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AN ASYMPTOTIC DOUBLE COMMUTANT THEOREM FOR C^*-ALGEBRAS

BY

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ABSTRACT. An asymptotic version of von Neumann's double commutant theorem is proved in which C^*-algebras play the role of von Neumann algebras. This theorem is used to investigate asymptotic versions of similarity, reflexivity, and reductivity. It is shown that every nonseparable, norm closed, commutative, strongly reductive algebra is selfadjoint. Applications are made to the study of operators that are similar to normal (subnormal) operators. In particular, if T is similar to a normal (subnormal) operator and π is a representation of the C^*-algebra generated by t, then $\pi(T)$ is similar to a normal (subnormal) operator.

1. Introduction. One of the reasons for the success of the theory of von Neumann algebras is J. von Neumann's double commutant theorem **[46]**, which gives an alternate description of the weak closure of a selfadjoint algebra of operators. It is the purpose of this paper to prove an asymptotic version of the double commutant theorem that gives an alternate description of the norm closure of a selfadjoint algebra of operators. This asymptotic double commutant theorem helps to unify asymptotic versions of various operator-theoretic concepts (e.g., similarity, reflexivity, reductivity). Applications are made to the study of operators that are similar to normal (or subnormal) operators. Also a proof is given that a strongly reductive, nonseparable, commutative, norm closed algebra of operators is selfadjoint.

Throughout, H denotes a separable, infinite-dimensional complex Hilbert space, $B(H)$ denotes the set of *operators* (bounded linear transformations) on H, and $\mathcal{K}(H)$ denotes the set of compact operators on H. Also $\mathcal{S}$ denotes a *separable*, nonempty subset of $B(H)$. However, in §8 the separability assumptions on H and $\mathcal{S}$ will be dropped. If $\mathcal{S} \subseteq B(H)$, then $\mathcal{S}^* = \{S^*: S \in \mathcal{S}\}$, $\mathcal{Q}_u(\mathcal{S})$ is the norm closed algebra generated by 1 and $\mathcal{S}$, $\mathcal{Q}_w(\mathcal{S})$ is the weakly closed algebra generated by 1 and $\mathcal{S}$, $C^*(\mathcal{S})$ is the C^*-algebra generated by 1 and $\mathcal{S}$, and $W^*(\mathcal{S})$ is the von Neumann algebra generated by

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1 and $\mathcal{S}$. If $T \in B(H)$, then ker T and ran T denote the kernel and range of T, respectively. If M is a closed subspace of H, then $M^{(n)}$ denotes the direct sum of n copies of M, and if $A \in B(M)$, then $A^{(n)}$ denotes the direct sum of n copies of A acting on $M^{(n)}$. If M is a subspace of H, then dim M is the cardinality of an orthonormal basis for M. If X is a compact Hausdorff space, then $C(X)$ denotes the set of continuous complex functions on X. The set of complex numbers is denoted by $\mathbf{C}$.

2. Double commutants. If $\mathcal{S} \subseteq B(H)$, then the *commutant* of $\mathcal{S}$, denoted by $\mathcal{S}'$, is the set $\{T \in B(H)\colon TS = ST$ for every S in $\mathcal{S}\}$. The *double commutant* of $\mathcal{S}$ is $\mathcal{S}''$. The *approximate double commutant* of $\mathcal{S}$ is the set of those operators T for which $\|A_nT - TA_n\| \to 0$ whenever $\{A_n\}$ is a bounded sequence such that $\|A_nS - SA_n\| \to 0$ for every S in $\mathcal{S}$. The approximate double commutant of $\mathcal{S}$ is denoted by appr$(\mathcal{S})''$. The (simple) proof of the following proposition is left to the reader.

PROPOSITION 2.1. *If* $\mathcal{S} \subseteq B(H)$, *then* appr$(\mathcal{S})''$ *is a* (*norm*) *closed subalgebra of* $\mathcal{S}''$.

The double commutant theorem says that if $\mathcal{S} \subseteq B(H)$ and $\mathcal{S} = \mathcal{S}^*$, then $\mathcal{S}'' = W^*(\mathcal{S})$. If $\mathcal{S} = \mathcal{S}^*$, then $\mathcal{S}'$ is a von Neumann algebra and is therefore generated by its projections (or unitary operators). Since the projections (unitary operators) in $\mathcal{S}'$ are in $(\mathcal{S}^*)'$, it follows that the double commutant theorem has the following reformulations:

(1) $W^*(\mathcal{S}) = \{T : UT = TU$ for every unitary operator U in $\mathcal{S}'\}$,

(2) $W^*(\mathcal{S}) = \{T : PT = TP$ for every projection P in $\mathcal{S}'\}$.

The following asymptotic double commutant theorem generalizes all three of these versions of the double commutant theorem. A nonseparable version of this theorem is proved in §8.

THEOREM 2.2. *Suppose* $\mathcal{S}$ *is a separable subset of* $B(H)$. *Then*

(1) $C^*(\mathcal{S}) = \text{appr}(\mathcal{S} \cup \mathcal{S}^*)''$,

(2) $C^*(\mathcal{S}) = \{T\colon \|U_nT - TU_n\| \to 0$ *whenever* $\{U_n\}$ *is a sequence of unitary operators such that* $\|U_nS - SU_n\| \to 0$ *for every* S *in* $\mathcal{S}\}$,

(3) $C^*(\mathcal{S}) = \{T\colon \|P_nT - TP_n\| \to 0$ *whenever* $\{P_n\}$ *is a sequence of projections such that* $\|P_nS - SP_n\| \to 0$ *for every* S *in* $\mathcal{S}\}$.

PROOF. It is clear that (1) follows from either (2) or (3). The proofs of (2) and (3) are so similar that only the proof of (3) is presented here.

Write $\mathcal{S}^\sim$ for the right-hand side of the equation in (3). It is easily shown that $\mathcal{S}^\sim$ is a C^*-algebra and $C^*(\mathcal{S}) \subseteq \mathcal{S}^\sim \subseteq W^*(\mathcal{S})$. Assume via contradiction that there is an operator T in $\mathcal{S}^\sim$ but not in $C^*(\mathcal{S})$. It follows (see the proof 4.7.8 in **[35]**, or the proof of Theorem III.7, p. 288 in **[1]**, or the proof of Theorem 1.8 in **[44]**) that there is a representation π of $C^*(\mathcal{S} \cup \{T\})$ such that $\pi(T) \notin W^*(\pi(\mathbb{S}))$ and ran π has a cyclic vector. It follows from a theorem of Voiculescu [44, Theorem 1.5] (see also [19, Proposition 2.5]) that we can assume that there is a sequence $\{U_n\}$ of unitary operators such that $\|U_n^* S U_n - \pi(S)\| \to 0$ for every S in $C^*(\mathbb{S} \cup \{T\})$. (This is because Voiculescu's theorem [44, Theorem 1.5] implies that π must be unitarily equivalent to a subrepresentation of an infinite direct sum of copies of such a representation.) Since $\pi(T) \notin W^*(\pi(\mathbb{S}))$, there is a projection P in $\pi(\mathbb{S})'$ such that $\pi(T)P \neq P\pi(T)$. Thus $\{U_n P U_n^*\}$ is a sequence of projections such that, for every S in $\mathbb{S}$,

$$\begin{aligned} \|U_n P U_n^* S - S U_n P U_n^*\| &= \|P U_n^* S U_n - U_n^* S U_n P\| \\ &\to \|P\pi(S) - \pi(S)P\| = 0. \end{aligned}$$

Since $T \in \mathbb{S}^\sim$, it follows that

$$\|\pi(T)P - P\pi(T)\| = \lim \|U_n P U_n^* T - T U_n P U_n^*\| = 0.$$

This is the desired contradiction. Hence $C^*(\mathbb{S}) = \mathbb{S}^\sim$.

Corollary 2.3. *If $\mathbb{S}$ is a separable subset of $B(H)$, then $\mathcal{Q}_u(\mathbb{S}) \subseteq$ appr$(\mathbb{S})''$ $\subseteq C^*(\mathbb{S})$.*

The author wishes to express his gratitude to John Bunce, whose valuable suggestions enabled the author to greatly simplify his original proof of Theorem 2.2.

The remainder of this section determines appr$(T)''$ for T in various classes of operators. T. R. Turner [41] studied the class of those operators T for which $\{T\}'' = \mathcal{Q}_w(T)$. In particular, A. F. Ruston [33] (see also [42], [43]) proved that every algebraic operator is in this class. It was shown by A. Brown and P. R. Halmos [9] that the unilateral shift operator is in this class, and it was shown by L. J. Wallen and A. L. Shields [37] that every weighted unilateral shift is in this class. Moreover, it was shown by A. Lambert [28], [29] that if an operator T in $B(H)$ is *strictly cyclic* (i.e., $\mathcal{Q}_w(T)f = H$ for some f in H), then $\{T\}' = \{T\}'' = \mathcal{Q}_u(T)$. A more general result (for certain types of strictly cyclic algebras) was proved by E. J. Rosenthal [32, Theorem 3.1.2]. The following proposition applies to both algebraic and strictly cyclic operators.

Proposition 2.4. *If $\{T\}'' = \mathcal{Q}_u(T)$, then* appr$(T)'' = \{T\}''$.

Proof. $\mathcal{Q}_u(T) \subseteq$ appr$(T)'' \subseteq \{T\}'' = \mathcal{Q}_u(T)$.

It seems that appr$(T)'' = \{T\}''$ is a rather severe restriction. However, D. A. Herrero [27, Corollary 4] has shown that the set of such operators is norm dense in $B(H)$. In the preceding proposition $\mathcal{Q}_u(T)$ could be replaced by the norm closed algebra $\mathcal{Q}_r(T)$ generated by the rational functions in T.

Question 2.5. Does appr$(T)'' = \{T\}''$ imply that $\{T\}'' = \mathcal{Q}_r(T)$?

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out the relevant properties of strictly cyclic operators.

PROPOSITION 2.6. *If T is the unilateral shift operator, then* $\operatorname{appr}(T)'' = \mathcal{Q}_u(T)$.

PROOF. Suppose $S \in \operatorname{appr}(T)''$. Thus, by [9], S is an analytic Toeplitz operator. Also $S \in C^*(T)$; hence, by [13, Theorem 7.23], S is a compact perturbation of a Toeplitz operator with continuous symbol. Since a nonzero Toeplitz operator cannot be compact [9], it follows that S is a Toeplitz operator whose symbol is in the disk algebra. Thus $S \in \mathcal{Q}_u(T)$.

If T is a normal operator, then Fuglede's theorem [13, Theorem 4.76] says that $T^* \in \{T\}''$; this is equivalent to $\{T\}'' = W^*(T)$. R. Moore [30] has proved an asymptotic version of Fuglede's theorem: if T is normal, then $T^* \in \operatorname{appr}(T)''$. This theorem is fundamental to the results in §6; an immediate consequence (using Corollary 2.3) is the following proposition.

PROPOSITION 2.7. *If T is normal, then* $\operatorname{appr}(T)'' = C^*(T)$.

The previous three propositions gave examples where $\operatorname{appr}(T)''$ is as small ($\mathcal{Q}_u(T)$) and as large ($C^*(T)$) as possible. However, in all three of these examples it is true that $\operatorname{appr}(T)'' = \{T\}'' \cap C^*(T)$. (Note that $\operatorname{appr}(T)'' \subseteq \{T\}'' \cap C^*(T)$ is always true.)

Question 2.8. If $\mathbb{S}$ is a separable subset of $B(H)$, then must it be true that $\operatorname{appr}(\mathbb{S})'' = \mathbb{S}'' \cap C^*(\mathbb{S})$?

3. Similarity. The author initiated a study [18] of an asymptotic version of unitary equivalence of operators. (Earlier special cases were studied by P. R. Halmos [24] and by R. Gellar and L. Page [17].) All of the questions raised in [18] were answered in a very deep and beautiful paper of D. Voiculescu [44] which contains a complete characterization of "approximately" (asymptotically) equivalent representations of a separable C^*-algebra. Two operators S and T are *approximately equivalent* if there is a sequence $\{U_n\}$ of unitary operators such that $\|U_n^* S U_n - T\| \to 0$. (Actually, Voiculescu [44] requires that $U_n^* S U_n - T$ be compact for $n = 1, 2, \ldots,$ but he then proves that the two notions coincide.) There are several choices for a concept of approximate similarity, but only one seems to fit into the framework of this paper. Two operators S and T are *approximately similar* if there is a sequence $\{V_n\}$ of invertible operators with $\sup(\|V_n\|, \|V_n^{-1}\|) < \infty$ (such a sequence will be called *invertibly bounded*) and $\|V_n^{-1} S V_n - T\| \to 0$. The following proposition lists some of the elementary properties of approximate similarity. The proof is omitted.

PROPOSITION 3.1. *The following statements are true*:

(1) *approximate similarity is an equivalence relation*,

(2) *if $\mathbb{S}$ is an open* (*closed*) *subset of $B(H)$ that is closed under similarity*,

then it is closed under approximate similarity,

(3) *if S and T are approximately similar and S is invertible (algebraic, Fredholm* [**13**, *p.* 127], *quasitriangular* [**23**]), *then so is T,*

(4) *approximately equivalent operators have the same spectrum (approximate point spectrum, essential spectrum, left essential spectrum).*

Although the previous proposition does not apply to the set of operators having closed range, the next proposition shows that this set is closed under approximate similarity. First we need a technical lemma.

LEMMA 3.2. *If T, $V \in B(H)$, V is invertible, $\varepsilon > 0$, and $\|Tf\| \geqslant \varepsilon\|f\|$ for every f in* $(\ker T)^{\perp}$, *then $\|V^{-1}TVg\| \geqslant (\varepsilon/\|V\|\,\|V^{-1}\|)\|g\|$ for every g in* $(\ker V^{-1}TV)^{\perp}$.

PROOF. Let P be the projection onto $(\ker T)^{\perp}$, and let $Q = V^{-1}PV$. Write $Q = \begin{pmatrix} 1 & 0 \\ A & 0 \end{pmatrix}$ relative to $H = (\ker Q) \oplus (\ker Q)^{\perp}$, and let $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\ker Q_1 = \ker Q = \ker V^{-1}PV = \ker V^{-1}TV$. Thus Q_1 is the orthogonal projection onto $(\ker V^{-1}TV)^{\perp}$. Suppose $g = Q_1 g$. Then $\|V^{-1}TVg\| = \|V^{-1}(TP)Vg\| = \|V^{-1}TVQg\| \geqslant (1/\|V\|)\|TVQg\|$. Since $VQ = PV$, we have $VQg \in (\ker T)^{\perp}$. Hence $\|TVQg\| \geqslant \varepsilon\|VQg\| \geqslant (\varepsilon/\|V^{-1}\|)\|Qg\|$. A simple matrix computation shows that $Q_1 g = g$ implies that $\|Qg\| \geqslant \|g\|$.

PROPOSITION 3.3. *Suppose S, $T \in B(H)$, T has closed range, and $\{V_n\}$ is an invertibly bounded sequence with $V_n^{-1}TV_n \to S$. Then*

(1) *S has closed range,*

(2) *the projections onto* $\ker V_n^{-1}TV_n$ *converge in norm to the projection onto* $\ker S$,

(3) $\dim \ker S = \dim \ker T$.

PROOF. Let $T_n = V_n^{-1}TV_n$ for $n = 1, 2, \ldots$. Since $T_n \to S$, it follows that $T_n^* T_n \to S^*S$. It follows from the preceding lemma that there is a positive number ε such that $\|T_n f\| \geqslant \varepsilon\|f\|$ for every f in $(\ker T_n)^{\perp}$. Choose a continuous complex function φ such that $\varphi(0) = 0$, $\varphi|[\varepsilon^2, \infty) = 1$, and $0 < \varphi(z) < 1$ for all other values of z. Then $\varphi(T_n^* T_n) \to \varphi(S^*S)$, and since $\varphi(T_n^* T_n)$ is the projection onto $(\ker T_n)^{\perp}$ for $n = 1, 2, \ldots$, it follows that $\varphi(S^*S)$ is the projection onto $(\ker S)^{\perp}$ and that $S^*S|(\ker S)^{\perp} \geqslant \varepsilon^2$. Hence (1) and (2) are true; clearly (2) implies (3).

One of the reasons that C^*-algebras play a central role in the study of approximate equivalence of operators (see [**19**]) is the fact that if S, $T \in B(H)$ and $\{U_n\}$ is a sequence of unitary operators such that $U_n^* T U_n \to S$, then $\pi(A) = \lim U_n^* A U_n$ defines a $*$-isomorphism from $C^*(T)$ onto $C^*(S)$ with $\pi(1) = 1$ and $\pi(T) = S$. In fact, it follows from part (2) of Theorem 2.2 that $C^*(T) = \{A\colon \{U_n^* A U_n\}$ is convergent whenever $\{U_n\}$ is a sequence of unitary operators such that $\{U_n^* T U_n\}$ is convergent$\}$. The following theorem shows that $\text{appr}(T)''$ plays the corresponding role for approximate similarity.

THEOREM 3.4. *Suppose* $\mathcal{S} \subseteq B(H)$. *Then*

(1) $\text{appr}(\mathcal{S})'' = \{T: \{V_n^{-1}TV_n\}$ *is convergent whenever* $\{V_n\}$ *is an invertibly bounded sequence such that* $\{V_n^{-1}SV\}$ *is convergent for every* S *in* $\mathcal{S}\}$,

(2) *if* $\{V_n\}$ *is an invertibly bounded sequence such that* $\{V_n^{-1}SV_n\}$ *is convergent for every* S *in* $\mathcal{S}$, *then* $\pi(A) = \lim V_n^{-1}AV_n$ *defines a homeomorphic isomorphism from* $\text{appr}(\mathcal{S})''$ *onto* $\text{appr}(\pi(\mathcal{S}))''$.

PROOF. Let $\mathfrak{B}$ denote the expression on the right-hand side of the equation in (1). Suppose $T \in \text{appr}(\mathcal{S})''$ and $\{V_n\}$ is an invertibly bounded sequence such that $\{V_n^{-1}SV_n\}$ is convergent for every S in $\mathcal{S}$. Since every convergent sequence is Cauchy, it follows that $\|V_mV_n^{-1}S - SV_mV_n^{-1}\| \to 0$ (as $m, n \to \infty$) for every S in $\mathcal{S}$. Therefore $\|V_mV_n^{-1}T - TV_mV_n^{-1}\| \to 0$. It follows that $\{V_n^{-1}TV_n\}$ is convergent.

Conversely, suppose $T \in \mathfrak{B}$ and $\{W_n\}$ is a bounded sequence such that $\|W_nS - SW_n\| \to 0$ for every S in $\mathcal{S}$. Choose a positive number λ such that $\lambda > 2\sup\|W_n\|$. Then, for every n, we have $\|(\lambda - W_n)^{-1}\| = \|\lambda^{-1}\sum_{n=0}^{\infty}(W_n/\lambda)^n\| < 2/\lambda$. Hence $\{(W_n - \lambda)\}$ is an invertibly bounded sequence and since $\|(W_n - \lambda)S - S(W_n - \lambda)\| = \|W_nS - SW_n\| \to 0$ for every S in $\mathcal{S}$, it follows that $(W_n - \lambda)^{-1}S(W_n - \lambda) \to S$ for every S in $\mathcal{S}$. Define a sequence $\{V_k\}$ whose even terms are the $(W_n - \lambda)$'s and whose odd terms are all 1. Then $\{V_k\}$ is an invertibly bounded sequence and $V_k^{-1}SV_k \to S$ for every S in $\mathcal{S}$. Since $T \in \mathfrak{B}$, it follows that $\{V_k^{-1}TV_k\}$ is convergent, and therefore convergent to T (consider the odd terms). Thus

$$\|(W_n - \lambda)^{-1}T(W_n - \lambda) - T\| \to 0,$$

and it follows that $\|W_nT - TW_n\| \to 0$. Whence $T \in \text{appr}(T)''$. Thus (1) is true.

It is clear that the mapping π in (2) is a homeomorphic isomorphism on $\text{appr}(\mathcal{S})''$. Since $\{V_nAV_n^{-1}\}$ is convergent for every A in $\pi(\mathcal{S})$, then, by symmetry, it suffices to show that $\text{ran}\,\pi \subseteq \text{appr}(\pi(\mathcal{S}))''$. Suppose that $\{W_n\}$ is an invertibly bounded sequence and $\{W_n^{-1}\pi(S)W_n\}$ is convergent for every S in $\mathcal{S}$. Then it follows that $\{W_n^{-1}V_n^{-1}SV_nW_n\}$ is convergent for every S in $\mathcal{S}$. Hence, by (1), $\{W_n^{-1}V_n^{-1}TV_nW_n\}$ is convergent for every T in $\text{appr}(\mathcal{S})''$. Therefore $\{W_n^{-1}AW_n\}$ is convergent for every A in $\text{ran}\,\pi$. It follows from (1) that $\text{ran}\,\pi \subseteq \text{appr}(\pi(\mathcal{S}))''$.

The following theorem characterizes approximate similarity for normal operators. Note the analogy with the result [25, Problem 152] that similar normal operators are unitarily equivalent. Also recall that two normal operators are approximately equivalent if and only if they have the same spectrum and their isolated eigenvalues have the same multiplicities (see [17]). This theorem was discovered independently by B. Chan (see §7).

THEOREM 3.5. *Two approximately similar normal operators are approximately equivalent. An operator S is approximately similar to a normal operator T if and only if S is similar to an operator that is approximately equivalent to T.*

PROOF. If two normal operators are approximately similar, then they must have the same spectrum, and by Proposition 3.3, their isolated eigenvalues have the same multiplicities; hence they are approximately equivalent.

Suppose $S, T \in B(H)$, T is normal, and $\{V_n\}$ is an invertibly bounded sequence such that $V_n^{-1}TV_n \to S$. Since T is normal, it follows (Proposition 2.7) that $\operatorname{appr}(T)'' = C^*(T)$. Since $C^*(T)$ is isometrically isomorphic to $C(\sigma(T))$, it follows from part (2) of Theorem 3.4 that the mapping $\varphi \to \lim V_n^{-1}\varphi(T)V_n$ is a homeomorphic isomorphism from $C(\sigma(T))$ onto $\operatorname{appr}(S)''$. It follows from [47] that S is similar to a normal operator, which, being approximately similar to T, must be approximately equivalent to T. The converse is trivial.

In [19, Corollary 3.7] it is shown that two operators are approximately equivalent if and only if there is a rank-preserving $*$-isomorphism between the C^*-algebras they generate that sends one of the operators onto the other. Although, in general, there seems to be no analogous result for approximate similarity, there is one in the case when one of the operators is normal.

PROPOSITION 3.6. *Suppose $S, T \in B(H)$ and T is normal. Then S, T are approximately similar if and only if there is an isomorphism π: $C^*(T) \to B(H)$ such that $\pi(1) = 1$, $\pi(T) = S$, and* $\operatorname{rank} A = \operatorname{rank} \pi(A)$ *for every A in $C^*(T)$.*

PROOF. Since $C^*(T)$ is isomorphic to $C(\sigma(T))$, it follows from [47] that S is similar to a normal operator W. Let $W = V^{-1}SV$ for some invertible operator V. Then $\sigma(T) = \sigma(W)$. If λ is an isolated eigenvalue of T, then the projection P onto $\ker(T - \lambda)$ is in $C^*(T)$. Hence $V^{-1}\pi(P)V$ is an idempotent whose range is $\ker(W - \lambda)$. Since $\operatorname{rank} P = \operatorname{rank} \pi(P) = \operatorname{rank} V^{-1}\pi(P)V$, it follows that λ has the same multiplicity for both T and W. Thus, by [17], T and W are approximately equivalent; hence S and T are approximately similar. The converse follows from Theorem 3.5.

It follows from Theorem 3.5 that the set of operators that are similar to normal operators is closed under approximate similarity. In §6 it will be shown (Theorem 6.12) that the set of operators that are similar to subnormal operators is closed under approximate similarity. Both of these sets of operators will be studied in more detail in §6.

There is an analogue of Theorem 3.5 for isometries. The proof is based on the characterization [40] of the set of operators similar to an isometry as the set of all operators T for which there is a positive number r such that $(1/r)\|f\| < \|T^nf\| < r\|f\|$ for every vector f and for every positive integer n. It was also shown by Halmos [24] that two nonunitary isometries are approximately equivalent if and only if they have the same Fredholm index [13, p. 127].

PROPOSITION 3.7. *Two approximately similar isometries are approximately equivalent. An operator S is approximately similar to an isometry T if and only if S is similar to an operator that is approximately equivalent to T.*

PROOF. Suppose that A, B are approximately similar isometries. Then A, B must have the same Fredholm index (because the Fredholm index is norm continuous and invariant under similarity [13, Theorem 5.36]). If the index is zero, then A, B are both unitary and, by Theorem 3.5, are approximately equivalent. Otherwise, A and B are approximately equivalent because they are nonunitary isometries with the same Fredholm index [24].

Next suppose that S is approximately similar to an isometry T. Choose an invertibly bounded sequence $\{V_n\}$ such that $V_n^{-1}TV_n \to S$, and let $r = \sup(\|V_n\| \cdot \|V_n^{-1}\|)$. Since $(1/r)\|f\| \leq \|V_n^{-1}T^kV_nf\| \leq r\|f\|$ for every vector f and for all positive integers k, n, it follows that $(1/r)\|f\| \leq \|S^kf\| \leq r\|f\|$ for every vector f and every positive integer k. It follows from [40] that S is similar to an isometry, which, being approximately similar to T, must be approximately equivalent to T. The converse is obvious.

The preceding proposition and Theorem 3.5 suggest the following questions.

Question 3.8. If S and T are approximately similar, then must S be similar to an operator that is approximately equivalent to T?

Question 3.9. If S is similar to an operator that is approximately equivalent to T, then must T be similar to an operator that is approximately equivalent to S?

Question 3.10. If a set of operators is closed under similarity and approximate equivalence, then must it be closed under approximate similarity?

All three of these questions are related to the general question of whether approximate similarity can be expressed in terms of similarity and approximate equivalence. Note that an affirmative answer to Question 3.8 would imply an affirmative answer to the other two questions. Another implication of an affirmative answer to Question 3.8 is the statement that if the *unitary equivalence class*, $U(T)$, of T is closed, then every operator that is approximately similar to T must be similar to T. If $U(T)$ is closed, then T is unitarily equivalent to $A^{(\infty)} \oplus B$ where A, B act on finite-dimensional Hilbert spaces. It follows from a result of C. Apostol and J. Stampfli [4] that an operator S is similar to an operator T with $U(T)$ closed if and only if S is algebraic and, for every polynomial p, $\operatorname{ran} p(S)$ is closed and $\operatorname{rank} p(S) = \operatorname{rank} p(T)$.

THEOREM 3.11. *If $U(T)$ is closed and S is approximately similar to T, then S is similar to T.*

PROOF. It follows from Proposition 3.3 that $p(S)$ has closed range for every polynomial p and that S is algebraic (because T has these properties). It is also clear that approximate similarity preserves rank. Hence S is similar to T.

A belief in an affirmative answer to Question 3.8 leads to the following conjecture.

Conjecture 3.12. The following are equivalent:

(1) S is similar to an operator T with $U(T)$ closed,

(2) every operator approximately similar to S is similar to S,

(3) every operator in $U(S)^-$ is similar to S.

Another question that deserves asking is whether approximate similarity implies similarity in the Calkin algebra $B(H)/\mathcal{K}(H)$.

4. Reflexivity. If $\mathbb{S} \subseteq B(H)$ and M is a subspace of H, then M is an *invariant subspace* of $\mathbb{S}$ if $S(M) \subseteq M$ for every S in $\mathbb{S}$. The set of all invariant subspaces of $\mathbb{S}$ is Lat $\mathbb{S}$, and Alg(Lat $\mathbb{S}$) $= \{T : \text{Lat } \mathbb{S} \subseteq \text{Lat } T\}$. In [3] C. Apostol, C. Foiaş, and D. Voiculescu introduced an asymptotic analogue of Alg(Lat $\mathbb{S}$); i.e., appr(Alg(Lat $\mathbb{S}$)) is the set of all operators T for which $\|(1 - P_n)TP_n\| \to 0$ whenever $\{P_n\}$ is a sequence of projections such that $\|(1 - P_n)SP_n\| \to 0$ for every S in $\mathbb{S}$. An operator T is *reflexive* if Alg(Lat T) $= \mathcal{Q}_w(T)$, and T is *approximately reflexive* if appr(Alg(Lat T)) $=$ $\mathcal{Q}_u(T)$. Reflexivity has been studied by various authors (see [31]).

It was proved in [3, Corollary 4] that if $\mathbb{S}$ is a separable subset of $B(H)$, then appr(Alg(Lat $\mathbb{S}$)) is a norm closed subalgebra of $[\mathcal{Q}_u(\mathbb{S}) + \mathcal{K}(H)]^-$. Combining these facts with part (3) of Theorem 2.2 we easily obtain the following proposition.

PROPOSITION 4.1. *If $\mathbb{S}$ is a separable subset of $B(H)$, then* appr(Alg(Lat $\mathbb{S}$)) $\subseteq C^*(\mathbb{S}) \cap [\mathcal{Q}_u(\mathbb{S}) + \mathcal{K}(H)]^-$.

COROLLARY 4.2. *If $C^*(\mathbb{S}) \cap \mathcal{K}(H) = 0$, then* appr(Alg(Lat $\mathbb{S}$)) $= \mathcal{Q}_u(\mathbb{S})$.

COROLLARY 4.3. *If T is approximately equivalent to $T \oplus T$, then T is approximately reflexive.*

PROOF. It follows from [44, Theorem 1.5] that T is approximately equivalent to $T \oplus T$ if and only if $C^*(T) \cap \mathcal{K}(H) = 0$.

Note that Corollary 4.3 is an asymptotic version of the well known fact [31, Corollary 9.19] that if T is unitarily equivalent to $T \oplus T$, then T is reflexive. Consideration of the "nonasymptotic" version of Corollary 4.2 leads to the following question.

Question 4.4. If $W^*(T) \cap \mathcal{K}(H) = 0$, then is T reflexive?

If P is an idempotent (not necessarily Hermitian), then ran P is an invariant subspace of an operator T if and only if $(1 - P)TP = 0$. Hence it seems that appr(Alg(Lat $\mathbb{S}$)) could be defined in terms of idempotents that are not necessarily projections. Note that Corollary 4.6 is not at all obvious from the original definition of appr(Alg(Lat $\mathbb{S}$)).

PROPOSITION 4.5. *If* $\mathbb{S} \subseteq B(H)$, *then*

$$\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S})) = \{T\colon \|(1-P_n)TP_n\| \to 0 \textit{ whenever } \{P_n\} \textit{ is a bounded sequence of idempotents such that } \|(1-P_n)SP_n\| \to 0 \textit{ for every } S \textit{ in } \mathbb{S}\}.$$

PROOF. Suppose that P is an idempotent and Q is the projection onto ran P. Write $P = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ relative to $H = (\operatorname{ran} P) \oplus (\operatorname{ran} P)^{\perp}$. Simple matrix calculations show that, for every operator S, we have

$$\begin{aligned} \|(1-P)SP\| &\geqslant \|(1-Q)(1-P)SPQ\| \\ &= \|(1-Q)SQ\| \geqslant (1/4\|P\|^2)\|(1-P)SP\|. \end{aligned}$$

Therefore if $\{P_n\}$ is a bounded sequence of idempotents, and if Q_n is the projection onto ran P_n for $n = 1, 2, \dots,$ then, for every operator S, we have that $\|(1-P_n)SP_n\| \to 0$ if and only if $\|(1-Q_n)SQ_n\| \to 0$. This fact clearly implies the validity of the proposition.

COROLLARY 4.6. *If* $\mathbb{S} \subseteq B(H)$ *and* V *is invertible, then*

$$V^{-1}\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S}))V = \operatorname{appr}\big(\operatorname{Alg}\big(\operatorname{Lat}(V^{-1}\mathbb{S}V)\big)\big).$$

An analogue of the preceding corollary with similarity replaced by approximate equivalence is also true.

PROPOSITION 4.7. *If* $\mathbb{S}$ *is a separable subset of* $B(H)$, *and if* π *is a representation of* $C^*(\mathbb{S})$ *defined by* $\pi(A) = \lim U_n^* A U_n$ *for some sequence* $\{U_n\}$ *of unitary operators, then* $\pi(\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S}))) = \operatorname{appr}(\operatorname{Alg}(\operatorname{Lat}(\pi(\mathbb{S}))))$.

PROOF. Suppose $\{P_n\}$ is a sequence of projections. The proposition follows from the fact that, for each T in $C^*(\mathbb{S})$, we have $\|(1-P_n)TP_n\| \to 0$ if and only if $\|(1-U_n^*P_nU_n)\pi(T)U_n^*P_nU_n\| \to 0$.

The reason that there is no analogue of Corollary 4.6 with similarity replaced by approximate similarity is that the homomorphism analogous to π in the preceding proposition may only be defined on $\operatorname{appr}(\mathbb{S})''$ (see Theorem 3.4) and it is possible appr(Alg(Lat($\mathbb{S}$))) is not contained in $\operatorname{appr}(\mathbb{S})''$ (see part (4) of Theorem 4.10). However, the proof of the preceding proposition is easily adapted to prove the following proposition.

PROPOSITION 4.8. *If* $\mathbb{S} \subseteq B(H)$ *and* $\{V_n\}$ *is an invertibly bounded sequence such that* $\{V_n^{-1}SV_n\}$ *is convergent for every* S *in* $\mathbb{S}$, *and if* π *is the mapping on* $\operatorname{appr}(\mathbb{S})''$ *defined by* $\pi(A) = \lim V_n^{-1}AV_n$, *then* $\pi(\operatorname{appr}(\mathbb{S}))'' \cap \operatorname{appr}(\operatorname{Alg}(\operatorname{Lat}(\mathbb{S}))) = \operatorname{appr}(\pi(\mathbb{S}))'' \cap \operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \pi(\mathbb{S})))$.

It has been conjectured that $\mathrm{Alg}(\mathrm{Lat}\, T) \cap \{T\}'' = \mathcal{Q}_w(T)$ for every operator T. The task of phrasing the "asymptotic analogue" is left to the reader. By now it should be clear to the reader that there is a definite parallel between various concepts in operator theory and their asymptotic counterparts. In the asymptotic version $C^*(\mathbb{S})$ plays the role of $W^*(\mathbb{S})$ and $\mathcal{Q}_u(\mathbb{S})$ plays the role of $\mathcal{Q}_w(\mathbb{S})$. Thus there is a two-way bridge which enables the asymptotic and nonasymptotic theories to enrich each other.

It follows from Corollary 4.6 and Proposition 4.7 that the set of approximately reflexive operators is closed under similarity and approximate equivalence. Thus Question 3.10 leads us to the following question.

Question 4.9. Is the set of approximately reflexive operators closed under approximate similarity?

It was shown by D. Sarason [36] that every normal operator and every analytic Toeplitz operator is reflexive, and J. Deddens proved that every isometry is reflexive [10]. Also J. Deddens and P. A. Fillmore gave a simple characterization [11] of reflexivity among the operators with finite rank. The following theorem shows that these operators are also approximately reflexive.

THEOREM 4.10. *The following statements are true.*

(1) *every normal operator is approximately reflexive,*

(2) *every analytic Toeplitz operator is approximately reflexive,*

(3) *every isometry is approximately reflexive,*

(4) *if* $\operatorname{rank} T < \infty$, *then* $\mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, T)) = \mathrm{Alg}(\mathrm{Lat}\, T)$,

(5) *a finite rank operator is approximately reflexive if and only if it is reflexive.*

PROOF. (1) Suppose φ is a continuous complex function and $\varphi(T) \in \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, T))$. Let A be a normal operator whose spectrum is the essential spectrum of T and such that A is unitarily equivalent to $A \oplus A$. It follows from [17] that T is approximately equivalent to $T \oplus A$. Hence, by Proposition 4.7, we have $\varphi(T \oplus A) \in \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, T \oplus A))$. Since $\varphi(T \oplus A) = \varphi(T) \oplus \varphi(A)$, it follows that $\varphi(A) \in \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, A))$. It follows from Corollary 4.3 that A is approximately reflexive. Therefore the restriction of φ to the essential spectrum of T is a uniform limit of polynomials. Since the rest of $\sigma(T)$ consists only of isolated points, it is an easy task (using Runge's theorem) to show that the restriction of φ to $\sigma(T)$ is a uniform limit of polynomials.

(2) Suppose T is an analytic Toeplitz operator and $S \in \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, T))$. Since T is reflexive, it follows that S must also be an analytic Toeplitz operator. It follows from Proposition 4.1 that there is a sequence $\{T_n\}$ of Toeplitz operators in $\mathcal{Q}_u(T)$ and a sequence $\{K_n\}$ of compact operators such

that $\|T_n + K_n - S\| \to 0$. Since the restriction of the quotient map from $B(H)$ into $B(H)/\mathcal{K}(H)$ is isometric when restricted to the class of analytic Toeplitz operators [13, 7.15, p. 182], it follows that $T_n \to S$. Hence $S \in \mathcal{Q}_u(T)$.

(3) Suppose T is an isometry. If T is unitary, then T is approximately reflexive. Otherwise, by [24], T is approximately equivalent to $S^{(n)}$ where S is the unilateral shift operator, and $1 < n < \infty$. Since S is an analytic Toeplitz operator, S must be approximately reflexive by (2). Thus appr(Alg(Lat $S^{(n)}$)) $\subseteq$ appr(Alg(Lat S))$^{(n)} = \mathcal{Q}_u(S)^{(n)} = \mathcal{Q}_u(S^{(n)})$.

(4) Suppose rank $T < \infty$ and write $T = A \oplus 0$ relative to $H = M \oplus M^{\perp}$ where $\dim M < \infty$. Suppose $S \in$ Alg(Lat T). We must prove that $S \in$ appr(Alg(Lat T)). Since $S|M^{\perp}$ is a scalar, we can assume that $S|M^{\perp} = 0$. Write $S = B \oplus 0$ relative to $H = M \oplus M^{\perp}$. Suppose that $\{P_n\}$ is a sequence of projections such that $\|(1 - P_n)TP_n\| \to 0$. If P_n has the operator matrix

$$\begin{pmatrix} C_n & D_n \\ D_n^* & E_n \end{pmatrix}$$

relative to $H = M \oplus M^{\perp}$, then a matrix calculation shows that

$$(1 - P_n)TP_n = \begin{pmatrix} (1 - C_n)C_n & (1 - C_n)AD_n \\ - D_n^*AC_n & - D_n^*AD_n \end{pmatrix}$$

and that the matrix of $(1 - P_n)SP_n$ is obtained from this one by replacing the A's by B's. Since $\{C_n\}$ is a bounded sequence on a finite-dimensional space, there is no harm in assuming that $C_n \to C$ for some operator C. The fact that $P_n^2 = P_n$ implies (using a matrix calculation) that $D_nD_n^* \to C(1 - C)$. Since $(1 - P_n)TP_n \to 0$, it follows that $(1 - C)AC = 0$. Since $0 < C < 1$ and $(\operatorname{ran} C)^- = (\operatorname{ran} C^{1/2})^-$ and $\ker(1 - C) = \ker(1 - C)^{1/2}$, it follows that $(1 - C)^{1/2}AC^{1/2} = 0$. Let

$$Q = \begin{bmatrix} C & C^{1/2}(1 - C)^{1/2} \\ C^{1/2}(1 - C)^{1/2} & 1 - C \end{bmatrix}$$

relative to $H = M \oplus M^{\perp}$. Matrix calculations show that Q is a projection and $(1 - Q)TQ = 0$. Hence $(1 - Q)SQ = 0$. It follows that $(1 - C)^{1/2}BC^{1/2} = 0$, and from this it follows that $(1 - P_n)SP_n \to 0$ (e.g., $(-D_nBC_n)^*(-D_nBC_n) \to CB^*C(1 - C)BC = 0$). Statement (5) follows immediately from (4).

An operator T is *reductive* if $T^* \in$ Alg(Lat T). It was shown by J. Dyer and P. Porcelli [15] that the reductive operator problem (Is every reductive operator normal?) is equivalent to the invariant subspace problem (Does every operator in $B(H)$ have a nontrivial invariant subspace?). An operator T is *strongly reductive* [26] if $T^* \in$ appr(Alg(Lat T)). It was shown by C. Apostol, C. Foiaş, and D. Voiculescu [2] that every strongly reductive operator is normal. It was shown by K. Harrison [26] (and it follows from part (1) of Theorem 4.10) that a normal operator T is reductive if and only if $\sigma(T)$ has no interior and does not separate the plane. It is obvious that a strongly reductive, approximately reflexive operator is normal. Thus a determination of which operators are approximately reflexive could be considered an extension of the theorem in [2]. This section is concluded with a question.

Question 4.11. If a compact perturbation of a strongly reductive operator is reductive, then must it be strongly reductive?

A theorem about strongly reductive algebras is proved in §8.

5. Decomposable functions. In [18] and [8] the author, A. Brown, and C. K. Fong introduced functions defined on operators that imitate the properties of Borel functions of normal operators. Let sub $B(H) = \bigcup \{B(M)\colon M$ is a subspace of $H\}$. A *decomposable function* is a function φ: sub $B(H) \to$ sub $B(H)$ such that

(1) $\varphi(B(M)) \subseteq B(M)$ for every subspace M of H.

(2) if $T \in B(H)$ and M reduces T, then M reduces $\varphi(T)$ and $\varphi(T|M) = \varphi(T)|M$,

(3) if M, N are subspaces of H and $U\colon M \to N$ is unitary and $T \in B(N)$, then $\varphi(U^*TU) = U^*\varphi(T)U$.

A decomposable function φ is (norm) *continuous* if $\varphi|B(M)$ is continuous for every subspace M of H. Note that (2) says that $\varphi(A \oplus B) = \varphi(A) \oplus \varphi(B)$.

Decomposable functions were used in [8] to establish a general theory of "parts" of operators (generalizing such concepts as unitary part). For a more complete account of the properties of decomposable functions the reader should consult [21].

It is the purpose of this section to give a simple characterization of continuous decomposable functions. Note that it follows from (2) and the double commutant theorem that $\varphi(T) \in W^*(T)$ for every decomposable function φ and every operator T.

THEOREM 5.1. *Suppose φ is a decomposable function. The following statements are equivalent.*

(1) *φ is continuous,*

(2) *$\varphi(T) \in C^*(T)$ for every operator T and $\pi(\varphi(T)) = \varphi(\pi(T))$ for every representation π of $C^*(T)$,*

(3) *there is a sequence $\{p_n(x, y)\}$ of noncommutative polynomials such that $p_n(T, T^*) \to \varphi(T)$ uniformly on bounded subsets of $B(H)$.*

PROOF. (1) $\Rightarrow$ (2). Suppose φ is continuous and $T \in B(H)$. Suppose also that $\{U_n\}$ is a sequence of unitary operators such that $\|U_nT - TU_n\| \to 0$.

Then $U_n^*TU_n \to T$. It follows that $U_n^*\varphi(T)U_n = \varphi(U_n^*TU_n) \to \varphi(T)$. Hence $\|U_n\varphi(T) - \varphi(T)U_n\| \to 0$. It follows from Theorem 2.2 that $\varphi(T) \in C^*(T)$. Suppose that π is a representation of $C^*(T)$. In view of [**44**, Theorem 1.5] there is no harm in assuming that there is a sequence $\{V_n\}$ of unitary operators such that $\pi(A) = \lim V_n^*AV_n$ for every A in $C^*(T)$. Whence $\pi(\varphi(T)) = \lim V_n^*\varphi(T)V_n = \lim \varphi(V_n^*TV_n) = \varphi(\pi(T))$.

(2) $\Rightarrow$ (3). Let S be unitarily equivalent to a direct sum of finite matrices so that $\|S\| = 1$, and, for each positive integer n, the $n \times n$ direct summands of S are dense in the unit ball in $B(\mathbf{C}^{(n)})$. For each positive integer n we have $\varphi(nS) \in C^*(nS)$; hence there is a noncommutative polynomial $p_n(x, y)$ such that $\|p_n(nS, nS^*) - \varphi(nS)\| < 1/n$. If $T \in B(H)$ and $n > \|T\|$, then there is a representation π of $C^*(nS)$ such that $\pi(1) = 1$ and $\pi(nS) = T$. Therefore $\|P_n(T, T^*) - \varphi(T)\| = \|\pi(p_n(nS, nS^*) - \varphi(nS))\| < 1/n$.

The implication (3) $\Rightarrow$ (1) is obvious.

6. Operators similar to normal operators. It was proved by J. Wermer [**47**] that an operator in $B(H)$ is similar to a normal operator if and only if it is a scalar type spectral operator in the sense of Dunford [**14**]. R. G. Douglas [**12**] has given sufficient conditions for a bounded homomorphism from a C^*-algebra into $B(H)$ to be similar to a $*$-homomorphism. In particular, every bounded homomorphism from a commutative C^*-algebra into $B(H)$ is similar to a $*$-homomorphism. It follows that an operator T in $B(H)$ is similar to a normal operator if and only if there is a compact Hausdorff space X and a bounded homomorphism τ: $C(X) \to B(H)$ such that $T \in \operatorname{ran} \tau$. More precisely, T is similar to a normal operator if and only if there is a homeomorphic isomorphism from $C(\sigma(T))$ into $B(H)$ that sends the function $f(z) = z$ onto T. The following proposition shows how approximate double commutants fit into this discussion. The corollary that follows is an asymptotic analogue of [**12**, Theorem 2].

PROPOSITION 6.1. *Suppose X is a compact subset of the plane, τ is a bounded homomorphism from $C(X)$ into $B(H)$ with $\tau(1) = 1$, and let $f(z) = z$ for every z in X. Then*

(1) $\operatorname{ran} \tau = \operatorname{appr}(\tau(f))''$,

(2) *if ν: $C(X) \to B(H)$ is a bounded homomorphism with $\nu(1) = 1$ and $\nu(f) = \tau(f)$, then $\nu = \tau$.*

PROOF. (1) Since τ is similar to a $*$-homomorphism, there is a normal operator S and an invertible operator V such that $\tau(g) = V^{-1}g(S)V$ for every g in $C(X)$. Thus, by Proposition 2.7, we have $\operatorname{ran} \tau = V^{-1}C^*(S)V = \operatorname{appr}(V^{-1}SV)'' = \operatorname{appr}(\tau(f))''$.

(2) If S, V are as in the proof of (1), then the mapping $g \to V\nu(g)V^{-1}$ is a homomorphism from $C(X)$ onto the commutative C^*-algebra $C^*(S)$

(because ν and τ have the same range); such a homomorphism must be a *-homomorphism. Therefore $V\nu(\bar{f})V^{-1} = S^*$. It follows that $\nu(\bar{f}) = \tau(\bar{f})$, and, by the Stone-Weierstrass theorem, $\nu = \tau$.

COROLLARY 6.2. *An operator T in $B(H)$ is similar to a normal operator if and only if the Gelfand map on* appr$(T)''$ *is* 1-1 *and onto.*

The following theorem is a basis for the proofs of most of the results in this section.

THEOREM 6.3. *If T is similar to a normal operator and π is a bounded homomorphism on $C^*(T)$, then $\pi(T)$ is similar to a normal operator.*

PROOF. Since T is similar to a normal operator, it follows that $C(\sigma(T))$ is isomorphic to appr$(T)''$. However, appr$(T)'' \subseteq C^*(T)$; thus the restriction of π to appr$(T)''$ induces a bounded homomorphism τ on $C(\sigma(T))$ such that $\pi(T) \in \operatorname{ran} \tau$. Hence $\pi(T)$ is similar to a normal operator.

Suppose T is similar to a normal operator. Write $T = V^{-1}SV$ where S is normal and V is invertible. Then the mapping τ: $C(\sigma(T)) \to \operatorname{appr}(T)''$ defined by $\tau(g) = V^{-1}g(S)V$ is called the *spectral homomorphism* of T. (Note that while S, V are not unique, the mapping τ is unique.) The mapping τ gives rise to a (unique) functional calculus: if φ is a continuous complex function, then $\varphi(T)$ is defined to be $\tau(\varphi|\sigma(T))$. Note that if $T = T_1 \oplus T_2$, then $\varphi(T) = \varphi(T_1) \oplus \varphi(T_2)$ for every continuous complex function φ.

DEFINITION 6.4. We define functions α, β: $B(H) \to [1, \infty]$ by

$$\alpha(T) = \begin{cases} \infty & \text{if } T \text{ is not similar to a normal operator,} \\ \|\tau\| & \text{if } \tau \text{ is the spectral homomorphism of } T, \end{cases}$$

$$\beta(T) = \inf\{\max(\|V\|, \|V^{-1}\|): V^{-1}TV \text{ is normal}\}$$

for every T in $B(H)$. Note that if T is not similar to a normal operator, then $\beta(T) = \inf \varnothing = \infty$.

PROPOSITION 6.5. *Suppose $T \in B(H)$. Then*

(1) $\beta(T) \leq \alpha(T) \leq \beta(T)^2$,

(2) *if $T = T_1 \oplus T_2 \oplus \cdots$, then $\alpha(T) = \sup \alpha(T_n)$ and $\beta(T) = \sup \beta(T_n)$,*

(3) *if S and T are approximately equivalent, then $\alpha(S) = \alpha(T)$,*

(4) *if π is a representation of $C^*(T)$, then $\alpha(\pi(T)) \leq \alpha(T)$,*

(5) $\alpha(T) = \sup\{\alpha(\pi(T))$: *$\pi$ is an irreducible representation of $C^*(T)$*$\}$.

PROOF. Statements (2), (3) are easily proved, and (4) follows from the fact that representations of C^*-algebras are contractive. Statement (5) follows from (2)–(4) and the fact [44, Proposition 2.1] that every operator is approximately equivalent to a direct sum of irreducible operators. Statement (1) follows from the proofs of XV.6.1 and XV.6.2 in [14].

For the notation and terminology used in the next corollary see [20].

COROLLARY 6.6. *If* $(X, \mathfrak{M}, \mu)$ *is a* σ*-finite measure space and* $\tau: X \to B(H)$ *is essentially bounded and weakly Borel measurable, then* $\alpha(\int_X^{\oplus} \tau(x)d\mu(x)) =$ ess-sup $\alpha(\tau(x))$.

PROOF. This follows immediately from parts (2), (3) of Proposition 6.5 and Theorem A in [20].

We can now prove some theorems about limits of operators that are similar to normal operators. The set of operators that are similar to normal operators is not closed; its closure was shown by D. Voiculescu to be the set of biquasitriangular operators [45]. The following eight results describe some closure properties of certain subsets of this set. The first theorem shows that α is lower semicontinuous.

THEOREM 6.7. *If* $\{T_n\}$ *is a sequence of operators with* $\sup \alpha(T_n) < \infty$ *and* $\|T_n - T\| \to 0$, *then*

(1) $\alpha(T) \leq \sup \alpha(T_n)$,

(2) $\|\varphi(T_n) - \varphi(T)\| \to 0$ *for every continuous complex function* φ *on* **C**.

PROOF. Let $S = T_1 \oplus T_2 \oplus \cdots$. Then $\|p(S, S^*)\| \geq \lim\|p(T_n, T_n^*)\| = \|p(T, T^*)\|$ for every noncommutative polynomial $p(x, y)$. Hence there is a representation π of $C^*(S)$ such that $\pi(1) = 1$ and $\pi(S) = T$. It follows from Proposition 6.5 that $\alpha(T) \leq \alpha(S) = \sup \alpha(T_n)$. Since $\mathfrak{B} = \{A_0 \oplus A_1 \oplus A_2 \oplus \cdots : \|A_n - A_0\| \to 0\}$ is a C^*-algebra, it follows that $\operatorname{appr}(T \oplus T_1 \oplus T_2 \oplus \cdots)'' \subseteq \mathfrak{B}$. If φ is a continuous complex function, then $\varphi(T) \oplus \varphi(T_1) \oplus \cdots = \varphi(T \oplus T_1 \oplus \cdots) \in \mathfrak{B}$. Therefore (2) is true.

The preceding theorem remains true when norm convergence is replaced by convergence in the *-strong operator topology. A net $\{T_n\}$ converges **-strongly* to an operator T if $T_n \to T$ strongly and $T_n^* \to T^*$ strongly. (For more information about this topology see [16].) All of the *-algebraic operations are (sequentially) *-strongly continuous. It therefore follows from Theorem 5.1 that a (norm) continuous decomposable function is (sequentially) *-strongly continuous. The following theorem should be compared with [6, Theorem 2.3]. Its proof can be obtained by replacing convergence by *-strong convergence in the proof of the preceding theorem.

THEOREM 6.8. *If* $\{T_n\}$ *is a sequence with* $\sup \alpha(T_n) < \infty$ *and if* $T_n \to T$ **-strongly, then*

(1) $\alpha(T) \leq \sup \alpha(T_n)$,

(2) $\varphi(T_n) \to \varphi(T)$ **-strongly for every continuous complex function* φ *on* **C**.

In [16, §2.1] J. Ernest indicated the importance of determining which important classes of operators are Borel sets in the *-strong operator topology.

COROLLARY 6.9. *The set of operators similar to normal operators is an F_σ in the *-strong operator topology.*

COROLLARY 6.10. *Suppose r, R are positive numbers and let $\mathfrak{B} = \{T: \alpha(T) < r$ and $\|T\| < R\}$. For each continuous complex function f there is a continuous decomposable function φ such that $\varphi(T) = f(T)$ for every T in $\mathfrak{B}$.*

PROOF. Choose a sequence $\{T_n\}$ that is *-strongly dense in $\mathfrak{B}$ and let $S = T_1 \oplus T_2 \oplus \cdots$. Since $f(S) \in C^*(S)$, it follows from **[21**, Theorem 2.2] that there is a continuous decomposable function φ such that $\varphi(S) = f(S)$. Hence $\varphi(T_n) = f(T_n)$ for $n = 1, 2, \ldots$. It follows from Theorem 6.8 and the density of the T_n's that $\varphi(T) = f(T)$ for every T in $\mathfrak{B}$.

Normality can be described in terms of an equation: $T^*T - TT^* = 0$. The following theorem is an analogue for operators similar to normal operators. Recall that sub $B(H) = \bigcup \{B(M)$: M is a subspace of $H\}$.

THEOREM 6.11. *Suppose r, R are positive numbers and let $\mathfrak{B} = \{T: \alpha(T) < r$ and $\|T\| < R\}$. Then there is a continuous decomposable function φ such that $\mathfrak{B} = \{T \in B(H): \varphi(T) = 0\}$.*

PROOF. Let $\mathfrak{B}_1 = \{T \in \text{sub}\, B(H): \alpha(T) < r$ and $\|T\| < R\}$. It follows from Proposition 6.5 that a direct sum of operators is in $\mathfrak{B}_1$ if and only if each summand is in $\mathfrak{B}_1$. Moreover, it follows from Theorem 6.8 that $\mathfrak{B}_1 \cap B(H)$ is *-strongly closed. Clearly $\mathfrak{B}_1$ is closed under unitary equivalence. It follows from **[21**, Theorem 5.1] that there is a continuous decomposable function φ such that $\mathfrak{B}_1 = \{T \in \text{sub}\, B(H): \varphi(T) = 0\}$. Thus $\mathfrak{B} = \mathfrak{B}_1 \cap B(H) = \{T \in B(H): \varphi(T) = 0\}$.

It should be noted that if it could be proved that $\beta(S) = \beta(T)$ whenever S, T are approximately equivalent operators, then all of the previous results of this section would remain true with α replaced by β. It would be interesting to know the precise relationship between α and β.

Although it seems that most of the results of this section are aimed at showing operators to be similar to normal operators, it should be noted that they can be used to show that an operator is not similar to a normal operator. To illustrate this idea consider the bilateral weighted shift T whose 2^nth weight is $1/n$ for $n = 1, 2, \ldots,$ and whose other weights are all 1. If U is the (unweighted) bilateral shift, then $(U^*)^{2^n}TU^{2^n}$ converges *-strongly to a bilateral weighted shift S with one 0 weight and the rest of the weights equal to 1. Since $\ker S \neq \ker S^2$, it follows that S is not similar to a normal operator. Thus, by Theorem 6.8, T is not similar to a normal operator. Techniques such as this one are used by the author and T. Hoover in a forthcoming paper that includes a characterization of those weighted translation operators that are similar to normal operators.

We now turn our attention to subnormal operators. If we replace *-strong convergence in Theorem 6.8 by strong convergence, then the limit operator T is not necessarily similar to a normal operator. In fact the strong closure of the set of normal operators is the set of subnormal operators [7, Theorem 3.3], [38]; the next theorem is an analogue of this fact. All of the remaining results in this section are related to the work of W. W. Saffern [34].

THEOREM 6.11. *If* $\{T_n\}$ *is a sequence with* $\sup \alpha(T_n) < \infty$, *and if* $T_n \to T$ *strongly, then T is similar to a subnormal operator.*

PROOF. Let $S = T_1 \oplus T_2 \oplus \cdots$ and define isometries $W_n\colon H \to H^{(\infty)}$ for $n = 1, 2, \ldots,$ by $W_1 f = f \oplus 0 \oplus 0 \oplus \cdots$, $W_2 f = 0 \oplus f \oplus 0 \oplus \cdots, \cdots$ for every f in H. Since $C^*(S)$ is separable and the unit ball of $B(H)$ is weakly compact, there is a subsequence $\{V_n\}$ of $\{W_n\}$ such that $\{V_n^* A V_n\}$ is weakly convergent for every A in $C^*(S)$. Let $\psi(A)$ denote the weak limit of $\{V_n^* A V_n\}$ for each A in $C^*(S)$. Then ψ is a completely positive map on $C^*(S)$ such that $\psi(1) = 1$, $\psi(S) = T$, and $\psi(S^*S) = \psi(S^*)\psi(S)$. It follows from a theorem of Stinespring [39] that there is a Hilbert space $\mathcal{H}$ containing H and a representation $\pi\colon C^*(S) \to B(\mathcal{H})$ such that $\psi(A) = P\pi(A)|H$ for every A in $C^*(T)$ (where P denotes the projection of $\mathcal{H}$ onto H). Since $\psi(S^*S) = \psi(S^*)\psi(S)$, it follows that H is an invariant subspace for $\pi(S)$; whence $T = \pi(S)|H$. Since $\sup \alpha(T_n) < \infty$, we have that S, and hence $\pi(S)$, is similar to a normal operator. Thus $\pi(S)|H$ is similar to a subnormal operator.

The following theorem shows that the set of operators similar to subnormal operators is closed under approximate similarity.

THEOREM 6.12. *If T is approximately similar to a subnormal operator, then T is similar to a subnormal operator.*

PROOF. Suppose S is subnormal, $\{V_n\}$ is an invertibly bounded sequence, and $V_n^{-1} S V_n \to T$. Since S is subnormal, there is a normal operator B and a sequence $\{U_n\}$ of unitary operators such that $U_n^* B U_n \to S$ strongly [7]. Thus there are subsequences $\{U_{n_k}\}$ and $\{V_{m_k}\}$ such that $(V_{m_k}^{-1} U_{n_k}^*) B (U_{n_k} V_{m_k}) \to T$ strongly. It follows from Theorem 6.11 that T is similar to a subnormal operator.

Define a function $\gamma\colon B(H) \to [1, \infty]$ by $\gamma(T) = \inf\{\max(\|V\|, \|V^{-1}\|)\colon V^{-1} T V \text{ is subnormal}\}$. It is easily shown that $\gamma(T_1 \oplus T_2 \oplus \cdots) = \sup \gamma(T_n)$, and that the same result holds for uncountable direct sums. Also if M is an invariant subspace for an operator T, then $\gamma(T|M) < \gamma(T)$.

THEOREM 6.13. *If T is similar to a subnormal operator and π is a representation of $C^*(T)$, then $\pi(T)$ is similar to a subnormal operator. Moreover,* $\sup\{\gamma(\pi(T))\colon \pi \textit{ is a representation of } C^*(T)\} < \infty$.

PROOF. First suppose that $\pi(T)$ acts on a separable Hilbert space. It follows from [44, Theorem 1.5] (s. e [19, Theorem 3.3]) that there is an operator S that is approximately equivalent to T such that $\pi(T)$ is unitarily equivalent to a direct summand of $S^{(\infty)}$. Clearly $\gamma(\pi(T)) \leq \gamma(S^{(\infty)}) = \gamma(S)$. It follows from the preceding theorem that $\gamma(S) < \infty$. Thus $\pi(T)$ is similar to a subnormal operator. If $\{\pi_n\}$ is a sequence of representations of $C^*(T)$ on separable Hilbert spaces, then $\pi_1 \oplus \pi_2 \oplus \cdots$ is also such a representation. Hence $\sup \gamma(\pi_n(T)) < \infty$. Since every representation of $C^*(T)$ is a direct sum of representations on separable Hilbert spaces, it follows that $\sup\{\gamma(\pi(T))\colon \pi$ is a representation of $C^*(T)\} < \infty$.

COROLLARY 6.14. *An operator T is similar to a subnormal operator if and only if* $\sup\{\gamma(\pi(T))\colon \pi$ *is an irreducible representation of* $C^*(T)\} < \infty$.

COROLLARY 6.15. *If T is similar to a subnormal operator, then* $\sup\{\gamma(\psi(T))\colon$ ψ *is a completely positive map on* $C^*(T)$ *such that* $\psi(T^*T) = \psi(T^*)\psi(T)\} < \infty$.

PROOF. Suppose ψ is a completely positive map and $\psi(T^*T) = \psi(T^*)\psi(T)$. As in the proof of Theorem 6.11 we can use Stinespring's theorem [39] to find a Hilbert space $\mathcal{H}$ and a representation $\pi\colon C^*(T) \to B(\mathcal{H})$ such that H is an invariant subspace of $\pi(T)$ and $\pi(T)|H = \psi(T)$. Hence $\gamma(\psi(T)) \leq \gamma(\pi(T))$.

THEOREM 6.16. *Suppose $\{T_n\}$ is a sequence with* $\sup \gamma(T_n) < \infty$ *and $T_n \to T$ strongly. Then T is similar to a subnormal operator.*

PROOF. Let $S = T_1 \oplus T_2 \oplus \cdots$, and imitate the proof of Theorem 6.11 to construct a completely positive map ψ on $C^*(S)$ such that $\psi(1) = 1$, $\psi(S) = T$, and $\psi(S^*S) = \psi(S^*)\psi(S)$. It follows from the preceding corollary that $T = \psi(S)$ is similar to a subnormal operator.

COROLLARY 6.17. *The set of operators similar to subnormal operators is an F_σ in the strong operator topology.*

It is possible to prove an analogue of Theorem 6.10 for subnormal operators using the strong closure of $\{T\colon \gamma(\tau) \leq r, \|T\| \leq R\}$ for $\mathfrak{B}$.

We conclude with an application. Since every compact subnormal operator is normal, it follows that $\beta(K) = \gamma(K)$ for every compact operator K. The following theorem extends this idea to direct integrals of compact operators; the set of all such operators includes the n-normal and K-normal operators (see [16, p. 181]).

THEOREM 6.18. *A direct integral of compact operators is similar to a subnormal operator if and only if it is similar to a normal operator.*

PROOF. It follows from [20] that a direct integral of compact operators is approximately equivalent to a direct sum $K_1 \oplus K_2 \oplus \cdots$ where $K_1, K_2, \ldots$ are compact operators. The theorem now follows from the fact that $\beta(K_1 \oplus K_2 \oplus \cdots) = \sup \beta(K_n) = \sup \gamma(K_n) = \gamma(K_1 \oplus K_2 \oplus \cdots)$.

7. Another viewpoint. There is a simple representation of $B(H)$ in which the asymptotic concepts previously considered translate back into their non-asymptotic counterparts. Let $l^\infty(B(H))$ denote the C^*-algebra of all bounded sequences in $B(H)$, and let $c_0(B(H))$ denote the ideal of those sequences that converge (in norm) to 0. Let $\mathcal{Q}$ denote the quotient $l^\infty(B(H))/c_0(B(H))$, and let ν: $B(H) \to \mathcal{Q}$ be the representation that sends an operator T onto the (equivalence class of the) sequence $(T, T, T, \ldots)$.

If $\mathbb{S} \subseteq B(H)$, then $\nu(\mathbb{S})'$ is the set of (equivalence classes of) all bounded sequences $\{A_n\}$ such that $\|A_nS - SA_n\| \to 0$ for every S in $\mathbb{S}$. Therefore the inverse image in $B(H)$ of $\nu(\mathbb{S})''$ is appr$(\mathbb{S})''$. Similarly, appr(Alg(Lat $\mathbb{S}$)) is the inverse image of Alg(Lat $\nu(\mathbb{S})$). Furthermore, two operators S, T are approximately equivalent (approximately similar) in $B(H)$ if and only if $\nu(S)$, $\nu(T)$ are unitarily equivalent (similar) in $\mathcal{Q}$.

While this representation does not yield immediate solutions to all of the asymptotic problems in this paper, there are some results that can be obtained very easily. As an illustrative example, we present a significant improvement of R. Moore's asymptotic Fuglede theorem [30]. The proof here was shown to me by W. Zame, and essentially the same proof was discovered earlier by M. Radjabalipour. Using similar techniques, B. Chan proved Theorem 3.5. A version of this theorem that is valid in every Banach algebra is contained in [22].

THEOREM 7.1. *If $\{T_n\}$, $\{S_n\}$ are bounded sequences of operators such that*
(1) $\|T_n^*T_n - T_nT_n^*\| \to 0$, *and*
(2) $\|S_nT_n - T_nS_n\| \to 0$,
then
(3) $\|S_nT_n^* - T_n^*S_n\| \to 0$.

PROOF. Statement (1) says that $\{T_n\}$ is normal in $\mathcal{Q}$, and (2) says that $\{S_n\}$ commutes in $\mathcal{Q}$ with $\{T_n\}$. Thus (3) is just what Fuglede's theorem implies.

One of the reasons that the preceding proof worked so easily is that Fuglede's theorem is a theorem about C^*-algebras. Most theorems about commutants, reflexivity, and similarity are not theorems about C^*-algebras, and it should be expected that proofs as simple as the preceding one do not exist for such theorems. However, there are many very interesting questions about commutants, reflexivity, and similarity in the algebra $\mathcal{Q}$ that deserve consideration.

The author is indebted to W. Zame for several stimulating conversations about the algebra $\mathcal{Q}$.

8. Nonseparable cases. It is the purpose of this section to extend some of the preceding results in the case when the set $\mathbb{S}$ of operators or the Hilbert space H is nonseparable. In the case when $\mathbb{S}$ is not separable it is necessary to define objects like $\operatorname{appr}(\mathbb{S})''$ in terms of nets rather than sequences; with that proviso, most of the theorems of the preceding sections remain true for arbitrary $\mathbb{S}$ and H. In this section we prove a general asymptotic double commutant theorem and a theorem about nonseparable, strongly reductive, commutative algebras.

Throughout this section $\mathbb{S}$ denotes a (possibly nonseparable) nonempty set of operators acting on a (possibly nonseparable) Hilbert space H.

The following lemma is useful in extending results from the case of separable $\mathbb{S}$ and separable H to the case of separable $\mathbb{S}$ and nonseparable H.

LEMMA 8.1. *If $\mathbb{S}$ is a separable, closed subset of $B(H)$ and $T \in B(H)$, then $T \in \mathbb{S}$ precisely when $T|M \in (\mathbb{S}|M)^{-}$ for every separable subspace M of H that reduces both $\mathbb{S}$ and T.*

PROOF. Assume that $T \notin \mathbb{S}$ and let d be the (norm) distance from T to $\mathbb{S}$. Let $S_1, S_2, \ldots$ be dense in $\mathbb{S}$. Since $\mathbb{S}$ is separable, it follows that H is an orthogonal direct sum of separable subspaces that reduce both $\mathbb{S}$ and T. Hence, for each positive integer n, we can choose a separable subspace M_n that reduces both $\mathbb{S}$ and T such that $\|(T - S_n)|M_n\| > d/2$. Let M be the closed linear span of $M_1, M_2, \ldots$. Then M is separable, M reduces both $\mathbb{S}$ and T, and $T|M \notin (\mathbb{S}|M)^{-}$. The converse is obvious.

We now extend the definitions of $\operatorname{appr}(\mathbb{S})''$ and $\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat}(\mathbb{S})))$ to the case when $\mathbb{S}$ is nonseparable. More precisely, $\operatorname{appr}(\mathbb{S})'' = \{T\colon \|A_nT - TA_n\| \to 0$ for every bounded *net* $\{A_n\}$ such that $\|A_nS - SA_n\| \to 0$ for each S in $\mathbb{S}\}$, $\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S})) = \{T\colon \|(1 - P_n)TP_n\| \to 0$ for every *net* $\{P_n\}$ of projections such that $\|(1 - P_n)SP_n\| \to 0$ for each S in $\mathbb{S}\}$.

It is an easy exercise to show that in the case when $\mathbb{S}$ is separable these definitions agree with those of previous sections.

The following theorem is the main tool used to extend results from the case of separable $\mathbb{S}$ to the case of nonseparable $\mathbb{S}$. First, we need some "distance" formulas analogous to those studied by W. Arveson [5] (see also [44, Remark following Corollary 1.9]).

If $\mathbb{S} \subseteq B(H)$ and $T \in B(H)$, then define $\delta(T, \mathbb{S}) = \sup\{\lim_n \sup\|A_nT - TA_n\|\colon \{A_n\}$ is a net of contractions such that $\|A_nS - SA_n\| \to 0$ for every S in $\mathbb{S}\}$, $\rho(T, \mathbb{S}) = \sup\{\lim_n \sup\|(1 - P_n)TP_n\|\colon \{P_n\}$ is a net of projections such that $\|(1 - P_n)SP_n\| \to 0$ for every S in $\mathbb{S}\}$.

It is easily shown that if $\mathbb{S}$ is separable, then "net" can be replaced by "sequence" in the preceding definitions. It is also clear that $T \in \operatorname{appr}(\mathbb{S})''$ precisely when $\delta(T, \mathbb{S}) = 0$, and $T \in \operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S}))$ precisely when

$\rho(T, \mathbb{S}) = 0$. Recall that an operator A is a *contraction* if $\|A\| \le 1$.

THEOREM 8.2. *If* $\mathbb{S} \subseteq B(H)$ *and* $T \in B(H)$, *then*
(1) $\operatorname{appr}(\mathbb{S})'' = \bigcup\{\operatorname{appr}(\mathfrak{R})'' \colon \mathfrak{R} \subseteq \mathbb{S}, \mathfrak{R}$ *separable*$\}$,
(2) $\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathbb{S})) = \bigcup\{\operatorname{appr}(\operatorname{Alg}(\operatorname{Lat} \mathfrak{R})) \colon \mathfrak{R} \subseteq \mathbb{S}, \mathfrak{R}$ *separable*$\}$,
(3) $\delta(T, \mathbb{S}) = \inf\{\delta(T, \mathfrak{R}) \colon \mathfrak{R} \subseteq \mathbb{S}, \mathfrak{R}$ *separable*$\}$,
(4) $\rho(T, \mathbb{S}) = \inf\{\rho(T, \mathfrak{R}) \colon \mathfrak{R} \subseteq \mathbb{S}, \mathfrak{R}$ *separable*$\}$.

PROOF. (3) It is clear that $\varnothing \neq \mathfrak{R} \subseteq \mathbb{S}$ implies $\delta(T, \mathbb{S}) \le \delta(T, \mathfrak{R})$. Hence $\delta(T, \mathbb{S}) \le d$ where d represents the infimum in (3). For each finite subset $\mathfrak{F}$ of $\mathbb{S}$ and each positive number ε it follows from $d \le \delta(T, \mathfrak{F})$ that there is a contraction $A = A_{(\mathfrak{F},\varepsilon)}$ such that $\|AT - TA\| \ge d - \varepsilon$ and $\|AF - FA\| \le \varepsilon$ for each F in $\mathfrak{F}$. If we define a partial ordering on pairs of the form $(\mathfrak{F}, \varepsilon)$ by $(\mathfrak{F}, \varepsilon) \le (\mathfrak{F}_1, \varepsilon_1)$ if $\mathfrak{F} \subseteq \mathfrak{F}_1$ and $\varepsilon_1 \le \varepsilon$, then $\{A_n \colon n = (\mathfrak{F}, \varepsilon), \varepsilon > 0, \mathfrak{F}$ is a finite subset of $\mathbb{S}\}$ is a net such that $\|A_n S - SA_n\| \to 0$ for every S in $\mathbb{S}$ and $\lim_n \sup\|A_n T - TA_n\| \ge d$. It follows that $\delta(T, \mathbb{S}) \ge d$.

(1) It is clear that $\operatorname{appr}(\mathbb{S})'' \supseteq \bigcup\{\operatorname{appr}(\mathfrak{R})'' \colon \mathfrak{R} \subseteq \mathbb{S}, \mathfrak{R}$ separable$\}$. To show the reverse inclusion suppose $T \in \operatorname{appr}(\mathbb{S})''$. Then $\delta(T, \mathbb{S}) = 0$. It follows from (3) that, for each positive integer n, there is a separable subset $\mathfrak{R}_n$ of $\mathbb{S}$ such that $\delta(T, \mathfrak{R}_n) \le 1/n$. Let $\mathfrak{R} = \bigcup_{n=1}^{\infty} \mathfrak{R}_n$. Then $\mathfrak{R}$ is separable and $\delta(T, \mathfrak{R}) = 0$; whence $T \in \operatorname{appr}(\mathfrak{R})''$.

The proofs of (4) and (2) are very similar to the proofs of (3) and (1), respectively.

We can now extend the asymptotic double commutant theorem to the nonseparable case. Note that we could define $\delta_u(T, \mathbb{S}) = \sup\{\lim_n \sup\|U_n T - TU_n\| \colon \{U_n\}$ is a net of unitary operators such that $\|U_n S - SU_n\| \to 0$ for every S in $\mathbb{S}\}$, and $\delta_p(T, \mathbb{S}) = \sup\{\lim_n \sup\|P_n T - TP_n\| \colon \{P_n\}$ is a net of projections such that $\|P_n S - SP_n\| \to 0$ for every S in $\mathbb{S}\}$, and prove theorems about these functions analogous to the preceding theorem.

THEOREM 8.3. *Suppose* $\varnothing \neq \mathbb{S} \subseteq B(H)$. *Then*
(1) $C^*(\mathbb{S}) = \operatorname{appr}(\mathbb{S} \cup \mathbb{S}^*)''$,
(2) $C^*(\mathbb{S}) = \{T \colon \|U_n T - TU_n\| \to 0$ *for every net* $\{U_n\}$ *of unitary operators such that* $\|U_n S - SU_n\| \to 0$ *for every* S *in* $\mathbb{S}\}$,
(3) $C^*(\mathbb{S}) = \{T \colon \|P_n T - TP_n\| \to 0$ *for every net* $\{P_n\}$ *of projections such that* $\|P_n S - SP_n\| \to 0$ *for every* S *in* $\mathbb{S}\}$.

PROOF. We only prove (1); the proofs of (2), (3) are similar (using δ_u and δ_p, respectively). First suppose $\mathbb{S}$ is separable and $T \in \operatorname{appr}(\mathbb{S} \cup \mathbb{S}^*)''$. If M is a separable subspace of H that reduces both $\mathbb{S}$ and T, then, by Theorem 2.2, $T|M \in \operatorname{appr}(\mathbb{S} \cup \mathbb{S}^*)''|M \subseteq \operatorname{appr}((\mathbb{S} \cup \mathbb{S}^*)''|M) = C^*(\mathbb{S}|M) = C^*(\mathbb{S})|M$. It therefore follows from Lemma 8.1 that $T \in C^*(\mathbb{S})$. If $\mathbb{S}$ is not separable, then it follows from Theorem 8.2 that $\operatorname{appr}(\mathbb{S} \cup \mathbb{S}^*)'' = \bigcup\{\operatorname{appr}(\mathfrak{R} \cup$

$\mathfrak{R}^*)''$: $\mathfrak{R}$ separable, $\mathfrak{R} \subseteq \mathbb{S}\} = \bigcup\{C^*(\mathfrak{R})$: $\mathfrak{R}$ separable, $\mathfrak{R} \subseteq \mathbb{S}\} = C^*(\mathbb{S})$.

A subset $\mathbb{S}$ of $B(H)$ is *strongly reductive* if $\mathbb{S}^* \subseteq \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, \mathbb{S}))$. The following theorem was proved by Apostol, Foiaş, and Voiculescu [3, Theorem 2] in the case when $\mathbb{S}$ and H are both separable.

THEOREM 8.4. *Suppose $\mathbb{S}$ is a strongly reductive, commutative algebra of operators in $B(H)$. Then the norm closure of $\mathbb{S}$ is $C^*(\mathbb{S})$.*

PROOF. Suppose $\mathbb{S}$ is separable and $T \in C^*(\mathbb{S})$. If M is a separable subspace of H that reduces both $\mathbb{S}$ and T, then $\mathbb{S}|M$ is strongly reductive, and by [3], $(\mathbb{S}|M)^- = C^*(\mathbb{S}|M) = C^*(\mathbb{S})|M$. Thus $T|M \in (\mathbb{S}|M)^-$. Therefore, by Lemma 8.1, $T \in \mathbb{S}^-$. Thus $\mathbb{S}^- = C^*(\mathbb{S})$. Next suppose $\mathbb{S}$ is not separable. We can assume that $\mathbb{S}$ is norm closed. We need only show that $\mathbb{S} = \mathbb{S}^*$. Suppose $S \in \mathbb{S}$, and let $\mathfrak{R}_0 = \{S\}$. It follows from Theorem 8.2 that there is a separable subset $\mathfrak{R}_1$ of $\mathbb{S}$ such that $\mathfrak{R}_0^* \subseteq \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, \mathfrak{R}_1))$. Similarly, there is a separable subset $\mathfrak{R}_2$ of $\mathbb{S}$ such that $\mathfrak{R}_1^* \subseteq \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, \mathfrak{R}_1))$. Proceed inductively to choose a sequence $\{\mathfrak{R}_n\}$ of separable subsets of $\mathbb{S}$ such that $\mathfrak{R}_n^* \subseteq \mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, \mathfrak{R}_{n+1}))$ for $n = 0, 1, \ldots$. Let $\mathfrak{R}$ be the union of all of the $\mathfrak{R}_n$'s. It follows that $\mathfrak{R}$ is separable and strongly reductive. Thus $S^* \in \mathcal{Q}_u(\mathfrak{R}) \subseteq \mathbb{S}$; whence, $\mathbb{S} = \mathbb{S}^* = C^*(\mathbb{S})$.

The following proposition shows why it is necessary to use nets when defining $\mathrm{appr}(\mathbb{S})''$ or $\mathrm{appr}(\mathrm{Alg}(\mathrm{Lat}\, \mathbb{S}))$ when $\mathbb{S}$ is not separable; in particular, Theorems 8.2 and 8.3 would no longer be true.

PROPOSITION 8.5. *Let $\mathfrak{D}$ be the C^*-algebra of all operators that are diagonal with respect to a fixed orthonormal basis $e_1, e_2, \ldots$ for H, and let $\varphi\colon \mathfrak{D} \to \mathbf{C}$ be a unital representation that annihilates $\mathfrak{D} \cap \mathcal{K}(H)$. Let $\mathbb{S}$ be the C^*-algebra of all operators of the form $D \oplus \varphi(D)$ acting on $H \oplus \mathbf{C}$. Then $\mathbb{S} \neq \{T\colon \|A_nT - TA_n\| \to 0$ for every bounded sequence $\{A_n\}$ such that $\|A_nS - SA_n\| \to 0$ for each S in $\mathbb{S}\}$.*

PROOF. Let $\mathfrak{B}$ denote the set on the right-hand side of the nonequation in the theorem. We shall show that $0 \oplus 1 \in \mathfrak{B}$. Assume via contradiction that $T = 0 \oplus 1 \notin \mathfrak{B}$. Let e_0 be a unit vector in ran T, and let Q_k be the projection onto the span of $e_1, \ldots, e_k$ for $k = 1, 2, \ldots$. Since $T \in \mathfrak{B}$ and $\mathbb{S} = \mathbb{S}^*$, it follows that there is a bounded sequence $\{A_n\}$ of Hermitian operators and a positive number ε such that $\|A_nS - SA_n\| \to 0$ for every S in $\mathbb{S}$ and $\|A_nT - TA_n\| > \varepsilon$ for $n = 1, 2, \ldots$. There is no loss of generality in assuming that $(A_ne_0, e_0) = 0$ for $n = 1, 2, \ldots$ (otherwise, replace each A_n by $A_n - (A_ne_0, e_0)$). Therefore, $\|A_nT - TA_n\| = \|A_ne_0\| > \varepsilon$ for $n = 1, 2, \ldots$. Let $n_1 = 1$, and choose k_1 so that $\|Q_{k_1}A_1e_0\| > \varepsilon$. Choose $n_2 > n_1$ and $k_2 > k_1$ so that $\|Q_{k_1}A_{n_2}e_0\| = \|(Q_{k_1}A_{n_2} - A_{n_2}Q_{k_1})e_0\| < \varepsilon$ and $\|(Q_{k_2} - Q_{k_1})A_{n_2}e_0\| > \varepsilon$. Proceed inductively to choose increasing sequences $\{n_j\}$ and $\{k_j\}$ of positive integers so that $\|Q_{k_j}A_{n_{j+1}}e_0\| < \varepsilon$ and $\|(Q_{k_{j+1}} - Q_{k_j})A_{n_{j+1}}e_0\| > \varepsilon$ for $j = 1, 2, \ldots$. Let $D = \Sigma(Q_{k_{2j+1}} - Q_{k_{2j}})$, and let $S = D \oplus \varphi(D)$. It is clear that $\varphi(D)$ is 0 or 1. There is no harm in assuming that $\varphi(D) = 0$ (otherwise, replace D by $1 - D$). However,

$$\|(SA_{n_{2j+1}} - A_{n_{2j+1}}S)e_0\| = \|SA_{n_{2j+1}}e_0\| > \|(Q_{k_{2j+1}} - Q_{k_{2j}})A_{n_{2j+1}}e_0\| > \varepsilon$$

for $j = 1, 2, \ldots$. However, $S \in \mathbb{S}$, and $\|A_nS - SA_n\| \to 0$ yields the desired contradiction.

Combining the techniques of this section with those of the proof of part (2) of Theorem 4.10 we can easily prove that every norm closed algebra of analytic Toeplitz operators is approximately reflexive (generalizing Sarason's result [36] that every weakly closed algebra of analytic Toeplitz operators is reflexive).

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