $G(\xi,\tau;x,t) = 0$ in V if $t-\tau+\xi-x < 0$ or $t-\tau-\xi+x < 0$. In figure 2.2.1 also the number of Bessel functions in each subregion of V is given.

From the fact that the Bessel function of the first kind of order zero is not periodic and is equal to one if its argument is equal to zero, and from (2.2.11) it follows that an estimate of $G(\xi,\tau;x,t)$ can be made which is linear in t. This estimate is based upon the number of Bessel functions in each subregion. For $0 \le \xi \le \pi$ and $0 \le \tau \le t$ it then follows that

$$\begin{aligned} |G(\xi,\tau;x,t)| &\le \frac{1}{2} (\text{'number of Bessel functions in each subregion'}) \le \\ &\le \frac{1}{2}\cdot\frac{2}{\pi} (t-\tau+\pi) = \frac{1}{\pi} (t-\tau+\pi). \end{aligned} \tag{2.2.14}$$

It should be noted that no sharper estimate of $G(\xi,\tau;x,t)$ could be made so far. Nevertheless, the question remains open if one can establish an estimate sharper than an estimate

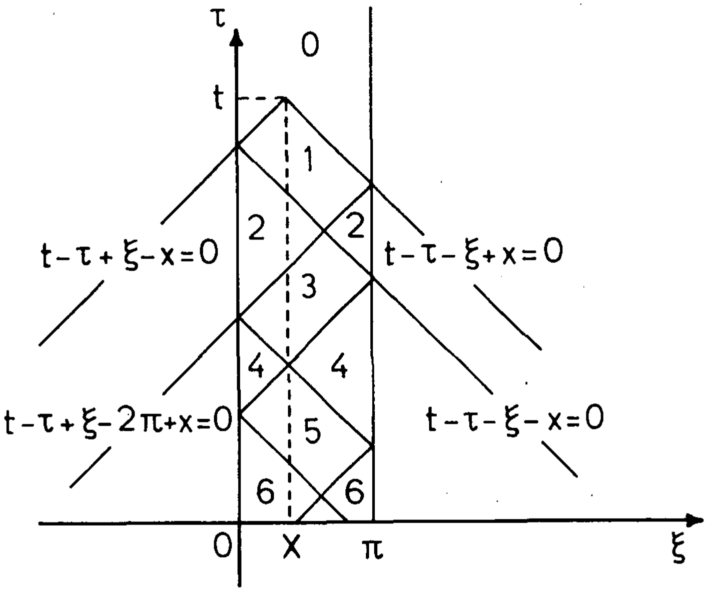

Figure 2.2.1.

linear in t.

Since u_0 and u_1 are sufficiently smooth and uniformly bounded for those values of x and ϵ under consideration, there exists a constant M_1 independent of ϵ such that

$$\| u_\ell \|_{D_L} \leq \frac{1}{2} M_1. \tag{2.2.15}$$

From the smoothness properties of F it follows that there are constants M_2 and M_3 both independent of ϵ such that

$$|F(x,t,w;\epsilon)| \leq M_2 \text{ and} \tag{2.2.16}$$

$$|F(x,t,w_1;\epsilon) - F(x,t,w_2;\epsilon)| \leq M_3 \| w_1 - w_2 \|_{D_L} \tag{2.2.17}$$

for all $(x,t) \in D_L$, $\epsilon \in [-\epsilon_0,\epsilon_0]$ and w, w_1 and $w_2 \in C_{M_1}(D_L)$.

Using (2.2.10), (2.2.13), (2.2.14), (2.2.15) and (2.2.16) it can be shown that T maps $C_{M_1}(D_L)$ into itself.

$$|(Tw)(x,t)| \leq \left|\epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)F(\xi,\tau,w(\xi,\tau);\epsilon)d\xi d\tau\right| + |u_\ell(x,t)| \leq$$

$$\leq |\epsilon|\ M_2 \int_0^t \int_0^\pi \frac{1}{\pi}(t-\tau+\pi)d\xi d\tau + \frac{1}{2} M_1 \leq$$

$$\leq |\epsilon| M_2 \left(\frac{1}{2} t^2 + \pi t\right) + \frac{1}{2} M_1 \leq M_2\left(\frac{1}{2} L^2 + \pi L|\epsilon|^{1/2}\right) + \frac{1}{2} M_1.$$

Taking the constant L such that $M_2\left(\frac{1}{2} L^2 + \pi L|\epsilon|^{1/2}\right) \leq \frac{1}{2} M_1$ it follows that

$$\| Tw \|_{D_L} \leq M_1 \qquad \text{for all } w \in C_{M_1}(D_L).$$

Hence, T maps $C_{M_1}(D_L)$ into itself. Using (2.2.10), (2.2.13), (2.2.14) and (2.2.17) it will be shown that T is a contraction on $C_{M_1}(D_L)$. Let w_1 and $w_2 \in C_{M_1}(D_L)$, then

$$\| Tw_1 - Tw_2 \|_{D_L} \leq$$

$$\leq \max_{(x,t)\in D_L} \left|\epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)\left\{F(\xi,\tau,w_1(\xi,\tau);\epsilon) - F(\xi,\tau,w_2(\xi,\tau);\epsilon)\right\} d\xi d\tau\right| \leq$$

$$\leq M_3\left(\frac{1}{2} L^2 + \pi L|\epsilon|^{1/2}\right) \| w_1 - w_2 \|_{D_L}.$$

Taking the constant L such that $M_2\left(\frac{1}{2} L^2 + \pi L|\epsilon|^{1/2}\right) \leq \frac{1}{2} M_1$ and $M_3\left(\frac{1}{2} L^2 + \pi L|\epsilon|^{1/2}\right) \leq k$ with $0 < k < 1$, it follows that

$$\| Tw_1 - Tw_2 \|_{D_L} \leq k \; \| w_1 - w_2 \|_{D_L} \quad \text{with } 0 < k < 1.$$

Then, Banach-Caccioppoli's fixed point theorem implies that T has a unique fixed point $w \in C_{M_1}(D_L)$, that is, a continuous function w on D_L satisfying the integral equation. From appendix 2A formula (2A.1) it easily follows that T: $C \rightarrow C^1$ and T: $C^1 \rightarrow C^2$. Hence, the solution u(x,t) of the integral equation (2.2.10) and so the solution of the initial-boundary value problem (2.2.1)-(2.2.9), is two times continuously differentiable on D_L. Now, it will be shown that the solution of the initial-boundary value problem depends continuously on the initial values. Let u(x,t) satisfy (2.2.1)-(2.2.4) and let $\bar{u}(x,t)$ satisfy (2.2.1), (2.2.4), $\bar{u}(x,0) = \bar{u}_0(x;\epsilon)$, $\bar{u}_t(x,0) = \bar{u}_1(x;\epsilon)$, where $\bar{u}_0$ and $\bar{u}_1$ satisfy (2.2.7) and (2.2.8). Using (2.2.10), (2.2.13), (2.2.14), (2.2.17) and assuming u and $\bar{u} \in C_{M_1}(D_L)$, one obtains

$$|u(x,t)-\bar{u}(x,t)| \leq \left| \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t) \left\{ F(\xi,\tau,u(\xi,\tau);\epsilon) - F(\xi,\tau,\bar{u}(\xi,\tau);\epsilon) \right\} d\xi d\tau \right| +$$

$$+ \| u_\ell - \bar{u}_\ell \|_{D_L} \leq |\epsilon| \; M_3(t+\pi) \int_0^t \max_{0 \leq \xi \leq \pi} |u(\xi,\tau)-\bar{u}(\xi,\tau)| d\tau + \| u_\ell - \bar{u}_\ell \|_{D_L},$$

where $\bar{u}_\ell(x,t) = \int_0^\pi \left\{ G_\tau(\xi,0;x,t)\bar{u}_0(\xi;\epsilon) - G(\xi,0;x,t)\bar{u}_1(\xi;\epsilon) \right\} d\xi.$

Since the right-hand side of the inequality is independent of x, the maximum for x on the left-hand side can be taken, then using Gronwall's lemma, and one obtains after taking the maximum for t that

$$\| u - \bar{u} \|_{D_L} \leq M_4 \; \| u_\ell - \bar{u}_\ell \|_{D_L},$$

where M_4 is a positive, bounded constant independent of ϵ. Since the solution u_ℓ of t linear initial-boundary value problem (2.2.1)-(2.2.4) with $F \equiv 0$ depends continuously on the initial values, it follows from this estimate that also the solution of the weakly non-linear initial-boundary value problem (2.2.1)-(2.2.4) depends continuously on the initial values. So, the following theorem on the well-posedness of the problem can be formulated.

Theorem 2.2.1

Suppose that F, u_0 and u_1 satisfy the assumptions (2.2.6)-(2.2.9). Then for any ϵ satisfying (2.2.5), the nonlinear initial-boundary value problem (2.2.1)-(2.2.4) and the equivalent nonlinear integral equation (2.2.10) have the same, unique and twice continuously differentiable solution for $0 \le x \le \pi$ and $0 \le t \le L|\epsilon|^{-1/2}$, in which L is a sufficiently small, positive constant independent of ϵ. Furthermore, this unique solution depends continuously on the initial values.

2.3. On the validity of formal approximations

Since the initial-boundary value problem (2.1.1)-(2.1.3) contains a small parameter ϵ perturbation methods can be applied in order to construct approximations. In most perturbation methods for weakly nonlinear problems a function is constructed that satisfies the differential equation and the initial conditions up to some order depending on the small parameter ϵ. Such a function is called a formal approximation. To show that this formal approximation is an asymptotic approximation (as $\epsilon \to 0$) requires an additional analysis. Suppose that a twice continuously differentiable function $v(x,t)$ is constructed on D_L (D_L is given by (2.2.13)) satisfying:

$$v_{tt} - v_{xx} + v + \epsilon F(x,t,v;\epsilon) = |\epsilon|^m c_1(x,t;\epsilon), \qquad 0 < x < \pi,\ t > 0,\ m > 1, \tag{2.3.1}$$

$$v(x,0) = u_0(x;\epsilon) + |\epsilon|^m c_2(x;\epsilon) = v_0(x;\epsilon), \qquad 0 < x < \pi, \tag{2.3.2}$$

$$v_t(x,0) = u_1(x;\epsilon) + |\epsilon|^m c_3(x;\epsilon) = v_1(x;\epsilon), \qquad 0 < x < \pi, \tag{2.3.3}$$

$$v(0,t) = v(\pi,t) = 0, \qquad t \geq 0, \tag{2.3.4}$$

where ϵ, F, u_0 and u_1 satisfy (2.2.5)-(2.2.9) and where c_1, c_2 and c_3 satisfy:

$$c_1,\ \frac{\partial c_1}{\partial x} \in C([0,\pi]\times[0,\infty)\times[-\epsilon_0,\epsilon_0],\ \mathbb{R})$$

with $c_1(0,t;\epsilon) = c_1(\pi,t;\epsilon) = 0$ for $t \geq 0$, (2.3.5)

$$c_2,\ \frac{\partial c_2}{\partial x},\ \frac{\partial^2 c_2}{\partial x^2} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\ \mathbb{R})$$

with $c_2(0;\epsilon) = c_2(\pi;\epsilon) = c_2''(0;\epsilon) = c_2''(\pi;\epsilon) = 0$, and (2.3.6)

$$c_3, \frac{\partial c_3}{\partial x} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0], \mathbb{R}) \text{ with } c_3(0;\epsilon) = c_3(\pi;\epsilon) = 0. \tag{2.3.7}$$

Furthermore, c_1 is assumed to be uniformly bounded for those values of t under consideration. (2.3.8)

Let $\tilde{F}(x,t,v;\epsilon) = F(x,t,v;\epsilon) - |\epsilon|^{m-1}c_1(x,t;\epsilon)$ (2.3.9)

and let v_ℓ be given by

$$v_\ell(x,t) = \int_0^\pi \left\{v_0(\xi;\epsilon)G_\tau(\xi,0;x,t) - v_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi. \tag{2.3.10}$$

Supposing that v_ℓ satisfies (2.2.15) and $\tilde{F}$ satisfies (2.2.16) and (2.2.17), it follows from theorem 2.2.1 that the initial-boundary value problem (2.3.1)-(2.3.4) has a unique, twice continuously differentiable solution v(x,t) on D_L. Using the result of appendix 2A the initial-boundary value problem (2.3.1)-(2.3.8) can be transformed into the equivalent integral equation

$$v(x,t) = \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)\tilde{F}(\xi,\tau,v(\xi,\tau);\epsilon)d\xi d\tau + v_\ell(x,t), \tag{2.3.11}$$

where G, $\tilde{F}$ and v_ℓ are given by (2.2.11), (2.3.9) and (2.3.10) respectively. Now, it will be shown that the formal approximation v is an asymptotic approximation (as $\epsilon \to 0$) for the solution of the initial-boundary value problem (2.2.1)-(2.2.9) if $m > 1$, that is, it will be proved that

$$\lim_{\epsilon\to 0} |u(x,t)-v(x,t)| = 0 \quad \text{for } m > 1 \text{ and } (x,t) \in D_L.$$

Since the functions c_1, c_2 and c_3 satisfy (2.3.5)-(2.3.8) there are bounded constants k_1,

k_2 and k_3 such that: $|c_1(x,t;\epsilon)| \leq k_1$, $|c_2(x;\epsilon)| \leq k_2$ and $|c_3(x;\epsilon)| \leq k_3$ for all x, t and ϵ under consideration. Subtracting the integral equation (2.3.11) from the integral equation (2.2.10), using (2.2.11), (2.2.13), (2.2.14), (2.2.17) and the fact that u and $v \in C_{M_1}(D_L)$, it follows that

$$\begin{aligned}
|u(x,t)-v(x,t)| &\leq \left|\epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)\left\{F(\xi,\tau,u(\xi,\tau);\epsilon) - F(\xi,\tau,v(\xi,\tau);\epsilon)\right\} d\xi d\tau\right| + \\
&+ \left||\epsilon|^m \int_0^t \int_0^\pi G(\xi,\tau;x,t)c_1(\xi,\tau;\epsilon)d\xi d\tau\right| + \\
&+ \left||\epsilon|^m \int_0^\pi \left\{c_2(\xi;\epsilon)G_\tau(\xi,0;x,t) - c_3(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi\right| \leq \\
&\leq |\epsilon|(t+\pi)M_3 \int_0^t \max_{0\leq\xi\leq\pi} |u(\xi,\tau)-v(\xi,\tau)|d\tau + |\epsilon|^m t(t+\pi)k_1 + \\
&+ |\epsilon|^m(4k_2\pi + 2k_2t(t+\pi) + 2k_3(t+\pi)) \leq \\
&\leq |\epsilon|(t+\pi)M_3 \int_0^t \max_{0\leq x\leq\pi} |u(x,\tau)-v(x,\tau)|d\tau + |\epsilon|^{m-1}k,
\end{aligned}$$

in which $k = L^2(k_1+2k_2) + |\epsilon|^{1/2}L(\pi k_1+2k_2\pi+2k_3) + 2\pi|\epsilon|(2k_2+k_3) < \infty$. Since the right-hand side of the inequality does not depend on x, the maximum for x on the left-hand side can be taken, and after using Gronwall's lemma it follows that

$$\max_{0\leq x\leq\pi} |u(x,t-v(x,t)| \leq |\epsilon|^{m-1}k \exp(|\epsilon|t(t+\pi)M_3) \leq \bar{k}|\epsilon|^{m-1} \quad \text{on } D_L,$$

where $\bar{k} = k \exp(L(L + \pi|\epsilon|^{1/2})M_3) < \infty$. Hence, $|u(x,t)-v(x,t)| = O(|\epsilon|^{m-1})$ for $0 \leq x \leq \pi$ and $0 \leq t \leq L|\epsilon|^{-1/2}$. So, for $m > 1$ the function v is an asymptotic approximation (as $\epsilon \to 0$) of the solution u of problem (2.2.1)-(2.2.9). The following theorem has now been proved.

Theorem 2.3.1

Let v satisfy (2.3.1)-(2.3.4) where ϵ, F, u_0 and u_1 are given by (2.2.5)-(2.2.9) and where c_1, c_2 and c_3 satisfy (2.3.5)-(2.3.8). Then for $m > 1$, the formal approximation v is an asymptotic approximation (as $\epsilon \to 0$) of the solution u of the nonlinear initial-boundary value problem (2.2.1)-(2.2.9). The asymptotic approximation v is valid for those values of the independent variables x and t for which problem (2.2.1)-(2.2.9) has been proved well-posed. That is,

$$|u(x,t)-v(x,t)| = O(|\epsilon|^{m-1}) \quad \text{for } 0 \le x \le \pi \text{ and } 0 \le t \le L|\epsilon|^{-1/2},$$

in which L is a sufficiently small, positive constant independent of ϵ.

2.4. An asymptotic approximation for a special case

In this section an asymptotic approximation of the solution of the initial-boundary value problem (2.1.1)-(2.1.3) with $F(x,t,u;\epsilon) \equiv u^3$ will be constructed using a two-timescales perturbation method. The following initial-boundary value problem will be considered:

$$u_{tt} - u_{xx} + u + \epsilon u^3 = 0, \qquad 0 < x < \pi,\ t > 0,\ 0 < |\epsilon| << 1, \qquad (2.4.1)$$

$$u(x,0) = u_0(x), \qquad 0 < x < \pi, \qquad (2.4.2)$$

$$u_t(x,0) = u_1(x), \qquad 0 < x < \pi, \qquad (2.4.3)$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \geq 0, \qquad (2.4.4)$$

with $u_0(0) = u_0(\pi) = u_0''(0) = u_0''(\pi) = u_0^{(iv)}(0) = u_0^{(iv)}(\pi) = 0,$

$u_1(0) = u_1(\pi) = u_1''(0) = u_1''(\pi) = 0$, and

$u_0 \in C^5([0,\pi],\mathbb{R})$ and $u_1 \in C^4([0,\pi],\mathbb{R})$. (2.4.5)

Since an approximation in the form of an infinite series will be constructed, (2.4.5) is required in order to get a convergent series representation for which summation and differentiation may be interchanged. In the sense of theorem 2.3.1 a function $\bar{u}$ will be constructed that satisfies (2.4.2) and (2.4.4) exactly and (2.4.1) and (2.4.3) up to order ϵ^2. From the theorems 2.2.1 and 2.3.1 it then follows

$$|u(x,t)-\bar{u}(x,t)| = O(|\epsilon|) \quad \text{for } 0 \leq x \leq \pi \text{ and } 0 \leq t \leq L|\epsilon|^{-1/2},$$

in which L is a sufficiently small, positive constant independent of ϵ. In constructing $\bar{u}$ a perturbation method will be used. The straightforward expansion $\bar{u}(x,t) = u_0(x,t) + \epsilon u_1(x,t) + \ldots$ will cause secular terms. However, from the energy equation (with $t_0 > 0$)

$$E(t_0) = \int_0^{\pi} \left\{u_t^2(x,t_0) + u_x^2(x,t_0) + u^2(x,t_0) + \frac{\epsilon}{2}\, u^4(x,t_0)\right\} dx = E(0),$$

which can be obtained by multiplying (2.4.1) by u_t and integrating over the region $0 \le x \le \pi$ and $0 \le t \le t_0$, it follows that if the initial values are bounded and if $\epsilon \ge 0$ that

$$\int_0^{\pi} u^2(x,t_0)\, dx < \infty \qquad \text{for all } t_0 > 0.$$

So, in that case secular terms should be avoided. For that reason a two-timescales perturbation method will be used. In using a two-timescales perturbation method u(x,t) is supposed to be a function of x, t and ϵt. Now let

$$\tau = \epsilon t, \text{ and} \tag{2.4.6}$$

$$u(x,t) = v(x,t,\tau;\epsilon). \tag{2.4.7}$$

By introducing (2.4.6) and (2.4.7) the initial-boundary value problem (2.4.1)-(2.4.4) becomes

$$v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau} - v_{xx} + v + \epsilon v^3 = 0, \qquad 0 < x < \pi,\ t > 0, \tag{2.4.8}$$

$$v(x,0,0;\epsilon) = u_0(x), \qquad 0 < x < \pi, \tag{2.4.9}$$

$$v_t(x,0,0;\epsilon) + \epsilon v_\tau(x,0,0;\epsilon) = u_1(x), \qquad 0 < x < \pi, \tag{2.4.10}$$

$$v(0,t,\tau;\epsilon) = v(\pi,t,\tau;\epsilon) = 0, \qquad t \ge 0. \tag{2.4.11}$$

Furthermore, it is assumed that v may be approximated by the formal expansion

$$v_0(x,t,\tau) + \epsilon v_1(x,t,\tau) + \epsilon^2 v_2(x,t,\tau) + \dots . \tag{2.4.12}$$

By substituting the expansion (2.4.12) into (2.4.8)-(2.4.11), and after equating the coefficients of like powers in ϵ, it follows from the powers 0 and 1 of ϵ respectively, that v_0 should satisfy

$$v_{0_{tt}} - v_{0_{xx}} + v_0 = 0, \qquad 0 < x < \pi,\ t > 0, \tag{2.4.13}$$
$$v_0(x,0,0) = u_0(x), \qquad 0 < x < \pi, \tag{2.4.14}$$
$$v_{0_t}(x,0,0) = u_1(x), \qquad 0 < x < \pi, \tag{2.4.15}$$
$$v_0(0,t,\tau) = v_0(\pi,t,\tau) = 0, \qquad t \geq 0, \tag{2.4.16}$$

and that v_1 should satisfy

$$v_{1_{tt}} - v_{1_{xx}} + v_1 = -2v_{0_{t\tau}} - v_0^3, \qquad 0 < x < \pi,\ t > 0, \tag{2.4.17}$$
$$v_1(x,0,0) = 0, \qquad 0 < x < \pi, \tag{2.4.18}$$
$$v_{1_t}(x,0,0) = -v_{0_\tau}(x,0,0), \qquad 0 < x < \pi, \tag{2.4.19}$$
$$v_1(0,t,\tau) = v_1(\pi,t,\tau) = 0, \qquad t \geq 0. \tag{2.4.20}$$

In the further analysis v_0 and v_1 will be determined, and it will be proved that $\bar{u}(x,t) = v_0(x,t,\tau) + \epsilon v_1(x,t,\tau)$ is an asymptotic approximation (as $\epsilon \to 0$) of the solution $u(x,t)$ of the initial-boundary value problem (2.4.1)-(2.4.5).

The solution v_0 of the initial-boundary value problem (2.4.13)-(2.4.16) is given by

$$v_0(x,t,\tau) = \sum_{n=1}^{\infty} \left\{A_n(\tau) \cos [(1 + n^2)^{1/2}\, t] + B_n(\tau) \sin [(1 + n^2)^{1/2}\, t]\right\} \sin nx. \tag{2.4.21}$$

From (2.4.14) and (2.4.15) it follows that

$$A_n(0) = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin nx dx, \tag{2.4.22}$$

$$B_n(0) = \frac{2}{\pi(1 + n^2)^{1/2}} \int_0^\pi u_1(x) \sin nx dx. \tag{2.4.23}$$

The functions $A_n(\tau)$ and $B_n(\tau)$ will be determined by the requirement that v_1 should not contain secular terms. From appendix 2B it follows that this condition can be satisfied if $A_n(\tau)$ and $B_n(\tau)$ satisfy

$$\frac{dA_n}{d\tau} - \frac{3B_n}{8(1 + n^2)^{1/2}} \left[-\frac{1}{4} \left(A_n^2 + B_n^2\right) + \sum_{k=1}^{\infty} \left(A_k^2 + B_k^2\right) \right] = 0, \tag{2.4.24}$$

$$\frac{dB_n}{d\tau} + \frac{3A_n}{8(1 + n^2)^{1/2}} \left[-\frac{1}{4} \left(A_n^2 + B_n^2\right) + \sum_{k=1}^{\infty} \left(A_k^2 + B_k^2\right) \right] = 0. \tag{2.4.25}$$

This system of infinitely many nonlinear first order ordinary differential equations for $A_n(\tau)$ and $B_n(\tau)$ ($n = 1,2,3,\ldots$) subject to the initial conditions (2.4.22) and (2.4.23) can be solved exactly. Multiplying (2.4.24) by A_n and (2.4.25) by B_n, adding the obtained equations and integrating with respect to τ, it follows that

$$A_n^2(\tau) + B_n^2(\tau) = A_n^2(0) + B_n^2(0), \qquad n = 1,2,3,\ldots . \tag{2.4.26}$$

Substituting (2.4.26) into (2.4.24) and (2.4.25) one obtains

$$\frac{dA_n}{d\tau} = \frac{-3C_n}{8(1 + n^2)^{1/2}} B_n \quad \text{and} \quad \frac{dB_n}{d\tau} = \frac{3C_n}{8(1 + n^2)^{1/2}} A_n,$$

$$\text{where } C_n = \frac{1}{4} \left(A_n^2(0) + B_n^2(0)\right) - \sum_{k=1}^{\infty} \left(A_k^2(0) + B_k^2(0)\right). \tag{2.4.27}$$

The solution of these differential equations for A_n and B_n is given by

$$A_n(\tau) = A_n(0) \cos\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}}\right] - B_n(0) \sin\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}}\right] =$$

$$= \left(A_n^2(0) + B_n^2(0)\right)^{1/2} \cos\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}} + \alpha_n\right], \tag{2.4.28}$$

$$B_n(\tau) = A_n(0) \sin\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}}\right] + B_n(0) \cos\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}}\right] =$$

$$= \left(A_n^2(0) + B_n^2(0)\right)^{1/2} \sin\left[\frac{3C_n\tau}{8(1+n^2)^{1/2}} + \alpha_n\right], \tag{2.4.29}$$

in which $\alpha_n = 0$ if $A_n^2(0) + B_n^2(0) = 0$ and else α_n is given by

$$\cos\alpha_n = A_n(0)\left(A_n^2(0) + B_n^2(0)\right)^{-1/2} \text{ and } \sin\alpha_n = B_n(0)\left(A_n^2(0) + B_n^2(0)\right)^{-1/2}. \tag{2.4.30}$$

Now the solution v_0 of the initial-boundary value problem (2.4.13)-(2.4.16) has been determined completely. Using (2.4.21), (2.4.28), (2.4.29) and some trigonometric relations, v_0 is given by

$$v_0(x,t,\tau) = \sum_{n=1}^{\infty} \left\{ A_n(0) \cos\left[(1+n^2)^{1/2}\, t - \frac{3C_n\tau}{8(1+n^2)^{1/2}} \right] + \right.$$

$$\left. + B_n(0) \sin\left[(1+n^2)^{1/2}\, t - \frac{3C_n\tau}{8(1+n^2)^{1/2}} \right] \right\} \sin nx =$$

$$= \sum_{n=1}^{\infty} \left(A_n^2(0) + B_n^2(0)\right)^{1/2} \cos\left[(1+n^2)^{1/2}\, t - \frac{3C_n\tau}{8(1+n^2)^{1/2}} - \alpha_n \right] \sin nx, \tag{2.4.31}$$

where $A_n(0)$, $B_n(0)$, C_n and α_n are given by (2.4.22), (2.4.23), (2.4.27) and (2.4.30) respectively. From (2.4.5), (2.4.22) and (2.4.23) it follows that the infinite series representation for v_0 is twice continuously differentiable with respect to x and t, and infinitely many times with respect to τ.

In appendix 2B the solution v_1 of the initial-boundary value problem (2.4.17)-(2.4.20) has been determined and it has been proved there that v_1 has the same convergence and differentiability properties as mentioned before for v_0. So far a function $\bar{u}(x,t) = v_0(x,t,\tau) + \epsilon v_1(x,t,\tau)$ has been constructed. It can easily be shown that $\bar{u}(x,t)$ satisfies (2.4.2) and (2.4.4) exactly, and (2.4.3) up to order ϵ^2 in the sense of theorem 2.3.1. Now it will be proved that $\bar{u}(x,t)$ satisfies (2.4.1) up to order ϵ^2 in the sense of theorem 2.3.1.

$$\begin{aligned}
\bar{u}_{tt} - \bar{u}_{xx} + \bar{u} + \epsilon\bar{u}^3 &= \left(v_{0_{tt}} - v_{0_{xx}} + v_0\right) + \epsilon\left(v_{1_{tt}} - v_{1_{xx}} + v_1 + 2v_{0_{t\tau}} + v_0^3\right) + \\
&\quad + \epsilon^2\left(v_{0_{\tau\tau}} + 2v_{1_{t\tau}} + 3v_0^2 v_1 + \epsilon v_{1_{\tau\tau}} + 3\epsilon v_0^2 v_1 + \epsilon^2 v_1^3\right) = \\
&= 0 + 0 + \epsilon^2\left(v_{0_{\tau\tau}} + 2v_{1_{t\tau}} + 3v_0^2 v_1 + \epsilon v_{1_{\tau\tau}} + 3\epsilon v_0^2 v_1 + \epsilon^2 v_1^3\right) \equiv \\
&\equiv \epsilon^2 c_1(x,t;\epsilon).
\end{aligned}$$

From the convergence and differentiability properties of the infinite series representations for v_0 and v_1 it follows that $c_1(x,t;\epsilon)$ is continuously differentiable and uniformly bounded on D_L and $c_1(0,t;\epsilon) = c_1(\pi,t;\epsilon) = 0$. Then it follows from theorem 2.3.1 that $\bar{u}(x,t)$ is an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $u(x,t)$ of the initial-boundary value problem (2.4.1)-(2.4.5) for $(x,t) \in D_L$, that is,

$$|u(x,t)-\bar{u}(x,t)| = O(|\epsilon|) \qquad \text{for } 0 \le x \le \pi \text{ and } 0 \le t \le L|\epsilon|^{-1/2}, \tag{2.4.32}$$

in which L is a sufficiently small, positive constant independent of ϵ.

Using (2.4.32) the following estimate can be obtained

$$|u - v_0| = |u - \bar{u} + \bar{u} - v_0| \le |u - \bar{u}| + |\epsilon v_1| = O(|\epsilon|) \qquad \text{on } D_L.$$

Hence, $v_0(x,t,\tau)$ given by (2.4.31) is also an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $u(x,t)$ of problem (2.4.1)-(2.4.5) for those values of x and t for which the problem has been proved well-posed.

2.5. Concluding remarks

The asymptotic theory as presented in this chapter provides a rigorous basis for a number of formal perturbation methods. An interesting aspect of the theory is that asymptotic validity has been established for a class of formal approximations.

The intriguing question whether the proof of the time-scale of asymptotic validity may be extended to an $O(|\epsilon|^{-1})$ time-scale remains open. It should be noticed that the extension to longer time-scales may depend on the estimate of the kernel G of the equivalent integral equation (2.2.10). More explicitly when the uniform boundedness in the sup-norm of G on $[0,\pi]\times\mathbb{R}^+$ could be established then an extension of the asymptotic theory to an $O(|\epsilon|^{-1})$ time-scale may easily be given.

For the explicit example studied in section 2.4, which may serve as a model for the vibrations of a taut string embedded in a nonlinear elastic medium, an $O(\epsilon)$ asymptotic approximation is given by (2.4.31). It is not difficult to show that when one excites this string initially with a finite number of modes, say N, the $O(\epsilon)$ approximation involves only these N modes on a time-scale of $O(|\epsilon|^{-1/2})$. This implies that in the $O(\epsilon)$ approximation no new modes will be excited up to $O(\epsilon)$. Moreover, only an energy transfer of $O(\epsilon)$ between the N modes may take place on a time-scale of $O(|\epsilon|^{-1/2})$.

An interesting phenomenon due to the nonlinearity may be noticed: the phase-shifts of the modes are determined by the coefficients C_n (as defined by (2.4.27)) which depend on the initial amplitudes of all modes initially present. In fact, the distorsion of the signals is determined by linear dispersion and dispersion due to the nonlinear term.

The approximation $v_0 + \epsilon v_1$ where v_1 is given in appendix 2B may be a good candidate for a second order, i.e. an $O(\epsilon^2)$ asymptotic approximation. However, in order to establish this, one has to show that v_2 satisfies certain smoothness conditions on the $O(|\epsilon|^{-1/2})$ time-scale, which requires additional tedious calculations.

Appendix 2A

In this section the equivalent integral equation for the initial-boundary value problem (2.2.1)-(2.2.9) will be derived. The initial-boundary value problem can be transformed into an initial value problem by extending the functions u, F, u_0 and u_1 in x to odd and 2π-periodic functions u^*, F^*, u_0^* and u_1^*. Then, an integral equation for the solution of the initial value problem can easily be obtained (see for instance [7, chapter 5] and [28, chapter 2]):

$$u^*(u,t) = \frac{-\epsilon}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} J_0([(\tau-t)^2 - (\xi-x)^2]^{1/2})\, F^*(\xi,\tau,u^*(\xi,\tau);\epsilon)d\xi d\tau +$$

$$+ \frac{1}{2} (u_0^*(x+t;\epsilon) + u_0^*(x-t;\epsilon)) + \frac{1}{2} \int_{x-t}^{x+t} \frac{\partial J_0([t^2 - (\xi-x)^2]^{1/2})}{\partial t} u_0^*(\xi;\epsilon)d\xi +$$

$$+ \frac{1}{2} \int_{x-t}^{x+t} J_0([t^2 - (\xi-x)^2]^{1/2})\, u_1^*(\xi;\epsilon)d\xi \equiv (Tu)(x,t), \tag{2A.1}$$

where J_0 is the Bessel function of the first kind of order zero.

Obviously, the solution of (2.2.1)-(2.2.9) is a fixed point of the integral opeator T (T: $C \to C^1$ and T: $C^1 \to C^2$). After some manipulations the integral equation (2A.1) can be rewritten as

$$u(x,t) = \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)F(\xi,\tau,u(\xi,\tau);\epsilon)d\xi d\tau +$$

$$+ \int_0^\pi \left\{u_0(\xi;\epsilon)G_\tau(\xi,0;x,t) - u_1(\xi;\epsilon)G(\xi,0;x,t)\right\} d\xi \equiv (Tu)(x,t), \tag{2A.2}$$

where G is given by

$$G(\xi,\tau;x,t) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \Big\{ H(t-\tau-\xi+2k\pi-x)H(t-\tau+\xi-2k\pi+x)J_0([(t-\tau)^2 - (\xi-2k\pi+x)^2]^{1/2}) +$$

$$- H(t-\tau+\xi+2k\pi-x)H(t-\tau-\xi-2k\pi+x)J_0([(t-\tau)^2 - (\xi+2k\pi-x)^2]^{1/2}) \Big\}, \qquad (2A.3)$$

in which H(a) is a step function which is equal to 1 for $a > 0$, $\frac{1}{2}$ for $a = 0$ and zero otherwise. In (2A.2) it is assumed that $\frac{d}{da} H(a) = \delta(a)$ in the sense of the theory of distributions.

In fact G as defined by (2A.3) is the Green's function for the differential operator $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1$ and the boundary conditions (2.2.4).

After some elementary calculations it can be shown that if v(x,t) is a twice continuously differentiable solution of (2.2.1)-(2.2.9) then v(x,t) is a solution of the integral equations (2A.1) and (2A.2). From section 2.2 it follows that under the assumptions (2.2.5)-(2.2.9) the solution of the integral equation (2A.2) is twice continuously differentiable on some subdomain D_L of the (x,t)-plane. It can easily be proved that if w(x,t) is a solution of (2A.1) or (2A.2) then w(x,t) is a solution of (2.2.1)-(2.2.9). Hence, the integral equation (2A.2) and the initial-boundary value problem (2.2.1)-(2.2.9) are equivalent on D_L.

Appendix 2B

In this appendix the solution v_1 of the initial-boundary value problem (2.4.17)-(2.4.20) will be determined such that this solution satisfies the requirement that it does not contain secular terms. It should be noted that the equations for v_0 and v_1 have been derived under the assumption that v_0, v_1 and their derivatives up to order two are O(1) on D_L. For that reason v_1 should not contain secular terms, that is, terms like $t \sin [(1+n^2)^{1/2}\, t] \sin nx$. To determine v_1 it is assumed that v_1 may be written as

$$v_1(x,t,\tau) = \sum_{n=1}^{\infty} D_n(t,\tau) \sin nx. \tag{2B.1}$$

By substituting (2B.1) and (2.4.21) into (2.4.17) one obtains

$$\sum_{n=1}^{\infty} \left(D_{n_{tt}} + (n^2+1) D_n\right) \sin nx = -2 \sum_{n=1}^{\infty} \tilde{H}_n \sin nx +$$

$$+ \frac{1}{4} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} H_k H_\ell H_m \left(\sin(k+\ell+m)x - \sin(k+\ell-m)x - \sin(k-\ell+m)x + \sin(k-\ell-m)x\right),$$

in which

$$\tilde{H}_n = -(1+n^2)^{1/2} \frac{dA_n}{d\tau} \sin [(1+n^2)^{1/2}\, t] + (1+n^2)^{1/2} \frac{dB_n}{d\tau} \cos [(1+n^2)^{1/2}\, t] \tag{2B.2}$$

and

$$H_n = A_n \cos [(1+n^2)^{1/2}\, t] + B_n \sin [(1+n^2)^{1/2}\, t]. \tag{2B.3}$$

Equating the coefficients of like functions in sin nx yields

$$D_{n_{tt}} + (n^2 + 1) D_n = -2\tilde{H}_n + \frac{1}{4} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} H_k H_\ell H_m (\delta_{n,k+\ell+m} + \delta_{n,-k-\ell+m} +$$

$$- \delta_{n,k+\ell-m} + \delta_{n,-k+\ell-m} - \delta_{n,k-\ell+m} + \delta_{n,k-\ell-m} - \delta_{n,-k+\ell+m}) =$$

$$= -2\tilde{H}_n + \frac{1}{4} \left[\sum_{\substack{k,\ell,m=1 \\ k+\ell+m=n}}^{\infty} - 3 \sum_{\substack{k,\ell,m=1 \\ k+\ell-m=n}}^{\infty} + 3 \sum_{\substack{k,\ell,m=1 \\ m-k-\ell=n}}^{\infty} \right] H_k H_\ell H_m \equiv R_n(t,\tau), \qquad (2B.4)$$

where $\delta_{i,j}$ is the Kronecker delta symbol that is defined to be zero for $i \neq j$ and unity for $i = j$. So, assuming (2B.1) problem (2.4.17)-(2.4.20) becomes

$$D_{n_{tt}} + (n^2 + 1) D_n = R_n(t,\tau), \qquad (2B.5)$$

$$D_n(0,0) = 0, \qquad (2B.6)$$

$$D_{n_t}(0,0) = \frac{dA_n(0)}{d\tau}, \qquad (2B.7)$$

where $R_n(t,\tau)$ is defined in (2B.4). The solution of problem (2B.5)-(2B.7) is given by

$$D_n(t,\tau) = P_n(\tau) \cos [(1 + n^2)^{1/2} t] + Q_n(\tau) \sin [(1 + n^2)^{1/2} t] + T_n(t,\tau) \qquad (2B.8)$$

with $P_n(0) = -T_n(0,0)$ and $Q_n(0) = -(1 + n^2)^{1/2} \left\{ T_{n_t}(0,0) + \frac{dA_n(0)}{d\tau} \right\}$. (2B.9)

In (2B.8) $T_n(t,\tau)$ can be regarded as the particular solution of (2B.5). First $T_n(t,\tau)$ will be determined. After evaluating the product $H_k H_\ell H_m$ in $R_n(t,\tau)$ it can easily be seen that secular terms in $T_n(t,\tau)$ can occur (for k, ℓ, m and $n \in \mathbb{Z}^+$) if

$$\pm\,(n^2+1)^{1/2} = (k^2+1)^{1/2} + (\ell^2+1)^{1/2} - (m^2+1)^{1/2},$$
$$\pm\,(n^2+1)^{1/2} = (\ell^2+1)^{1/2} + (m^2+1)^{1/2} - (k^2+1)^{1/2},$$
$$\pm\,(n^2+1)^{1/2} = (m^2+1)^{1/2} + (k^2+1)^{1/2} - (\ell^2+1)^{1/2},$$
$$(n^2+1)^{1/2} = (k^2+1)^{1/2} + (\ell^2+1)^{1/2} + (m^2+1)^{1/2}.$$

Only two of these cases are essentially different. These two cases are

$$\pm\,(n^2+1)^{1/2} = (k^2+1)^{1/2} + (\ell^2+1)^{1/2} - (m^2+1)^{1/2}. \tag{2B.10}$$

So the problem is to determine the values of k, ℓ, m and $n \in \mathbb{Z}^+$ for which (2B.10) can be satisfied. To solve this problem the following inequality is used

$$j < (j^2+1)^{1/2} \le j - 1 + \sqrt{2} \qquad \text{for all } j \in \mathbb{Z}^+. \tag{2B.11}$$

Case I (plus sign in (2B.10)):

$$(n^2+1)^{1/2} = (k^2+1)^{1/2} + (\ell^2+1)^{1/2} - (m^2+1)^{1/2}. \tag{2B.12}$$

Using (2B.11) and (2B.12) one obtains

$$n < (n^2+1)^{1/2} < k - 1 + \sqrt{2} + \ell - 1 + \sqrt{2} - m = k + \ell - m - 2 + 2\sqrt{2}, \text{ and}$$
$$n - 1 + \sqrt{2} \ge (n^2+1)^{1/2} > k + \ell - m + 1 - \sqrt{2}. \text{ Hence,}$$
$$k + \ell - m + 2 - 2\sqrt{2} < n < k + \ell - m - 2 + 2\sqrt{2}.$$

Since k, ℓ, m and $n \in \mathbb{Z}^+$ it then follows that $n = k + \ell - m$. Then (2B.12) becomes

$$((k + \ell - m)^2 + 1)^{1/2} = (k^2 + 1)^{1/2} + (\ell^2 + 1)^{1/2} - (m^2 + 1)^{1/2}. \tag{2B.13}$$

By squaring equation (2B.13) two times and rearranging terms one obtains

$$(k\ell - 1 - (k^2 + 1)^{1/2}(\ell^2 + 1)^{1/2})(m^2 - km - \ell m + k\ell) = 0, \text{ hence}$$
$$k\ell - 1 - (k^2 + 1)^{1/2}(\ell^2 + 1)^{1/2} = 0 \quad \text{or} \quad m^2 - km - \ell m + k\ell = 0. \tag{2B.14}$$

By squaring the first equation in (2B.14) one obtains after rearranging terms that $(k + \ell)^2 = 0$. However, this contradicts the assumption k and $\ell \in \mathbb{Z}^+$. The solution of the second equation in (2B.14) is given by $m = \frac{1}{2}(k + \ell) \pm \frac{1}{2}((k + \ell)^2 - 4k\ell)^{1/2}$. Hence, $m = k$ or $m = \ell$. Combining this result with $n = k + \ell - m$ yields

$$m = k \quad \text{and} \quad n = 1 \quad \text{or} \tag{2B.15}$$
$$m = \ell \quad \text{and} \quad n = k. \tag{2B.16}$$

By substituting (2B.15) and (2B.16) into (2B.12) it can be verified that both (2B.15) and (2B.16) satisfy (2B.12).

Case II (minus sign in (2B.10)):

$$-(n^2 + 1)^{1/2} = (k^2 + 1)^{1/2} + (\ell^2 + 1)^{1/2} - (m^2 + 1)^{1/2}. \tag{2B.17}$$

Using (2B.11) and (2B.17) one obtains

$$n < (n^2 + 1)^{1/2} < -k - \ell + m - 1 + \sqrt{2}\text{ , and}$$
$$n - 1 + \sqrt{2} \geq (n^2 + 1)^{1/2} > -k + 1 - \sqrt{2} - \ell + 1 - \sqrt{2} + m = m - k - \ell + 2 - 2\sqrt{2}, \text{ hence}$$
$$m - k - \ell + 3 - 3\sqrt{2} < n < m - k - \ell - 1 + \sqrt{2}\,.$$

From k, ℓ, m and n $\in \mathbb{Z}^+$ it then follows that $n = m - k - \ell - 1$ or $n = m - k - \ell$. Since no summation in (2B.4) involves $n = m - k - \ell - 1$ one only has to consider $n = m - k - \ell$. Then (2B.17) becomes

$$-((m - k - \ell)^2 + 1)^{1/2} = -((k + \ell - m)^2 + 1)^{1/2} =$$
$$= (k^2 + 1)^{1/2} + (\ell^2 + 1)^{1/2} - (m^2 + 1)^{1/2}. \qquad (2B.18)$$

By squaring (2B.18) two times and after rearranging terms one obtains the equations (2B.14). These equations have the solutions (2B.15) and (2B.16). However, the solutions (2B.15) and (2B.16) do not satisfy (2B.17). So, in case II there do not exist integers k, ℓ, m and n $\in \mathbb{Z}^+$ giving rise to secular terms.

Hence, the integers k, ℓ, m and n $\in \mathbb{Z}^+$ given by (2B.15) and (2B.16) will cause secular terms in $T_n(t,\tau)$. Taking apart those terms in (2B.4) one obtains:

$$D_{n_{tt}} + (n^2 + 1)\, D_n =$$
$$= \sin\,[(1+n^2)^{1/2}t] \left[2(1+n^2)^{1/2} \frac{dA_n}{d\tau} - \frac{3}{4} B_n \Big(-\frac{1}{4} \big(A_n^2 + B_n^2\big) + \sum_{k=1}^{\infty} \big(A_k^2 + B_k^2\big) \Big) \right] +$$
$$-\cos\,[(1+n^2)^{1/2}t] \left[2(1+n^2)^{1/2} \frac{dB_n}{d\tau} + \frac{3}{4} A_n \Big(-\frac{1}{4} \big(A_n^2 + B_n^2\big) + \sum_{k=1}^{\infty} \big(A_k^2 + B_k^2\big) \Big) \right] +$$
$$+ \frac{1}{4} \left[\sum_{\substack{k,\ell,m=1\\ k+\ell+m=n}}^{\infty} - 3 \sum_{\substack{k,\ell,m=1\\ k+\ell-m=n}}^{\infty *} + 3 \sum_{\substack{k,\ell,m=1\\ m-k-\ell=n}}^{\infty} \right] H_k H_\ell H_m. \qquad (2B.19)$$

where the * in (2B.19) indicates that terms in $H_k H_\ell H_m$ giving rise to secular terms in

$T_n(t,\tau)$ are excluded.

In order to avoid secular terms $A_n(\tau)$ and $B_n(\tau)$ have to satisfy

$$2(1+n^2)^{1/2}\frac{dA_n}{d\tau} - \frac{3}{4}B_n\left(-\frac{1}{4}\left(A_n^2+B_n^2\right)+\sum_{k=1}^{\infty}\left(A_k^2+B_k^2\right)\right)=0, \qquad (2B.20)$$

$$2(1+n^2)^{1/2}\frac{dB_n}{d\tau} + \frac{3}{4}A_n\left(-\frac{1}{4}\left(A_n^2+B_n^2\right)+\sum_{k=1}^{\infty}\left(A_k^2+B_k^2\right)\right)=0. \qquad (2B.21)$$

After lengthy but rather elementary calculations it follows that the particular solution $T_n(t,\tau)$ of (2B.19) is given by

$$T_n(t,\tau)=\frac{1}{4}\left[\sum_{\substack{k,\ell,m=1\\ k+\ell+m=n}}^{\infty} - 3\sum_{\substack{k,\ell,m=1\\ k+\ell-m=n}}^{\infty} + 3\sum_{\substack{k,\ell,m=1\\ m-k-\ell=n}}^{\infty}\right]R_{k\ell m}(t,\tau) \qquad (2B.22)$$

with $R_{k\ell m}(t,\tau) = \left(\left(A_k^2+B_k^2\right)\left(A_\ell^2+B_\ell^2\right)\left(A_m^2+B_m^2\right)\right)^{1/2}\times$

$$\times\sum_{i=1}^{4}{}^{*}\ \frac{\cos\left[d_{k\ell m}^{(i)}t - e_{k\ell m}^{(i)}\tau - f_{k\ell m}^{(i)}\right]}{n^2+1-\left(d_{k\ell m}^{(i)}\right)^2}, \qquad (2B.23)$$

where

$d_{k\ell m}^{(1)} = sq_k + sq_\ell - sq_m$ with $sq_n = (n^2+1)^{1/2}$,

$d_{k\ell m}^{(2)} = d_{\ell mk}^{(1)}$, $d_{k\ell m}^{(3)} = d_{mk\ell}^{(1)}$ and $d_{k\ell m}^{(4)} = d_{k\ell m}^{(1)} + d_{k\ell m}^{(2)} + d_{k\ell m}^{(3)}$,

$e_{k\ell m}^{(1)} = \frac{3}{8}\left(C_k sq_k^{-1} + C_\ell sq_\ell^{-1} - C_m sq_m^{-1}\right)$ with C_n given by (2.4.27),

$e_{k\ell m}^{(2)} = e_{\ell mk}^{(1)}$, $e_{k\ell m}^{(3)} = e_{mk\ell}^{(1)}$ and $e_{k\ell m}^{(4)} = e_{k\ell m}^{(1)} + e_{k\ell m}^{(2)} + e_{k\ell m}^{(3)}$,

$f_{k\ell m}^{(1)} = \alpha_k + \alpha_\ell - \alpha_m$ with α_n given by (2.4.30),

$f_{k\ell m}^{(2)} = f_{\ell mk}^{(1)}$, $f_{k\ell m}^{(3)} = f_{mk\ell}^{(1)}$ and $f_{k\ell m}^{(4)} = f_{k\ell m}^{(1)} + f_{k\ell m}^{(2)} + f_{k\ell m}^{(3)}$.

The * in (2B.23) indicates that the summation over i for i = 1, 2 or 3 is excluded if $(n^2+1)^{1/2} = d^{(i)}_{k\ell m}$.

It should be noted that all the denominators in (2B.22) are nonzero because of the non-secularity requirement. In order to obtain some properties of $T_n(t,\tau)$ the following results will be needed

(a) if $k + \ell + m = n$ then

$$\frac{2}{n^2+1-\left(d^{(4)}_{k\ell m}\right)^2} =$$

$$= \frac{k\ell m - k - \ell - m + k(\ell^2+1)^{1/2}(m^2+1)^{1/2} + m(k^2+1)^{1/2}(\ell^2+1)^{1/2} + \ell(k^2+1)^{1/2}(m^2+1)^{1/2}}{-2(k+m)(k+\ell)(\ell+m)}.$$

(b) if $k + \ell + m = n$ then

$$\frac{2}{n^2+1-\left(d^{(1)}_{k\ell m}\right)^2} =$$

$$= \frac{k\ell m - k - \ell - m - k(\ell^2+1)^{1/2}(m^2+1)^{1/2} + m(k^2+1)^{1/2}(\ell^2+1)^{1/2} - \ell(k^2+1)^{1/2}(m^2+1)^{1/2}}{-2(k+m)(k+\ell)(\ell+m)}.$$

and similar formulas for $d^{(1)}_{\ell mk}$ and $d^{(1)}_{mk\ell}$.

(c) if $k + \ell - m = n$ then

$$\frac{2}{n^2+1-\left(d^{(4)}_{k\ell m}\right)^2} =$$

$$= \frac{k\ell m + k + \ell - m - k(\ell^2+1)^{1/2}(m^2+1)^{1/2} + m(k^2+1)^{1/2}(\ell^2+1)^{1/2} - \ell(k^2+1)^{1/2}(m^2+1)^{1/2}}{2(k+\ell)(k-m)(\ell-m)}.$$

(d) if $k + \ell - m = n$ then

$$\frac{2}{n^2 + 1 - \left(d^{(1)}_{k\ell m}\right)^2} =$$

$$= \frac{k\ell m + k + \ell - m + k(\ell^2+1)^{1/2}(m^2+1)^{1/2} + m(k^2+1)^{1/2}(\ell^2+1)^{1/2} + \ell(k^2+1)^{1/2}(m^2+1)^{1/2}}{2(k+\ell)(k-m)(\ell-m)}.$$

and similar formulas for $d^{(1)}_{\ell mk}$ and $d^{(1)}_{mk\ell}$.

(e) if $m - k - \ell = n$ then formulas analogous to (c) and (d) can be obtained because $n^2 = (m - k - \ell)^2 = (k + \ell - m)^2$.

From (2.4.5), (2.4.22), (2.4.23), (2.4.28) and (2.4.29) it follows that there exists a uniformly bounded constant K such that for all $n \in \mathbb{Z}^+$ and for all $\tau \in \mathbb{R}$:

$$|A_n(\tau)| \le \frac{K}{n^5} \quad \text{and} \quad |B_n(\tau)| \le \frac{K}{n^5}. \tag{2B.24}$$

Using (2B.24) and the properties (a)-(e) of the denominators of $R_{k\ell m}(t,\tau)$ in $T_n(t,\tau)$, the following estimates can be made:

$$\left| \sum_{\substack{k,\ell,m=1 \\ k+\ell+m=n}}^{\infty} R_{k\ell m}(t,\tau) \right| \le 4 \sum_{\substack{k,\ell,m=1 \\ k+\ell+m=n}}^{\infty} \frac{2\sqrt{2}\ K^3}{k^5 \ell^5 m^5} \frac{4k\ell m}{(k+\ell)(k+m)(\ell+m)} \le$$

$$\le 32\sqrt{2}\ K^3 \sum_{\substack{k,\ell,m=1 \\ k+\ell+m=n}}^{\infty} \frac{1}{k^2\ell^2m^2} \frac{9}{(k+\ell+m)^2} \frac{3}{2n^2} \le \frac{32\sqrt{2}\ K^3\, 27}{2n^4} \left(\frac{\pi^2}{6}\right)^3 \text{ and}$$

$$\left| \sum_{\substack{k,\ell,m=1 \\ k+\ell-m=n}}^{\infty} R_{k\ell m}(t,\tau) \right| \leq 4 \sum_{\substack{k,\ell,m=1 \\ k+\ell-m=n}}^{\infty} \frac{2\sqrt{2}\ K^3}{k^5 \ell^5 m^5} \frac{4k\ell m}{(k+\ell)} \leq$$

$$\leq 32\sqrt{2}\ K^3 \sum_{\substack{k,\ell=1 \\ k+\ell\geq n+1}}^{\infty} \frac{2^4}{(k+\ell-n)^4 (k+\ell)^5} \leq$$

$$\leq 32\sqrt{2}\ K^3 \sum_{k+\ell=n+1}^{\infty} \frac{2^4(k+\ell-1)}{(k+\ell-n)^4 (k+\ell)^5} \leq \frac{32\sqrt{2}\ K^3 2^4}{n^4} \frac{\pi^4}{90}.$$

Since $(k + \ell - m)^2 = (m - k - \ell)^2$ the estimate of the third sum with summation index $m - k - \ell = n$ in (2B.22) is similar to the estimate of the second sum.

Hence, there is a uniformly bounded constant c such that for all $n \in \mathbb{Z}^+$ and $t,\tau \in \mathbb{R}$

$$|T_n(t,\tau)| \leq cn^{-4}. \tag{2B.25}$$

Estimating and arguing by analogy it can easily be obtained that for all $n \in \mathbb{Z}^+$ and for all $t,\tau \in \mathbb{R}$ there exists a uniformly bounded constant c such that

$$\left|T_{n_t}(t,\tau)\right| \leq \frac{c}{n^3},\ \left|T_{n_{t\tau}}(t,\tau)\right| \leq \frac{c}{n^3},\ \left|T_{n_{tt}}(t,\tau)\right| \leq \frac{c}{n^2} \text{ and } \left|T_{n_{\tau\tau}}(t,\tau)\right| \leq \frac{c}{n^4}. \tag{2B.26}$$

Considering the solution $v_1(x,t,\tau)$ it should be noted that the functions $P_n(\tau)$ and $Q_n(\tau)$ in (2B.8) have to be used in order to avoid secular terms in the solution $v_2(x,t,\tau)$. However, it is our purpose to construct a function $\bar{u}(x,t)$ that satisfies the differential equation up to order ϵ^2. Therefore $P_n(\tau)$ and $Q_n(\tau)$ are taken to be equal to their initial values (2B.9), that is,

$$P_n(\tau) = P_n(0) \quad \text{and} \quad Q_n(\tau) = Q_n(0). \tag{2B.27}$$

Now it easily follows from (2B.8), (2B.9), (2B.22), (2B.25), (2B.26) and (2B.27) that the series representation (2B.1) for v_1 is twice continuously differentiable with respect to x and t and infinitely many times with respect to τ.

CHAPTER 3

ASYMPTOTICS FOR A SYSTEM OF NONLINEARLY COUPLED WAVE EQUATIONS WITH AN APPLICATION TO THE GALLOPING OSCILLATIONS OF OVERHEAD TRANSMISSION LINES

Abstract

In this chapter an asymptotic theory for a class of initial-boundary value problems for systems of weakly and nonlinearly coupled wave equations is presented. The theory implies the well-posedness of the problem in the classical sense and the asymptotic validity of formal approximations on long time-scales.

As an application of the theory an initial-boundary value problem for a system of weakly and nonlinearly coupled wave equations is studied in detail using a two-timescales perturbation method. From an aero-elastic analysis it is shown that this initial-boundary value problem may be regarded as a model describing the galloping oscillations of overhead transmission lines in the vertical and in the horizontal direction.

3.1. Introduction

In this chapter an asymptotic theory is presented for the following initial-boundary value problem for a system of nonlinearly perturbed wave equations

$$\underline{u}_{tt} - C\underline{u}_{xx} + \epsilon \underline{f}(x,t,\underline{u},\underline{u}_t,\underline{u}_x;\epsilon) = \underline{0}, \qquad 0 < x < \pi,\ t > 0, \tag{3.1.1}$$

$$\underline{u}(x,0;\epsilon) = \underline{u}_0(x;\epsilon) \text{ and } \underline{u}_t(x,0;\epsilon) = \underline{u}_1(x;\epsilon), \qquad 0 < x < \pi, \tag{3.1.2}$$

$$\underline{u}(0,t;\epsilon) = \underline{u}(\pi,t;\epsilon) = \underline{0}, \qquad t \geq 0, \tag{3.1.3}$$

with $\underline{u} = (u_1,u_2,...,u_n)^T$, $\underline{f} = (f_1,f_2,...,f_n)^T$ and $0 < |\epsilon| \leq \epsilon_0 << 1$. The (nxn) diagonal-matrix C has ϵ-independent diagonal elements c_{ii}^2 (i = 1,2,...,n) with $c_{ii} > 0$, and the functions $\underline{f}$, $\underline{u}_0$ and $\underline{u}_1$ have to satisfy certain smoothness properties, which are mentioned in section 3.2. As usual the derivative of a matrix-valued function is obtained by taking the derivative of each element of the matrix-valued function. The asymptotic theory presented here implies the well-posedness in the classical sense of the initial-boundary value problem (3.1.1)-(3.1.3) as well as the asymptotic validity of formal approximations. In this chapter formal approximations are defined to be vector-valued functions satisfying the differential equations and the initial conditions up to some order depending on the small parameter ϵ.

For scalar-valued functions similar asymptotic theories have been developed in [11] for an initial-boundary value problem for the weakly semi-linear telegraph equation $u_{tt} - u_{xx} + u + \epsilon f(x,t,u;\epsilon) = 0$, and in [12] for an initial-boundary value problem for the weakly nonlinear wave equation $u_{tt} - u_{xx} + \epsilon f(x,t,u,u_t,u_x;\epsilon) = 0$. Both type of equations were considered subject to the initial values $u(x,0;\epsilon) = u_0(x;\epsilon)$ and $u_t(x,0;\epsilon) = u_1(x;\epsilon)$ and the boundary values $u(0,t;\epsilon) = u(\pi,t;\epsilon) = 0$. The well-posedness in the classical sense and the asymptotic validity of a class of formal approximations could be obtained on a time-scale of order $|\epsilon|^{-1/2}$ for the problem for the weakly semi-linear telegraph equation and could be established on a time-scale of order $|\epsilon|^{-1}$ for the problem for the weakly nonlinear wave equation. For the initial-boundary value problem (3.1.1)-(3.1.3) it will be shown that a time-scale of order $|\epsilon|^{-1}$ can be obtained.

The asymptotic theory in [11,12] and the asymptotic theory presented in this chapter can be regarded as an extension of the asymptotic theory for ordinary differential equations as for instance described in [1,2,8,25]. Moreover, the asymptotic results presented in this chapter can be seen as a generalization of the asymptotic results obtained in [12].

This chapter, being an attempt to contribute to the foundations of the asymptotic methods for (systems of) weakly nonlinear hyperbolic partial differential equations, is organized as follows. In section 3.2 the well-posedness of the problem is investigated and established on a time-scale of order $|\epsilon|^{-1}$ and in section 3.3 the asymptotic validity of formal approximations is studied, that is, estimates of the differences between the solution and the formal approximations are given on a time-scale for which the problem has been shown to be well-posed. The asymptotic theory is applied in section 3.5 to the initial-boundary value problem (3.1.1)-(3.1.3) with $\underline{u} = (v,w)^T$ and

$$\underline{f}(x,t,\underline{u},\underline{u}_t,\underline{u}_x;\epsilon) \equiv \begin{pmatrix} a_{10}v_t + a_{01}w_t + a_{20}v_t^2 + a_{11}v_tw_t + a_{02}w_t^2 + a_{03}w_t^3 \\ b_{01}w_t + b_{11}v_tw_t + b_{02}w_t^2 + b_{03}w_t^3 \end{pmatrix},$$

where $a_{10},a_{01},\ldots,b_{03}$ are constants independent of ϵ. In section 3.4 it follows from an aero-elastic analysis that this initial-boundary value problem may be regarded as a model which describes the growth of wind-induced oscillations of overhead transmission lines in the vertical and in the horizontal direction. In fact this initial-boundary value problem is an extension of a model (describing only the vertical displacements of the transmission lines) which has been postulated in the early seventies in [22,23] and which recently has been derived in [12]. From a practical point of view it is interesting to investigate the vertical as well as the horizontal displacements of the transmission line, since one or both of the displacements may give rise to conductor damage due to impact of conductor lines or due to flashover as a result of a phase-difference between conductor lines, for which the mutual distance has become too small.

Using a two-timescales perturbation method, as for instance successfully used in [4,11, 12,17,18], an asymptotic approximation of the solution of the aforementioned initial-boundary value problem will be constructed. In some sense it is remarkable that the two-timescales perturbation method as developed in [4,18] and applied in [4,11,12,17,18] to an initial-boundary value problem for a single perturbed wave equation may also be used for an initial-boundary value problem for the aforementioned system of perturbed wave equations. Finally, in section 3.6 some of the results obtained in this chapter will be discussed.

3.2. The well-posedness of the problem

In this section a weakly nonlinear initial-boundary value problem for a vector-valued function $\underline{u}(x,t;\epsilon)$ will be considered. As usual a derivative of a vector-valued function is obtained by taking the derivative of each element of the vector-valued function. Furthermore, a vector-valued function is said to be continuous (or differentiable) if and only if all the elements of the vector-valued function are continuous (or differentiable). Let $\underline{u}(x,t;\epsilon) \equiv (u_1(x,t;\epsilon),u_2(x,t;\epsilon),\ldots,u_n(x,t;\epsilon))^T$ with $1 \leq n < \infty$ and let $\underline{u},\underline{u}_t,\underline{u}_x,\underline{u}_{tt},\underline{u}_{tx} = \underline{u}_{xt}$ and $\underline{u}_{xx}$ be continuous on $0 \leq x \leq \pi$ and $t \geq 0$. The following weakly nonlinear initial-boundary value problem for the vector-valued function $\underline{u}(x,t;\epsilon)$ will now be considered:

$$\underline{u}_{tt} - C\underline{u}_{xx} + \epsilon\underline{F}(\underline{u};\epsilon) = \underline{0}, \qquad 0 < x < \pi,\ t > 0, \tag{3.2.1}$$

$$\underline{u}(x,0;\epsilon) = \underline{u}_0(x;\epsilon), \qquad 0 < x < \pi, \tag{3.2.2}$$

$$\underline{u}_t(x,0;\epsilon) = \underline{u}_1(x;\epsilon), \qquad 0 < x < \pi, \tag{3.2.3}$$

$$\underline{u}(0,t;\epsilon) = \underline{u}(\pi,t;\epsilon) = \underline{0}, \qquad t \geq 0, \tag{3.2.4}$$

where C is a (n×n) diagonal-matrix with ϵ-independent diagonal elements c_{ii}^2 $(i = 1,2,..,n)$ and $c_{ii} > 0$,

$$\underline{F}(\underline{u};\epsilon)(x,t) \equiv \underline{f}(x,t,\underline{u}(x,t;\epsilon),\underline{u}_t(x,t;\epsilon),\underline{u}_x(x,t;\epsilon);\epsilon), \tag{3.2.5}$$

$0 < |\epsilon| \leq \epsilon_0 << 1$, and where $\underline{u}_0(x;\epsilon) = (u_{01}(x;\epsilon),\ldots,u_{0n}(x;\epsilon))^T$, $\underline{u}_1(x;\epsilon) = (u_{11}(x;\epsilon),\ldots, u_{1n}(x;\epsilon))^T$ and $\underline{f}(x,t,\underline{u},\underline{p},\underline{q};\epsilon) = (f_1(x,t,\underline{u},\underline{p},\underline{q};\epsilon),\ldots,f_n(x,t,\underline{u},\underline{p},\underline{q};\epsilon))^T$ with $\underline{u} = (u_1,\ldots,u_n)^T$, $\underline{p} = (p_1,\ldots,p_n)^T$ and $\underline{q} = (q_1,\ldots,q_n)^T$ satisfy:

$$u_{0i}, \frac{\partial u_{0i}}{\partial x}, \frac{\partial^2 u_{0i}}{\partial x^2} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R}) \quad \text{for } i = 1,2,\dots,n,$$

$$\text{with } \underline{u}_0(0;\epsilon) = \underline{u}_0(\pi;\epsilon) = \frac{\partial^2 \underline{u}_0(0;\epsilon)}{\partial x^2} = \frac{\partial^2 \underline{u}_0(\pi;\epsilon)}{\partial x^2} = \underline{0}, \tag{3.2.6}$$

$$u_{1i}, \frac{\partial u_{1i}}{\partial x} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R}) \text{ for } i = 1,2,\dots,n, \text{ with } \underline{u}_1(0;\epsilon) = \underline{u}_1(\pi;\epsilon) = \underline{0}, \text{ and} \tag{3.2.7}$$

$$f_i, \frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial u_j}, \frac{\partial f_i}{\partial p_j}, \frac{\partial f_i}{\partial q_j} \in C([0,\pi]\times[0,\infty>\times\mathbb{R}^{3n}\times[-\epsilon_0,\epsilon_0],\mathbb{R}) \text{ for } i,j = 1,2,\dots,n,$$

$$\text{with } \underline{F}(\underline{u};\epsilon)(0,t) = \underline{F}(\underline{u};\epsilon)(\pi,t) = \underline{0} \quad \text{for } t \geq 0. \tag{3.2.8}$$

Furthermore, $\underline{f}(x,t,u,p,q;\epsilon)$ and its partial derivatives with respect to x,u,p and q are assumed to be uniformly bounded for those values of t under consideration.

To prove in the classical sense existence and uniqueness of the solution of the initial-boundary value problem (3.2.1)-(3.2.4) an equivalent system of coupled integral equations will be used. In order to derive this system of integral equations the initial-boundary value problem is transformed into an initial value problem by extending the vector-valued functions $\underline{u}$, $\underline{f}$, $\underline{u}_0$ and $\underline{u}_1$ in x to odd and 2π-periodic functions (see for instance [28, chapter 2]). The extensions of $\underline{u}$, $\underline{f}$, $\underline{u}_0$ and $\underline{u}_1$ are denoted by $\underline{u}^*$, $\underline{f}^*$, $\underline{u}_0^*$ and $\underline{u}_1^*$ respectively. Then, assuming that $\underline{u}^*, \underline{u}_t^*, \underline{u}_x^*, \underline{u}_{tt}^*, \underline{u}_{tx}^* = \underline{u}_{xt}^*$ and $\underline{u}_{xx}^*$ are continuous on $-\infty < x < \infty$ and $t \geq 0$, an integral equation for the solution $\underline{u}^*$ of the initial value problem is given by

$$\underline{u}^*(x,t;\epsilon) = -\frac{\epsilon}{2}\int_0^t \underline{If}^*(\tau;x,t)d\tau + \underline{u}_\ell^*(x,t;\epsilon), \tag{3.2.9}$$

where $\underline{If}^*(\tau;x,t)$ is a vector with elements $If_i^*(\tau;x,t)$ defined by

$$If_i^*(\tau;x,t) = \frac{1}{c_{ii}} \int_{x-c_{ii}(t-\tau)}^{x+c_{ii}(t-\tau)} f_i^*(\xi,\tau,\underline{u}^*(\xi,\tau;\epsilon),\underline{u}_t^*(\xi,\tau;\epsilon),\underline{u}_\xi^*(\xi,\tau;\epsilon);\epsilon)d\xi$$

for $i = 1,2,...,n$, and where $\underline{u}_\ell^*(x,t;\epsilon)$ is a vector with elements $u_{\ell i}^*(x,t;\epsilon)$ defined by

$$u_{\ell i}^*(x,t;\epsilon) = \frac{1}{2}\, u_{0i}^*(x+c_{ii}t;\epsilon) + \frac{1}{2}\, u_{0i}^*(x-c_{ii}t;\epsilon) + \frac{1}{2c_{ii}} \int_{x-c_{ii}t}^{x+c_{ii}t} u_{1i}^*(\xi;\epsilon)d\xi$$

for $i = 1,2,...,n$. As usual the integral of a matrix-valued function is obtained by taking the integral of each element of the matrix-valued function. Using reflection principles (3.2.9) can be rewritten as a system of coupled integral equations on the semi-infinite strip $0 \le x \le \pi$, $0 \le t < \infty$, yielding

$$\underline{u}(x,t;\epsilon) = \frac{\epsilon}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t)\underline{F}(\underline{u};\epsilon)(\xi,\tau)d\xi d\tau + \underline{u}_\ell(x,t;\epsilon), \tag{3.2.10}$$

where $G(\xi,\tau;x,t)$ is the (nxn) diagonal-matrix with diagonal elements

$$\begin{aligned} g_{ii}(\xi,\tau;x,t) = \frac{1}{c_{ii}} \sum_{k\in\mathbb{Z}} \Big\{ & H(c_{ii}(t-\tau)-\xi+2k\pi-x)H(c_{ii}(t-\tau)+\xi-2k\pi+x) + \\ & -H(c_{ii}(t-\tau)+\xi+2k\pi-x)H(c_{ii}(t-\tau)-\xi-2k\pi+x) \Big\} \end{aligned} \tag{3.2.11}$$

for $i = 1,2,...,n$, and where $\underline{u}_\ell$ is given by

$$\underline{u}_\ell(x,t;\epsilon) = \frac{1}{2} \int_0^\pi \left\{ \frac{\partial G}{\partial \tau}(\xi,0;x,t)\underline{u}_0(\xi;\epsilon) - G(\xi,0;x,t)\underline{u}_1(\xi;\epsilon)\right\} d\xi. \tag{3.2.12}$$

The function $H(a)$ on $\mathbb{R}$ is equal to 1 for $a > 0$, $\frac{1}{2}$ for $a = 0$ and zero otherwise. In (3.2.12) it is assumed that g_{ii} is differentiated according to the rule $\frac{d}{d\tau}\{ H(f(\tau))H(g(\tau)) \} = \delta_0(f(\tau)) \frac{df(\tau)}{d\tau} H(g(\tau)) + H(f(\tau))\delta_0(g(\tau)) \frac{dg(\tau)}{d\tau}$, where δ_0 is the Dirac delta function. In fact, g_{ii} as defined by (3.2.11) is the Green's function for the differential operator $\frac{\partial^2}{\partial t^2} - c_{ii}^2 \frac{\partial^2}{\partial x^2}$ and the Dirichlet boundary conditions. And so, G can be identified with the matrix-valued function of Green for the differential operator $\frac{\partial^2}{\partial t^2} - C \frac{\partial^2}{\partial x^2}$ and the boundary conditions (3.2.4). It is also worth noticing that the solution of the (linear) initial-boundary value problem (3.2.1)-(3.2.4) with $\underline{F} \equiv \underline{0}$ is given by $\underline{u}_\ell(x,t;\epsilon)$.

Elementary calculations show that if $\underline{v}(x,t;\epsilon)$ is a twice continuously differentiable solution of the initial-boundary value problem (3.2.1)-(3.2.4) then $\underline{v}(x,t;\epsilon)$ is a solution of the system of integral equations (3.2.10). And if $\underline{w}(x,t;\epsilon)$ is a twice continuously differentiable solution of the system of integral equations (3.2.10) then it can easily be shown that $\underline{w}(x,t;\epsilon)$ is a solution of the initial-boundary value problem (3.2.1)-(3.2.4). Hence, the system of integral equations (3.2.10) and the initial-boundary value problem (3.2.1)-(3.2.4) are equivalent if twice continuously differentiable solutions exist, that is, if vector-valued functions exist of which all elements are twice continuously differentiable. Now it will be proved that a unique, twice continuously differentiable solution of the system of integral equations (3.2.10) exists in a strip Ω_L of the (x,t)-plane. And so, a unique and twice continuously differentiable solution exists for the initial-boundary value problem (3.2.1)-(3.2.4) on Ω_L.

In order to prove existence and uniqueness in the classical sense of the solution of the system of nonlinear integral equations (3.2.10) a fixed point theorem will be used. Let Ω_L be given by

$$\Omega_L = [0,\pi]\times[0,L|\epsilon|^{-1}] \tag{3.2.13}$$

in which L is a sufficiently small, positive constant independent of ϵ. Let $C_M^2(\Omega_L,\mathbb{R}^n)$ be the space of all real-valued and twice continuously differentiable functions $\underline{w}(x,t) = (w_1(x,t),w_2(x,t),\dots,w_n(x,t))^T$ on Ω_L with norm $\|\cdot\|_{C_M^2}$ defined by:

$$\| \underline{w} \|_{C^2_M} = \sum_{i=1}^{n} \sum_{\substack{k,\ell=0 \\ k+\ell\le 2}}^{2} \max_{(x,t)\in\Omega_L} \left| \frac{\partial^{k+\ell} w_i(x,t)}{\partial x^k \partial t^\ell} \right| \le M.$$

From the smoothness properties of $\underline{u}_0$ and $\underline{u}_1$ it follows that (for fixed $\underline{u}_0$ and $\underline{u}_1$) there exists a positive constant M_1 independent of ϵ such that,

$$\| \underline{u}_\ell \|_{C^2_{M_1}} \le \frac{1}{2} M_1, \tag{3.2.14}$$

and from the smoothness properties of $\underline{F}$ (as given by (3.2.5) and (3.2.8)) it follows that there exist ϵ-independent constants M_2 and M_3 such that,

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} f_i(x,t,\underline{v}(x,t),\underline{v}_t(x,t),\underline{v}_x(x,t);\epsilon) \right| \le M_2, \tag{3.2.15}$$

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} \Big(f_i(x,t,\underline{v}(x,t),\underline{v}_t(x,t),\underline{v}_x(x,t);\epsilon) - f_i(x,t,\underline{w}(x,t),\underline{w}_t(x,t),\underline{w}_x(x,t);\epsilon) \Big) \right| \le$$

$$\le M_3 \| \underline{v}-\underline{w} \|_{C^2_{M_1}}, \tag{3.2.16}$$

for all $(x,t) \in \Omega_L$ and $i = 1,2,..,n$, $\epsilon \in [-\epsilon_0,\epsilon_0]$ and $\underline{v},\underline{w} \in C^2_{M_1}(\Omega_L,\mathbb{R}^n)$. Now let the integral operator T: $C^2(\Omega_L,\mathbb{R}^n) \to C^2(\Omega_L,\mathbb{R}^n)$, which is related to the system of integral equations (3.2.10), be defined by

$$(T\underline{w})(x,t) \equiv \frac{\epsilon}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t)\underline{F}(\underline{w};\epsilon)(\xi,\tau)d\xi d\tau + \underline{u}_\ell(x,t;\epsilon), \tag{3.2.17}$$

where G, $\underline{F}$ and $\underline{u}_\ell$ are given by (3.2.11), (3.2.5) and (3.2.12), respectively. According to Banach's fixed point theorem the integral operator T has a unique fixed point in

$C^2_{M_1}(\Omega_L, \mathbb{R}^n)$ if the operator T maps $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$ into itself and if T is a contraction on $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$. Now, it will be proved that the integral operator T satisfies these two conditions. It is not difficult to show that T maps $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$ into the space of (real- and vector-valued) twice continuously differentiable functions on Ω_L. In order to prove that T maps $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$ into itself estimates of the elements $g_{ii}(\xi,\tau;x,t)$ of the diagonal matrix G should be obtained for $0 \le \xi \le \pi$, $0 \le \tau \le t$, fixed x and t, and i = 1,2,...,n. It can be shown (see also [11]) that $|g_{ii}(\xi,\tau;x,t)| \le \frac{1}{c_{ii}}$ for $0 \le \xi < \pi$, $\tau \ge 0$, fixed x and t, and i = 1,2,...,n. Now let $(T\underline{v}-\underline{u}_\ell)_i$ be the i-th element (i = 1,2,...,n) of the (n×1)-matrix $T\underline{v}-\underline{u}_\ell$. Using (3.2.13)-(3.2.15) and (3.2.17) and putting $b = \max(c_{11},c_{22},...,c_{nn})$ and $c = \min(1,c_{11},c_{22},...,c_{nn})$ the following estimate can be made

$$\begin{aligned}
\|T\underline{v}\|_{C^2_{M_1}} &\le \|T\underline{v} - \underline{u}_\ell\|_{C^2_{M_1}} + \|\underline{u}_\ell\|_{C^2_{M_1}} = \\
&= \sum_{i=1}^{n} \sum_{\substack{k,\ell=0\\k+\ell\le 2}}^{2} \max_{(x,t)\in\Omega_L} \left| \frac{\partial^{k+\ell}}{\partial x^k \partial t^\ell} \Big((T\underline{v})(x,t) - \underline{u}_\ell(x,t;\epsilon)\Big)_i \right| + \|\underline{u}_\ell\|_{C^2_{M_1}} \le \\
&\le \frac{n}{c} \Big(\Big(\frac{\pi}{2} + 4 + b\Big) M_2 L + \epsilon_0 M_2 \Big) + \frac{1}{2} M_1
\end{aligned}$$

for all $\underline{v} \in C^2_{M_1}(\Omega_L, \mathbb{R}^n)$. Now ϵ_0 has been assumed to be sufficiently small and so, there exists an ϵ-independent constant L such that $\frac{n}{c}\Big(\Big(\frac{\pi}{2} + 4 + b\Big)M_2 L + \epsilon_0 M_2\Big) \le \frac{1}{2} M_1$. Hence, $\|T\underline{v}\|_{C^2_{M_1}} \le M_1$ for all $\underline{v} \in C^2_{M_1}(\Omega_L, \mathbb{R}^n)$. So, T maps $C^2_{M_1}$ into itself. Using (3.2.13), (3.2.16) and (3.2.17) it will be shown that T is a contraction on $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$. Let $\underline{v},\underline{w} \in C^2_{M_1}(\Omega_L, \mathbb{R}^n)$, then the following estimate can be obtained

$$\|T\underline{v} - T\underline{w}\|_{C^2_{M_1}} \le \frac{n}{c}\Big(\Big(\frac{\pi}{2} + 4 + b\Big) M_3 L + \epsilon_0 M_3\Big) \|\underline{v} - \underline{w}\|_{C^2_{M_1}} .$$

It is clear that there exists an ϵ-independent constant L such that $\frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)M_3L+\right.$ $\left.+\epsilon_0M_3\right) \le k < 1$. Since there always exists a constant L independent of ϵ such that $\frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)M_2L+\epsilon_0M_2\right) \le \frac{1}{2}M_1$ and $\frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)M_3L+\epsilon_0M_3\right) \le k < 1$, it follows that T maps $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$ into itself and is a contraction. Banach's fixed point theorem then implies that T has a unique fixed point in $C^2_{M_1}(\Omega_L, \mathbb{R}^n)$, that is, a unique and twice continuously differentiable function on Ω_L. Hence, the solution of the system of the integral equations (3.2.10) is unique and twice continuously differentiable on Ω_L. And so, on Ω_L a unique and twice continuously differentiable solution exists for the initial-boundary value problem (3.2.1)-(3.2.4).

Next it will be shown that the solution of the initial-boundary value problem (3.2.1)-(3.2.4) depends continuously on the initial values. Let $\underline{u}(x,t;\epsilon)$ satisfy (3.2.1)-(3.2.4) and let $\underline{\tilde{u}}(x,t;\epsilon)$ satisfy (3.2.1), (3.2.4), $\underline{\tilde{u}}(x,0;\epsilon) = \underline{\tilde{u}}_0(x;\epsilon)$ and $\underline{\tilde{u}}_t(x,0;\epsilon) = \underline{\tilde{u}}_1(x;\epsilon)$, where $\underline{\tilde{u}}_0$ and $\underline{\tilde{u}}_1$ satisfy (3.2.6) and (3.2.7). Let $\underline{\tilde{u}}_\ell$ be given by

$$\underline{\tilde{u}}_\ell(x,t;\epsilon) = \frac{1}{2}\int_0^\pi \left\{ \frac{\partial G}{\partial \tau}(\xi,0;x,t)\underline{\tilde{u}}_0(\xi;\epsilon) - G(\xi,0;x,t)\underline{\tilde{u}}_1(\xi;\epsilon)\right\} d\xi.$$

After subtracting the integral equation for $\underline{u}$ from the integral equation for $\underline{\tilde{u}}$, using (3.2.10), (3.2.13) and (3.2.16), assuming $\underline{u}$ and $\underline{\tilde{u}} \in C^2_{M_1}(\Omega_L, \mathbb{R}^n)$, one obtains the estimate

$$\|\underline{u}-\underline{\tilde{u}}\|_{C^2_{M_1}} \le \frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)M_3L+\epsilon_0M_3\right)\|\underline{u}-\underline{\tilde{u}}\|_{C^2_{M_1}} + \|\underline{u}_\ell-\underline{\tilde{u}}_\ell\|_{C^2_{M_1}} \le$$

$$\le k\,\|\underline{u}-\underline{\tilde{u}}\|_{C^2_{M_1}} + \|\underline{u}_\ell-\underline{\tilde{u}}_\ell\|_{C^2_{M_1}} \qquad \text{with } 0 \le k < 1.$$

This inequality implies $\|\underline{u}-\underline{\tilde{u}}\|_{C^2_{M_1}} \le \frac{1}{1-k}\|\underline{u}_\ell-\underline{\tilde{u}}_\ell\|_{C^2_{M_1}}$ with $0 \le k < 1$.

Since the solution $\underline{u}_\ell$ of the linear initial-boundary value problem (3.2.1)-(3.2.4) with $\underline{F} \equiv \underline{0}$ depends continuously on the initial values it follows from this inequality that small differences between the initial values generate small differences between the solutions $\underline{u}$ and $\tilde{\underline{u}}$ on Ω_L. In other words the solution of the initial-boundary value problem depends continuously on the initial values. The following theorem on the well-posedness of the problem can now be formulated.

Theorem 3.2.1

Suppose that $\underline{u}_0$ and $\underline{u}_1$ and $\underline{F}$ satisfy the assumptions (3.2.6)-(3.2.8). Then for any ϵ satisfying $0 < |\epsilon| \leq \epsilon_0 << 1$, the nonlinear initial-boundary value problem (3.2.1)-(3.2.4) and the equivalent system of nonlinear integral equations (3.2.10) have the same, unique and twice continuously differentiable solution for $0 \leq x \leq \pi$ and $0 \leq t \leq L|\epsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ϵ. Furthermore, this unique solution depends continuously on the initial values.

3.3. On the validity of formal approximations

Since the initial-boundary value problem (3.2.1)-(3.2.4) contains a small parameter ϵ perturbation methods may be applied to construct approximations of the solution. In most perturbation methods for weakly nonlinear problems a function is constructed that satisfies the differential equation(s) and the initial conditions up to some order depending on the small parameter ϵ. Such a function is usually called a formal approximation. It requires an additional analysis to show that this formal approximation is an asymptotic approximation as ϵ tends to zero. Therefore suppose that on Ω_L (given by (3.2.13)) a twice continuously differentiable function $\underline{v}(x,t;\epsilon)$ is constructed satisfying

$$\underline{v}_{tt} - C\underline{v}_{xx} + \epsilon\underline{F}(\underline{v};\epsilon) = |\epsilon|^m \underline{c}_1(x,t;\epsilon), \qquad m > 1, \tag{3.3.1}$$

$$\underline{v}(x,0;\epsilon) = \underline{u}_0(x;\epsilon) + |\epsilon|^{m-1} \underline{c}_2(x;\epsilon) \equiv \underline{v}_0(x;\epsilon), \qquad 0 < x < \pi, \tag{3.3.2}$$

$$\underline{v}_t(x,0;\epsilon) = \underline{u}_1(x;\epsilon) + |\epsilon|^{m-1} \underline{c}_3(x;\epsilon) \equiv \underline{v}_1(x;\epsilon), \qquad 0 < x < \pi, \tag{3.3.3}$$

$$\underline{v}(0,t;\epsilon) = \underline{v}(\pi,t;\epsilon) = \underline{0}, \qquad 0 \le t \le L|\epsilon|^{-1}, \tag{3.3.4}$$

where ϵ, $\underline{u}_0$, $\underline{u}_1$ and $\underline{F}$ satisfy (3.2.5)-(3.2.8) and where $\underline{c}_1(x,t;\epsilon) = (c_{11}(x,t;\epsilon),c_{12}(x,t;\epsilon),\ldots, c_{1n}(x,t;\epsilon))^T$, $\underline{c}_2(x;\epsilon) = (c_{21}(x;\epsilon),c_{22}(x;\epsilon),\ldots,c_{2n}(x;\epsilon))^T$ and $\underline{c}_3(x;\epsilon) = (c_{31}(x;\epsilon),c_{32}(x;\epsilon),\ldots, c_{3n}(x;\epsilon))^T$ satisfy

$$c_{1i}, \frac{\partial c_{1i}}{\partial x} \in C(\Omega_L \times [-\epsilon_0,\epsilon_0],\mathbb{R}) \qquad \text{for } i = 1,2,\ldots,n \tag{3.3.5}$$

with $\underline{c}_1(0,t;\epsilon) = \underline{c}_1(\pi,t;\epsilon) = \underline{0}$ for $0 \le t \le L|\epsilon|^{-1}$,

$$c_{2i}, \frac{\partial c_{2i}}{\partial x}, \frac{\partial^2 c_{2i}}{\partial x^2} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R}) \qquad \text{for } i = 1,2,\ldots,n \tag{3.3.6}$$

with $\underline{c}_2(0;\epsilon) = \underline{c}_2(\pi;\epsilon) = \dfrac{\partial^2 \underline{c}_2(0;\epsilon)}{\partial x^2} = \dfrac{\partial^2 \underline{c}_2(\pi;\epsilon)}{\partial x^2} = \underline{0}$, and for $i = 1,2,\ldots,n$

$$c_{3i}, \frac{\partial c_{3i}}{\partial x} \in C([0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R}) \qquad \text{with } \underline{c}_3(0;\epsilon) = \underline{c}_3(\pi;\epsilon) = \underline{0}. \tag{3.3.7}$$

Furthermore, (for $i = 1,2,\ldots,n$) $c_{1i}(x,t;\epsilon)$ and its derivative with respect to x are supposed to be uniformly bounded for those values of t and ϵ under consideration. From theorem 3.2.1 it follows that the initial-boundary value problem (3.3.1)-(3.3.4) has a unique, twice continuously differentiable solution $\underline{v}$ on a time-scale of $O(|\epsilon|^{-1})$. This initial-boundary value problem can then be transformed into the equivalent system of integral equations

$$\underline{v}(x,t;\epsilon) = \frac{\epsilon}{2} \int_0^t \int_0^\pi G(\xi,\tau;x,t)\tilde{\underline{F}}(\underline{v};\epsilon)(\xi,\tau)\, d\xi d\tau + \underline{v}_\ell(x,t;\epsilon), \tag{3.3.8}$$

where G is given by (3.2.11) and where $\tilde{\underline{F}}$ and $\underline{v}_\ell$ are respectively given by

$$\tilde{\underline{F}}(\underline{v};\epsilon)(x,t) \equiv \underline{F}(\underline{v};\epsilon)(x,t) - |\epsilon|^{m-1}\, \underline{c}_1(x,t;\epsilon) \qquad \text{and}$$

$$\underline{v}_\ell(x,t;\epsilon) = \frac{1}{2} \int_0^\pi \left\{ \frac{\partial G}{\partial \tau}(\xi,0;x,t)\underline{v}_0(\xi;\epsilon) - G(\xi,0;x,t)\underline{v}_1(\xi;\epsilon) \right\} d\xi.$$

Now, it will be shown that the formal approximation $\underline{v}$ is an asymptotic approximation (as $\epsilon \to 0$) of the solution of the initial-boundary value problem (3.2.1)-(3.2.4) if $m > 1$, that is, it will be proved that

$$\| \underline{u}-\underline{v} \|_{C^2_{M_1}} = O(|\epsilon|^{m-1}) \qquad \text{as } \epsilon \to 0.$$

This result implies that

$$\lim_{\epsilon\to 0} \left| u_i(x,t;\epsilon) - v_i(x,t;\epsilon) \right| = 0 \qquad \text{for } i = 1,2,\dots,n \text{ and } (x,t) \in \Omega_L.$$

Subtracting the system of integral equations (3.3.8) from the system of integral equations (3.2.10), supposing that $\underline{v}_\ell$ satisfies (3.2.14) and that $\tilde{\underline{F}}$ satisfies (3.2.15) and (3.2.16), using (3.2.13), (3.2.16), and the fact that $\underline{u},\underline{v} \in C^2_{M_1}(\Omega_L,\mathbb{R}^n)$, the following estimate is obtained

$$\|\underline{u}-\underline{v}\|_{C^2_{M_1}} \le \frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)M_3L+\epsilon_0M_3\right)\|\underline{u}-\underline{v}\|_{C^2_{M_1}} + \|\underline{c}\|_{C^2_{M_1}} + \|\underline{u}_\ell-\underline{v}_\ell\|_{C^2_{M_1}} \le$$

$$\le k\,\|\underline{u}-\underline{v}\|_{C^2_{M_1}} + \|\underline{c}\|_{C^2_{M_1}} + \|\underline{u}_\ell-\underline{v}_\ell\|_{C^2_{M_1}},$$

with $0 \le k < 1$, $b = \max(c_{11},c_{22},\dots,c_{nn})$, $c = \min(1,c_{11},c_{22},\dots,c_{nn})$ and where $\underline{c}$ is given by

$$\underline{c}(x,t;\epsilon) = \frac{|\epsilon|^m}{2}\int_0^t\int_0^\pi G(\xi,\tau;x,t)\underline{c}_1(\xi,\tau;\epsilon)\,d\xi d\tau,$$

and where $\underline{u}_\ell - \underline{v}_\ell$ is given by

$$\underline{u}_\ell(x,t;\epsilon) - \underline{v}_\ell(x,t;\epsilon) = -\frac{|\epsilon|^{m-1}}{2}\int_0^\pi \left\{\frac{\partial G}{\partial\tau}(\xi,0;x,t)\underline{c}_2(\xi;\epsilon) - G(\xi,0;x,t)\underline{c}_3(\xi;\epsilon)\right\} d\xi.$$

Hence,

$$\|\underline{u}-\underline{v}\|_{C^2_{M_1}} \le \frac{1}{1-k}\left\{\|\underline{c}\|_{C^2_{M_1}} + \|\underline{u}_\ell-\underline{v}_\ell\|_{C^2_{M_1}}\right\} \quad \text{with } 0 \le k < 1.$$

From the smoothness properties of $\underline{c}_1$, $\underline{c}_2$ and $\underline{c}_3$ it follows that there exists a constant K independent of ϵ, such that

$$\|\underline{c}\|_{C^2_{M_1}} \leq \frac{n}{c}\left(\left(\frac{\pi}{2}+4+b\right)KL + |\epsilon|K\right)|\epsilon|^{m-1} \text{ and}$$

$$\|\underline{u}_\ell-\underline{v}_\ell\|_{C^2_{M_1}} \leq \frac{n}{c}\left(\frac{\pi}{2}+7+3b+b^2\right) K|\epsilon|^{m-1}.$$

So, $$\|\underline{u}-\underline{v}\|_{C^2_{M_1}} \leq \frac{n|\epsilon|^{m-1}K}{c(1-k)}\left\{\left(\frac{\pi}{2}+4+b\right)L + |\epsilon| + \frac{\pi}{2}+7+3b+b^2\right\}$$

For $m > 1$ this inequality implies the asymptotic validity (as $\epsilon \to 0$) of the formal approximation $\underline{v}$. The following theorem has now been established.

Theorem 3.3.1

Let the formal approximation $\underline{v}$ satisfy (3.3.1)-(3.3.4), where ϵ, $\underline{u}_0$, $\underline{u}_1$ and $\underline{F}$ are given by (3.2.5)-(3.2.8) and where $\underline{c}_1$, $\underline{c}_2$ and $\underline{c}_3$ satisfy (3.3.5)-(3.3.7). Then for $m > 1$, the formal approximation $\underline{v}$ is an asymptotic approximation (as $\epsilon \to 0$) of the solution $\underline{u}$ of the nonlinear initial-boundary value problem (3.2.1)-(3.2.4). The asymptotic approximation $\underline{v}$ is valid for those values of the independent variables x and t for which problem (3.2.1)-(3.2.4) has been proved well-posed. That is,

$$\|\underline{u}-\underline{v}\|_{C^2_{M_1}} = O(|\epsilon|^{m-1}), \text{ implying } |u_i(x,t;\epsilon) - v_i(x,t;\epsilon)| = O(|\epsilon|^{m-1})$$

for $i = 1,2,\ldots,n$ and $0 \leq x \leq \pi$, $0 \leq t \leq L|\epsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ϵ.

3.4. A model of the galloping oscillations of overhead transmission lines

In this section a model describing the galloping oscillations of overhead transmission lines will be derived. Galloping is a low frequency, large amplitude phenomenon involving an almost purely vertical oscillation of single-conductor lines on which for instance ice has accreted. The frequencies involved are so low that the assumption can be made that the aerodynamic forces are as in steady flow. Another consequence of these low frequencies is that structural damping may be neglected. In severe cases galloping may give rise to conductor damage due to impact of conductor lines and due to flashover as a result of a phase-difference between conductor lines, for which the mutual distance has become too small. The usual conditions (see [26]) causing galloping are those of incipient icing in a stable atmospheric environment implying uniform (but not necessarily high velocity) airflows.

In [1] an oscillator with two degrees of freedom has been considered to describe the oscillations of a rigid circular cylinder with a small ice ridge. In that approach a system of two coupled, ordinary differential equations is obtained, describing the displacements of the cylinder in two directions. In [12] a cylinder-shaped transmission line has been considered to describe the vertical displacement of the conductor due to galloping. In this section the vertical as well as the horizontal displacements of the transmission line will be taken into account. To describe the galloping oscillations a circular cylinder-shaped conductor will be considered to be situated in a horizontal airflow. Such a symmetric circular conductor cannot exhibit galloping because there cannot be generated a force that lifts the conductor against gravity. On the other hand, a conductor on which ice has accreted may gallop if it adopts a suitable attitude to the wind. To describe this phenomenon a right-handed coordinate system is set up where one of the endpoints of the conductor is the origin. Through this point three mutually perpendicular axes (the x-, y- and z-axis) are drawn, where the z-axis coincides with the direction of gravity and the y-axis with the direction of the airflow. The three coordinate axes span the three coordinate planes in space, the (x,y)-, (x,z)- and (y,z)-planes. On each coordinate axis a unit vector is fixed: on the x-axis the vector $\underline{e}_x$, on the y-axis the vector $\underline{e}_y$ and on the z-axis the vector $\underline{e}_z$, which has a direction opposite to gravity. The coordinate axes are directed by these vectors, such that a right-handed coordinate system is obtained. The coordinates of the endpoints of the conductor are supposed to be (0,0,0) and (ℓ,0,0), where ℓ is the distance between the endpoints. To model galloping a cross-section (perpendicular to the x-axis) of the conductor with ice ridge is considered. Assume that all cross-sectional shapes are congruent and have only one axis of symmetry. Along the axis of symmetry of a cross-section a unit vector $\underline{e}_s$ is defined to be directing away from the ice ridge and starting in the centre of the uniced cross-section. In figure 3.4.1 the centre of the cross-section is considered to be at $x = x_0$, $y = y_0$ and $z = z_0$ with $0 < x_0 < \ell \leq \ell_c$, where ℓ_c is the length of the conductor.

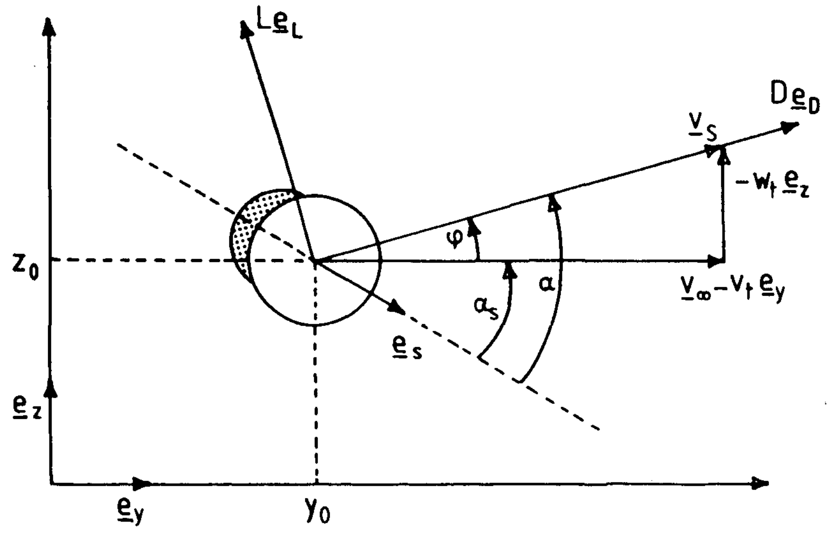

Figure 3.4.1. Cross-section of the circular cylinder-shaped conductor with ice ridge in a uniform airflow $\underline{v}_\infty = v_\infty \underline{e}_y$.

At $x = x_0$ and time t the y-coordinate and the z-coordinate of the centre of the cross-section are denoted by $v(x_0,t)$ and $w(x_0,t)$ respectively. Assume that every cross-section perpendicular to the x-axis oscillates in the (y,z)-plane. Furthermore, the torsion of the conductor is not taken into account. Let the static angle of attack α_s (assumed to be constant and identical for all cross-sections) be the angle between $\underline{e}_s$ and the uniform airflow $\underline{v}_\infty = v_\infty \underline{e}_y (v_\infty > 0)$, that is, $\alpha_s := \angle(\underline{e}_s, \underline{v}_\infty)$ with $|\alpha_s| \le \pi$. In this uniform airflow with flow velocity $\underline{v}_\infty = v_\infty \underline{e}_y$ the conductor may oscillate due to the lift force $L\underline{e}_L$ and the drag force $D\underline{e}_D$. It should be noted that the drag force $D\underline{e}_D$ has the direction of the virtual windvelocity $\underline{v}_s \equiv \underline{v}_\infty - v_t \underline{e}_y - w_t \underline{e}_z$ and that the lift force $L\underline{e}_L$ has a direction perpendicular to the virtual windvelocity $\underline{v}_s$ ($\underline{e}_L$ is chosen perpendicular and anti-clockwise to $\underline{e}_D$). In figure 3.4.1 the forces $L\underline{e}_L$ and $D\underline{e}_D$ acting on the cross-section are given. Now the conductor is considered to be an one-dimensional continuum in which the only interaction between different parts is due to a tension T, which is assumed to be constant in space and time. The validity of this assumption will be discussed in section 3.6. The equations describing the horizontal and the vertical motion of the conductor are then given by

$$\rho_c A v_{tt} - TA \frac{\partial}{\partial x} \left[\frac{v_x}{(1+v_x^2+w_x^2)^{1/2}} \right] = D\cos\phi - L\sin\phi, \tag{3.4.1}$$

$$\rho_c A w_{tt} - TA \frac{\partial}{\partial x} \left[\frac{w_x}{(1+v_x^2+w_x^2)^{1/2}} \right] = -\rho_c A g + D\sin\phi + L\cos\phi, \tag{3.4.2}$$

where D and L are the magnitudes of the drag and lift force acting on the conductor per unit length of the conductor respectively, ρ_c the mass-density of the conductor (including the small ice ridge), A the constant cross-sectional area of the conductor (including the small ice ridge), ϕ the angle between $\underline{v}_\infty - v_t \underline{e}_y$ and $\underline{v}_s$ (that is, $\phi := \angle(\underline{v}_\infty - v_t \underline{e}_y, \underline{v}_s)$

with $|\phi| \leq \pi$) and g the gravitational acceleration. The magnitudes D and L of the aerodynamic forces may be given by

$$D = \frac{1}{2} \rho_a d c_D(\alpha) v_s^2, \tag{3.4.3}$$

$$L = \frac{1}{2} \rho_a d c_L(\alpha) v_s^2, \tag{3.4.4}$$

where ρ_a is the air-density, d the diameter of the cross-section of the uniced conductor, $v_s^2 = (v_\infty - v_t)^2 + w_t^2$, α the angle between $\underline{e}_s$ and $\underline{v}_s$ (that is, $\alpha := \angle(\underline{e}_s, \underline{v}_s)$ with $|\alpha| \leq \pi$), and $c_D(\alpha)$ and $c_L(\alpha)$ the quasi-steady drag- and lift-coefficients, which may be obtained from wind-tunnel measurements. For a certain range of values of v_∞ some characteristic results from wind-tunnel experiments are given in figure 3.4.2 (see also [1,12,24]). According to the Den Hartog criterion [10] a two-dimensional section is aerodynamically unstable if

$$c_D(\alpha) + \frac{dc_L(\alpha)}{d\alpha} < 0.$$

From figure 3.4.2 it follows that this condition is likely to be satisfied for some interval in α with $\alpha_0 < \alpha < \alpha_2$, where α_0 and α_2 are determined by $c_D(\alpha) + \dfrac{dc_L(\alpha)}{d\alpha} = 0$. For certain range of values of α (including those values which satisfy the Den Hartog criterion) the drag- and lift-coefficients $c_D(\alpha)$ and $c_L(\alpha)$ can be approximated by

$$c_{D0} + c_{D1}(\alpha-\alpha_1) + c_{D2}(\alpha-\alpha_1)^2 + c_{D3}(\alpha-\alpha_1)^3, \tag{3.4.5}$$

$$c_{L1}(\alpha-\alpha_1) + c_{L2}(\alpha-\alpha_1)^2 + c_{L3}(\alpha-\alpha_1)^3, \tag{3.4.6}$$

with $c_{D0} > 0$, $c_{L1} < 0$, $c_{L3} > 0$, $\alpha_0 < \alpha_1 < \alpha_2$ where $c_L(\alpha_1) = 0$. Since galloping is a low frequency phenomenon it may be assumed that $|v_t| << v_\infty$ and $|w_t| << v_\infty$. The

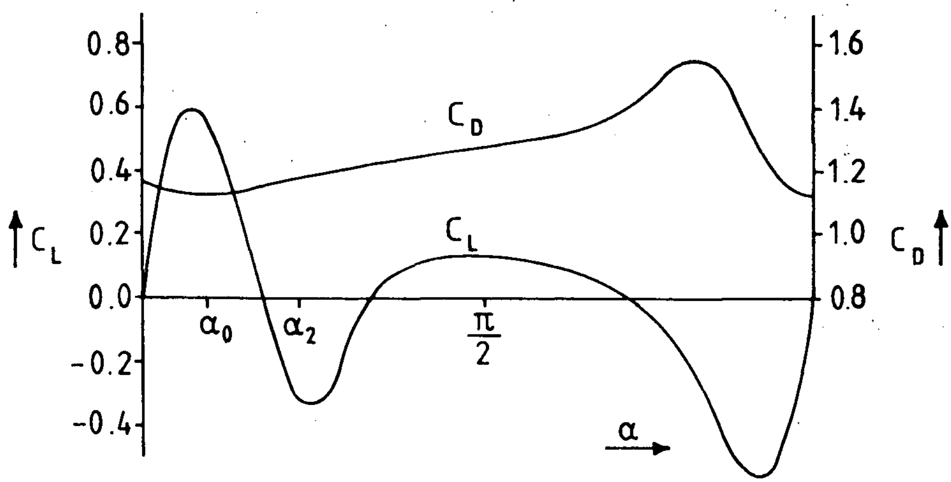

Figure 3.4.2. Typical variation of the drag and lift-coefficients c_D and c_L with angle of attack for a symmetric profile with small icy nose.

right-hand sides of the equations (3.4.1) and (3.4.2) can be considered as functions of $\frac{v_t}{v_\infty}$ and $\frac{w_t}{v_\infty}$, and so these right-hand sides are be expanded in Taylor series near $\frac{v_t}{v_\infty} = 0$ and $\frac{w_t}{v_\infty} = 0$. To obtain these Taylor series the approximations (3.4.5) and (3.4.6) for c_D and c_L respectively are used. Since the amplitudes of the oscillations are small compared to the length of the conductor and since galloping is an oscillation described by the lower modes of vibration, it may be assumed that $|v_x| << 1$ and $|w_x| << 1$. And so, the left-hand sides of the equations (3.4.1) and (3.4.2) are expanded in Taylor series near $v_x = 0$ and $w_x = 0$. Neglecting terms of degree four and higher equations (3.4.1) and (3.4.2) become

$$v_{tt} - c^2 \left\{ \left(1 - \frac{3}{2} v_x^2 - \frac{1}{2} w_x^2\right) v_{xx} - v_x w_x w_{xx} \right\} =$$

$$= \frac{\rho_a d v_\infty^2}{2\rho_c A} \left\{ a_{00} + a_{10} \frac{v_t}{v_\infty} + a_{01} \frac{w_t}{v_\infty} + a_{20} \frac{v_t^2}{v_\infty^2} + a_{11} \frac{v_t w_t}{v_\infty^2} + a_{02} \frac{w_t^2}{v_\infty^2} + a_{03} \frac{w_t^3}{v_\infty^3} \right\},$$

(3.4.7)

$$w_{tt} - c^2 \left\{ -v_x w_x v_{xx} + \left(1 - \tfrac{1}{2} v_x^2 - \tfrac{3}{2} w_x^2\right) w_{xx} \right\} =$$

$$= \frac{\rho_a d v_\infty^2}{2\rho_c A} \left\{ b_{00} + b_{10}\frac{v_t}{v_\infty} + b_{01}\frac{w_t}{v_\infty} + b_{20}\frac{v_t^2}{v_\infty^2} + b_{11}\frac{v_t w_t}{v_\infty^2} + b_{02}\frac{w_t^2}{v_\infty^2} + b_{03}\frac{w_t^3}{v_\infty^3} \right\}, \tag{3.4.8}$$

where $c = (T\rho_c^{-1})^{1/2}$,

$$a_{00} = c_D(\alpha_s), \qquad a_{10} = -2c_D(\alpha_s),$$

$$a_{01} = c_L(\alpha_s) - \frac{dc_D(\alpha_s)}{d\alpha}, \qquad a_{20} = c_D(\alpha_s),$$

$$a_{11} = -c_L(\alpha_s) + \frac{dc_D(\alpha_s)}{d\alpha}, \qquad a_{02} = \frac{1}{2} c_D(\alpha_s) - \frac{dc_L(\alpha_s)}{d\alpha} + \frac{1}{2}\frac{d^2c_D(\alpha_s)}{d\alpha^2},$$

$$a_{03} = \frac{1}{2} c_L(\alpha_s) - \frac{1}{6}\frac{dc_D(\alpha_s)}{d\alpha} + \frac{1}{2}\frac{d^2c_L(\alpha_s)}{d\alpha^2} - \frac{1}{6}\frac{d^3c_D(\alpha_s)}{d\alpha^3}, \tag{3.4.9}$$

$$b_{00} = c_L(\alpha_s) - \frac{2\rho_c A g}{\rho_a d v_\infty^2}, \qquad b_{10} = -2c_L(\alpha_s),$$

$$b_{01} = -c_D(\alpha_s) - \frac{dc_L(\alpha_s)}{d\alpha}, \qquad b_{20} = c_L(\alpha_s),$$

$$b_{11} = c_D(\alpha_s) + \frac{dc_L(\alpha_s)}{d\alpha}, \qquad b_{02} = \frac{1}{2} c_L(\alpha_s) + \frac{dc_D(\alpha_s)}{d\alpha} + \frac{1}{2}\frac{d^2c_L(\alpha_s)}{d\alpha^2},$$

$$b_{03} = -\frac{1}{2} c_D(\alpha_s) - \frac{1}{6}\frac{dc_L(\alpha_s)}{d\alpha} - \frac{1}{2}\frac{d^2c_D(\alpha_s)}{d\alpha^2} - \frac{1}{6}\frac{d^3c_L(\alpha_s)}{d\alpha^3}.$$

In (3.4.9) the approximations (3.4.5) and (3.4.6) for c_D and c_L respectively should be substituted. Applying the transformation $v(x,t) = \tilde{v}(x,t) - \dfrac{\rho_a d v_\infty^2 a_{00}}{4TA} x(x-\ell)$ and

$w(x,t) = \tilde{w}(x,t) - \frac{\rho_a d v_\infty^2 b_{00}}{4TA} x(x-\ell)$ and using the dimensionless variables $\bar{v} = \frac{\pi c}{\ell v_\infty} \tilde{v}$, $\bar{w} = \frac{\pi c}{\ell v_\infty} \tilde{w}$, $\bar{x} = \frac{\pi}{\ell} x$ and $\bar{t} = \frac{\pi c}{\ell} t$ the equations (3.4.7) and (3.4.8) become

$$\bar{v}_{\bar{t}\bar{t}} - \bar{v}_{\bar{x}\bar{x}} + \left(\frac{v_\infty}{c}\right)^2 \left\{ \left(\frac{3}{2} \left(\bar{v}_{\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right)^2 + \right.\right.$$

$$\left. + \frac{1}{2} \left(\bar{w}_{\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right)^2 \right) \left(\bar{v}_{\bar{x}\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{2\pi\rho_c Ac}\right) +$$

$$\left. + \left(\bar{v}_{\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right) \left(\bar{w}_{\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right) \left(\bar{w}_{\bar{x}\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{2\pi\rho_c Ac}\right) \right\} =$$

$$= \frac{\rho_a d\ell}{2\pi\rho_c A} \left(\frac{v_\infty}{c}\right) \left\{ a_{10}\bar{v}_{\bar{t}} + a_{01}\bar{w}_{\bar{t}} + a_{20}\bar{v}_{\bar{t}}^2 + a_{11}\bar{v}_{\bar{t}}\bar{w}_{\bar{t}} + a_{02}\bar{w}_{\bar{t}}^2 + a_{03}\bar{w}_{\bar{t}}^3 \right\}, \qquad (3.4.10)$$

$$\bar{w}_{\bar{t}\bar{t}} - \bar{w}_{\bar{x}\bar{x}} + \left(\frac{v_\infty}{c}\right)^2 \left\{ \left(\bar{v}_{\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right) \left(\bar{w}_{\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right) \times \right.$$

$$\times \left(\bar{v}_{\bar{x}\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{2\pi\rho_c Ac}\right) + \left(\frac{1}{2} \left(\bar{v}_{\bar{x}} - \frac{\rho_a d v_\infty \ell a_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right)^2 + \right.$$

$$\left.\left. + \frac{3}{2} \left(\bar{w}_{\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{4\pi\rho_c Ac} (2\bar{x}-\pi)\right)^2 \right) \left(\bar{w}_{\bar{x}\bar{x}} - \frac{\rho_a d v_\infty \ell b_{00}}{2\pi\rho_c Ac}\right) \right\} =$$

$$= \frac{\rho_a d\ell}{2\pi\rho_c A} \left(\frac{v_\infty}{c}\right) \left\{ b_{10}\bar{v}_{\bar{t}} + b_{01}\bar{w}_{\bar{t}} + b_{20}\bar{v}_{\bar{t}}^2 + b_{11}\bar{v}_{\bar{t}}\bar{w}_{\bar{t}} + b_{02}\bar{w}_{\bar{t}}^2 + b_{03}\bar{w}_{\bar{t}}^3 \right\}, \qquad (3.4.11)$$

where the dimensionless constants $a_{00}, a_{10}, \ldots, b_{03}$ are given by (3.4.9).

Typical values of the physical quantities in a practical application are: $\ell = 400$ m, $d = 0.04$ m, $A = \pi\left(\frac{d}{2}\right)^2 = 4\pi.10^{-4}$ m^2, $\rho_c = 4000$ kg/m^3, $\rho_a = 1.25$ kg/m^3, $g = 10$ m/s^2 and $v_\infty = 10$ m/s. The tension T in the conductor is estimated by $\frac{1}{2}\rho_c g \left(\frac{\ell}{2}\right)^2 s_0^{-1}$, where s_0 (usually 2 or 3 per cent of ℓ) is the sag of the conductor. Let s_0 be 10 m, then $T = 8.10^7$ kg/ms^2 and consequently $c = 140$ m/s (c may be identified with the speed of propagation of transversal waves in the conductor). By putting $\tilde{\epsilon} = \frac{v_\infty}{c}$ and by assuming that the static angle of attack α_s is such that galloping may set in according to the instability criterion of Den Hartog [10], that is, by assuming that $\alpha_s = \alpha_1 + O(\tilde{\epsilon})$, it then follows that: $(a_{00} \approx \frac{12}{10})$

$$\tilde{\epsilon} = \frac{v_\infty}{c} \approx \frac{1}{14}, \quad \frac{\rho_a d\ell}{2\pi\rho_c A} \approx \frac{5}{8}, \quad \frac{\rho_a d v_\infty \ell a_{00}}{4\pi\rho_c Ac} \approx \frac{1}{37} \text{ and } \frac{\rho_a d v_\infty \ell b_{00}}{4\pi\rho_c Ac} \approx \frac{5}{11}.$$

The equations (3.4.10) and (3.4.11) now become up to order $\tilde{\epsilon}$

$$\bar{v}_{\bar{t}\bar{t}} - \bar{v}_{\bar{x}\bar{x}} = \epsilon\left\{a_{10}\bar{v}_{\bar{t}} + a_{01}\bar{w}_{\bar{t}} + a_{20}\bar{v}_{\bar{t}}^2 + a_{11}\bar{v}_{\bar{t}}\bar{w}_{\bar{t}} + a_{02}\bar{w}_{\bar{t}}^2 + a_{03}\bar{w}_{\bar{t}}^3\right\}, \quad (3.4.12)$$

$$\bar{w}_{\bar{t}\bar{t}} - \bar{w}_{\bar{x}\bar{x}} = \epsilon\left\{b_{10}\bar{v}_{\bar{t}} + b_{01}\bar{w}_{\bar{t}} + b_{20}\bar{v}_{\bar{t}}^2 + b_{11}\bar{v}_{\bar{t}}\bar{w}_{\bar{t}} + b_{02}\bar{w}_{\bar{t}}^2 + b_{03}\bar{w}_{\bar{t}}^3\right\}, \quad (3.4.13)$$

where $\epsilon = \frac{\rho_a d\ell}{2\pi\rho_c A}\tilde{\epsilon}$ is a small, positive parameter and where the constants $a_{10}, a_{01}, \ldots, b_{03}$ are given by (3.4.9) with $\alpha_s = \alpha_1$, that is, $a_{10} = -2c_{D0}$, $a_{01} = -c_{D1}$, $a_{20} = c_{D0}$, $a_{11} = c_{D1}$, $a_{02} = \frac{1}{2}c_{D0} + c_{D2} - c_{L1}$, $a_{03} = -\frac{1}{6}c_{D1} - c_{D3} + c_{L2}$, $b_{10} = 0$, $b_{01} = -c_{D0} - c_{L1}$, $b_{20} = 0$, $b_{11} = c_{D0} + c_{L1}$, $b_{02} = c_{D1} + c_{L2}$ and $b_{03} = -\frac{1}{2}c_{D0} + - c_{D2} - \frac{1}{6}c_{L1} - c_{L3}$. For the cross-sectional shape of the conductor with small ice ridge under consideration the aerodynamic coefficients $c_{D0}, c_{D1}, c_{D2}, c_{D3}, c_{L1}, c_{L2}$ and c_{L3} may be determined from wind-tunnel measurements (as for instance given in figure 3.4.2). Figure 3.4.2 suggests that $c_{D0} > 0$, $c_{D1} > 0$, $c_{L1} < 0$, $c_{L3} > 0$, $c_{D0} + c_{L1} < 0$ and $\frac{1}{2}c_{D0} + c_{D2} + \frac{1}{6}c_{L1} + c_{L3} > 0$.

If a conductor with fixed endpoints is considered the boundary conditions $\bar{v}(0,\bar{t}) = \bar{v}(\pi,\bar{t}) = \bar{w}(0,\bar{t}) = \bar{w}(\pi,\bar{t}) = 0$ are obtained. In the next section the partial differential equations (3.4.12) and (3.4.13) subject to these Dirichlet boundary conditions and the initial values $\bar{v}(\bar{x},0) = \bar{v}_0(\bar{x})$, $\bar{v}_{\bar{t}}(\bar{x},0) = \bar{v}_1(\bar{x})$, $\bar{w}(\bar{x},0) = \bar{w}_0(\bar{x})$ and $\bar{w}_{\bar{t}}(\bar{x},0) = \bar{w}_1(\bar{x})$ will be studied, where $\bar{v}_0$ and $\bar{w}_0$ can be regarded as the initial displacement of the conductor in y- and z-direction respectively, and where $\bar{v}_1$ and $\bar{w}_1$ represent the initial velocity of the conductor in y- and z-direction respectively.

3.5. An asymptotic approximation of the solution of a system of nonlinear wave equations

In this section the following initial-boundary value problem for a twice continuously differentiable and vector-valued function $\underline{u}(x,t;\epsilon) = (\bar{v}(x,t;\epsilon),\bar{w}(x,t;\epsilon))^T$ will be considered.

$$\bar{v}_{tt} - \bar{v}_{xx} = \epsilon(a_{10}\bar{v}_t + a_{01}\bar{w}_t + a_{20}\bar{v}_t^2 + a_{11}\bar{v}_t\bar{w}_t + a_{02}\bar{w}_t^2 + a_{03}\bar{w}_t^3),$$
$$0 < x < \pi,\ t > 0, \qquad (3.5.1)$$
$$\bar{w}_{tt} - \bar{w}_{xx} = \epsilon(b_{01}\bar{w}_t + b_{11}\bar{v}_t\bar{w}_t + b_{02}\bar{w}_t^2 + b_{03}\bar{w}_t^3),\quad 0 < x < \pi,\ t > 0, \qquad (3.5.2)$$
$$\bar{v}(x,0;\epsilon) = \bar{v}_0(x;\epsilon) \equiv (V_{00} + \epsilon V_{01})\sin(mx), \quad 0 < x < \pi, \qquad (3.5.3)$$
$$\bar{w}(x,0;\epsilon) = \bar{w}_0(x;\epsilon) \equiv (W_{00} + \epsilon W_{01})\sin(nx), \quad 0 < x < \pi, \qquad (3.5.4)$$
$$\bar{v}_t(x,0;\epsilon) = \bar{v}_1(x;\epsilon) \equiv (V_{10} + \epsilon V_{11})\sin(mx), \quad 0 < x < \pi, \qquad (3.5.5)$$
$$\bar{w}_t(x,0;\epsilon) = \bar{w}_1(x;\epsilon) \equiv (W_{10} + \epsilon W_{11})\sin(nx), \quad 0 < x < \pi, \qquad (3.5.6)$$
$$\underline{u}(0,t;\epsilon) = \underline{u}(\pi,t;\epsilon) = \underline{0}, \quad t \geq 0, \qquad (3.5.7)$$

where a_{10}, a_{01},..., b_{03}, V_{00}, V_{01},..., W_{10}, W_{11} are constants independent of ϵ, m and n integers, and $0 < \epsilon << 1$. From theorem 3.2.1 it follows that this initial-boundary value problem is well-posed on Ω_L (given by (3.2.13)).

For arbitrary m, n, a_{10}, a_{01},..., b_{03}, V_{00},..., W_{10} and W_{11} an asymptotic approximation (as ϵ tends to zero) of the solution of (3.5.1)-(3.5.7) will be constructed in this section. In view of computational difficulties (as also has been noticed in [18]) whenever one assumes an infinite series representation for the solution of the nonlinear initial-boundary value problem, one may alternatively investigate the problem in the characteristic coordinates $\sigma = x-t$ and $\xi = x+t$. In this approach the initial-boundary value problem (3.5.1)-(3.5.7) has to be replaced by an initial value problem. This replacement requires to extend the dependent variable $\underline{u}(x,t)$, the right-hand sides of the equations (3.5.1) and (3.5.2) as well as the functions $\bar{v}_0$, $\bar{v}_1$, $\bar{w}_0$ and $\bar{w}_1$ in x to odd and 2π-periodic functions. For simplicity the extended functions $\underline{u}$, $\bar{v}_0$, $\bar{v}_1$, $\bar{w}_0$ and $\bar{w}_1$ will be denoted by the same symbols. In constructing an approximation of the solution $\underline{u}(x,t) = \underline{u}^*(\sigma,\xi)$ of this initial value problem a two-timescales perturbation method will be used, because the straightforward perturbation expansion $\underline{u}_0^*(\sigma,\xi) + \epsilon\underline{u}_1^*(\sigma,\xi) + \ldots$ causes secular terms. Applying the two-timescales perturbation method $\underline{u}(x,t)$ is supposed to be a function of $\sigma = x-t$, $\xi = x+t$ and $\tau = \epsilon t$. By putting $\underline{u}(x,t) = \tilde{\underline{u}}(\sigma,\xi,\tau) = (\tilde{v}(\sigma,\xi,\tau),\tilde{w}(\sigma,\xi,\tau))^T$ the following initial value problem is obtained

$$-4\tilde{v}_{\sigma\xi} + 2\epsilon(\tilde{v}_{\xi\tau}-\tilde{v}_{\sigma\tau}) + \epsilon^2\tilde{v}_{\tau\tau} = \epsilon\Big(a_{10}(-\tilde{v}_\sigma+\tilde{v}_\xi+\epsilon\tilde{v}_\tau) + a_{01}(-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau) +$$
$$+ p_1(\sigma+\xi,-\tilde{v}_\sigma+\tilde{v}_\xi+\epsilon\tilde{v}_\tau,-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau) + a_{03}(-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau)^3\Big),$$
$$\text{for } -\infty < \sigma < \xi < \infty,\ \tau > 0 \qquad (3.5.8)$$

$$-4\tilde{w}_{\sigma\xi} + 2\epsilon(\tilde{w}_{\xi\tau}-\tilde{w}_{\sigma\tau}) + \epsilon^2\tilde{w}_{\tau\tau} = \epsilon\Big(b_{01}(-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau) + p_2(\sigma+\xi,\tilde{v}_\sigma+\tilde{v}_\xi+\epsilon\tilde{v}_\tau,$$
$$-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau) + b_{03}(-\tilde{w}_\sigma+\tilde{w}_\xi+\epsilon\tilde{w}_\tau)^3\Big), \quad \text{for } -\infty < \sigma < \xi < \infty,\ \tau > 0 \qquad (3.5.9)$$

$$\tilde{v}(\sigma,\xi,\tau) = (V_{00}+\epsilon V_{01})\sin(m\sigma), \quad \text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0, \qquad (3.5.10)$$

$$\tilde{w}(\sigma,\xi,\tau) = (W_{00}+\epsilon W_{01})\sin(n\sigma), \quad \text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0, \qquad (3.5.11)$$

$$-\tilde{v}_\sigma(\sigma,\xi,\tau) + \tilde{v}_\xi(\sigma,\xi,\tau) + \epsilon\tilde{v}_\tau(\sigma,\xi,\tau) = v_1(\sigma) = (V_{10}+\epsilon V_{11})\sin(m\sigma),$$
$$\text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0, \qquad (3.5.12)$$

$$-\tilde{w}_\sigma(\sigma,\xi,\tau) + \tilde{w}_\xi(\sigma,\xi,\tau) + \epsilon\tilde{w}_\tau(\sigma,\xi,\tau) = w_1(\sigma) = (W_{10}+\epsilon W_{11})\sin(n\sigma),$$
$$\text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0, \qquad (3.5.13)$$

where $p_1(a,b,c) = E\left(\frac{a}{2}\right)\left\{a_{20}b^2 + a_{11}bc + a_{02}c^2\right\}$ and $p_2(a,b,c) = E\left(\frac{a}{2}\right) \times \times \left\{b_{11}bc + b_{02}c^2\right\}$ with $E(x) = 1$ for $0 < x < \pi$, $E(x) = -1$ for $-\pi < x < 0$, $E(0) = = E(\pi) = 0$ and $E(x)$ is 2π-periodic in x. Furthermore, $\tilde{v} = \tilde{w} = 0$ if $\sigma = k\pi - \theta$, $\xi = k\pi + \theta$ and $\tau = \epsilon\theta$ with $k \in \mathbb{Z}$ and $\theta \geq 0$. Now it is assumed that $\tilde{v}$ and $\tilde{w}$ may be approximated by the formal perturbation expansions $v_0(\sigma,\xi,\tau) + \epsilon v_1(\sigma,\xi,\tau) + + \epsilon^2 v_2(\sigma,\xi,\tau) + \ldots$ and $w_0(\sigma,\xi,\tau) + \epsilon w_1(\sigma,\xi,\tau) + \epsilon^2 w_2(\sigma,\xi,\tau) + \ldots$ respectively. By substituting these expansions into (3.5.8)-(3.5.13), and after equating the coefficients of like powers in ϵ, it follows from the powers 0 and 1 of ϵ that v_0 and w_0 should satisfy

$$-4v_{0_{\sigma\xi}} = 0, \qquad -\infty < \sigma < \xi < \infty,\ \tau > 0, \tag{3.5.14}$$

$$-4w_{0_{\sigma\xi}} = 0, \qquad -\infty < \sigma < \xi < \infty,\ \tau > 0, \tag{3.5.15}$$

$$v_0(\sigma,\xi,\tau) = V_{00}\sin(m\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.16}$$

$$w_0(\sigma,\xi,\tau) = W_{00}\sin(n\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.17}$$

$$-v_{0_\sigma}(\sigma,\xi,\tau) + v_{0_\xi}(\sigma,\xi,\tau) = V_{10}\sin(m\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.18}$$

$$-w_{0_\sigma}(\sigma,\xi,\tau) + w_{0_\xi}(\sigma,\xi,\tau) = W_{10}\sin(n\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.19}$$

and that v_1 and w_1, respectively, should satisfy

$$\begin{aligned} -4v_{1_{\sigma\xi}} = {} & 2v_{0_{\sigma\tau}} - 2v_{0_{\xi\tau}} + a_{10}(-v_{0_\sigma}+v_{0_\xi}) + a_{01}(-w_{0_\sigma}+w_{0_\xi}) + p_1(\sigma+\xi,-v_{0_\sigma}+v_{0_\xi}, \\ & -w_{0_\sigma}+w_{0_\xi}) + a_{03}(-w_{0_\sigma}+w_{0_\xi})^3, \qquad -\infty < \sigma < \xi < \infty,\ \tau > 0, \end{aligned} \tag{3.5.20}$$

$$\begin{aligned} -4w_{1_{\sigma\xi}} = {} & 2w_{0_{\sigma\tau}} - 2w_{0_{\xi\tau}} + b_{01}(-w_{0_\sigma}+w_{0_\xi}) + p_2(\sigma+\xi,-v_{0_\sigma}+v_{0_\xi},-w_{0_\sigma}+w_{0_\xi}) + \\ & + b_{03}(-w_{0_\sigma}+w_{0_\xi})^3, \qquad -\infty < \sigma < \xi < \infty,\ \tau > 0, \end{aligned} \tag{3.5.21}$$

$$v_1(\sigma,\xi,\tau) = V_{01}\sin(m\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.22}$$

$$w_1(\sigma,\xi,\tau) = W_{01}\sin(n\sigma), \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \tag{3.5.23}$$

$$\begin{aligned} & -v_{1_\sigma}(\sigma,\xi,\tau) + v_{1_\xi}(\sigma,\xi,\tau) = -v_{0_\tau}(\sigma,\xi,\tau) + V_{11}\sin(m\sigma), \\ & \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0, \end{aligned} \tag{3.5.24}$$

$$\begin{aligned} & -w_{1_\sigma}(\sigma,\xi,\tau) + w_{1_\xi}(\sigma,\xi,\tau) = -w_{0_\tau}(\sigma,\xi,\tau) + W_{11}\sin(n\sigma), \\ & \qquad -\infty < \sigma = \xi < \infty,\ \tau = 0. \end{aligned} \tag{3.5.25}$$

Furthermore, $v_0 = w_0 = v_1 = w_1 = 0$ if $\sigma = k\pi - \theta$, $\xi = k\pi + \theta$ and $\tau = \epsilon\theta$ with $k \in \mathbb{Z}$ and $\theta \geq 0$. In the further analysis v_0, w_0, v_1 and w_1 will be determined, and it will be shown that the formal approximation $\underline{u}_A(x,t;\epsilon) \equiv (v_0(x-t,x+t,\epsilon t) + \epsilon v_1(x-t,x+t,\epsilon t), w_0(x-t,x+t,\epsilon t) + \epsilon w_1(x-t,x+t,\epsilon t))^T$ is an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $\underline{u}(x,t)$ of the initial-boundary value problem (3.5.1)-(3.5.7) for $0 \leq x \leq \pi$ and $0 \leq t \leq L|\epsilon|^{-1}$.

The general solutions of the partial differential equations (3.5.14) and (3.5.15) are given by $v_0(\sigma,\xi,\tau) = h_0(\sigma,\tau) + k_0(\xi,\tau)$ and $w_0(\sigma,\xi,\tau) = f_0(\sigma,\tau) + g_0(\xi,\tau)$ respectively. The initial values (3.5.16)-(3.5.19) imply that h_0, k_0, f_0 and g_0 have to satisfy $h_0(\sigma,0) + k_0(\sigma,0) = V_{00}\sin(m\sigma)$, $-h_0'(\sigma,0) + k_0'(\sigma,0) = V_{10}\sin(m\sigma)$, $f_0(\sigma,0) + g_0(\sigma,0) = W_{00}\sin(n\sigma)$ and $-f_0'(\sigma,0) + g_0'(\sigma,0) = W_{10}\sin(n\sigma)$, where the prime denotes differentiation with respect to the first argument. From the odd and 2π-periodic extension in x it follows indirectly that h_0, k_0, f_0 and g_0 have to satisfy $k_0(\sigma,\tau) = -h_0(-\sigma,\tau)$, $h_0(\sigma,\tau) = h_0(\sigma+2\pi,\tau)$, $g_0(\sigma,\tau) = -f_0(-\sigma,\tau)$ and $f_0(\sigma,\tau) = f_0(\sigma+2\pi,\tau)$ for $-\infty < \sigma < \infty$ and $\tau \geq 0$. The undetermined behaviour of h_0 and f_0 with respect to τ will be used to avoid secular terms in v_1 and w_1. From the well-posedness theorem it followed that $\underline{u}$, $\underline{u}_t$ and $\underline{u}_x$ are O(1) on Ω_L. So, $\tilde{\underline{u}}$ and its first derivatives have to remain O(1) on $-\infty < x < \infty$ and $0 \leq t \leq L|\epsilon|^{-1}$. Furthermore, it should be noticed that the equations for v_0, w_0, v_1 and w_1 have been derived under the assumption that v_0, w_0, v_1, w_1 and their derivatives up to order two are O(1). These boundedness conditions on v_0, w_0, v_1 and w_1 determine the behaviour of h_0 and f_0 with respect to τ. From (3.5.20)-(3.5.25) v_{1_σ}, v_{1_ξ}, w_{1_σ} and w_{1_ξ} may be obtained easily. For instance,

$$-4w_{1_\sigma}(\sigma,\xi,\tau) = -4w_{1_\sigma}(\sigma,\sigma,\tau) + (\xi-\sigma)\left(2f_{0_{\sigma\tau}}(\sigma,\tau) - b_{01}f_{0_\sigma}(\sigma,\tau) - b_{03}f^3_{0_\sigma}(\sigma,\tau)\right) +$$
$$- 3b_{03}f_{0_\sigma}(\sigma,\tau)\int_\sigma^\xi g^2_{0_\theta}(\theta,\tau)d\theta + \int_\sigma^\xi \Big\{-2g_{0_{\theta\tau}}(\theta,\tau) + b_{01}g_{0_\theta}(\theta,\tau) + 3b_{03}f^2_{0_\sigma}(\sigma,\tau)g_{0_\theta}(\theta,\tau) +$$
$$+ b_{03}g^3_{0_\theta}(\theta,\tau)\Big\}\,d\theta + \int_\sigma^\xi p_2(\sigma+\theta,-h_{0_\sigma}(\sigma,\tau)+k_{0_\theta}(\theta,\tau),-f_{0_\sigma}(\sigma,\tau)+g_{0_\theta}(\theta,\tau))d\theta + h^*(\sigma,\tau), \quad (3.5.26)$$

where h^* will be determined later. In (3.5.26) the integral with integrand p_2 is of O(1) for all values of σ and ξ, because the function p_2 of O(1) is 4π-periodic in θ and the integral over such a period is equal to zero. Since the first integral in (3.5.26) contains a non-negative and 2π-periodic integrand it follows that this integral will grow with the length $\xi-\sigma$ of the integration interval. It turns out that this integral can be written in a part which is of O(1) for all values of σ and ξ, and in a part which is linear in $\xi-\sigma$:

$$\int_\sigma^\xi g_{0_\theta}^2(\theta,\tau)\,d\theta = \int_\sigma^\xi \left\{ g_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} g_{0_\psi}^2(\psi,\tau)\,d\psi \right\} d\theta + $$
$$+ \frac{\xi-\sigma}{2\pi}\int_0^{2\pi} g_{0_\psi}^2(\psi,\tau)\,d\psi.$$

Noticing that $\xi-\sigma = 2t$ it follows that $\xi-\sigma$ is of $O(|\epsilon|^{-1})$ on a time-scale of $O(|\epsilon|^{-1})$. So, w_{1_σ} will be of $O(|\epsilon|^{-1})$ unless f_0 and g_0 are such that in (3.5.26) the terms of $O(|\epsilon|^{-1})$ (that is, terms linear in $\xi-\sigma$) disappear. It turns out that $w_{1_\sigma}, w_{1_\xi}, v_{1_\sigma}$ and v_{1_ξ} are all O(1) on a time-scale of $O(|\epsilon|^{-1})$ if $f_0(\sigma,\tau)$, $g_0(\xi,\tau)$, $h_0(\sigma,\tau)$ and $k_0(\xi,\tau)$ satisfy the following four conditions

$$2f_{0_{\sigma\tau}} - b_{01}f_{0_\sigma} - b_{03}f_{0_\sigma}^3 - \frac{3b_{03}}{2\pi} f_{0_\sigma}\int_0^{2\pi} g_{0_\theta}^2(\theta,\tau)d\theta = 0,$$

$$2g_{0_{\xi\tau}} - b_{01}g_{0_\xi} - b_{03}g_{0_\xi} - \frac{3b_{03}}{2\pi} g_{0_\xi}\int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)d\theta = 0,$$

$$2h_{0_{\sigma\tau}} - a_{10}h_{0_\sigma} - a_{01}f_{0_\sigma} - a_{03}f_{0_\sigma}^3 - \frac{3a_{03}}{2\pi} f_{0_\sigma}\int_0^{2\pi} g_{0_\theta}^2(\theta,\tau)d\theta = 0,$$

$$2k_{0_{\xi\tau}} - a_{10}k_{0_\xi} - a_{01}g_{0_\xi} - a_{03}g_{0_\xi}^3 - \frac{3a_{03}}{2\pi} g_{0_\xi}\int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)d\theta = 0.$$

From $g_0(\theta,\tau) = -f_0(-\theta,\tau)$ and from $k_0(\theta,\tau) = -h_0(-\theta,\tau)$ it follows that the first and second condition as well as the third and fourth condition are equivalent. So, $w_{1_\sigma}, w_{1_\xi}, v_{1_\sigma}$ and v_{1_ξ} are all O(1) on a time-scale of $O(|\epsilon|^{-1})$ if f_0 and h_0 satisfy

$$2f_{0_{\sigma\tau}} - b_{01}f_{0_\sigma} - b_{03}f_{0_\sigma}^3 - \frac{3b_{03}}{2\pi} f_{0_\sigma} \int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)d\theta = 0, \text{ and} \tag{3.5.27}$$

$$2h_{0_{\sigma\tau}} - a_{10}h_{0_\sigma} - a_{01}f_{0_\sigma} - a_{03}f_{0_\sigma}^3 - \frac{3a_{03}}{2\pi} f_{0_\sigma} \int_0^{2\pi} f_{0_\theta}^2(\theta,\tau)d\theta = 0. \tag{3.5.28}$$

In [4,12] an equation similar to equation (3.5.27) has been solved. If the method introduced in [4] is applied to equation (3.5.27) one obtains after some calculations $f_0(\sigma,\tau)$, and so $w_0(\sigma,\xi,\tau) = f_0(\sigma,\tau) - f_0(-\xi,\tau)$. It turns out that f_0 and w_0 are given by

$$f_0(\sigma,\tau) = \frac{\lambda(\tau)}{n\phi^{1/2}(\tau)} \arcsin\left[\left[\frac{W_n\phi(\tau)}{1 + W_n\phi(\tau)}\right]^{1/2} \sin(\alpha+n\sigma)\right] + k^*(\tau), \tag{3.5.29}$$

$$w_0(\sigma,\xi,\tau) = \frac{\lambda(\tau)}{n\phi^{1/2}(\tau)} \left\{\arcsin\left[\left[\frac{W_n\phi(\tau)}{1 + W_n\phi(\tau)}\right]^{1/2} \sin(\alpha+n\sigma)\right] + \right.$$
$$\left. - \arcsin\left[\left[\frac{W_n\phi(\tau)}{1 + W_n\phi(\tau)}\right]^{1/2} \sin(\alpha-n\xi)\right]\right\}, \tag{3.5.30}$$

where $k^*(\tau)$ is an arbitrary function in τ with $k^*(0) = 0$, $\sigma = x-t$, $\xi = x+t$, $\tau = \epsilon t$, $W_n = n^2W_{00}^2 + W_{10}^2$, α is given by $\cos\alpha = nW_{00}W_n^{-1/2}$ and $\sin\alpha = W_{10}W_n^{-1/2}$, where $\lambda(\tau)$ and $\phi(\tau)$ are implicitly given by $\lambda(\tau) = 4m^{-3}(\tau)\exp\left(\frac{b_{01}}{2}\tau\right)$ and $\phi(\tau) = \frac{m(\tau)}{W_n} \times$ $\times (m(\tau)-2)$ with $m(\tau)$ determined by $m^8(\tau) - \frac{8}{7}m^7(\tau) = 2^6W_nb_{03}b_{01}^{-1}(1 - \exp(b_{01}\tau)) +$ $+ \frac{3.2^8}{7}$. Now the (with respect to h_0) linear partial differential equation (3.5.28) can be solved and one obtains after some calculations h_0, and so $v_0(\sigma,\xi,\tau) = h_0(\sigma,\tau) - h_0(-\xi,\tau)$. It turns out that v_0 is given by

$$v_0(\sigma,\xi,\tau) = \frac{1}{2}\exp\left(\frac{a_{10}}{2}\tau\right)\left\{V_{00}\sin(m\sigma) + \frac{1}{m}V_{10}\cos(m\sigma) + V_{00}\sin(m\xi) - \frac{1}{m}V_{10}\cos(m\xi)\right\} +$$

$$+ \frac{a_{03}}{b_{03}}\left\{w_0(\sigma,\xi,\tau) - \frac{1}{2}\exp\left(\frac{a_{10}}{2}\tau\right)\left(W_{00}\sin(n\sigma) + \frac{1}{n}W_{10}\cos(n\sigma) + W_{00}\sin(n\xi) +\right.\right.$$

$$\left.\left.- \frac{1}{n}W_{10}\cos(n\xi)\right)\right\} + \frac{1}{2}\left(a_{01} - \frac{a_{03}}{b_{03}}b_{01} + \frac{a_{03}}{b_{03}}a_{10}\right)\int_0^{\tau}\exp\left(\frac{a_{10}}{2}(\tau-\tau')\right)w_0(\sigma,\xi,\tau')d\tau', \tag{3.5.31}$$

where w_0 is given by (3.5.30). Now the linear initial value problems (3.5.20)-(3.5.25) for v_1 and w_1 can be solved, yielding

$$v_1(\sigma,\xi,\tau) = -\frac{1}{4}\int_{\xi}^{\sigma}\int_{\psi}^{\xi} p_1\left(\psi+\phi, -h_{0_\psi}(\psi,\tau)+k_{0_\phi}(\phi,\tau), -f_{0_\psi}(\psi,\tau)+g_{0_\phi}(\phi,\tau)\right)d\phi d\psi +$$

$$+ \frac{3}{4}a_{03}\left(f_0(\sigma,\tau)+g_0(\xi,\tau)\right)\int_{\sigma}^{\xi}\left(f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)d\psi\right)d\theta - \frac{3}{4}a_{03}\times$$

$$\times\int_{\sigma}^{\xi}\left[\left(f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)d\psi\right)\left(f_0(\theta,0)+g_0(\theta,0)\right)\right]d\theta + h_1(\sigma,\tau) + k_1(\xi,\tau), \tag{3.5.32}$$

and

$$w_1(\sigma,\xi,\tau) = -\frac{1}{4}\int_{\xi}^{\sigma}\int_{\psi}^{\xi} p_2\left(\psi+\phi, -h_{0_\psi}(\psi,\tau)+k_{0_\phi}(\phi,\tau), -f_{0_\psi}(\psi,\tau)+g_{0_\phi}(\phi,\tau)\right)d\phi d\psi +$$

$$+ \frac{3}{4}b_{03}\left(f_0(\sigma,\tau)+g_0(\xi,\tau)\right)\int_{\sigma}^{\xi}\left(f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)d\psi\right)d\theta - \frac{3}{4}b_{03}\times$$

$$\times\int_{\sigma}^{\xi}\left[\left(f_{0_\theta}^2(\theta,\tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi,\tau)d\psi\right)\left(f_0(\theta,0)+g_0(\theta,0)\right)\right]d\theta + f_1(\sigma,\tau) + g_1(\xi,\tau), \tag{3.5.33}$$

where (for $\sigma = \xi$ and $\tau = 0$) h_1+k_1 and f_1+g_1 are determined by the initial values

(3.5.22)-(3.5.25). The undetermined behaviour of f_1, g_1, h_1 and k_1 with respect to τ can be used to avoid secular terms in v_2 and w_2. However, in this analysis v_2 and w_2 will not be determined. For that reason it may be assumed that $f_1 = f_1(\sigma)$, $g_1 = g_1(\xi)$, $h_1 = h_1(\sigma)$ and $k_1 = k_1(\xi)$, and then $f_1(\sigma)+g_1(\xi) = -\frac{1}{2}\int_\sigma^\xi w_{0_\tau}(\theta,\theta,0)d\theta + \frac{1}{2}W_{01}\left(\sin(n\sigma)+\sin(n\xi)\right) + \frac{1}{2n}W_{11}\left(\cos(n\sigma)-\cos(n\xi)\right)$ and $h_1(\sigma)+k_1(\xi) = -\frac{1}{2}\int_\sigma^\xi v_{0_\tau}(\theta,\theta,0)d\theta + \frac{1}{2}V_{01}\left(\sin(m\sigma)+\sin(m\xi)\right) + \frac{1}{2m}V_{11}\left(\cos(m\sigma)-\cos(m\xi)\right)$. It can be shown from (3.5.30)-(3.5.33) that v_0, v_1, w_0, w_1 and their derivatives up to order two are of $O(1)$ for $-\infty < x < \infty$ and $0 \le t \le L|\epsilon|^{-1}$. So, the assumptions under which the equations for v_0, v_1, w_0 and w_1 have been derived, are justified. So far a vector-valued function $\underline{u}_A(x,t;\epsilon) \equiv (v_0(x-t,x+t,\epsilon t) + \epsilon v_1(x-t,x+t,\epsilon t),\ w_0(x-t,x+t,\epsilon t) + \epsilon w_1(x-t,x+t,\epsilon t))^T$ has been constructed. It can easily be seen that $\underline{u}_A$ satisfies (3.5.3), (3.5.4) and (3.5.7) exactly, and (3.5.5) and (3.5.6) up to order ϵ^2 in the sense of theorem 3.3.1. After lengthy calculations it can also be shown that $\underline{u}_A$ satisfies (3.5.1) and (3.5.2) up to $\epsilon^2\underline{c}_1(x,t;\epsilon) \equiv \epsilon^2(c_{11}(x,t;\epsilon),c_{12}(x,t;\epsilon))^T$, where $c_{1i}, \frac{\partial c_{1i}}{\partial x} \in C(\Omega_L \times [-\epsilon_0,\epsilon_0], \mathbb{R})$ for $i = 1,2$ with $\underline{c}_1(0,t;\epsilon) = \underline{c}_1(\pi,\tau;\epsilon) = \underline{0}$ for $0 \le t \le L|\epsilon|^{-1}$. Furthermore, it can be shown that $\underline{c}_1(x,t;\epsilon)$ and its derivative with respect to x are uniformly bounded in t and ϵ. Then it follows from theorem 3.3.1 that $\underline{u}_A(x,t;\epsilon)$ is an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution of the initial-boundary value problem (3.5.1)-(3.5.7) for $0 \le x \le \pi$ and $0 \le t \le L|\epsilon|^{-1}$, that is, $\|\underline{u}-\underline{u}_A\|_{C^2_{M_1}} = O(\epsilon)$. From this estimate the following estimate can be obtained with $\underline{u}_0(x,t;\epsilon) \equiv (v_0(x-t,x+t,\epsilon t),w_0(x-t,x+t,\epsilon t))^T$:

$$\|\underline{u}-\underline{u}_0\|_{C^2_{M_1}} = \|\underline{u}-\underline{u}_A+\underline{u}_A-\underline{u}_0\|_{C^2_{M_1}} \le \|\underline{u}-\underline{u}_A\|_{C^2_{M_1}} + \|\underline{u}_A-\underline{u}_0\|_{C^2_{M_1}} = O(\epsilon).$$

Hence, $(v_0(x-t,x+t,\epsilon t),w_0(x-t,x+t,\epsilon t))^T$, where v_0 and w_0 are given by (3.5.31) and (3.5.30) respectively, is also an order ϵ asymptotic approximation (as $\epsilon \to 0$) of the solution $\underline{u}(x,t;\epsilon)$ of problem (3.5.1)-(3.5.7) for $0 \le x \le \pi$ and $0 \le t \le L|\epsilon|^{-1}$, in which L is an ϵ-independent, positive constant.

3.6. Some general remarks

The asymptotic theory presented in the sections 3.2 and 3.3 can readily be extended to other types of initial-boundary value problems. For instance, if the initial-boundary value problem (3.2.1)-(3.2.4) is considered where the boundary conditions (3.2.4) are replaced by $\underline{u}(0,t) = \underline{0}$ and $\underline{u}_x(\pi,t) = \underline{0}$ (a fixed end condition at $x = 0$ and a free end condition at $x = \pi$) then an integral equation equivalent with this problem is needed to prove the well-posedness of the problem and the asymptotic validity of a class of formal approximations. To obtain this equivalent integral equation the initial-boundary value problem should be extended to an initial value problem. This can be accomplished by extending the dependent variable $\underline{u}$, the nonlinearity and the initial values in x, such that these functions are odd about $x = 0$, even about $x = \pi$ and 4π-periodic with respect to x. In this way the equivalent integral equation is obtained and the techniques applied in sections 3.2 and 3.3 can again be used to prove the well-posedness of the problem and the asymptotic validity of formal approximations on ϵ-dependent time-scales.

In section 3.4 the assumption is made that the cross-section of the circular conductor with small ice ridge is symmetric. However, this assumption is not necessary. In fact, for an arbitrary profile galloping may occur if the lift- and drag-coefficients $c_L(\alpha)$ and $c_D(\alpha)$ are such that there exists an interval in α with $\alpha_0 < \alpha < \alpha_2$ for which the Den Hartog-criterion $c_D(\alpha) + \frac{dc_L(\alpha)}{d\alpha} < 0$ is satisfied. Then, the static angle of attack α_s can be chosen such that galloping may set in. It should be noted that the analysis in sections 3.4 and 3.5 is correct if there exists an angle α_1 with $c_L(\alpha_1) = 0$ and $\alpha_0 < \alpha_1 < \alpha_2$.

In section 3.5 the model is studied, which has been derived in section 3.4. It is assumed in section 3.5 that $\underline{u}(x,t)$ is a twice continuously differentiable function. This assumption can be justified as follows. The velocities v_t and w_t of the conductor in horizontal and in vertical direction are continuous in x and t, and the aerodynamic coefficients $c_D(\alpha)$ and

$c_L(\alpha)$ are continuous in α. Since $\alpha = \alpha_s + \phi = \alpha_s + \arctan(-w_t(v_\infty - v_t)^{-1})$ with $|w_t| < v_\infty$ and $|v_t| < v_\infty$ it follows that the right-hand sides of the equations (3.4.1) and (3.4.2) are continuous in x and t. In the left-hand sides of the equations (3.4.1) and (3.4.2) the terms $\frac{\partial}{\partial x}\left\{v_x(1+v_x^2+w_x^2)^{-1/2}\right\}$ and $\frac{\partial}{\partial x}\left\{w_x(1+v_x^2+w_x^2)^{-1/2}\right\}$ represent in fact the curvature of the transmission line. Since it is natural to assume that the curvature is continuous in x and t, it follows that v_{tt} and w_{tt} should be continuous in x and t. And so, it is more or less natural to assume that $\underline{u}(x,t) = (v(x,t),w(x,t))^T$ should be twice continuously differentiable with respect to x and t.

In section 3.5 monochromatic initial values have been considered, because the galloping oscillations often affect only a single mode of vibration [25]. To obtain some information about the oscillation amplitudes the following formulas can be used

$$v(x,t) = \frac{-\rho_a d v_\infty^2 a_{00}}{4TA}\, x(x-\ell) + \frac{\ell v_\infty}{\pi c}\, \bar{v}\left(\frac{\pi}{\ell}x, \frac{\pi c}{\ell}t\right),$$

$$w(x,t) = \frac{-\rho_a d v_\infty^2 b_{00}}{4TA}\, x(x-\ell) + \frac{\ell v_\infty}{\pi c}\, \bar{w}\left(\frac{\pi}{\ell}x, \frac{\pi c}{\ell}t\right),$$

where $v(x,t)$, $w(x,t)$, ρ_a, d, v_∞, c, T, A, ℓ, $\bar{v}$, $\bar{w}$, a_{00} and b_{00} are defined as in section 3.4. The first terms in these formulas may be considered as the position of the conductor in rest, whereas the second terms represent the change of the position of the conductor due to galloping. For large values of t (that is, $\epsilon t = \tau \to \infty$) it can be shown from (3.5.30) that w_0 (the first order approximation of $\bar{w}$) tends to a standing triangular wave with amplitude $\frac{\pi}{2n}\left(\frac{b_{01}}{-b_{02}}\right)^{1/2}$ and period $\frac{2\pi}{n}$. It should be noted that for these values of t little can be said about the asymptotic validity of the results, since only for finite values of ϵt, that is, for $0 \le \tau = \epsilon t \le L < \infty$ the asymptotic validity of the results could be established. It is also interesting to mention that it can be shown from (3.5.31) that

$$\lim_{\tau\to\infty} v_0(\sigma,\xi,\tau) = \left(\frac{a_{03}b_{01}-a_{01}b_{03}}{a_{10}b_{03}}\right) \lim_{\tau\to\infty} w_0(\sigma,\xi,\tau). \tag{3.6.1}$$

These results imply that the maximum oscillation amplitudes due to galloping may be approximated by

$$\frac{\ell v_\infty}{2nc}\left\{\frac{-c_{D0}-c_{L1}}{\frac{1}{2}c_{D0}+\frac{1}{6}c_{L1}+c_{D2}+c_{L3}}\right\}^{1/2}$$ in the vertical direction, and by

$$\frac{\ell v_\infty}{2nc}\left\{\frac{-c_{D0}-c_{L1}}{\frac{1}{2}c_{D0}+\frac{1}{6}c_{L1}+c_{D2}+c_{L3}}\right\}^{1/2}\times$$

$$\times\left\{\frac{-c_{D1}(\frac{1}{3}c_{D0}+c_{D2}+c_{L3})+(c_{D3}-c_{L2})(c_{D0}+c_{L1})}{2c_{D0}(\frac{1}{2}c_{D0}+\frac{1}{6}c_{L1}+c_{D2}+c_{L3})}\right\}$$

in the horizontal direction, where c_{D0}, c_{D1},..., c_{L3} are the aërodynamic coefficients, which may be obtained from wind-tunnel measurements. In a practical application the quantity $(a_{03}b_{01}-a_{01}b_{03})(a_{10}b_{03})^{-1}$ is small compared to one. This implies that the amplitude of the horizontal oscillation is small compared to the amplitude of the vertical oscillation. This phenomenon, that galloping is an almost purely vertical oscillation, has also been noticed in nature [26].

In section 3.4 it has been assumed that the tension T in the conductor is constant. In [16] it has been shown for the free vibrations of a suspended cable for which the sag to span ratio is small that the assumption is valid if the cable oscillates in a so-called anti-symmetric in-plane mode. For the monochromatic initial values considered in section 3.5 this implies that n should be even. If the cable oscillates in a symmetric in-plane mode (that is, n is odd) the assumption that T is constant is only valid for the higher modes of vibration. For the lower modes of vibrations with n odd the validity of the assumption that T is constant, heavily depends on the elastic properties of the conductor and the sag-span ratios (see [16]). However, in [26] it has been remarked that the most troublesome galloping mode is the S-shaped vertical mode of the conductor catenary, that is, n is equal to 2. So, it may be concluded that for this S-shaped mode the model (describing the galloping oscillations of overhead transmission lines) can be justified. However, if n = 1 the assumption that T is constant, is incorrect or at least doubtful.

In section 3.5 monochromatic initial values have been considered in the vertical and in the horizontal direction. It can be shown that the analysis given in section 3.5 also can be applied if the initial values in the horizontal direction consist of an arbitrary number of modes, that is, if $\bar{v}_0(x;\epsilon) = \sum_{k=1}^{\infty} a_k(\epsilon)\sin(kx)$ and $\bar{v}_1(x;\epsilon) = \sum_{k=1}^{\infty} b_k(\epsilon)\sin(kx)$, where $a_k(\epsilon)$ and $b_k(\epsilon)$ are such that differentiation and summation, and integration and summation may be interchanged as often as is required. It then turns out that $v_0(\sigma,\xi,\tau)$ is given by

$$
\begin{aligned}
v_0(\sigma,\xi,\tau) &= \frac{1}{2}\exp\left(\frac{a_{10}}{2}\tau\right)\left\{\sum_{k=1}^{\infty}\left(a_k(0)\sin(k\sigma) + \frac{b_k(0)}{k}\cos(k\sigma) + \right.\right. \\
&+ a_k(0)\sin(k\xi) - \left.\left.\frac{b_k(0)}{k}\cos(k\xi)\right)\right\} + \frac{a_{03}}{b_{03}}\left\{w_0(\sigma,\xi,\tau) - \frac{1}{2}\exp\left(\frac{a_{10}}{2}\tau\right) \times \right. \\
&\times \left.\left(W_{00}\sin(n\sigma) + \frac{W_{10}}{n}\cos(n\sigma) + W_{00}\sin(n\xi) - \frac{W_{10}}{n}\cos(n\xi)\right)\right\} + \\
&+ \frac{1}{2}\left(a_{01} - \frac{a_{03}}{b_{03}}b_{01} + \frac{a_{03}}{b_{03}}a_{10}\right)\int_0^{\tau}\exp\left(\frac{a_{10}}{2}(\tau-\tau')\right)w_0(\sigma,\xi,\tau')d\tau',
\end{aligned}
$$

where $w_0(\sigma,\xi,\tau)$ is given by (3.5.30). Since $a_{10} < 0$ it follows from this formula that (for $\tau \to \infty$) the modes initially present in the horizontal direction disappear, and that v_0 tends to a standing triangular wave (see (3.6.1)), which is in fact determined by the vibration mode initially present in the vertical direction. After having obtained the approximation for the oscillation-amplitudes (and the velocities of the conductor) it should always be checked if the values of $\alpha = \alpha_1 + \arctan\left(-\dot{w}_t(v_\infty - v_t)^{-1}\right)$ are such that the approximations (3.4.5) and (3.4.6) for $c_D(\alpha)$ and $c_L(\alpha)$ are still valid. It turns out that for the c_D- and c_L-curve given in figure 3.4.2 and for the typical values of the physical quantities ℓ, d, v_∞, c, etc. (given in section 3.4) the values of α are such that the approximations (3.4.5) and (3.4.6) for $c_D(\alpha)$ and $c_L(\alpha)$ are valid, and so these approximations are justified.

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SUMMARY

In this thesis a class of initial-boundary value problems for (systems of) weakly nonlinear hyperbolic equations of order two is studied. These problems contain a small parameter ϵ, which precedes the nonlinear terms in the partial differential equation(s). To obtain a classical solution the initial values and the nonlinear terms in the partial differential equation(s) have to satisfy certain smoothness conditions. In order to prove existence and uniqueness of the solution of an initial-boundary value problem for a (system of) hyperbolic equation(s) an equivalent (system of) integral equation(s) is used. By applying Banach's fixed point theorem to this (system of) integral equation(s) existence of a unique classical solution is shown on an ϵ-dependent time-scale.

Since the initial-boundary value problems contain a small parameter ϵ perturbation methods can be applied to construct formal asymptotic approximations of the solutions. In this thesis the asymptotic validity (as ϵ tends to zero) of a class of formal approximations is shown on time-scales for which the initial-boundary value problems have been shown to be well-posed.

As application of the theory several inititial-boundary value problems have been formulated and studied. From an aero-elastic analysis it is shown that an initial-boundary value problem for the Rayleigh wave equation can be regarded as a simple model describing the galloping oscillations of overhead transmission lines in the vertical direction. Furthermore, an initial-boundary value problem for a system of weakly nonlinear and weakly coupled wave equations has been derived to describe these oscillations in the vertical and horizontal direction. These initial-boundary value problems have been investigated in detail using characteristic coordinates and a two-timescales perturbation method.

The well-posedness of these problems and the asymptotic validity of the constructed approximation of the solutions of these problems have been shown using the developed asymptotic theory. Also an initial-boundary value problem for a telegraph equation perturbed with a cubic nonlinearity has been investigated by means of a Fourier series expansion of the solution and a two-timescales perturbation method. For this problem the asymptotic validity of the approximation has also been demonstrated.

SAMENVATTING

In dit proefschrift wordt een klasse van begin-randwaarde problemen voor (stelsels) zwak niet-lineaire, hyperbolische vergelijkingen van de tweede orde bestudeerd. De problemen bevatten een kleine parameter ϵ, die de niet-lineaire termen in de partiële differentiaalvergelijking(en) voorafgaat. Om een klassieke oplossing te verkrijgen, moeten de beginwaarden en de niet-lineaire termen in de partiële differentiaalvergelijking(en) aan bepaalde gladheidseisen voldoen. Door het begin-randwaarde probleem voor een (stelsel) hyperbolische vergelijking(en) te herschrijven in een (stelsel) integraalvergelijking(en) worden existentie en eenduidigheid van de oplossing van het begin-randwaarde probleem aangetoond met behulp van de dekpuntstelling van Banach.

Aangezien de begin-randwaarde problemen een kleine parameter ϵ bevatten, kunnen perturbatiemethoden worden toegepast om formele benaderingen van de oplossingen te construeren. In dit proefschrift wordt de asymptotische geldigheid (voor $\epsilon \to 0$) van een klasse van formele benaderingen aangetoond op tijdschalen voor welke de begin-randwaarde problemen goed-gesteld zijn.

Als toepassing van de theorie worden verscheidene begin-randwaarde problemen geformuleerd en bestudeerd. Uit een aero-elastische analyse volgt dat een begin-randwaarde probleem voor de Rayleigh golfvergelijking beschouwd kan worden als een eenvoudig model beschrijvende de wind-geïnduceerde, verticale trillingen ('galloping') van hoogspanningsleidingen. Om zowel de verticale als de horizontale trillingen van deze hoogspanningsleidingen te beschrijven is bovendien een begin-randwaarde probleem voor een stelsel zwak niet-lineaire en zwak gekoppelde golfvergelijkingen afgeleid. Deze begin-randwaarde problemen zijn in detail onderzocht met behulp van karakteristieke coördinaten en een twee-tijdschalen perturbatiemethode. Gebruik makende van de ontwikkelde theorie zijn de goed-gesteldheid van deze problemen en de asymptotische geldigheid van de geconstrueerde benaderingen aangetoond. Verder is een begin-randwaarde probleem voor een telegraafvergelijking met een kubische niet-lineariteit onderzocht met behulp van een twee-tijdschalen perturbatiemethode en een Fourier reeksontwikkeling van de oplossing. Voor dit probleem is eveneens de asymptotische geldigheid van de geconstrueerde benadering aangetoond.

CURRICULUM VITAE

De auteur van dit proefschrift werd op 9 december 1960 geboren te Delft. Op 1 juni 1979 behaalde hij het V.W.O.-diploma aan het Christelijk Lyceum te Delft. In september van datzelfde jaar ving hij aan met de studie wiskunde aan de Technische Hogeschool te Delft (thans genoemd Technische Universiteit Delft). Het kandidaatsexamen werd op 8 oktober 1982 met goed gevolg afgelegd. Onder begeleiding van Prof.dr.ir. J.W. Reyn en Ir. P.G. Bakker werd in de afstudeerfase onderzoek gedaan naar de structuur van kegelstromingen in de omgeving van conische stuwpunten. Op 31 augustus 1984 werd met lof het diploma wiskundig ingenieur behaald. Als onderzoek-medewerker bij de Stichting Mathematisch Centrum (SMC) trad hij op 1 september 1984 in dienst van de Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (ZWO). Het onderzoek werd op de faculteit der wiskunde en informatica van de Technische Universiteit Delft uitgevoerd onder begeleiding van Prof.dr.ir. J.W. Reyn en Dr.ir. A.H.P. van der Burgh. In het kader van dit onderzoek werden congressen in Canada, Frankrijk, Hongarije, Italië en Nederland bezocht. Tijdens de onderzoeksperiode gaf hij bovendien enkele colleges differentiaalvergelijkingen en matrixrekening aan de Technische Universiteit Delft.

Stelling 1.

Laat f: $\mathbb{R} \to \mathbb{R}$ een continu differentieerbare, oneven en 2π–periodieke functie zijn. Dan geldt voor alle $x \in \mathbb{R}$ en $t \in \mathbb{R}$ dat

$$\left| \int_{x-t}^{x+t} J_0\left[\sqrt{t^2 - (x-\xi)^2}\right] f(\xi)\, d\xi \right| \leq 2\pi \max_{0 \leq \xi \leq \pi} |f(\xi)| \quad ,$$

waarin J_0 de Bessel functie is van de orde nul.

Stelling 2.

Gegeven zijn k, l, m, n en p die voldoen aan:

(i) $\quad k, l, m, n \in \mathbb{Z}^+$ en $p \in \mathbb{R}\backslash\{0\}$,

(ii) $\quad \sqrt{n^2 + p^2} = \sqrt{k^2 + p^2} + \sqrt{l^2 + p^2} - \sqrt{m^2 + p^2}$,

(iii) $\quad n = k + l - m$.

Door (ii) op een handige manier tweemaal te kwadrateren kan worden aangetoond dat de voorwaarden (i), (ii) en (iii) equivalent zijn met:

$$\left\{k, l, m, n \in \mathbb{Z}^+, p \in \mathbb{R}\backslash\{0\} \;\middle|\; k = m \wedge l = n \;\vee\; k = n \wedge l = m\right\}.$$

Indien voorwaarde (iii) vervangen wordt door $n = k + l + m$ of door $n = m - k - l$ kan bewezen worden dat er geen $k, l, m, n \in \mathbb{Z}^+$ bestaan, die aan de voorwaarden (i), (ii) en (iii) voldoen.

Stelling 5.

Ten onrechte wordt bij de behandeling van gewone differentiaal vergelijkingen weinig en soms zelfs geen aandacht besteed aan de methode van de integrerende factor ter bepaling van oplossingen van eerste orde vergelijkingen. Niet alleen is deze methode ook toepasbaar als de overige methoden voor eerste orde vergelijkingen (zoals methoden voor lineaire vergelijkingen, methode voor vergelijkingen met te scheiden veranderlijken, etc.) toepasbaar zijn, maar deze methode is bovendien uitbreidbaar naar hogere orde gewone differentiaal vergelijkingen.

Stelling 6.

Voor de opleiding tot wiskundig ingenieur behoort het een vanzelfsprekendheid te zijn dat een college partiële differentiaal vergelijkingen in het onderwijsprogramma is opgenomen. Het is betreurenswaardig dat dit niet het geval is aan de faculteit der Technische Wiskunde en Informatica van de Technische Universiteit Delft.

Stelling 7.

Als toetsingsvormen in het wiskunde–onderwijs zijn meerkeuzetoetsen en toetsen waarbij de kandidaten alleen het antwoord moeten vermelden ongeschikt om wiskundige kennis en vaardigheid van de kandidaten te testen.

Stelling 8.

Het is verwerpelijk om op de middelbare school naast Nederlands en een moderne taal alleen het vak wiskunde verplicht te stellen. Voor zowel de scholier als voor de maatschappij heeft een breed georiënteerde schoolopleiding niet te onderschatten voordelen. Het verplicht stellen en examineren (eventueel op verschillende niveaus) van alle op de middelbare school gedoceerde vakken is een overweging waard.

Stelling 9.

Bij de aanstelling van voetbalverslaggevers (m/v) dienen de Nederlandse omroep verenigingen en/of stichtingen slechts die personen aan te nemen die met goed gevolg een test hebben afgelgd waaruit blijkt dat die personen tenminste de voetbalspelregels kennen en de Nederlandse taal beheersen.