Tôhoku Math. J.
50 (1998), 229–242

AN ASYMPTOTIC EXPANSION OF THE p-ADIC GREEN FUNCTION

YOUNGHO JANG

(Received October 3, 1996, revised November 29, 1996)

Abstract. Using the functional equation of the local zeta function attached to the quadratic form due to Rallis and Schiffmann and the t-representation introduced by Bikulov, we obtain an asymptotic expansion of the Green function defined on the even-dimensional space of p-adic numbers.

Introduction. Let $\boldsymbol{Q}$ be the field of rational numbers, and p a fixed prime number. The completion of $\boldsymbol{Q}$ with respect to the p-adic norm gives the field of p-adic numbers $\boldsymbol{Q}_p$. Any $x \in \boldsymbol{Q}_p$ can be expressed as $x = p^v \sum_{j=0}^{\infty} a_j p^j$ with integers a_j satisfying $0 \leq a_j \leq p-1$, $a_p \neq 0$. To define the Fourier transform, the standard character $\chi_p(kx) = \exp(2\pi i \{kx\}_p)$ is used. Here $\{x\}_p = p^v \sum_{j=0}^{-v-1} a_j p^j$ is the decimal part of a p-adic number x. We use the theory of $\boldsymbol{C}$-valued distributions on $\boldsymbol{Q}_p$. For example, the distribution $|x|_p^{\alpha}$, $(\alpha \in \boldsymbol{C})$ and the p-adic Dirac δ-distribution $\delta(x)$ are defined. Their Fourier transforms are

$$\int_{\boldsymbol{Q}_p} |x|_p^{\alpha} \chi_p(kx) dx = \Gamma_p(\alpha+1) |k|_p^{-\alpha-1} \quad \text{and} \quad \int_{\boldsymbol{Q}_p} \delta(x) \chi_p(kx) dx = 1 ,$$

where $\Gamma_p(\alpha) = (1-p^{\alpha-1})/(1-p^{-\alpha})$ is the p-adic Γ-function and dx is the Haar measure on $\boldsymbol{Q}_p$ such that the volume of the unit ball $\{x \in \boldsymbol{Q}_p \mid |x|_p \leq 1\}$ is 1.

The n-dimensional p-adic space $\boldsymbol{Q}_p^n$ has the standard norm $|x|_p = \max_{1 \leq j \leq n} |x_j|_p$, $x = (x_1, \ldots, x_n) \in \boldsymbol{Q}_p^n$. The Fourier transform is defined with respect to the character $\chi_p((k, x)) = \prod_{j=1}^{n} \chi_p(k_j x_j)$, where $(k, x) = \sum_{j=1}^{n} k_j x_j$. We consider a propagator which is the inverse Fourier transform of a kinetic operator $(\square + m^2)$, $m \in \boldsymbol{R}$. We have the following possible choice of the scalar propagators in $\boldsymbol{Q}_p^n$:

(1.1)
$$\frac{1}{|(k,k)+m^2|_p}, \quad \frac{1}{|(k,k)|_p + m^2}, \quad \frac{1}{|k_1|_p^2 + \cdots + |k_n|_p^2 + m^2}, \quad \frac{1}{|k|_p^2 + m^2}, \quad \frac{1}{|k, m|_p^2},$$

where $k \in \boldsymbol{Q}_p^n$ and $|k, m|_p = \max(|k|_p, |m|_p)$.

In the one-dimensional case, the second, third and fourth propagators coincide; it is this version that was applied in quantum mechanics [8]. In the massless 2-dimensional case, the fourth version was proposed in [5], using another p-adic norm $|k|_p := |\sum_j k_j|_p$. The fifth version was calculated in [7].

1991 *Mathematics Subject Classification.* Primary 11S80; Secondary 11S85, 11S99.

In particular, the second version was proposed for p-adic quantum field theory: Let Δ_p be Vladimirov's operator in [8], which is defined by

$$(1.2)\qquad (\Delta_p\varphi)(x)=\int_{\boldsymbol{Q}_p^n} |(k,k)|_p\chi_p((k,x))\tilde{\varphi}(k)dk\,,\qquad \varphi\in S(\boldsymbol{Q}_p^n)\,,$$

where $S(\boldsymbol{Q}_p^n)$ is the space of Schwartz-Bruhat functions on $\boldsymbol{Q}_p^n$ and $\tilde{\varphi}$ is the Fourier transform of φ. Vladimirov and Volovich [9] proposed the *Green Function* $G(x)$ that satisfies $(\Delta_p+m^2)G(x)=\delta(x)$:

$$(1.3)\qquad G(x)=\int_{\boldsymbol{Q}_p^n}\frac{\chi_p((k,x))}{|(k,k)|_p+m^2}dk\,,\qquad m\in\boldsymbol{R}_{>0}\,.$$

The properties of the Green function for $n=1$ are studied in [8]. Since

$$\frac{1}{|(k,k)|_p+m^2}=\lim_{\varepsilon\to\infty}\int_0^\varepsilon \exp(-m^2\theta-|(k,k)|_p\theta)d\theta\,,\qquad \theta\in\boldsymbol{R}_{>0}\,,$$

we have

$$G(x)=\lim_{N\to\infty}\int_{(p^{-N}\boldsymbol{Z}_p)^n}\chi_p((k,x))\left(\lim_{\varepsilon\to\infty}\int_0^\varepsilon \exp(-m^2\theta-|(k,k)|_p\theta)d\theta\right)dk\,.$$

Since $|\chi_p((k,x))\int_0^\varepsilon\exp(-m^2\theta-|(k,k)|_p\theta)d\theta|\leq 1/(|(k,k)|_p+m^2)\in L^1((p^{-N}\boldsymbol{Z}_p)^n)$, by Lebesgue's theorem and Fubini's theorem, we obtain

$$G(x)=\lim_{N,\varepsilon\to\infty}\int_0^\varepsilon \exp(-m^2\theta)\int_{(p^{-N}\boldsymbol{Z}_p)^n}\chi_p((k,x))\exp(-|(k,k)|_p\theta)dkd\theta\,.$$

Expanding $\exp(-|(k,k)|_p\theta)$ into the Taylor series and using Weierstrass' criterion, we obtain

$$(1.4)\quad G(x)=\lim_{N,\varepsilon\to\infty}\int_0^\varepsilon \exp(-m^2\theta)\sum_{\alpha=0}^{\infty}\frac{(-\theta)^\alpha}{\alpha!}\left(\int_{(p^{-N}\boldsymbol{Z}_p)^n}|(k,k)|_p^\alpha\chi_p((k,x))dk\right)d\theta\,.$$

For convenience, we put

$$(1.5)\qquad J=J(\alpha,n)=\int_{(p^{-N}\boldsymbol{Z}_p)^n}|(k,k)|_p^\alpha\chi_p((k,x))dk\,.$$

Bikulov [1] studied the properties of the Green function for $n=2$ and $p\geq 3$ by calculating (1.5) in a new method (he call it the *t-representation*). More generally, Kochubei [4] introduced the Green function of the pseudodifferential operator with the symbol $|Q(\xi)|_p^\alpha$, where $\alpha>0$, $p\neq 2$, and $Q(\xi)=Q(\xi_1,\ldots,\xi_n)$ is a nondegenerate quadratic form on $\boldsymbol{Q}_p^n$ with coefficients in $\boldsymbol{Q}_p$ that satisfies the condition

$$(1.6)\qquad Q(\xi)\neq 0\qquad \text{if}\quad |\xi_1|_p+\cdots+|\xi_n|_p\neq 0\,.$$

It is given by the inverse Fourier transform of the function $(Q(\xi)+\lambda)^{-1}$, $\lambda\in\boldsymbol{R}_{>0}$. However, as is well known, quadratic forms that satisfy the condition (1.6) exist only for $n\leq 4$. Thus he gave the asymptotic expansion of the Green function (1.3) for $n=2$ and $n=4$.

On the other hand, Rallis and Schiffmann [6] investigated a distribution

$$\varphi\mapsto Z_Q(\varphi,\chi,\alpha)=\int_E\varphi(x)\chi(Q(x))|Q(x)|^{\alpha-n/2}dx\,,$$

where $\alpha\in\boldsymbol{C}$, E is an n-dimensional vector space over the local field K of characteristic different from 2, Q is the quadratic form on E, and χ is a unitary character of $K^*=K\setminus\{0\}$.

In this paper, using the functional equation (2.11) of the local zeta function $Z_Q(\varphi,\chi,s)$, we calculate (1.5) for any even dimension n and prime number $p\geq 3$. Furthermore, using the t-representation, we directly calculate (1.5) for any even dimension n and $p=2$. By using the results of (1.5), we obtain an asymptotic expansion of $G(x)$ for any even-dimensional space. In §2, we summarize the fundamental properties of the local zeta function. We prove the main theorems in §3 and §4.

In the original manuscript, the author used the method of t-representation and proved Lemma 3.3 by estimating a complicated integral. Then, Professor Fumihiro Sato suggested to simplify the proof by using the local functional equation of the prehomogeneous vector space. His advice gave a new proof of Lemma 3.3, a nice perspective and the possibility of a generalization. The author is very grateful to Professor Sato.

Finally, thanks are due to Professor Yasuo Morita for invaluable advice.

2. The functional equation of the local zata function. In this section, we summarize well-known classical results on the local zeta function attached to a quadratic form. For the proofs and more details, see [2], [6] and [11].

Let G be a locally compact abelian group and G^* the Pontrjagin dual of G. For $x\in G$ and $x^*\in G^*$, we write $\langle x,x^*\rangle=x^*(x)$. Let dx be the Haar measure on G and dx^* the Haar measure on G^* which is dual to dx. A continuous mapping φ from G to the group $T=\{z\in\boldsymbol{C}\,|\,|z|=1\}$ is a *quadratic character* of G if the mapping

$$(x,y)\mapsto\varphi(x+y)\varphi(x)^{-1}\varphi(y)^{-1}\,,\qquad x,y\in G \tag{2.1}$$

is a bicharacter of $G\times G$. Then we can put

$$\varphi(x+y)=\varphi(x)\varphi(y)\langle x,\rho y\rangle\,, \tag{2.2}$$

where $\rho=\rho_\varphi$ is a symmetric continuous homomorphism of G to G^*. The quadratic character φ is *nondegenerate* if ρ is an isomorphism of G onto G^*. If φ is nondegenerate, the *modulus* $|\rho|$ of ρ is defined by the formula

$$(2.3)\qquad |\rho|\int_G u(\rho x)dx=\int_{G^*} u(x^*)dx^*\,,\qquad u\in L^1(G^*)\,.$$

Note that the modulus of ρ depends on the choice of dx.

Let $A(G)$ be the space consisting of continuous functions u in $L^1(G)$ such that the Fourier transform $\hat{u}$ is in $L^1(G^*)$.

THEOREM 2.1 (cf. [11, p. 161], [2, p. 95]). *If φ is a nondegenerate quadratic character of G, then there exists a complex constant $r(\varphi)$ of modulus* 1 (*called the Weil constant*) *such that*

$$(2.4)\qquad \int_G \varphi(x)\hat{u}(\rho x)dx=r(\varphi)|\rho|^{-1/2}\int_G \overline{\varphi(x)}u(x)dx\,,\quad \textit{for any}\ u\in A(G)\,.$$

This means that the Fourier transform of the quadratic character φ is $r(\varphi)|\rho|^{-1/2}\overline{\varphi(x)}$. From now on, we choose the unique Haar measure dx such that $|\rho|=1$; this measure is said to be *adapted for* φ. We identify G with G^* by means of ρ.

PROPOSITION 2.2 (cf. [11, p. 170]). *Let G_1 (resp. G_2) be a locally compact group and φ_1(resp. φ_2) a nondegenerate quadratic character of G_1 (resp. G_2). Then the mapping*

$$\varphi_1\otimes\varphi_2:(x_1,x_2)\mapsto\varphi_1(x_1)\varphi_2(x_2)$$

is a nondegenerate quadratic character of $G_1\times G_2$, and $r(\varphi_1\otimes\varphi_2)=r(\varphi_1)r(\varphi_2)$.

Now, let K be a local field of characteristic different from 2, and τ a nontrivial additive character of K. Let E be an n-dimensional vector space over K, and E^* the algebraic dual of E. If Q is a nondegenerate quadratic form on E, then $\tau\circ Q$ is a nondegenerate quadratic character of E. Let $B(x,y)=\{Q(x+y)-Q(x)-Q(y)\}$ be the nondegenerate symmetric bilinear form associated with Q. Then the isomorphism ρ of E onto E^* with respect to $\tau\circ Q$ is defined by $\langle x,\rho y\rangle=\tau(B(x,y))$. Let dx be the Haar measure on E which is adapted for $\tau\circ Q$. Then the Fourier transform is defined by

$$(2.5)\qquad \hat{u}(y)=\int_E u(x)\tau(B(x,y))dx\,,\qquad u\in L^1(E)\,.$$

By Theorem 2.1, there exists a constant $r(Q)=r(\tau\circ Q)$ such that

$$(2.6)\qquad \int_E \hat{u}(x)\tau(Q(x))dx=r(Q)\int_E u(x)\tau(-Q(x))dx\,.$$

This formula is valid for any $u\in A(E)$ and, in particular, for any Schwartz-Bruhat function u on E. The constant $r(Q)$ depends on the choice of τ. Let $(\ ,\ )_H$ be the Hilbert symbol. If we put

$$h_a(b)=(a,b)_H\,,$$

then $a\mapsto h_a$ is an isomorphism of the finite abelian group $K^*/(K^*)^2$ onto its dual. We can find a coordinate system on E such that

$$Q(x)=a_1x_1^2+\cdots+a_nx_n^2 \quad (a_j\in K^*, j=1,\dots,n)\,. \tag{2.7}$$

Suppose K is ultrametric. Then the quadratic form Q is characterized by three invariants: The dimension n, the discriminant $D=a_1\cdots a_n(K^*)^2$ and the Hasse-Minkowski character $\prod_{k<j}(a_k, a_j)_H$. We put $\triangle=(-1)^{[n/2]}D$, where the symbol $[x]$ denotes the greatest integer not exceeding x. By Proposition 2.2 and (2.6), we have the following proposition.

PROPOSITION 2.3 (cf. [6, pp. 499–504]). *Let $q(x)=x^2$ be the quadratic form on K; put $f(a)=r(aq)$ for $a\in K^*$; and let Q be as in* (2.7). *Then we have*:

(i) $\varphi(x)=f(x)/f(1)$ *is a nondegenerate quadratic character of $K^*/(K^*)^2$ associated to the isomorphism $a\mapsto h_a$*;

(ii) $r(\varphi)^{-1}=\sum_{a\in K^*/(K^*)^2}\overline{\varphi(a)}$;

(iii) $r(Q)=f(1)^{n-1}f(D)\prod_{k<j}(a_k, a_j)_H$.

For $t\in K^*$, we calculate the number $r(tQ)$. As a function of t, $r(tQ)$ is invariant under the subgroup $(K^*)^2$ of K^*. Thus we can put

$$r(tQ)=\sum_{a\in K^*/(K^*)^2}\beta_a(Q)h_a(t)\,,\quad (\beta_a(Q)\in \boldsymbol{C})\,. \tag{2.8}$$

PROPOSITION 2.4 (cf. [6, p. 505]). *If K is ultrametric, then we have*

$$r(tQ)=\begin{cases} r(Q)h_\triangle(t) & \text{if } n \text{ is even} \\ r(Q)r(\varphi)f(1)\sum_{a\in K^*/(K^*)^2}\overline{f(a\triangle)}h_a(t) & \text{if } n \text{ is odd}\,. \end{cases} \tag{2.9}$$

Let χ be a unitary character of K^* and α a complex number. For $\varphi\in S(E)$, we define the local zeta function $Z_Q(\varphi, \chi, \alpha)$ by

$$Z_Q(\varphi, \chi, \alpha)=\int_E \varphi(x)\chi(Q(x))|Q(x)|^{\alpha-n/2}dx\,. \tag{2.10}$$

THEOREM 2.5 (cf. [6, p. 521]). *The integral* (2.10) *is absolutely convergent for* $\mathrm{Re}(\alpha)>0$ (*resp.* $\mathrm{Re}(\alpha)>n/2-1$) *if Q is anisotropic* (*resp. if Q is isotropic*). *Further, as a function of α, $Z_Q(\varphi, \chi, \alpha)$ has an analytic continuation to a meromorphic function on* $\boldsymbol{C}$, *and satisfies the functional equation*

(2.11)

$$Z_Q(\varphi, \chi, \alpha)=\rho(\chi, \alpha-n/2+1)\sum_{a\in K^*/(K^*)^2}\overline{\beta_a(Q)}h_a(-1)\rho(\chi h_a, \alpha)Z_Q(\hat{\varphi}, \chi^{-1}h_a^{-1}, n/2-\alpha)\,,$$

where $\beta_a(Q)$ is defined in (2.8) *and $\rho(\chi, \alpha)$ is the gamma factor of Tate. Hence for all $\varphi\in S(K)$, we have*

$$\int_{K^*} \varphi(t)\chi(t)|t|^{\alpha} d^*t = \rho(\chi, \alpha) \int_{K^*} \hat{\varphi}(t)\chi^{-1}(t)|t|^{1-\alpha} d^*t, \qquad 0 < \mathrm{Re}(\alpha) < 1. \tag{2.12}$$

3. Calculation of $J=J(\alpha, n)$ for an odd prime p. In this section, we use the functional equation of the local zeta function and calculate J. From now on, we choose the standard quadratic form $Q(x)=(x, x)$ on $\boldsymbol{Q}_p^n$, and apply the results of the preceding section.

For a unitary character χ of $\boldsymbol{Q}_p^*$ and a test function $\varphi \in S(\boldsymbol{Q}_p^n)$, the local zeta function $Z_Q(\varphi, \chi, \alpha)$ is given by

$$Z_Q(\varphi, \chi, \alpha) = \int_{\{k \in \boldsymbol{Q}_p^n | (k,k) \neq 0\}} \varphi(k)\chi((k, k))|(k, k)|_p^{\alpha - n/2} dk. \tag{3.1}$$

When χ is trivial, we simply write $Z_Q(\varphi, \alpha)$. For any integer N and $y \in \boldsymbol{Q}_p$, let $\mathrm{ch}_{N,y}(k)$ denote the characteristic function of $y + (p^{-N}\boldsymbol{Z}_p)^n$. Fix an element $x \in \boldsymbol{Q}_p^n$ and put

$$\psi_{N,x}(k) = \chi_p((k, x))\, \mathrm{ch}_{N,0}(k). \tag{3.2}$$

Then $\psi_{N,x}(k)$ is in $S(\boldsymbol{Q}_p^n)$ and we have

$$J = J(\alpha, n) = Z_Q(\psi_{N,x}, \alpha + n/2). \tag{3.3}$$

By the functional equation (2.11), we have

$$J = \rho(1, \alpha+1) \sum_{a \in \boldsymbol{Q}_p^*/(\boldsymbol{Q}_p^*)^2} \overline{\beta_a(Q)} h_a(-1)\rho(h_a, \alpha + n/2) Z_Q(\hat{\psi}_{N,x}, h_a^{-1}, -\alpha). \tag{3.4}$$

Note that

$$\hat{\psi}_{N,x}(k) = p^{nN} \times \mathrm{ch}_{-N,-x}(k).$$

Hence we have

$$\begin{aligned}
Z_Q(\hat{\psi}_{N,x}, h_a^{-1}, -\alpha) &= \int_{\{k \in \boldsymbol{Q}_p^n | (k,k) \neq 0\}} \hat{\psi}_{N,x}(k) h_a^{-1}((k, k))|(k, k)|_p^{-(\alpha+n/2)} dk \\
&= \int_{\{k \in \boldsymbol{Q}_p^n | (k,k) \neq 0\}} p^{nN} \mathrm{ch}_{-N,-x}(k) h_a^{-1}((k, k))|(k, k)|_p^{-(\alpha+n/2)} dk \\
&= p^{nN} \int_{\{k \in -x + (p^N \boldsymbol{Z}_p)^n\}} h_a^{-1}((k, k))|(k, k)|_p^{-(\alpha+n/2)} dk \\
&= h_a((x, x))|(x, x)|_p^{-(\alpha+n/2)} \quad \text{for any } N \text{ sufficiently large}.
\end{aligned}$$

On the other hand, by calculating (2.12) for the trivial character χ, we easily obtain $\rho(1, \alpha+1) = \Gamma_p(\alpha+1)$. Thus, for any N sufficiently large, we have

$$J = \Gamma_p(\alpha+1)|(x, x)|_p^{-(\alpha+n/2)} \sum_{a \in \boldsymbol{Q}_p^*/(\boldsymbol{Q}_p^*)^2} \overline{\beta_a(Q)} h_a(-(x, x))\rho(h_a, \alpha + n/2). \tag{3.5}$$

PROPOSITION 3.1 (cf. [10, p. 130]). *Let $p\neq 2$ and let ε be a unit, $\varepsilon\notin(\boldsymbol{Q}_p^*)^2$. Then*

$$h_\varepsilon(x)=(x,\varepsilon)_H=1 \quad \textit{if and only if } v(x) \textit{ is even},$$

where $|x|_p=p^{v(x)}$, $v(x)\in\boldsymbol{Z}$.

PROPOSITION 3.2. *For the trivial character χ, we have*

$$h_{-1}(t)=(t,-1)_H=\begin{cases}1 & \textit{if} \quad p\equiv 1 \pmod 4\\ -1 & \textit{if} \quad p\equiv 3 \pmod 4\end{cases} \quad \textit{for any } t\in\boldsymbol{Q}_p^*, \tag{3.6}$$

$$\rho(h_{-1},\alpha)=\begin{cases}\Gamma_p(\alpha) & \textit{if} \quad p\equiv 1 \pmod 4\\ -(1+p^{\alpha-1})/(1+p^{-\alpha}) & \textit{if} \quad p\equiv 3 \pmod 4).\end{cases} \tag{3.7}$$

PROOF. Since $p\equiv 1 \pmod 4$ if and only if $-1\in(\boldsymbol{Q}_p^*)^2$, $h_{-1}(t)=1$ and $\rho(h_{-1},\alpha)=\Gamma_p(\alpha)$. Assume $p\equiv 3 \pmod 4$. If $h_{-1}(t)=1$, then $t\in(\boldsymbol{Q}_p^*)^2$ and $t=a^2+b^2$ for some $a,b\in\boldsymbol{Q}_p^*$. Thus $a_0^2+b_0^2\equiv 0 \pmod p$, i.e., -1 is a quadratic residue modulo p. Thus the Legendre symbol $(-1/p)=1$. This is a contradiction. Hence $h_{-1}(t)=-1$. Next let $g_\alpha(t)=|t|_p^{\alpha-1}h_{-1}(t)$. Then $g_\alpha(t)$ is a multiplicative character of $\boldsymbol{Q}_p^*$ and is a homogeneous generalized function of degree $g_\alpha(t)$. Since $\hat{g}_\alpha(tk)=|t|_p^{-1}g_\alpha(1/t)\hat{g}_\alpha(k)=|t|_p^{-\alpha}h_{-1}(t)\hat{g}_\alpha(k)$, the Fourier transform $\hat{g}_\alpha$ of g_α is a homogeneous generalized function of degree $|t|_p^{-\alpha}h_{-1}(t)$, i.e., $\hat{g}_\alpha(k)$ is proportional to degree $|k|_p^{-\alpha}h_{-1}(k)$. Hence we can write

$$\hat{g}_\alpha(k)=\Gamma_p(g_\alpha)|k|_p^{-\alpha}h_{-1}(k) \quad (\Gamma_p(g_\alpha)\in\boldsymbol{C}). \tag{3.8}$$

Putting $k=1$ in (3.8), we obtain

$$\Gamma_p(g_\alpha)=-\hat{g}_\alpha(1)=-\int_{\boldsymbol{Q}_p} g_\alpha(t)\chi_p(t)dt.$$

Since $h_{-1}(t)=-1$ for all $t\in\boldsymbol{Q}_p^*$, $g_1(t)\equiv -1$ and by Proposition 3.1, we can write $g_\alpha(t)=|t|_p^{\alpha-1+\pi i/\ln p}$. Therefore

$$\begin{aligned}\Gamma_p(g_\alpha)&=-\int_{\boldsymbol{Q}_p}|t|_p^{\alpha-1+\pi i/\ln p}\chi_p(t)dt\\ &=-\Gamma_p(\alpha+\pi i/\ln p)=-(1+p^{\alpha-1})/(1+p^{-\alpha}).\end{aligned}$$

In the formula (2.12), let $\varphi(t)=\chi_p(t)\in S(\boldsymbol{Q}_p)$. Then

$$\begin{aligned}\int_{\boldsymbol{Q}_p^*}\hat{\chi}_p(t)h_{-1}(t)|t|_p^{1-\alpha}d^*t&=\int_{\boldsymbol{Q}_p}\chi_p(t)\hat{g}_{-\alpha+1}(t)dt\\ &=\Gamma(g_{-\alpha+1})\int_{\boldsymbol{Q}_p}\chi_p(t)h_{-1}(t)|t|_p^{\alpha-1}dt\\ &=-(1+p^{-\alpha})/(1+p^{\alpha-1})\int_{\boldsymbol{Q}_p^*}\chi_p(t)h_{-1}(t)|t|_p^{\alpha}d^*t.\end{aligned}$$

Thus $\rho(h_{-1}, \alpha) = -(1+p^{\alpha-1})/(1+p^{-\alpha})$. ■

Now we calculate $J = J(\alpha, n)$. From (2.8) and (2.9), we observe the following: If n is even, we have $\beta_a(Q) = 0$ if $a \neq \triangle$ and $\beta_\triangle(Q) = r(Q)$, where $\triangle = (-1)^{n/2}$; if n is odd, we have $\beta_a(Q) = r(Q)r(\varphi)f(1)\overline{f(a\triangle)}$, where $\triangle = (-1)^{[n/2]}$. Thus, for any N sufficiently large, (3.5) can be rewritten as follows: If n is even,

$$J = \overline{r(Q)}\Gamma_p(\alpha+1)h_\triangle(-(x,x))\rho(h_\triangle, \alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)}; \tag{3.9}$$

if n is odd,

$$J = \overline{r(Q)} \cdot \overline{r(\varphi)}\Gamma_p(\alpha+1)h_\triangle(-(x,x))\rho(h_\triangle, \alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)} + \Phi((x,x)), \tag{3.10}$$

where

$$\Phi((x,x)) = \overline{r(Q)} \cdot \overline{r(\varphi)} \cdot \overline{f(1)}\Gamma_p(\alpha+1)|(x,x)|_p^{-(\alpha+n/2)}$$
$$\times \sum_{a \in \mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2;\, a \neq \triangle} f(a\triangle)h_a(-(x,x))\rho(h_a, \alpha+n/2).$$

By Proposition 3.2, we obtain the following lemma.

Lemma 3.3. *Let Q be the standard quadratic form (x,x) on $\mathbf{Q}_p^n$. For an arbitrary $\alpha \in \mathbf{C}$ and any N sufficiently large,*

(a) *if either* $n \equiv 0 \pmod 4$ *or* $[n \equiv 2 \pmod 4$ *and* $p \equiv 1 \pmod 4]$, *then*

$$J(\alpha, n) = \overline{r(Q)}\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)};$$

(b) *if* $n \equiv 2 \pmod 4$ *and* $p \equiv 3 \pmod 4$, *then*

$$J(\alpha, n) = \overline{r(Q)}\Gamma_p(\alpha+1)\frac{1+p^{\alpha+n/2-1}}{1+p^{-(\alpha+n/2)}}|(x,x)|_p^{-(\alpha+n/2)}.$$

For convenience, we denoted by Cond. 1 the condition either $n \equiv 0 \pmod 4$ or $[n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4]$; Cond. 2 the condition $n \equiv 2 \pmod 4$ and $p \equiv 3 \pmod 4$.

Theorem 3.4. *For any even dimension n and $p \geq 3$, the Green function $G(x)$ defined by* (1.3) *has the following asymptotic expansion*:

$$G(x) \sim \begin{cases} \dfrac{-p^{n/2}(p^{n/2}-1)}{p(p+1)(p^{n/2+1}-1)}\left(\dfrac{p}{m}\right)^4 \dfrac{1}{|(x,x)|_p^{1+n/2}} & \text{Cond. 1} \\ \dfrac{p^{n/2}(p^{n/2}+1)}{p(p+1)(p^{n/2+1}+1)}\left(\dfrac{p}{m}\right)^4 \dfrac{1}{|(x,x)|_p^{1+n/2}} & \text{Cond. 2}. \end{cases}$$

Proof. Suppose that n and p satisfy Cond. 1. We substitute the formula (a) of Lemma 3.3 into the expression for the Green function (1.4):

$$G(x) = \overline{r(Q)}\int_0^\infty \exp(-m^2\theta)\sum_{\alpha=0}^{\infty}\frac{(-\theta)^\alpha}{\alpha!}\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)|(x,x)|_p^{-(\alpha+n/2)}d\theta. \tag{3.11}$$

For further simplification of (3.11), we substitute the following expansion

$$\Gamma_p(\alpha+1)\Gamma_p(\alpha+n/2)=\sum_{r=0}^{\infty} a_r p^{-nr/2}[p^{-r\alpha}-(1+p^{n/2-1})p^{-(r-1)\alpha}+p^{n/2-1}p^{-(r-2)\alpha}],$$

where $a_r=\sum_{j=0}^{r} p^{(n/2-1)j}$, into the expression (3.11). We can change the order of summations because the double series of α and r are absolutely convergent. Thus we have

$$G(x)=\overline{r(Q)}|(x,x)|_p^{-n/2}\int_0^\infty \exp(-m^2\theta)\sum_{r=0}^{\infty}\frac{a_r}{p^{nr/2}}$$
$$\times\left[\exp\left(\frac{-\theta p^{-r}}{|(x,x)|_p}\right)-(1+p^{n/2-1})\exp\left(\frac{-\theta p^{-(r-1)}}{|(x,x)|_p}\right)+p^{n/2-1}\exp\left(\frac{-\theta p^{-(r-2)}}{|(x,x)|_p}\right)\right]d\theta .$$

The above series converges uniformly, so that by term by term integration and passage to limit, we obtain

$$G(x)=\overline{r(Q)}|(x,x)|_p^{1-n/2}\sum_{r=0}^{\infty} a_r p^{-nr/2}$$
$$\times\left[\frac{1}{m^2|(x,x)|_p+p^{-r}}-\frac{1+p^{n/2-1}}{m^2|(x,x)|_p+pp^{-r}}+\frac{p^{n/2-1}}{m^2|(x,x)|_p+p^2p^{-r}}\right]$$
$$=\overline{r(Q)}|(x,x)|_p^{1-n/2}(p-1)\sum_{r=0}^{\infty} a_r p^{-(n/2+1)r}$$
$$\times\frac{p^{-r}(p^2-p^{n/2})-(p^{n/2}-1)m^2|(x,x)|_p}{(m^2|(x,x)|_p+p^{-r})(m^2|(x,x)|_p+pp^{-r})(m^2|(x,x)|_p+p^2p^{-r})}.$$

Thus we have

$$\lim_{|(x,x)|_p\to\infty} r(Q)|(x,x)|_p^{1+n/2}G(x)$$
$$=\frac{(p-1)(1-p^{n/2})}{m^4}\sum_{r=0}^{\infty} a_r p^{-(n/2+1)r}$$
$$=\frac{(p-1)(1-p^{n/2})}{(1-p^{n/2-1})m^4}\left(\sum_{r=0}^{\infty} p^{-(n/2+1)r}-p^{(n/2-1)}\sum_{r=0}^{\infty} p^{-2r}\right)$$
$$=\frac{-p^{n/2}(p^{n/2}-1)}{p(p+1)(p^{n/2+1}-1)}\left(\frac{p}{m}\right)^4 .$$

Similarly, in the case Cond. 2, we obtain the desired result if we use the expansion

$$\Gamma_p(\alpha+1)\frac{1+p^{\alpha+n/2-1}}{1+p^{-(\alpha+n/2)}}=\sum_{r=0}^{\infty} b_r p^{-nr/2}[p^{-r\alpha}-(1-p^{n/2-1})p^{-(r-1)\alpha}-p^{n/2-1}p^{-(r-2)\alpha}],$$

where $b_r=\sum_{j=0}^{r}(-1)^{r-j}p^{(n/2-1)j}$. ∎

4. An alternative method for $p=2$. In this section, we calculate $J(\alpha, n)$ for any even dimension n and $p=2$ by using the t-representation introduced by Bikulov [1] and obtain an asymptotic expansion of the Green function.

4.1. Gaussian integrals on an arbitrary locally abelian group were considered by Weil in 1964. In the theory of p-adic quantum mechanics which is based on the calculation of Gaussian integrals, explicit calculations in special cases were performed by Vladimirov, Volovich, Zelenov, etc. in 1988.

Integrals of the form $\int \chi_p(ax^2+br)dx$ are called *Gaussian integrals.* In order to calculate Gaussian integrals on $\boldsymbol{Q}_p$, we will use the following formulas, (see [8], [9], [10], [12] for the proofs):

$$\int_{|x|_p \le p^r} dx = p^r ; \tag{4.1}$$

$$\int_{|x|_p \le p^r} \chi_p(kx)dx = p^r \Omega(p^r |k|_p) , \tag{4.2}$$

where $\Omega(x)$ is 1 if $0 \le x \le 1$ and 0 if $x > 1$;

$$\int_{|x|_p = p^r} \chi_p(kx)dx = \begin{cases} p^r(1-p^{-1}) & \text{for } |k|_p \le p^{-r} \\ -p^{r-1} & \text{for } |k|_p = p^{-r+1} \\ 0 & \text{for } |k|_p > p^{-r+1} ; \end{cases} \tag{4.3}$$

and we use an arithmetic function $\lambda_p : \boldsymbol{Q}_p^* \to \boldsymbol{C}$ defined as follows: If $p \ne 2$,

$$\lambda_p(a) = \begin{cases} 1 & \text{if } r \text{ is even} \\ (a_0/p) & \text{if } r \text{ is odd}, p \equiv 1 \pmod 4 \\ i(a_0/p) & \text{if } r \text{ is odd}, p \equiv 3 \pmod 4 , \end{cases} \tag{4.4}$$

where $a = p^r(a_0 + a_1 p + a_2 p^2 + \cdots)$, $i = \sqrt{-1}$ and (a_0/p) is the Legendre symbol; if $p=2$,

$$\lambda_2(a) = \begin{cases} 2^{-1/2}(1+(-1)^{a_1} i) & \text{if } r \text{ is even} \\ 2^{-1/2}(-1)^{a_1+a_2}(1+(-1)^{a_1} i) & \text{if } r \text{ is odd} , \end{cases} \tag{4.5}$$

where $a = 2^r(1 + a_1 2 + a_2 2^2 + \cdots)$. This symbol $\lambda_p(a)$ has the following properties: For $a, b \in \boldsymbol{Q}_p^*$,

(i) $|\lambda_p(a)| = 1$ and $\lambda_p(a)\lambda_p(-a) = 1$; (ii) $\lambda_p(a^2 b) = \lambda_p(a)$;

(iii) $\lambda_p(a)\lambda_p(b) = \lambda_p(a+b)\lambda_p\left(\frac{1}{a} + \frac{1}{b}\right)$; (iv) $\prod_{p=2}^{\infty} \lambda_p(a) = 1$.

REMARK. A function similar to $\lambda_p(a)$ was considered by Weil for locally compact fields, and the function $\lambda_p(a)$ is connected with the Hilbert symbol $(\, , \,)_H$ by

$$\lambda_p(a)\lambda_p(b) = (a, b)_H \lambda_p(ab) \quad \text{for } a, b \in \boldsymbol{Q}_p^*, \ p \ne 2 .$$

A Gaussian integral on the disc $|x|_p \le p^r$ is given as follows: If $p \ne 2$ and $a \ne 0$,

(4.6)

$$\int_{|x|_p \le p^r} \chi_p(ax^2+bx)dx = \begin{cases} p^r \Omega(p^r |b|_p) & \text{for } |a|_p \le p^{-2r} \\ \lambda_p(a)|2a|_p^{-1/2}\chi_p(-b^2/4a)\Omega(p^{-r}|b/2a|_p) & \text{for } |4a|_p > p^{-2r}; \end{cases}$$

if $p=2$ and $a \ne 0$,

$$(4.7)\qquad \int_{|x|_2 \le 2^r} \chi_2(ax^2+bx)dx$$

$$= \begin{cases} 2^r \Omega(2^r|b|_2) & \text{for } |a|_2 \le 2^{-2r} \\ \lambda_2(a)|2a|_2^{-1/2}\chi_2(-b^2/4a)\delta(|b|_2 - 2^{1-r}) & \text{for } |a|_2 = 2^{-2r+1} \\ \lambda_2(a)|2a|_2^{-1/2}\chi_2(-b^2/4a)\Omega(2^r|b|_2) & \text{for } |a|_2 = 2^{-2r+2} \\ \lambda_2(a)|2a|_2^{-1/2}\chi_2(-b^2/4a)\Omega(2^{-2r}|b/2a|_2) & \text{for } |a|_2 \ge 2^{-2r+3}, \end{cases}$$

where $\delta(|b|_p - p^r)$ is 1 if $|b|_p = p^r$ and 0 if $|b| \ne p^r$.

REMARK. The Gaussian integrals on $\boldsymbol{Q}_p$ are derived from (4.6) and (4.7) by $r \to \infty$. Thus

$$(4.8)\qquad \int_{\boldsymbol{Q}_p} \chi_p(ax^2+bx)dx = \lambda_p(a)|2a|_p^{-1/2}\chi_p\left(-\frac{b^2}{4a}\right).$$

4.2. Bikulov [1] used the following formula to split the double integral $J(\alpha, n)$ into two one-dimensional Gaussian integrals: For $\alpha > 0$ and $p^{-M} < |z|_p < p^m$ ($z \in \boldsymbol{Q}_p$, $M \in \boldsymbol{Z}$),

$$(4.9)\qquad |z|_p^\alpha = \Gamma_p(\alpha+1) \lim_{M,m\to\infty} \int_{p^{-m} \le |t|_p \le p^M} |t|_p^{-(\alpha+1)}(\chi_p(zt)-1)dt.$$

His method called the t-representation can be used for any prime number p. We use it to calculate the integral $J(\alpha, n)$ for any even dimension n and $p=2$. The results are given by the following lemma.

LEMMA 4.1. *For an even n, $p=2$ and $\alpha \in \boldsymbol{C}$.*

(a) *if $n \equiv 0 \pmod 4$, then*

$$J(\alpha, n) = (2i)^{n/2} 2^{-1} \Gamma_2(\alpha+1) \frac{2^{-(2\alpha+n)+1} - 2^{-(\alpha+n/2)}}{1 - 2^{-(\alpha+n/2)}} |(x,x)|_2^{-(\alpha+n/2)};$$

(b) *if $n \equiv 2 \pmod 4$, then*

$$J(\alpha, n) = (-1)^{-y_1} i(2i)^{n/2} 2^{-1} \Gamma_2(\alpha+1)|(x,x)|_p^{-(\alpha+n/2)},$$

where y_1 is the second digit of the canonical representation of $(x,x) \in \boldsymbol{Q}_2$, i.e., $(x,x) = 2^{-\beta}(1 + y_1 2 + \cdots)$, $0 \le y_j \le 1$, $\beta \in \boldsymbol{Z}$.

PROOF. Let n be even and $p=2$. In order to use the t-representation, setting $z=(k,k)\in \boldsymbol{Q}_2^*$ in (4.9) and substituting it into (1.5), we obtain

$$\Gamma_2(\alpha+1)\int_{(2^{-N}\boldsymbol{Z}_2)^n}\left(\lim_{M,m\to\infty}\int_{2^{-m}\le|t|_2\le 2^M}|t|_2^{-(\alpha+1)}(\chi_2(zt)-1)dt\right)\chi_2((k,x))dk\,, \tag{4.10}$$

for $2^{-M}<|z|_2<2^m$. Since $\chi_2((k,x))\int_{2^{-m}\le|t|_2\le 2^M}|t|_2^{-(\alpha+1)}(\chi_2(zt)-1)dt$ (see (4.9)) converges uniformly as $M\to\infty$ for any $k\in(2^{-N}\boldsymbol{Z}_2)^n$, (4.10) can be rewritten in the form

$$\Gamma_2(\alpha+1)\lim_{M,m\to\infty}\int_{2^{-m}\le|t|_2\le 2^M}|t|_2^{-(\alpha+1)}$$
$$\times\left\{\int_{|k_1|_2\le 2^N}\cdots\int_{|k_n|_2\le 2^N}\left(\prod_{j=1}^n\chi_2(tk_j^2+x_jk_j)-\prod_{j=1}^n\chi_2(x_jk_j)\right)dk_1\dots dk_n\right\}dt\,.$$

Using the expressions (4.2), (4.7) and integrating it with respect to t, and taking the limit for $M\to\infty$, we obtain

$$J=J(\alpha,n)=\int_{(2^{-N}Z_2)^n}|(k,k)|_2^\alpha\chi_2((k,x))dk$$
$$=\Gamma_2(\alpha+1)\sum_{r\ge -2N+1}2^{-(\alpha+1)r+n(1-r)/2}$$
$$\times\begin{cases}\prod_{j=1}^n\delta(|x_j|_2-2^{-N+1})\int_{|t|_2=2^r}\lambda_2^n(t)\chi_2\left(\frac{(x,x)}{-4t}\right)dt\,, & |t|_2=2^{-2N+1}\\ \prod_{j=1}^n\Omega(2^N|x_j|_2)\int_{|t|_2=2^r}\lambda_2^n(t)\chi_2\left(\frac{(x,x)}{-4t}\right)dt\,, & |t|_2=2^{-2N+2}\\ \prod_{j=1}^n\Omega(2^{-N-r+1}|x_j|_2)\int_{|t|_2=2^r}\lambda_2^n(t)\chi_2\left(\frac{(x,x)}{-4t}\right)dt\,, & |t|_2\ge 2^{-2N+3}\end{cases}$$
$$-2^{nN}\Gamma_2(\alpha+1)\prod_{j=1}^n\Omega(2^N|x_j|_2)\sum_{r\ge -2N+1}2^{-(\alpha+1)r}\int_{|t|_2=2^r}dt\,.$$

If $2^{-N+1}<|x|_2=\max_{1\le j\le n}|x_j|_2=2^l\le 2^{N+r-1}$, we obtain

$$J=2^{n/2}\Gamma_2(\alpha+1)\sum_{r\ge -N+l+1}2^{-(\alpha+1+n/2)r}\int_{|t|_2=2^r}\lambda_2^n(t)\chi_2\left(\frac{(x,x)}{-4t}\right)dt\,. \tag{4.11}$$

Let $(x,x)=2^{-\beta}(y,y)$, $|(y,y)|_2=1$. After the change of variable $t=-(y,y)/2^rs$ ($|s|_2=1$, $dt=2^rds$), we obtain

$$J=2^{n/2}\Gamma_2(\alpha+2)\sum_{r\ge -N+l+1}2^{-(\alpha+n/2)r}\int_{|s|_2=1}\lambda_2^n\left(\frac{(y,y)}{-2^rs}\right)\chi_2(2^{-\beta+r-2}s)ds\,. \tag{4.12}$$

Since $|-(y,y)/2^rs|_2=2^r$, we can write

$$\frac{(y, y)}{-2^r s} = 2^{-r}(1+t_1 2+t_2 2^2+\cdots), \qquad 0 \le t_j \le 1 . \tag{4.13}$$

Since n is even, by the definition of λ_2 (see (4.5)), we have

$$\lambda_2^n\left(\frac{(y, y)}{-2^r s}\right) = \frac{(1+(-1)^{t_1} i)^n}{2^{n/2}} .$$

On the other hand, comparison of the second digits of the canonical representation on both sides in (4.13) gives $t_1 \equiv -(y_1+s_1) \pmod 2$, where s_1 and y_1 are the second digits of the canonical representation of s and (y, y), respectively. So we have

$$\lambda_2^n\left(\frac{(y, y)}{-2^r s}\right) = \frac{(1+(-1)^{t_1} i)^n}{2^{n/2}} = \frac{(1+(-1)^{-(y_1+s_1)} i)^n}{2^{n/2}} . \tag{4.14}$$

Substitution of the value (4.14) into (4.12) and the change of variable $s=1+2s_1+s'$ ($|s'|_2 \le 2^{-2}$, $0 \le s_1 \le 1$ and $ds'=ds$) gives

$$J = \Gamma_2(\alpha+1) \sum_{r \ge -N+l+1} 2^{-(\alpha+n/2)r}[(1+(-1)^{-y_1} i)^n C_1 + (1-(-1)^{-y_1} i)^n C_3] X , \tag{4.15}$$

where, by (4.2),

$$X = \int_{|s'|_2 \le 2^{-2}} \chi_2(2^{-\beta+r-2} s') ds' = \begin{cases} 1/4 & \text{for } r \ge \beta , \\ 0 & \text{for } r < \beta , \end{cases}$$

$$C_1 = \chi_2(2^{-\beta+r-2}) = \exp(2\pi i \{2^{-\beta+r-2}\}_2) = \begin{cases} 1 & \text{for } r \ge \beta+2 \\ -1 & \text{for } r = \beta+1 \\ i & \text{for } r = \beta , \end{cases}$$

$$C_3 = \chi_2(2^{-\beta+r-2} 3) = \exp(2\pi i \{2^{-\beta+r-2} 3\}_2) = \begin{cases} 1 & \text{for } r \ge \beta+2 \\ -1 & \text{for } r = \beta+1 \\ -i & \text{for } r = \beta . \end{cases}$$

Consider the condition $-N+l+1 < \beta$ (since $-N+1 < l$, we have $2^{-2N} < |(x, x)|_2$). Substitution of the values X and C_j ($j=1, 3$) into (4.15) gives

$$J = \Gamma_2(\alpha+1)\left\{\left(\sum_{r \ge \beta+2} 2^{-(\alpha+n/2)r} - 2^{-(\alpha+n/2)(\beta+1)}\right) A + 2^{-(\alpha+n/2)\beta} B\right\} , \tag{4.16}$$

where

$$A = \frac{(1+(-1)^{-y_1} i)^n + (1-(-1)^{-y_1} i)^n}{4} = \begin{cases} (2i)^{n/2} 2^{-1} , & n \equiv 0 \pmod 4 , \\ 0 , & n \equiv 2 \pmod 4 , \end{cases}$$

$$B = \frac{(1+(-1)^{-y_1} i)^n - (1-(-1)^{-y_1} i)^n}{4} i = \begin{cases} 0, & n \equiv 0 \pmod 4) \\ (-1)^{-y_1} i (2i)^{n/2} 2^{-1} , & n \equiv 2 \pmod 4 . \end{cases}$$

Substitution of the values A and B into (4.16) gives the formulas (a) and (b). ∎

THEOREM 4.2. *For any even dimension n and $p=2$, the Green function $G(x)$ has the asymptotic expansion*

$$G(x)\sim\begin{cases}\dfrac{-2}{3}\dfrac{i^{n/2}}{m^4}\left(\dfrac{2^{n/2}-1}{2^{n/2+1}-1}\right)\dfrac{1}{|(x,x)|_2^{1+n/2}} & \text{for } n\equiv 0 \pmod 4\\ \dfrac{(-1)^{-\gamma_1}(2i)^{n/2+1}}{3m^4}\dfrac{1}{|(x,x)|_2^{1+n/2}} & \text{for } n\equiv 2 \pmod 4\,.\end{cases}$$

PROOF. We substitute the formulas (a) and (b) in Lemma 4.1 into the expression for the Green function (1.4) and use the expansions

$$\Gamma_2(\alpha+1)\frac{2^{-(2\alpha+n)+1}-2^{-(\alpha+n/2)}}{1-2^{-(\alpha+n/2)}}$$
$$=2^{-n/2}\sum_{r=0}^{\infty}c_r2^{-nr/2}[2^{-r\alpha}-(1+2^{-n/2+1})2^{-(r+1)\alpha}+2^{-n/2+1}2^{-(r+2)\alpha}]\,,$$

where $c_r=\sum_{j=0}^{r}2^{(n/2-1)j}$; $\Gamma_2(\alpha+1)=\sum_{r=0}^{\infty}2^{-r}(2^{-r\alpha}-2^{-(r-1)\alpha})$. Then the proof of the theorem follows the same process as in Theorem 3.4. ∎

REFERENCES

[1] A. Kh. BIKULOV, Investigation of the p-adic Green function, Theoret. and Math. Phys. 87 (1991), 376–390.

[2] P. CARTIER, Uber einige Integralformen in der Theorie der quadratischen Formen, Math. Z. 84 (1964), 93–100.

[3] HARISH-CHANDRA, Invariant distributions on Lie algebras, Amer. J. Math. 86 (1964), 271–309.

[4] A. N. KOCHUBEI, On p-adic Green's functions, Theoret. and Math. Phys. 96 (1993), 854–865.

[5] G. PARISI, On the p-adic functional integrals, Modern Phys. Lett. A 3 (1988), 639–643.

[6] S. RALLIS AND G. SCHIFFMANN, Distributions invariantes par le group orthogonal, Lecture Notes in Math. 497, Springer-Verlag, Berlin (1973–75), 494–642.

[7] V. A. SMIRNOV, Renormalization in p-adic quantum field theory, Modern Phys. Lett. A 6 (1991), 1421–1427.

[8] V. S. VLADIMIROV, Generalized functions over p-adic number field, Uspekhi Mat. Nauk 43 : 5 (1988), 17–53; Russian Math. Surveys 43 : 5 (1988), 19–64.

[9] V. S. VLADIMIROV AND I. V. VOLOVICH, p-adic quantum mechanics, Comm. Math. Phys. 123 (1989), 659–676.

[10] V. S. VLADIMIROV, I. V. VOLOVICH AND E. I. ZELENOV, p-adic Analysis and Mathematical Physics, World Scientific Publishing, Singapore, 1994.

[11] A. WEIL, Sur certains groups d'operateurs unitaires, Acta Math. 111 (1964), 143–211.

[12] E. I. ZELENOV, p-adic quantum mechanics for $p=2$, Theoret. and Math. Phys. 80 (1989), 253–263.

TGRC
KYUNGPOOK UNIVERSITY
TAEGU 702–701
KOREA