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An Asymptotic Expansion Scheme for Optimal Investment Problems⋆

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Abstract. We shall propose a new computational scheme for the evaluation of the optimal portfolio for investment. Our method is based on an extension of the asymptotic expansion approach which has been recently developed for pricing problems of the contingent claims' analysis by Kunitomo and Takahashi (1992, 1995, 2001, 2003), Yoshida (1992), Takahashi (1995, 1999), Takahashi and Yoshida (2001). In particular, we will explicitly derive a formula of the optimal portfolio associated with maximizing utility from terminal wealth in a financial market with Markovian coefficients, and give a numerical example for a power utility function.

Key words:

1. Introduction

We shall propose a new computational scheme for the evaluation of the optimal portfolios for investment. Our method is based on the asymptotic expansion approach, a unified method of efficient computation justified by Malliavin-Watanabe (1987) theory, which has been recently developed for pricing problems of the contingent claims' analysis by Kunitomo and Takahashi (1992, 1995, 2001, 2003), Yoshida (1992), Takahashi (1995, 1999), Kim and Kunitomo (1999) and Takahashi and Yoshida (2001). They have developed the method through deriving formulas for practical examples such as average options, basket options, and options with stochastic volatility and with stochastic interest rates in a Markovian setting, as well as bond options (swaptions), average options on interest rates, and average options on foreign exchange rates with stochastic interest rates in the Heath–Jarrow–Morton (1992) framework. In this paper, we extend the method to portfolio problems. In particular, we will explicitly derive the formula of the optimal portfolio associated with maximizing utility from terminal wealth in a complete market, where the short term risk-free rate and the market price of risk

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are described by some functions of a random vector following a multi-dimensional Markov process. Moreover, we provide numerical examples for power utility functions. In general, it is quite difficult to compute an optimal portfolio explicitly when the investment opportunity is stochastic in a multiperiod setting. The stochastic control approach initiated by Merton (1969, 1971) gives a solution in terms of the derivatives of the value function: While the solution can be evaluated numerically based on the Hamilton–Jacobi–Bellman equation, the implementation is not easy especially for the case of multiple assets. In the martingale approach initiated by Karatzas et al. (1987) and Cox and Huang (1989), Ocone and Karatzas (1991) proposed the representation of optimal portfolios by utilizing the Clark formula. Although their representation formulas were derived in general setting, explicit evaluation was obtained only for logarithmic utility functions or a financial market with deterministic coefficients, which were already known without their formulas. Starting with the Clark formula, we will present an explicit expression for the optimal portfolio in a concrete and important setting where key variables such as the short term risk-free rate and the market price of risk are some functions of a random vector whose evolution is described by a multi-dimensional Markov process. Moreover, our method can be easily extended to the optimal portfolios associated with maximizing utility from both consumption and terminal wealth, and to the hedging portfolios associated with contingent claims. Regarding the related works, Detemple et al. (2000) utilizes Monte Carlo simulations to investigate optimal portfolios for a power utility function in several Markovian examples. The organization of this paper is as follows. In Section 2 we explain the problem of the optimal portfolio for investment and restate the problem in a Markovian setting. In Section 3 we briefly explain basic tools for an asymptotic expansion approach. In Section 4, we illustrate our method using a power utility function, and derive the second-order scheme explicitly. In Section 5 we also derive the second-order scheme for general utility functions. In Section 6 we give a numerical example. Finally, in appendix, we provide proofs of lemmas, show the result of the third-order scheme for power utility functions, and discuss the validity of our method for the numerical example considered in Section 6.

2. Representation of Optimal Portfolio

2.1. REPRESENTATION OF OPTIMAL PORTFOLIO FOR INVESTMENT

We will briefly describe the financial market and introduce the representation of the optimal portfolio for investment derived by Ocone and Karatzas (1991).

We start with basic setup of the financial market. Let $(\Omega, \mathcal{F}, P)$ probability space and $T \in (0, \infty)$ denotes some fixed time horizon of the economy. $w(t) = (w^1(t), \ldots, w^r(t))^*$, $0 \leqslant t \leqslant T$ is $\mathbf{R}^r$-valued Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}$, $0 \leqslant t \leqslant T$ stands for P-augmentation of the natural filtration, $\mathcal{F}_t^w = \sigma(w(s); 0 \leqslant s \leqslant t)$. Here, we use the notation of x^* as the transpose of x. $S_i(t)$, $i = 1, \ldots, r$ and $S_0(t)$ denote the prices at time $t \in [0, T]$ of the risky asset i and of the risless asset, respectively. The prices are assumed to follow the stochastic processes: For $t \in [0, T]$,

$$\begin{aligned} dS_i &= S_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{r} \sigma_{ij}(t)\,dw_j(t)\right], \quad S_i(0) = s_i, \quad i = 1, \dots, r, \\ dS_0 &= r(t)S_0(t)\,dt, \quad S_0(0) = 1, \end{aligned} \tag{1}$$

where we suppose that $r(t)$, $b_i(t)$ and $\sigma_{ij}(t)$, $i, j = 1, \dots, r$ are bounded and progressively measurable with respect to $\{\mathcal{F}_t\}$. We also assume the nondegeneracy condition; for the $r \times r$ matrix $\sigma(t) \equiv \{\sigma_{ij}(t)\}_{1 \leqslant i,j \leqslant r}$ there exists a real number $\varepsilon > 0$ such that

$$\xi^* \sigma(t, \omega)\sigma(t, \omega)^* \xi \geqslant \epsilon|\xi|^2; \ \forall \xi \in \mathbf{R}^r, \quad (t, \omega) \in [0, T] \times \Omega.$$

Then, the stochastic process of an investor's wealth denoted by $W(t)$ are expressed as

$$dW(t) = [r(t)W(t) - c(t)]\,dt + \pi(t)^*[(b(t) - r(t)\mathbf{1})\,dt + \sigma(t)\,dw(t)], \tag{2}$$

where $W(0) = W > 0$ is the initial capital, $\mathbf{1}$ denotes the vector in $\mathbf{R}^r$ with all elements equal to 1, $c(t)$ denotes the consumption rate, $b(t) = (b_1(t), \dots, b_r(t))^*$, and $\pi(t) = \{\pi_i(t)\}^*_{i=1,\dots,r}$ denotes the portfolio. $c(t)$ and $\pi(t)$ satisfy the integrability condition;

$$\int_0^T \{|\pi(t)|^2 + c(t)\}\,dt < \infty \text{ a.s.}$$

Next, let $\mathcal{A}(W)$ denote the set of stochastic processes (π, c) which generate $W(t) \geqslant 0$ for all $t \in [0, T]$ given $W(0) = W$. We call (π, c) is admissible for W if $(\pi, c) \in \mathcal{A}(W)$.

The problem of *maximizing utility from terminal wealth* is formulated as follows:

$$\sup_{(\pi,c)\in\mathcal{A}(W)} E[U(W(T))], \tag{3}$$

where $E[\cdot]$ denotes the expectation operator under P, and U denotes a utility function such that

$$\begin{aligned} &U\colon (0, \infty) \to \mathbf{R}, \\ &\text{a strictly increasing, strictly concave function of class } \mathbf{C}^2 \\ &\text{with } U(0+) \equiv \lim_{c\downarrow 0} U(c) \in [-\infty, \infty),\ U'(0+) \equiv \lim_{c\downarrow 0} U'(c) = \infty \\ &\text{and } U'(\infty) \equiv \lim_{c\to\infty} U'(c) = 0. \end{aligned} \tag{4}$$

Let the market price of risk $\theta(t)$ for $t \in [0, T]$ an $\mathbf{R}^r$-valued progressively measurable bounded process defined by

$$\theta(t) = \sigma(t)^{-1}[b(t) - r(t)\mathbf{1}].$$

Then, the martingale measure denoted by P_0 is defined as $P_0(A) = E[1_A Z(T)]$ for all $A \in \mathcal{F}_T$ where

$$Z(t) = \exp\left(-\int_0^t \theta(s)^* \,\mathrm{d}w(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 \,\mathrm{d}s \right), \quad 0 \leqslant t \leqslant T.$$

We note that $w_0(t) \equiv w(t) + \int_0^t \theta(u)\,\mathrm{d}u$ for $0 \leqslant t \leqslant T$ is a standard Brownian motion under P_0.

Regarding the problem of *maximizing utility from terminal wealth*, it is well known that the optimal wealth level of terminal wealth given by $W(T) = I(\mathcal{Y}(W)H_0(T))$ and that the value function $V(W) := \sup_{(\pi,c)\in\mathcal{A}(W)} E[U(W(T))]$ can be computed as $V(W) = G(\mathcal{Y}(W))$, where $G(y) := E[U(I(yH_0(T)))]$; $0 < y < \infty$ (see for instance Theorem 7.6 in Karatzas and Shreve, 1998, p. 114). Here, a continuously differentiable function $I : (0,\infty) \to (0,\infty)$ (which is expressed as $I \in C^1((0,\infty); (0,\infty)))$ denotes the inverse of $U'(\cdot)$, and $\mathcal{Y}(\cdot)$ denotes the inverse of the continuous decreasing function:

$$\mathcal{X}(y) = \mathbf{E}_0[\beta(T)I(yH_0(T))] = \mathbf{E}[H_0(T)I(yH_0(T))], \quad 0 < y < \infty$$

which we assume maps $(0,\infty)$ into $(0,\infty)$, where $\beta(t) \equiv 1/S_0(t)$, $H_0(t) \equiv \beta(t)Z(t)$ denotes the state price density at t and $E_0[\cdot]$ denotes the expectation operator under P_0.

Ocone and Karatzas (1991) provides the following theorem by utilizing the Clark formula regarding the problem of the optimal portfolio for investment associated with *maximizing utility from terminal wealth*.

THEOREM (OCONE AND KARATZAS, 1991). *Suppose that a utility function satisfies the conditions* (4) *and that*

$$I(y) + |I'(y)| \leqslant K(y^\alpha + y^{-\beta}), \quad 0 < y < \infty$$

holds for some real positive constants α, β *and* K. *Then the optimal portfolio admits the representation*;

$$\begin{aligned}\pi^*(t)\sigma(t) \;=\; & -\frac{1}{\beta(t)}\bigg\{\theta^*(t)\mathbf{E}_0[\beta(T)\mathcal{Y}(W)H_0(T)I'(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t] + \\ & + \mathbf{E}_0\bigg[\beta(T)\phi'(\mathcal{Y}(W)H_0(T))\bigg(\int_t^T D_t r(u)\,\mathrm{d}u + \\ & + \sum_{\alpha=1}^{r}\int_t^T \{D_t\theta_\alpha(u)\}\,\mathrm{d}w_0^\alpha(u)\bigg)\bigg|\mathcal{F}_t\bigg]\bigg\}, \end{aligned} \tag{5}$$

where $\phi(y) \equiv yI(y)$, $0 < y < \infty$, *and* $D_t r(u)$ *and* $D_t\theta_\alpha(u)$ *for* $\alpha = 1, 2, \ldots, r$ *denote the Malliavin derivatives of* $r(u)$ *and* $\theta_\alpha(u)$.

Here we suppose that θ and r satisfy the following conditions:

1. **R**-*valued progressively measurable process r is bounded*; *for a.e. $s \in [0, T]$ $r(s, \cdot) \in D_{1,1}$, where $D_{1,1}$ denotes the Sobolev space $D_{p,s}$ with $(p, s) = (1, 1)$, $(s, \omega) \to Dr(s, \omega) \in (L^2([0, T]))^r$ admits progressively measurable version, and*
$$\|r\|^a_{1,1} \equiv \mathbf{E}\left[\left(\int_0^T |r(s)|^2\,\mathrm{d}s\right)^{1/2} + \left(\int_0^T \|Dr(s)\|^2\,\mathrm{d}s\right)^{1/2}\right] < \infty,$$
where $\|\cdot\|$ denotes the $L^2([0, T])$ norm, and $\|Dr(s)\|^2 \equiv \sum_{i=1}^r \|D^i r(s)\|^2$.
2. $\mathbf{R}^r$-*valued progressively measurable process θ is bounded*; *for a.e. $s \in [0, T]$ $\theta(s, \cdot) \in (\mathbf{D}_{1,1})^r$, $(s, \omega) \to D\theta(s, \omega) \in (L^2([0, T]))^{r^2}$ admits a progressively measurable version, and*
$$\|\theta\|^a_{1,1} \equiv \mathbf{E}\left[\left(\int_0^T |\theta(s)|^2\,\mathrm{d}s\right)^{1/2} + \left(\int_0^T \|D\theta(s)\|^2\,\mathrm{d}s\right)^{1/2}\right] < \infty,$$
where $\|D\theta(s)\|^2 \equiv \sum_{i,j=1}^r \|D^i\theta_j(s)\|^2$.
3. *For some $p > 1$ we have*
$$\mathbf{E}\left[\left(\int_0^T \|Dr(s)\|^2\,\mathrm{d}s\right)^{p/2}\right] < \infty,$$
$$\mathbf{E}\left[\left(\int_0^T \|D\theta(s)\|^2\,\mathrm{d}s\right)^{p/2}\right] < \infty.$$

Proof. See Theorem 4.2 of Ocone and Karatzas (1991). □

More intuitive formula can be obtained under original measure P.

THEOREM 1. *Under the same conditions as in theorem* (*Ocone and Karatzas*, 1991), *the optimal portfolio has the representation under measure P*;

$$\begin{aligned}\pi^*\sigma(t) \;=\; & \left\{W(t) - \mathbf{E}\left[\frac{H_0(T)}{H_0(t)}\phi'(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t\right]\right\}\theta^*(t) - \\ & - \mathbf{E}\left[\frac{H_0(T)}{H_0(t)}\phi'(\mathcal{Y}(W)H_0(T))\left(\int_t^T D_t r(u)\,\mathrm{d}u + \right.\right. \\ & \left.\left. + \sum_{\alpha=1}^r \left\{\int_t^T \{D_t\theta_\alpha(u)\}\,\mathrm{d}w^\alpha(u) + \int_t^T \{D_t\theta_\alpha(u)\}\theta_\alpha(u)\,\mathrm{d}u\right\}\right)\middle|\mathcal{F}_t\right],\end{aligned} \tag{6}$$

where $W(t)$ denotes the optimal wealth at time t, and is determined by

$$W(t) = \mathbf{E}\left[\frac{H_0(T)}{H_0(t)} I(\mathcal{Y}(W)H_0(T))|\mathcal{F}_t\right]. \tag{7}$$

Proof. The relation (7) is well known. (See Theorem 7.6 of Karatzas and Shreve, 1998 for instance.) Rewrite Equation (5) under P by using (7), $\phi'(y) = I(y) + yI'(y)$, and $w_0(t) = w(t) + \int_0^t \theta(u)\,\mathrm{d}u$ to obtain the result. □

In the optimal portfolio of Equation (6), the first term on the right-hand side represents the optimal portfolio in one-period setting that is sometimes called mean-variance portfolio, while the second term is specific to multi-period setting, which Merton (1971) named *hedging demand* in a sense that the term represents demand for hedging against randomness in the future; specifically, the terms $D_t r(u)$ and $D_t\theta_\alpha(u)$ express the changes of the riskless interest rate and the market price of risk in the future, respectively.

It is well known that the optimal portfolio $\pi(t)$ is easily derived for two simple cases: (See for instance Chapter 3 in Karatzas and Shreve, 1998.) For the case of a log utility function $U(x) = \log x$,

$$\pi^*(t) = \theta^*(t)\sigma(t)^{-1}W(t),$$

where $\theta(t) = \sigma(t)^{-1}[b(t) - r(t)\mathbf{1}]$, since $\phi \equiv 1$ and hence $\phi' \equiv 0$. This is exactly the same as mean-variance portfolio in one-period portfolio problem; that is, the optimal portfolio per wealth is given by the vector of the excess expected returns of risky assets over the riskless asset multiplied by the inverse of *variance–covariance* matrix. In this sense, an investor with a log utility is sometimes called a myopic investor. For the case of a power utility function defined by $U(x) = x^\delta/\delta$, $\delta < 1$, $\delta \neq 0$ for $x \in (0, \infty)$, if $r(\cdot)$ and $\theta(\cdot)$ are deterministic,

$$\pi^*(t) = \frac{1}{1-\delta}\theta^*(t)\sigma(t)^{-1}W(t) \tag{8}$$

because $D_t r(u) \equiv 0$ and $D_t\theta_\alpha(u) \equiv 0$. However, if $r(\cdot)$ and $\theta(\cdot)$ are *not* deterministic, it is not easy to evaluate $\pi(t)$ explicitly for a power utility function.

2.2. OPTIMAL PORTFOLIO FOR INVESTMENT IN A MARKOVIAN SETTING

In the spirit of Ocone and Karatzas (1991), we will consider more concrete and important setting for practical purpose in the sequel.

From now on, we will consider a Wiener space on $[t, T]$ for some fixed $t \in [0, T]$ and assume that all random variables will be defined on it. Let X_u^ϵ be a d-dimensional diffusion process defined by the stochastic differential equation:

$$\mathrm{d}X_u^\epsilon = V_0(X_u^\epsilon, \epsilon)\,\mathrm{d}u + V(X_u^\epsilon, \epsilon)\,\mathrm{d}w_u, \quad X_t^\epsilon = x \tag{9}$$

for $u \in [t, T]$. Here we suppose that $\epsilon \in (0, 1]$ denotes a parameter used as the asymptotic expansion, $V_0 \in C_b^\infty(\mathbf{R}^d \times (0, 1]; \mathbf{R}^d)$ and $V = (V_\beta)_{\beta=1}^r \in C_b^\infty(\mathbf{R}^d \times (0, 1]; \mathbf{R}^d \otimes \mathbf{R}^r)$, where $C_b^\infty(R^d \times (0, 1]; E)$ denotes a class of smooth mappings f: $\mathbf{R}^d \times (0, 1] \to E$ whose derivatives $\partial_x^{\mathbf{n}}\partial_\epsilon^m f(x, \epsilon)$ are all bounded for $\mathbf{n} \in \mathbf{Z}_+^d$ such that $|\mathbf{n}| \geqslant 1$ and $m \in \mathbf{Z}_+$. Note that time-dependent-coefficient diffusion processes are included in the above equation if we enlarge the process to a higher-dimensional one. We also assume the bounded processes $r(u)$ and $\theta(u)$ to be $r(u) = r(X_u^\epsilon)$ and $\theta(u) = \theta(X_u^\epsilon)$, where $r \in C_b^\infty(\mathbf{R}^d; \mathbf{R}_+)$ and $\theta \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^r)$. We remark that our framework includes a financial market with Markovian coefficients of return processes as a special case, in which not only $r(u)$ but also $b(u)$ and $\sigma(u)$ are some functions of X_u^ϵ.

Let $Y_{t,u}^\epsilon$ be a unique solution of the $d \times d$-matrix valued stochastic differential equation:

$$\mathrm{d}Y_{t,u}^\epsilon = \sum_{\alpha=0}^{r} \partial_x V_\alpha(X_u^\epsilon, \epsilon) Y_{t,u}^\epsilon \, \mathrm{d}w_u^\alpha, \qquad Y_{t,t}^\epsilon = \underline{I} \tag{10}$$

Then, we have the representation of the optimal portfolio $\pi(t)$ in our Markovian setting, which is stated as a corollary of Theorem 1.

COROLLARY 1. *The optimal portfolio under the Markovian setting* (9) *and* (10) *is represented as follows*:

$$\begin{aligned}\pi^*(t)\sigma(x) \;=\; & \Big\{W - \mathbf{E}\Big[H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})\Big]\Big\}\theta^*(x) - \\ & - \mathbf{E}\Big[H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})\Big(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u + \\ & + \sum_{\alpha=1}^{r}\int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u) + \\ & + \sum_{\alpha=1}^{r}\int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u\Big)\Big],\end{aligned} \tag{11}$$

where W is a given wealth at time t, $H_{0,t,T}$ is defined by

$$\begin{aligned}H_{0,t,T} &\equiv \frac{H_0(T)}{H_0(t)} \\ &= \exp\left(-\int_t^T \theta(X_u^\epsilon)^*\,\mathrm{d}w(u) - \frac{1}{2}\int_t^T |\theta(X_u^\epsilon)|^2\,\mathrm{d}u - \int_t^T r(X_u^\epsilon)\,\mathrm{d}u\right),\end{aligned}$$

and $\mathcal{Y}$ is determined by the equation:

$$W = \mathbf{E}[H_{0,t,T} I(\mathcal{Y}H_{0,t,T})]. \tag{12}$$

Proof. It is well known that

$$D_t X_u^\epsilon = Y_{t,u}^\epsilon V(X_t^\epsilon, \epsilon) = Y_{t,u}^\epsilon V(x_t, \epsilon), \quad u \geqslant t,$$

and that

$$D_t f(X_u^\epsilon) = \partial f(X_u^\epsilon)[D_t X_u^\epsilon] = \partial f(X_u^\epsilon) Y_{t,u}^\epsilon V(x_t, \epsilon), \quad u \geqslant t,$$

for $f \in C_b^\infty(\mathbf{R}^d; \mathbf{R})$. Apply those facts to Equation (6) with $f(\cdot) \equiv r(\cdot)$ *or* $\theta(\cdot)$. □

Our objective is to evaluate $\pi(t)$ explicitly. It is possible to compute $\pi(t)$ based on a Monte Carlo simulation. However, it is path-dependent, besides, the functions

ϕ' is often irregular, and the error bounds are not yet fully investigated in such a situation, while Kohatsu and Yoshida (2001) recently provided an error bound to the Euler–Maruyama scheme for path-dependent functionals. Moreover, Monte Carlo methods are not so useful from a computational viewpoint when a family of stochastic differential equations is treated; especially, the sensitivity analysis which controls the underlying stochastic differential equations is the case.

In the present article, we will propose a practical and more efficient scheme for computing the optimal portfolio by utilizing the asymptotic expansion approach. On the asymptotic expansion approach, first-order asymptotics was proposed in Kunitomo and Takahashi (1992) for geometric Brownian motion. In order to obtain more precise approximation, the asymptotic expansion method was introduced with the Malliavin calculus and investigated for the evaluation of path-dependent contingent claims in Yoshida (1992), Takahashi (1995, 1999), Kunitomo and Takahashi (1995, 2001, 2003), Kim and Kunitomo (1999), and Takahashi and Yoshida (2001).

3. An Asymptotic Expansion Scheme

We will introduce basic tools for an asymptotic expansion scheme. First, we will derive the asymptotic expansions of X_u^ϵ and $Y_{t,u}^\epsilon$ in (9) and (10), respectively, which will provide the basis for the subsequent analysis. We start with a basic assumption, the **deterministic limit condition**:

$[A1] \qquad V(\cdot, 0) \equiv 0.$

It follows from $[A1]$ that the limit process $(X_u^0)_{u\in[t,T]}$ is a unique deterministic solution of the ordinary differential equation:

$$X_u^0 = x + \int_t^u V_0(X_s^0, 0)\,\mathrm{d}s. \tag{13}$$

We further assume $\sigma(X_u^0)$ is nonsingular for all $u \in [t, T]$. Next, put $Y_{t,s} := Y_{t,s}^0$ and then clearly, $Y_{t,s}$ is a unique deterministic solution of the ordinary differential equation:

$$\mathrm{d}Y_{t,s} = \partial_x V_0(X_s^0, 0) Y_{t,s}\,\mathrm{d}s, \quad s \in [t, T], \qquad Y_{t,t} = \underline{I}, \tag{14}$$

where $Y_{t,s} \in GL(d, \mathbf{R})$. Next, let $D(t; u) = \partial X_u^\epsilon / \partial\epsilon|_{\epsilon=0}$, $E(t; u) = \partial^2 X_u^\epsilon / \partial\epsilon^2|_{\epsilon=0}$ and $Y_{t,u}^{[1]} = \partial Y_{t,u}^\epsilon / \partial\epsilon|_{\epsilon=0}$. Then $D(t; u)$, $E(t, u)$ and $Y_{t,u}^{[1]}$ ($u \in [t, T]$) are determined by the following stochastic differential equations:

$$\begin{aligned} &\mathrm{d}D(t; u) = \partial_x V_0(X_u^0, 0) D(t; u)\,\mathrm{d}u + \sum_{\alpha=0}^{r} \partial_\epsilon V_\alpha(X_u^0, 0)\,\mathrm{d}w^\alpha, \\ &D(t; t) = 0, \end{aligned} \tag{15}$$

$$
\begin{aligned}
\mathrm{d}E(t;u) &= \partial_x V_0(X^0_u,0)E(t;u)\,\mathrm{d}u + \partial_x^2 V_0(X^0_u,0)[D(t;u),D(t;u)]\,\mathrm{d}u + \\
&\quad + 2\sum_{\alpha=0}^{r}\partial_x\partial_\epsilon V_\alpha(X^0_u,0)D(t;u)\,\mathrm{d}w^\alpha + \sum_{\alpha=0}^{r}\partial_\epsilon^2 V_\alpha(X^0_u,0)\,\mathrm{d}w^\alpha, \\
E(t;t) &= 0
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
\mathrm{d}Y^{[1]}_{t,s} &= \partial_x V_0(X^0_s,0)Y^{[1]}_{t,s}\,\mathrm{d}s + \partial_x^2 V_0(X^0_s,0)[D(t;s)]Y_{t,s}\,\mathrm{d}s + \\
&\quad + \sum_{\alpha=0}^{r}\partial_\epsilon\partial_x V_\alpha(X^0_s,0)Y_{t,s}\,\mathrm{d}w^\alpha_s, \\
Y^{[1]}_{t,t} &= 0.
\end{aligned}
\tag{17}
$$

Here we used the fact that $\partial_x V_\alpha(\cdot,0)=0$ for $\alpha = 1,\dots,r$. Moreover, we used conventions $\mathrm{d}w^0=\mathrm{d}u$, $\partial_x=\mathrm{grad}_x$, $\partial_\epsilon=\partial/\partial\epsilon$, and notations;

$$\partial_x^2 V_0(X^0_u,0)[D(t;u),D(t;u)] = \sum_{i=1}^{d}\sum_{j=1}^{d}\partial_{x^i}\partial_{x^j}V_0(X^0_u,0)D^{(i)}(t;u)D^{(j)}(t;u),$$

and

$$\partial_x^2 V_0(X^0_s,0)[D(t;s)]Y_{t,s}\,\mathrm{d}s = \sum_{i=1}^{d}\sum_{j=1}^{d}\partial_{x^i}\partial_{x^j}V_0(X^0_s,0)D^{(j)}(t;s)(Y_{t,s})^{(i,\cdot)}\mathrm{d}s.$$

where $D^{(i)}(t;s)$ denotes the ith element of $D(t;s)$ and $(Y_{t,s})^{(i,\cdot)}$ denotes the ith row of $Y_{t,s}$. Further, we will use the following abbreviations:

$$
\begin{aligned}
&X_u = X^0_u, \qquad Y_u = Y^0_u, \qquad V_{\alpha u} = V^{[0]}_{\alpha u} = V_\alpha(X_u,0), \quad \alpha = 0,1,\dots,r, \\
&\partial = \partial_x, \qquad \partial_i = \partial_{x^i}.
\end{aligned}
$$

Then, we obtain the asymptotic expansions of X^ϵ_u and $Y^\epsilon_{t,u}$ upto the order explictly used in the later sections.

LEMMA 1. *The asymptotic expansions of* X^ϵ_u *and* $Y^\epsilon_{t,u}$ *are obtained as follows*:

$$
\begin{aligned}
X^\epsilon_u &= X_u + \epsilon D(t;u) + \tfrac{1}{2}\epsilon^2 E(t;u) + o(\epsilon^2), \\
Y^\epsilon_{t,u} &= Y_{t,u} + \epsilon Y^{[1]}_{t,u} + o(\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
D(t;u) &= Y_{t,u}\int_t^u Y^{-1}_{t,s}\sum_{\alpha=0}^{r}\partial_\epsilon V_{\alpha s}\,\mathrm{d}w^\alpha_s, \\
E(t;u) &= Y_{t,u}\int_t^u Y^{-1}_{t,s}\Bigg\{\partial^2 V_{0s}[D(t;s),D(t;s)]\,\mathrm{d}s + \\
&\quad + 2\sum_{\alpha=0}^{r}\partial\partial_\epsilon V_{\alpha s}D(t;s)\,\mathrm{d}w^\alpha + \sum_{\alpha=0}^{r}\partial_\epsilon^2 V_{\alpha s}\,\mathrm{d}w^\alpha\Bigg\},
\end{aligned}
$$

$$Y^{[1]}_{t,u} = Y_{t,u} \int_t^u (Y_{t,s})^{-1} \left[\partial^2 V_{0s}[D(t;s)] Y_{t,s}\, \mathrm{d}s + \sum_{\alpha=0}^{r} \partial_\epsilon \partial V_{\alpha s} Y_{t,s}\, \mathrm{d}w_s^\alpha \right].$$

Proof. See appendix. □

Next, we will consider the asymptotic expansion of the following functional which will appear frequently in the sequel. Define

$$\zeta^\epsilon_{t,u} := \exp\left(\int_t^u a_0(X^\epsilon_s)\, \mathrm{d}s + \int_t^u a(X^\epsilon_s)\, \mathrm{d}w_s \right), \tag{18}$$

where $a_0 \in C^\infty_\uparrow(\mathbf{R}^d; \mathbf{R})$ and $a \in C^\infty_\uparrow(\mathbf{R}^d; \mathbf{R}^r)$. Here, $C^\infty_\uparrow(\mathbf{R}^d; \mathbf{R})(C^\infty_\uparrow(\mathbf{R}^d; \mathbf{R}^r))$ denotes a class of smooth functions $f\colon \mathbf{R}^d \to \mathbf{R}$ ($f\colon \mathbf{R}^d \to \mathbf{R}^r$) whose derivatives are of polynomial growth orders. In addition, we assume the following integrability condition for $\zeta^\epsilon_{t,T}$:

[$A2$] For any $p \in (1, \infty)$, $\sup_{\epsilon \in (0,1]} \|\zeta^\epsilon_{t,T}\|_p < \infty$.

Then, we easily obtain the next lemma.

LEMMA 2. *Under condition* [$A2$], $\zeta^\epsilon_{t,T}$ *has an asymptotic expansion*:

$$\zeta^\epsilon_{t,T} \sim \zeta^0_{t,T} + \epsilon \zeta^{[1]}_{t,T} + \tfrac{1}{2}\epsilon^2 \zeta^{[2]}_{t,T} + \cdots \tag{19}$$

in L^p *for every* $p > 1$ (*or in* $\mathbf{D}^\infty$) *as* $\epsilon \downarrow 0$. *The first three coefficients are given by*

$$\begin{aligned}
\zeta^0_{t,T} &= \exp\left(\int_t^T a_0(X_s)\, \mathrm{d}s + \int_t^T a(X_s)\, \mathrm{d}w_s \right),\\
\zeta^{[1]}_{t,T} &= \zeta^0_{t,T}\left(\int_t^T \partial_x a_0(X_s) D(t;s)\, \mathrm{d}s + \int_t^T \partial_x a(X_s) D(t;s)\, \mathrm{d}w_s \right),\\
\zeta^{[2]}_{t,T} &= \zeta^0_{t,T}\Bigg\{ \left(\int_t^T \partial_x a_0(X_s) D(t;s)\, \mathrm{d}s + \int_t^T \partial_x a(X_s) D(t;s)\, \mathrm{d}w_s \right)^2 + \\
&\quad + \int_t^T \partial_x a_0(X_s) E(t;s)\, \mathrm{d}s + \int_t^T \partial_x a(X_s) E(t;s)\, \mathrm{d}w_s + \\
&\quad + \int_t^T \partial_x^2 a_0(X_s)[D(t;s), D(t;s)]\, \mathrm{d}s + \\
&\quad + \int_t^T \partial_x^2 a(X_s)[D(t;s), D(t;s)]\, \mathrm{d}w_s \Bigg\}.
\end{aligned}$$

Proof. Taylor expansion under the integrability condition [$A2$] gives the result. □

Finally in this section, we will derive an expansion of the following functional which will be frequently used in the subsequent analysis:

$$g^{\alpha,\epsilon} = \int_t^T \partial f(X^\epsilon_u) Y^\epsilon_{t,u} V(x_t, \epsilon)\, \mathrm{d}w^\alpha_u, \quad \alpha = 0, 1, \ldots, r, \tag{20}$$

where $f \in C^\infty_\uparrow(\mathbf{R}^d; \mathbf{R})$.

LEMMA 3. *The asymptotic expansion of* $g^{\alpha,\epsilon}$ *is given by*

$$g^{\alpha,\epsilon} \sim \epsilon g^{\alpha,[1]} + \frac{1}{2}\epsilon^2 g^{\alpha,[2]} + \cdots$$

in L^p *for every* $p > 1$ (*or in* $\mathbf{D}^\infty(\mathbf{R}^d)$) *as* $\epsilon \downarrow 0$. *Here,* $g^{\alpha,[j]}$, $j = 1, 2$ *denote the first derivative* $g^{\alpha,[1]} \equiv \partial g^{\alpha,\epsilon}/\partial\epsilon|_{\epsilon=0}$ *and the second derivative* $g^{\alpha,[2]} \equiv \partial^2 g^{\alpha,\epsilon}/\partial\epsilon^2|_{\epsilon=0}$ *of* $g^{\alpha,\epsilon}$, *respectively, and they are expressed as follows*:

$$\begin{aligned}
g^{\alpha,[1]} &= \int_t^T \partial_x f(X_u)[Y_{t,u}\partial_\epsilon V(x_t, 0)]\,\mathrm{d}w_u^\alpha, \\
g^{\alpha,[2]} &= 2\int_t^T \sum_{i=1}^d \sum_{j=1}^d \partial_i\partial_j f(X_u) D^{(j)}(t;u) Y_{t,u}^{(i,\cdot)} \partial_\epsilon V(x_t, 0)\,\mathrm{d}w_u^\alpha + \\
&\quad + 2\int_t^T \sum_{i=1}^d \partial_i f(X_u) Y_{t,u}^{[1],(i,\cdot)} \partial_\epsilon V(x_t, 0)\,\mathrm{d}w_u^\alpha + \\
&\quad + \int_t^T \sum_{i=1}^d \partial_i f(X_u) Y_{t,u}^{(i,\cdot)} \partial_\epsilon^2 V(x_t, 0)\,\mathrm{d}w_u^\alpha. \qquad (21)
\end{aligned}$$

Proof. See appendix. □

4. The Scheme for Power Utility Functions

4.1. POWER UTILITY FUNCTIONS

In this section, we will illustrate our approach through an asymptotic expansion of the optimal portfolio for power utility functions. We first assume that a utility function in (4) is specified as so called a power function; that is

$$U(x) = \frac{x^\delta}{\delta}, \quad x \in (0, \infty), \ \ \delta < 1, \ \ \delta \neq 0. \qquad (22)$$

Then, $I(y)$ and $\phi(y)$ are given by $I(y) = y^{-1/(1-\delta)}$, $\phi(y) = y^{-\delta/(1-\delta)}$ and $\phi'(y) = [-\delta/(1-\delta)]I(y)$.

Hence, in this case the optimal portfolio given in Equation (11) is expressed as follows:

$$\begin{aligned}
\pi^*(t)\sigma(x) &= \frac{1}{1-\delta} W\theta(x)^* + \frac{\delta}{1-\delta}(\mathcal{Y})^{-1/(1-\delta)} \mathbf{E}\Big[(H_{0,t,T})^{-\delta/(1-\delta)} \times \\
&\quad \times \Big(\int_t^T \partial r(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u + \\
&\quad + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u) + \\
&\quad + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u\Big)\Big], \qquad (23)
\end{aligned}$$

where

$$W = (\mathcal{Y})^{-1/(1-\delta)}\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}].$$

Here, we use the abbreviations $r(u) = r(X_u^\epsilon)$ and $\theta_\alpha(u) = \theta_\alpha(X_u^\epsilon)$. We also notice that $(\mathcal{Y})^{-1/(1-\delta)}$ is expressed explicitly in terms of the current wealth W at time t. Then, alternatively, the optimal proportions of risky assets in wealth denoted by $\pi^*(t)/W$ are given as follows:

$$\begin{aligned}\pi^*(t)/W &= \frac{1}{1-\delta}\theta(x)^*\sigma^{-1}(x) + \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}]} \times \\ &\quad \times \mathbf{E}\Bigg[(H_{0,t,T})^{-\delta/(1-\delta)}\bigg(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u + \\ &\quad + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u) + \\ &\quad + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u\bigg)\Bigg]\sigma^{-1}(x). \end{aligned} \tag{24}$$

We remark that Equation (8) which represents the optimal portfolio when $r(\cdot)$ and $\theta(\cdot)$ are not random does not have the second term on the right-hand side of Equation (24), because in that case,

$$\partial r(u) = \partial\theta_\alpha(u) \equiv 0.$$

Hereafter, our objective is to derive an asymptotic expansion of Equation (24); because the first term on the right-hand side of Equation (24) is already obtained explicitly by the current information at time t, we will derive an asymptotic expansion of the second term applying the technique prepared in the previous section. More specifically, we will consider the term E defined as follows:

$$\begin{aligned}E &\equiv \frac{1}{\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}]}\mathbf{E}\Bigg[(H_{0,t,T})^{-\delta/(1-\delta)}\bigg(\int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u + \\ &\quad + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u) + \\ &\quad + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u\bigg)\Bigg]. \end{aligned} \tag{25}$$

4.2. PREPARATIONS

First, we notice that we can directly apply the expression (19) in Lemma 2 to $(H_{0,t,T})^{-\delta/(1-\delta)}$ in Equation (24) if we set $\zeta_{t,u}^\epsilon \equiv (H_{0,t,T})^{-\delta/(1-\delta)}$, and specify $a_0(X_s^\epsilon)$

and $a(X_s^\epsilon)$ as

$$a_0(X_s^\epsilon) = \left(\frac{\delta}{1-\delta}\right) r(X_s^\epsilon) + \frac{\delta}{2(1-\delta)}|\theta(X_s^\epsilon)|^2,$$
$$a(X_s^\epsilon) = \left(\frac{\delta}{1-\delta}\right)\theta^*(X_s^\epsilon).$$

Here, we note that [A2] is satisfied in this case because of the boundedness assumptions of $r(\cdot)$ and $\theta(\cdot)$.

Next, we will show the following expansions:

$$g_r^\epsilon \equiv \int_t^T \partial r(u) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u, \qquad g_\theta^{\alpha,\epsilon} \equiv \int_t^T \partial\theta_\alpha(u) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u),$$
$$g_{\theta^2}^{\alpha,\epsilon} \equiv \int_t^T \theta_\alpha(u)\partial\theta_\alpha(u) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u, \tag{26}$$

which appear in the second term of Equation (24).

LEMMA 4. *The asymptotic expansion of* g_r^ϵ, $g_\theta^{\alpha,\epsilon}$, *and* $g_{\theta^2}^{\alpha,\epsilon}$ *defined in* (26) *upto the* ϵ^2*-order are obtained as follows*:

$$g_r^\epsilon = \epsilon g_r^{[1]} + \tfrac{1}{2}\epsilon^2 g_r^{[2]} + o(\epsilon^2), \qquad g_\theta^{\alpha,\epsilon} = \epsilon g_\theta^{\alpha,[1]} + \tfrac{1}{2}\epsilon^2 g_\theta^{\alpha,[2]} + o(\epsilon^2),$$
$$g_{\theta^2}^{\alpha,\epsilon} = \epsilon g_{\theta^2}^{\alpha,[1]} + \tfrac{1}{2}\epsilon^2 g_{\theta^2}^{\alpha,[2]} + o(\epsilon^2), \tag{27}$$

where the coefficients of ϵ*-order*, *that is*, $g_r^{[1]}$, $g_\theta^{\alpha,[1]}$, *and* $g_{\theta^2}^{\alpha,[1]}$ *are given by*

$$g_r^{[1]} = \int_t^T \partial r^{[0]}(u) Y_{t,u} \partial_\epsilon V(x,0)\,\mathrm{d}u,$$
$$g_\theta^{\alpha,[1]} = \int_t^T \partial\theta_\alpha^{[0]}(u) Y_{t,u} \partial_\epsilon V(x,0)\,\mathrm{d}w^\alpha(u),$$
$$g_{\theta^2}^{\alpha,[1]} = \int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u) Y_{t,u} \partial_\epsilon V(x,0)\,\mathrm{d}u, \tag{28}$$

and the coefficients of ϵ^2*-order*, *that is*, $g_r^{[2]}$, $g_\theta^{\alpha,[2]}$, *and* $g_{\theta^2}^{\alpha,[2]}$ *are given by*

$$g_r^{[2]} = 2\left(\int_t^T \partial^2 r^{[0]}(u)[D(t;u)] Y_{t,u}\,\mathrm{d}u + \int_t^T \partial r^{[0]}(u) Y_{t,u}^{[1]}\,\mathrm{d}u\right)\partial_\epsilon V(x,0) +$$
$$+ \left(\int_t^T \partial r^{[0]}(u) Y_{t,u}\,\mathrm{d}u\right)\partial_\epsilon^2 V(x,0),$$
$$g_\theta^{\alpha,[2]} = 2\left(\int_t^T \partial^2\theta_\alpha^{[0]}(u)[D(t;u)] Y_{t,u}\,\mathrm{d}w^\alpha(u) + \right.$$
$$\left. + \int_t^T \partial\theta_\alpha^{[0]}(u) Y_{t,u}^{[1]}\,\mathrm{d}w^\alpha(u)\right)\partial_\epsilon V(x,0) +$$
$$+ \left(\int_t^T \partial\theta_\alpha^{[0]}(u) Y_{t,u}\,\mathrm{d}w^\alpha(u)\right)\partial_\epsilon^2 V(x,0),$$

$$g_{\theta^2}^{\alpha,[2]} = 2\Biggl(\int_t^T \Bigl\{ (\partial \theta_\alpha^{[0]}(u))^2 + \theta_\alpha^{[0]}(u) \partial^2 \theta_\alpha^{[0]}(u) \Bigr\} [D(t;u)] Y_{t,u} \, \mathrm{d}u + \\ + \int_t^T \theta_\alpha^{[0]}(u) \partial \theta_\alpha^{[0]}(u) Y_{t,u}^{[1]} \, \mathrm{d}u \Biggr) \partial_\epsilon V(x,0) + \\ + \left(\int_t^T \theta_\alpha^{[0]}(u) \partial \theta_\alpha^{[0]}(u) Y_{t,u} \, \mathrm{d}u \right) \partial_\epsilon^2 V(x,0). \tag{29}$$

We use the following notations in (29):

$$(\partial \theta_\alpha^{[0]}(u))^2 [D(t;u)] Y_{t,u} \equiv \sum_{i=1}^d \sum_{j=1}^d (\partial_i \theta_\alpha^{[0]}(u))(\partial_j \theta_\alpha^{[0]}(u)) \{D^{(j)}(t;u)\} Y_{t,u}^{(i,\cdot)},$$

$$\partial^2 \theta_\alpha^{[0]}(u) [D(t;u)] Y_{t,u} \equiv \sum_{i=1}^d \sum_{j=1}^d \partial_i \partial_j \theta_\alpha^{[0]}(u) \{D^{(j)}(t;u)\} Y_{t,u}^{(i,\cdot)}.$$

Proof. See appendix. □

4.3. THE SECOND-ORDER SCHEME

Finally, we will explicitly derive an asymptotic expansion of the optimal portfolio upto the ϵ-order. We will also show the third-order scheme, that is an asymptotic expansion formula upto the ϵ^2-order in appendix.

We first define $E(1)$ as

$$E(1) \equiv \mathbf{E}\Biggl[(H_{0,t,T})^{-\delta/(1-\delta)} \times \Biggl(\int_t^T \partial r(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon) \, \mathrm{d}u + \\ + \sum_{\alpha=1}^r \int_t^T \partial \theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon) \, \mathrm{d}w^\alpha(u) + \\ + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon) \partial \theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon) \, \mathrm{d}u \Biggr) \Biggr].$$

Then, we have the following lemma.

LEMMA 5. *$E(1)$'s expansion upto the ϵ-order is given by*

$$E(1) = \mathrm{e}^{\delta/(1-\delta) \int_t^T r^{[0]}(u) \, \mathrm{d}u} \, \mathrm{e}^{[\delta/2(1-\delta)^2] \int_t^T |\theta^{[0]}(u)|^2 \, \mathrm{d}u} \epsilon \Biggl(\int_t^T \partial r^{[0]}(u) Y_{t,u} \, \mathrm{d}u + \\ + \frac{1}{1-\delta} \sum_{\alpha=1}^r \int_t^T \theta_\alpha^{[0]}(u) \partial \theta_\alpha^{[0]}(u) Y_{t,u} \, \mathrm{d}u \Biggr) \partial_\epsilon V(x,0) + o(\epsilon). \tag{30}$$

Proof. See appendix. □

Then, using this result, we have the asymptotic expansion upto the ϵ-order of E defined by Equation (25).

LEMMA 6. *E's expansion upto the ϵ-order is given by*

$$E = \epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u + \frac{1}{1-\delta}\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right)\times \times \partial_\epsilon V(x,0) + o(\epsilon). \tag{31}$$

Proof. See appendix. □

Finally, we have the following asymptotic expansion scheme of optimal portfolios for power utilities.

THEOREM 2. *Under assumptions* [$A1$] *and* [$A2$], *an asymptotic expansion upto ϵ-order of the optimal portfolio for the power utility function* (22) *is given by*

$$\pi^*(t)/W = \frac{1}{1-\delta}\theta^*(x)\sigma^{-1}(x) + \frac{\delta}{1-\delta}\epsilon\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u + \frac{1}{1-\delta}\times \times \sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right)\partial_\epsilon V(x,0)\sigma^{-1}(x) + o(\epsilon). \tag{32}$$

Proof. Applying Lemma 6 to Equation (24) yields the result. □

Comparing to Equation (24) which represents the optimal portfolio for power utility functions, we easily see the second term on the right-hand side in (32) provides an approximation of *the hedging demand* that is specific to multiperiod setting, and that appears on the second term on the right-hand side in Equation (24).

5. General Case

In this section, we will derive an approximation formula for more general class of utility functions where the Malliavin calculus will be applied. That is, we will consider an asymptotic expansion scheme of Equation (11). First, we remark that the similar argument as in the power utility functions can be applied when $\mathbf{T} \equiv \phi'$ is smooth and for $n \in \mathbf{Z}_+$,

$$|\mathbf{T}^{(n)}(y)| \leqslant K_n(y^{\alpha_n} + y^{-\beta_n}), \quad 0 < y < \infty,$$

where $\mathbf{T}^{(n)}(y) \equiv \partial^n\mathbf{T}/\partial y^n$, and K_n, α_n and β_n are positive constants.

Thus, in this section we will concentrate on an asymptotic expansion scheme when $\mathbf{T} \equiv \phi'$ is nonsmooth. We also notice that in practical computation of optimal portfolios for the smooth case we may apply the formula (48) in Theorem 3. We first consider the functional $\zeta^{\epsilon}_{t,u}$ defined in (18) which is equivalent to $H_{0,t,u}$ in Equation (11). That is,

$$\zeta^{\epsilon}_{t,u} \equiv H_{0,t,u}, \text{ where}$$
$$a_0(X^{\epsilon}_u) \equiv -\tfrac{1}{2}|\theta(X^{\epsilon}_u)|^2 - r(X^{\epsilon}_u), \qquad a(X^{\epsilon}_u) \equiv -\theta(X^{\epsilon}_u)^*.$$

We put the following nondegeneracy condition on a:

[$A3$] For some $s \in [t, T]$, $a(X^0_s) \neq 0$.

Then the Malliavin covariance of $\zeta^{\epsilon}_{t,T}$ is given by the formula:

$$\sigma_{\zeta^{\epsilon}_{t,T}} = (\zeta^{\epsilon}_{t,T})^2 \int_t^T |\eta^{\epsilon}(u)|^2\, \mathrm{d}s,$$

where the r-dimensional process η^{ϵ} is expressed in:

$$\eta^{\epsilon}(u) = a(X^{\epsilon}_u) + \sum_{\alpha=0}^{r} \int_u^T \partial_x a_\alpha(X^{\epsilon}_s) Y^{\epsilon}_{t,s} (Y^{\epsilon}_{t,u})^{-1} V(X^{\epsilon}_u, \epsilon)\, \mathrm{d}w^{\alpha}_s$$

with $a = (a_\alpha)^r_{\alpha=1}$. Take a smooth function φ: $\mathbf{R} \to [0, 1]$ such that $\varphi(x) = 1$ if $|x| \leqslant 1/2$ and that $\varphi(x) = 0$ if $|x| \geqslant 1$. Define ξ^{ϵ} by

$$\xi^{\epsilon} = 4|\zeta^{\epsilon}_{t,T}/\zeta^{0}_{t,T} - 1|^2 + 4\left(\int_t^T |a(X^0_u)|^2\, \mathrm{d}u\right)^{-1} \int_t^T |\eta^{\epsilon}(u) - a(X^0_u)|^2\, \mathrm{d}u$$

and let $\psi^{\epsilon} = \varphi(\xi^{\epsilon})$. Obviously, $\psi^{\epsilon} \in \mathbf{D}^{\infty}$. The exponent of $\zeta^0_{t,T}$ is a Gaussian random variable, therefore [$A3$] yields the nondegeneracy of $\zeta^0_{t,T}$:

$$\sigma^{-1}_{\zeta^0_{t,T}} \in \cap_{p>1} L^p.$$

By using this fact and [$A1$], it is not difficult to show that

$$\sup_{\epsilon \in (0,1]} \|\psi^{\epsilon} \sigma^{-p}_{\zeta^{\epsilon}_{t,T}}\|_1 < \infty$$

for any $p > 1$, and that

$$\limsup_{\epsilon \downarrow 0} \epsilon^{-n} P[\psi^{\epsilon} < 1] < \infty$$

for any $n \in \mathbf{N}$. In fact, the first inequality is trivial by the definition of η^{ϵ}. The second inequality follows from the L^p-estimate $\| \sup_{u \in [t,T]} |X^{\epsilon}_u - X^0_u| \|_p = O(\epsilon)$ and [$A2$] with Taylor's formula.

We will in the sequel assume the boundedness of a_α ($\alpha = 0, 1, \ldots, r$). It will be sufficient for our use. Clearly, Condition [$A2$] is satisfied, in fact, $(\zeta^{\epsilon}_{t,T})^s$ is integrable for any $s \in \mathbf{R}$. We will consider the pull-back of $\zeta^{\epsilon}_{t,T}$ by a function $\mathbf{T}$. Under nondegeneracy of $\zeta^{\epsilon}_{t,T}$ the composition $\mathbf{T}(\zeta^{\epsilon}_{t,T})$ with a (roughly speaking) locally integrable function $\mathbf{T}$ is well-defined in $\tilde{\mathbf{D}}^{-\infty}$ (under truncation, if necessary), and the stochastic expansion of $\mathbf{T}(\zeta_{t,T}^{\epsilon})$ is valid. However, for practical purpose, we need to extend the class of functions $\mathbf{T}$ below. It would be more natural to consider Schwartz distributions on $(0, \infty)$ but we will not pursue it here. It requires global modification of the theory since it is necessary to replace spaces of $\mathbf{T}$ and to prepare another smoothing operator A^{-1} and estimates. The modification in our discussion is like the *finite part* of $x_+^{-\lambda}$ $(\lambda > 1)$ in the sense that it removes the difficulty of the lack of the local integrability around zero.

Let $\mathbf{T}$ is a measurable function such that

$$|T(y)| \leqslant K(y^{\alpha} + y^{-\beta}) \text{ for some } K > 0,\ \alpha > 0 \text{ and } \beta \in (0, 1). \tag{33}$$

We will consider utility functions of which ϕ' belong to class of $\mathbf{T}$. The function $\mathbf{T}$ may not be a Schwartz distribution but the composite function $\mathbf{T}(\zeta_{t,T}^{\epsilon})$ is of course well-defined in $\tilde{\mathbf{D}}^{-\infty}$:

$$\psi^{\epsilon}\mathbf{T}(\zeta_{t,T}^{\epsilon}) \text{ in } \tilde{\mathbf{D}}^{-\infty}. \tag{34}$$

It should be noted that $(\zeta_{t,T}^{\epsilon})$ is nondegenerate under truncation by ψ^{ϵ}. Let $\hat{g}^{\alpha,\epsilon}$ be a family of Wiener functionals admitting a stochastic expansion:

$$\hat{g}^{\alpha,\epsilon} \sim \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} \hat{g}^{\alpha,[j]}$$

in $\mathbf{D}^{\infty}$ as $\epsilon \downarrow 0$. Then one obtains the stochastic expansion:

$$\hat{g}^{\alpha,\epsilon}\psi^{\epsilon}T(\zeta_{t,T}^{\epsilon}) \sim \sum_{j=0}^{\infty} \epsilon^j \tilde{\Phi}_j^{\alpha}[\hat{g}] \tag{35}$$

in $\tilde{D}^{-\infty}$ as $\epsilon \downarrow 0$. Here the coefficients $\tilde{\Phi}_j^{\alpha}[\hat{g}] \in \tilde{D}^{-\infty}$ are determined by the formal Taylor expansion of the left-hand side after removing ψ^{ϵ}. Expectation of (35) yields the ordinary asymptotic expansion:

$$\begin{aligned} \langle \hat{g}^{\alpha,\epsilon}\psi^{\epsilon}T(\zeta_{t,T}^{\epsilon}), 1\rangle &\sim \sum_{j=0}^{\infty} \epsilon^j \langle \tilde{\Phi}_j^{\alpha}[\hat{g}], 1\rangle \\ &\sim \sum_{j=0}^{\infty} \epsilon^j \int_{-\infty}^{\infty} \mathbf{T}(z) p_j(z; \hat{g})\, \mathrm{d}z, \end{aligned} \tag{36}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized expectation, and $p_j(z; \hat{g})$ are integrable smooth functions, which can be described as the transform of the derivatives of the density of $\zeta_{t,T}^0$ multiplied by the conditional expectation of certain smooth functionals given $\zeta_{t,T}^0$. On the other hand, if $\mathbf{T} \in \mathcal{S}(\mathbf{R})$, that is $\mathbf{T}$ belongs to the Schwartz space of rapidly decreasing C^{∞} functions on $\mathbf{R}$, then with no problem we obtain

$$\hat{g}^{\alpha,\epsilon}\psi^{\epsilon}\mathbf{T}(\zeta_{t,T}^{\epsilon}) \sim \sum_{j=0}^{\infty} \epsilon^j \Phi_j^{\alpha}[\hat{g}] \tag{37}$$

and hence

$$\langle \hat{g}^{\alpha,\epsilon}\psi^{\epsilon}\mathbf{T}(\zeta^{\epsilon}_{t,T}), 1\rangle \sim \sum_{j=0}^{\infty}\epsilon^{j}\int_{-\infty}^{\infty}\mathbf{T}(z)\tilde{p}_{j}(z;\hat{g})\,\mathrm{d}z \tag{38}$$

for some smooth functions $\tilde{p}_j(z;\hat{g})$. Clearly, the expansion (36) coincides with the expansion (38) for $\mathbf{T}\in\mathcal{S}(\mathbf{R})$, and hence $p_j(z;\hat{g})=\tilde{p}_j(z;\hat{g})$.

Summarizing,

$$\langle \hat{g}^{\alpha,\epsilon}\psi^{\epsilon}\mathbf{T}(\zeta^{\epsilon}_{t,T}), 1\rangle \sim \sum_{j=0}^{\infty}\epsilon^{j}\int_{-\infty}^{\infty}\mathbf{T}(z)p_{j}(z;\hat{g})\,\mathrm{d}z \tag{39}$$

as $\epsilon\downarrow 0$, for measurable functions $\mathbf{T}$ admitting the representation $\mathbf{T}(y)=\tilde{T}(y^{\gamma})$, where $p_j(z;\hat{g})$ are determined as follows. By the formal Taylor expansion

$$\left(\sum_{j=0}^{\infty}\frac{\epsilon^{j}}{j!}\hat{g}^{\alpha,[j]}\right)\mathbf{T}\left(\zeta^{0}_{t,T}+\sum_{j=1}^{\infty}\frac{\epsilon^{j}}{j!}\zeta^{[j]}_{t,T}\right)=\sum_{j=0}^{\infty}\epsilon^{j}\Phi^{\alpha}_{j}[\hat{g}],$$

one has an expression of $\Phi^{\alpha}_{j}[\hat{g}]$:

$$\Phi^{\alpha}_{j}[\hat{g}]=\sum_{k=0}^{j}J_k\partial^{k}\mathbf{T}(\zeta^{0}_{t,T}),$$

for some functionals J_k, $k=0,\ldots,j$. Then $p_j(z;\hat{g})$ is given by

$$p_j(z;\hat{g})=\sum_{k=0}^{j}(-\partial_z)^{k}\{E[J_k|\zeta^{0}_{t,T}=z]p^{\zeta^{0}_{t,T}}(z)\},$$

where $p^{\zeta^{0}_{t,T}}(z)$ denotes the density function of $\zeta^{0}_{t,T}$.

Next, we will explicitly derive the asymptotic expansion for the general case based on the previous validity argument. We first note that we are able to express $\mathcal{Y}$ in terms of W and x through a function $\mathcal{Y}=\mathcal{Y}(W,x)$ because $W=\mathbf{E}[H_{0,t,T}I(\mathcal{Y}H_{0,t,T})]$ and $I(\cdot)$ is strictly decreasing. Hereafter, $\mathcal{Y}$ is regarded as a constant since we will derive a general formula given W, x and a function $\mathcal{Y}(\cdot,\cdot)$. Once a concrete utility function is determined, $\mathcal{Y}(W,x)$ can be also evaluated by an asymptotic expansion even if explicit evaluation is difficult. See an example of a power utility function in the previous section.

We need to evaluate the following terms in Equation (11):

$$E(2)\equiv\mathbf{E}[H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})], \tag{40}$$

$$\begin{aligned}E(3)\;\equiv\;\mathbf{E}\bigg[&H_{0,t,T}\phi'(\mathcal{Y}H_{0,t,T})\bigg(\int_t^T\partial r(X^{\epsilon}_u)Y^{\epsilon}_{t,u}V(x,\epsilon)\,\mathrm{d}u\,+\\&+\sum_{\alpha=1}^{r}\int_t^T\partial\theta_\alpha(X^{\epsilon}_u)Y^{\epsilon}_{t,u}V(x,\epsilon)\,\mathrm{d}w^{\alpha}(u)\,+\\&+\sum_{\alpha=1}^{r}\int_t^T\theta_\alpha(X^{\epsilon}_u)\partial\theta_\alpha(X^{\epsilon}_u)Y^{\epsilon}_{t,u}V(x,\epsilon)\,\mathrm{d}u\bigg)\bigg].\end{aligned} \tag{41}$$

We note that the validity of the expansions for $E(2)$ and $E(3)$ are guaranteed by the discussions on $\mathbf{T}(\zeta^{\epsilon}_{t,T})\hat{g}^{\alpha,\epsilon}$ if we suppose that ϕ' belongs to class of $\mathbf{T}$. Moreover, we remark that $E(2)$ exhibits the part to be evaluated in *the mean-variance* component, and $E(3)$ represents *the hedging demand*.

We start with the stochastic expansion of $H_{0,t,T}$ around $\epsilon = 0$.

LEMMA 7. *The asymptotic expansion of $H_{0,t,T}$ upto ϵ^2-order is given by*

$$H_{0,t,T} = H^{[0]}_{0,t,T}[1 + \epsilon H^{[1]}_{0,t,T} + \tfrac{1}{2}\epsilon^2 H^{[2]}_{0,t,T}] + o(\epsilon^2),$$

where

$$\begin{aligned} H^{[0]}_{0,t,T} &= \exp\left(-\int_t^T \theta^{[0]}(u)^* \,\mathrm{d}w(u) - \frac{1}{2}\int_t^T |\theta^{[0]}(u)|^2 \,\mathrm{d}u - \right.\\ &\quad \left. - \int_t^T r^{[0]}(u)\,\mathrm{d}u \right),\\ H^{[1]}_{0,t,T} &= R_1 + \Theta_{21} + \Theta_1, \qquad H^{[2]}_{0,t,T} = (H^{[1]}_{0,t,T})^2 + R_2 + \Theta_{22} + \Theta_2. \end{aligned}$$

Here we define R_1, Θ_{21}, Θ_1, R_2, Θ_{22} and Θ_2 as follows:

$$\begin{aligned} R_1 &\equiv -\int_t^T \partial r^{[0]}(u) D(t;u)\,\mathrm{d}u,\\ \Theta_{21} &\equiv -\sum_{\alpha=1}^{r} \int_t^T \theta^{[0]}_{\alpha}(u)\partial\theta^{[0]}_{\alpha}(u) D(t;u)\,\mathrm{d}u,\\ \Theta_1 &\equiv -\sum_{\alpha=1}^{r} \int_t^T \partial\theta^{[0]}_{\alpha}(u) D(t;u)\,\mathrm{d}w^{\alpha}(u),\\ R_2 &\equiv -\int_t^T \{\partial^2 r^{[0]}(u)[D(t;u), D(t;u)] + \partial r^{[0]}(u) E(t;u)\}\,\mathrm{d}u,\\ \Theta_{22} &\equiv -\sum_{r=1}^{\alpha} \int_t^T \{\partial\theta_{\alpha}(u) D(t;u)\}^2 \,\mathrm{d}u - \sum_{r=1}^{\alpha} \int_t^T \theta^{[0]}_{\alpha}(u)\{\partial^2\theta^{[0]}_{\alpha}(u) \times\\ &\qquad \times\ [D(t;u), D(t;u)] + \partial\theta_{\alpha}(u) E(t;u)\}\,\mathrm{d}u,\\ \Theta_2 &\equiv -\int_t^T \{\partial^2\theta^{[0]}(u)[D(t;u), D(t;u)] + \partial\theta^{[0]}(u) E(t;u)\}\,\mathrm{d}u. \end{aligned}$$

Proof. Set $\zeta^{\epsilon}_{t,u} \equiv H_{0,t,T}$, and apply (19) in Lemma 2. □

Next, we will explicitly derive the expansions of $E(2)$ and $E(3)$ upto the ϵ-order. We provide the following lemma for the expansion of $E(2)$.

LEMMA 8. *The asymptotic expansion of* $E(2)$ *defined by* (40) *upto the* ϵ*-order is given by*

$$\begin{aligned} E(2) &= \mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\left(\int_{-\infty}^{\infty}\phi'(\xi^{(1)}_{t,T,\mathcal{Y}}\mathrm{e}^z)p(z)\,\mathrm{d}z\right)+\\ &\quad+\epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\left[\left(\int_{-\infty}^{\infty}(c_{12}z^2+c_{11}z+c_{10})\phi'(\xi^{(1)}_{t,T,\mathcal{Y}}\mathrm{e}^z)p(z)\,\mathrm{d}z\right)-\right.\\ &\quad-\mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\left(\int_{-\infty}^{\infty}\phi'(\xi^{(2)}_{t,T,\mathcal{Y}}\mathrm{e}^z)\partial_z\{(c_{22}z^2+c_{21}z+c_{20})p(z)\}\times\right.\\ &\quad\left.\left.\times\,\mathrm{d}z\right)\right]+o(\epsilon), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \xi^{(1)}_{t,T,\mathcal{Y}} &\equiv \mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\,\mathrm{e}^{(1/2)\int_t^T|\theta^{[0]}(u)|^2\,\mathrm{d}u},\\ \xi^{(2)}_{t,T,\mathcal{Y}} &\equiv \mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\,\mathrm{e}^{(3/2)\int_t^T|\theta^{[0]}(u)|^2\,\mathrm{d}u}, \end{aligned} \tag{43}$$

and

$$p(z)\equiv\frac{1}{\sqrt{2\pi\Sigma_z}}\mathrm{e}^{-z^2/2\Sigma_z},\qquad \Sigma_z\equiv\int_t^T|\theta^{[0]}(u)|^2\,\mathrm{d}u. \tag{44}$$

Moreover, c_{i0}, c_{i1} *and* c_{i2} *for* $i=1,2$ *are defined as follows*:

$$\begin{aligned} c_{i0} &\equiv i\sum_{\alpha=1}^r\int_t^T\theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha\tilde{D}_i(t;u)\,\mathrm{d}u-\int_t^T\partial r^{[0]}(u)\tilde{D}_i(t;u)\,\mathrm{d}u-\\ &\quad-\sum_{\alpha=1}^r\int_t^T\theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha\tilde{D}_i(t;u)\,\mathrm{d}u+\frac{1}{\Sigma_z}\left\{\sum_{\alpha=1}^r\int_t^T\theta_\alpha(u)\partial\theta_\alpha(u)Y_{t,u}\times\right.\\ &\quad\left.\times\left(\int_t^u Y^{-1}_{t,u}\sum_{\alpha=1}^r\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s\right)\mathrm{d}u\right\},\\ c_{i1} &\equiv \frac{1}{\Sigma_z}\sum_{\alpha=1}^r\int_t^T\theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha Y_{t,u}\int_t^u Y^{-1}_{t,s}\partial_\epsilon V_{0s}\,\mathrm{d}s\,\mathrm{d}u+\frac{1}{\Sigma_z}\int_t^T\partial r^{[0]}(u)Y_{t,u}\times\\ &\quad\times\int_t^u Y^{-1}_{t,s}\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s\,\mathrm{d}u+\frac{1}{\Sigma_z}\sum_{\alpha=1}^r\int_t^T\theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha Y_{t,u}\times\\ &\quad\times\int_t^u Y^{-1}_{t,s}\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s\,\mathrm{d}u,\\ c_{i2} &\equiv -\frac{1}{\Sigma_z^2}\left\{\sum_{\alpha=1}^r\int_t^T\theta_\alpha(u)\partial\theta_\alpha(u)Y_{t,u}\left(\int_t^u Y^{-1}_{t,u}\sum_{\alpha=1}^r\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s\right)\mathrm{d}u\right\}, \end{aligned} \tag{45}$$

where

$$\tilde{D}_i(t;u) \equiv Y_{t,u}\int_t^u Y_{t,s}^{-1}\partial_\epsilon V_{0s}\,\mathrm{d}s - \mathrm{i}Y_{t,u}\int_t^u Y_{t,s}^{-1}\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s. \tag{46}$$

Proof. See appendix. □

In the similar manner, we obtain the expansion of $E(3)$.

LEMMA 9. *The asymptotic expansion of* $E(3)$ *defined by* (41) *upto the* ϵ*-order is given by*

$$\begin{aligned} E(3) \;=\; & \epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\Bigg[\left(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)p(z)\,\mathrm{d}z\right)\times\\ & \times\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u + \sum_{\alpha=1}^r\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right)+\\ & +\left(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)(c_{31}z+c_{30})p(z)\,\mathrm{d}z\right)\Bigg]\partial_\epsilon V(x,0)+o(\epsilon), \end{aligned}$$

where c_{30} *and* c_{31} *are constants which are defined as follows*:

$$c_{30} \equiv -\sum_{\alpha=1}^r\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u, \qquad c_{31} \equiv \frac{1}{\Sigma_z}c_{30}. \tag{47}$$

Proof. See appendix. □

Finally, we obtain the following theorem.

THEOREM 3. *Suppose that a utility function satisfies the conditions* (4) *and that* ϕ' *belongs to class of* (33). *Then*, *under* [A1], [A2] *and* [A3] *the asymptotic expansion of the optimal portfolio for investment is given by*

$$\begin{aligned} \pi^*(t)\sigma(x) \;=\; & \Bigg[\Bigg\{W - \mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\left(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)p(z)\mathrm{d}z\right)\Bigg\} -\\ & -\epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\Bigg\{\Bigg(\int_{-\infty}^{\infty}(c_{12}z^2+c_{11}z+c_{10})\phi'\times\\ & \times(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)p(z)\,\mathrm{d}z\Bigg) - \mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\Bigg(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(2)}\mathrm{e}^z)\partial_z\times\\ & \times\{(c_{22}z^2+c_{21}z+c_{20})p(z)\}\,\mathrm{d}z\Bigg)\Bigg\}\Bigg]\theta^*(x) -\\ & -\epsilon\,\mathrm{e}^{-\int_t^u r^{[0]}(u)\,\mathrm{d}u}\Bigg[\left(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)p(z)\,\mathrm{d}z\right)\times \end{aligned}$$

$$\times\left(\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u + \sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right) +$$
$$+\left(\int_{-\infty}^{\infty}\phi'(\xi_{t,T,\mathcal{Y}}^{(1)}\mathrm{e}^z)(c_{31}z+c_{30})p(z)\,\mathrm{d}z\right)\Bigg]\partial_\epsilon V(x,0)+o(\epsilon), \tag{48}$$

where $\xi_{t,T,\mathcal{Y}}^{(i)}$, $i = 1, 2$ *are defined by* (43), $p(z)$ *is defined by* (44), c_{i0}, c_{i1} *and* c_{i2} *for* $i = 1, 2$ *are defined by* (45) *and by* (46), *and* c_{30} *and* c_{31} *are defined by* (47).

Proof. Applying Lemmas 8 and 9, we obtain the result. □

We notice that the term multiplied by $\theta^*(x)$ approximates *the mean-variance* component of the optimal portfolio while the term multiplied by $\partial_\epsilon V(x, 0)$ provides an approximation of *the hedging demand*.

In the above theorems, we used the boundedness assumption of θ. On the other hand, it is possible to relax such a boundedness condition due to the following localizing arguments. Let us consider the computation of the expectation $\mathbf{E}[\hat{g}^{\alpha,\epsilon}\mathbf{T}(\zeta_{t,T}^\epsilon)]$ discussed before. We start with θ which is smooth but not necessarily bounded. Given a large number M such that $\sup_{t\in[0,T]}|X_t^0| < M$, we choose smooth modifications $\tilde{\theta}$ so that $\tilde{\theta}$ is bounded, and that $\tilde{\theta} = \theta$ over the region $\{x\colon |x| < M\}$. We also define $\tilde{\zeta}_{t,T}^\epsilon$ with those modifications. Condition $[A3]$ for the modified functions is the same as that for the original functions. So under $[A3]$ for the original functions, we already have the asymptotic expansion of $\mathbf{E}[\hat{g}^{\alpha,\epsilon}\mathbf{T}(\tilde{\zeta}_{t,T}^\epsilon)]$. On the other hand,

$$\begin{aligned}
&|\mathbf{E}[\hat{g}^{\alpha,\epsilon}\mathbf{T}(\tilde{\zeta}_{t,T}^\epsilon)] - \mathbf{E}[\hat{g}^{\alpha,\epsilon}\mathbf{T}(\zeta_{t,T}^\epsilon)]| \\
&\quad = \mathbf{E}[|\hat{g}^{\alpha,\epsilon}||\mathbf{T}(\tilde{\zeta}_{t,T}^\epsilon) - \mathbf{T}(\zeta_{t,T}^\epsilon)|1_{\{\sup_{t\in[0,T]}|X_t^\epsilon|>M\}}] \\
&\quad \leqslant \sup_\epsilon \|\hat{g}^{\alpha,\epsilon}\|_{p_1}\sup_\epsilon\{\|\mathbf{T}(\tilde{\zeta}_{t,T}^\epsilon)\|_{p_2} + \|\mathbf{T}(\zeta_{t,T}^\epsilon)\|_{p_3}\}P\left[\sup_{t\in[0,T]}|X_t^\epsilon| > M\right]^{1/p_3},
\end{aligned}$$

where $p_1, p_2, p_3 \in (1,\infty)$ $(p_1^{-1} + p_2^{-1} + p_3^{-1} = 1)$. Thus, under the finiteness of the second factor on the right-hand side, we will obtain an asymptotic expansion of $\mathbf{E}[\hat{g}^{\alpha,\epsilon}\mathbf{T}(\zeta_{t,T}^\epsilon)]$.

6. A Numerical Example

In this section, we will illustrate our method through a numerical example. In particular, following Theorem 2, we will provide *analytic* approximations of optimal portfolios for an example investigated by Detemple et al. (2000) which relies on naive Monte Carlo simulations as numerical technique. First, we divide the optimal portfolio for a power utility function in Equation (24) into *the mean variance*, *the interest rate hedge* (IR-hedge) and *the market price of risk hedge* (MPR-hedge) components defined as follows:

$$\text{mean variance} \equiv \frac{1}{1-\delta}\theta(x)^*\sigma^{-1}(x),$$

$$\text{IR-hedge} \equiv \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}]}\mathbf{E}\bigg[(H_{0,t,T})^{-\delta/(1-\delta)} \times \\ \times \int_t^T \partial r(X_u^\epsilon)Y_{t,u}^\epsilon \,\mathrm{d}u\bigg]V(x,\epsilon)\sigma^{-1}(x),$$

$$\text{MPR-hedge} \equiv \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}]}\mathbf{E}\bigg[(H_{0,t,T})^{-\delta/(1-\delta)} \times \\ \times \bigg(\sum_{\alpha=1}^{r}\int_t^T \partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon \,\mathrm{d}w^\alpha(u) + \\ + \sum_{\alpha=1}^{r}\int_t^T +\theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon)Y_{t,u}^\epsilon \,\mathrm{d}u\bigg)\bigg]V(x,\epsilon)\sigma^{-1}(x).$$

We remark that *hedging demand* is further divided into *IR-hedge* and *MPR-hedge* components above. Next, the corresponding components for the asymptotic expansion in Theorem 2 are given as follows:

$$\text{mean variance} \equiv \frac{1}{1-\delta}\theta^*(x)\sigma^{-1}(x),$$

$$\text{IR-hedge} \equiv \epsilon\frac{\delta}{1-\delta}\bigg(\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u\bigg)\partial_\epsilon V(x,0)\sigma^{-1}(x),$$

$$\text{MPR-hedge} \equiv \epsilon\frac{\delta}{(1-\delta)^2}\bigg(\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\bigg)\partial_\epsilon V(x,0)\sigma^{-1}(x).$$

In this example, we suppose that $d = 2$, that is, $X_u^\epsilon = (X_u^{\epsilon(1)}, X_u^{\epsilon(2)})^*$ and that they satisfy the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}X_u^{\epsilon(1)} &= \kappa_1(\bar{X}^{\epsilon(1)} - X_u^{\epsilon(1)})\,\mathrm{d}u - \epsilon(X_u^{\epsilon(1)})^{1/2}\,\mathrm{d}w_u, \quad X_0^{\epsilon(1)} = r_0, \\ \mathrm{d}X_u^{\epsilon(2)} &= \kappa_2(\bar{X}^{\epsilon(2)} - X_u^{\epsilon(2)})\,\mathrm{d}u + \epsilon\sigma_2\,\mathrm{d}w_u, \quad X_0^{\epsilon(2)} = \theta_0, \end{aligned}$$

where w denotes one-dimensional Brownian motion ($r = 1$).[1] We also suppose that there exist one risky asset and a locally riskless asset, and that $\theta_u = X_u^{\epsilon(2)}$ and r_u is a smooth modification of $\min\{X_u^{\epsilon(1)}, M\}$, where M is a positive large number. Then, the dynamics of both assets are described by

$$\begin{aligned} &\mathrm{d}S_u^\epsilon = S_u^\epsilon(X_u^{\epsilon(1)} + \sigma X_u^{\epsilon(2)})\,\mathrm{d}u + S_u^\epsilon\sigma\,\mathrm{d}w_u, \quad S^\epsilon(0) = s, \\ &\mathrm{d}S_{0u}^\epsilon = S_{0u}^\epsilon r(X_u^{\epsilon(1)})\,\mathrm{d}u, \quad S_0^\epsilon(0) = 1. \end{aligned}$$

In appendix, we will discuss the validity of the asymptotic expansions for this setting in detail. Further, we set the values of the parameters for X_u^ϵ following

[1] The volatility function of $X^{\epsilon(1)}$ is not smooth at the origin and we need to use a smoothed version of the square root process at the origin. However, we can show that the smoothing does not make significant differences and the effects are negligible in the *small disturbance asymptotic theory*.

Detemple et al. (2000), which were obtained by statistical estimation; $\kappa_1 = 0.0824$, $r_0 = \bar{X}^{\epsilon(1)} = 0.06$, $\epsilon = 0.03637$, $\kappa_2 = 0.6950$, $\bar{X}^{\epsilon(2)} = 0.0871$, $\sigma_2 = 0.21/0.03637$, $\theta_0 = 0.1$, $\sigma = 0.2$. For comparative purpose, we also compute each component of the optimal portfolios by using Monte Carlo simulations based on the Euler–Maruyama approximation; the discretized time step Δt is $1/365$ and the number of trials is 100 000 in each Monte Carlo simulation.

The percentage-shares in total wealth of mean variance, IR-hedge, MPR-hedge and the total demand which are sum of those three components are listed in Tables I–IV; the results for the asymptotic expansion are listed in Tables I and II while the results for the Monte Carlo simulation are listed in Tables III and IV. In addition, Tables I and III show the results for *investment horizons* $T = 1, 2, 3, 4, 5$ when *the Arrow-Pratt measure of relative risk aversion* $R(\equiv 1 - \delta)$ is fixed at 2, and Tables II and IV show the results for $R = 0.5, 1, 1.5, 4, 5$ when $T = 1$. We remark that *total demand* means the demand for the risky asset and hence it may not be 100% because the remaining shares (100%−*total demand*) are invested into the riskless asset. We also note that it may exceed 100% since selling (borrowing) riskless asset is admitted. We can observe that the results of asymptotic expansion and of

Table I. Asymptotic expansion (%, $R = 2.0$)

T (investment horizon)	1	2	3	4	5
Total demand	25.31	26.41	27.80	29.26	30.70
Mean variance	25.00	25.00	25.00	25.00	25.00
IR-hedge	2.14	4.11	5.92	7.59	9.13
MPR-hedge	−1.83	−2.70	−3.12	−3.33	−3.43

Table II. Asymptotic expansion (%, $T = 1.0$)

$R\ (\equiv 1 - \delta)$	0.5	1	1.5	4	5
Total demand	110.37	50	33.13	14.34	12.25
Mean variance	100.00	50.00	33.33	12.50	10.00
IR-hedge	−4.28	0	1.43	3.21	3.42
MPR-hedge	14.65	0	−1.63	−1.37	−1.17

Table III. Monte Carlo simulation (%, $R = 2.0$)

T (investment horizon)	1	2	3	4	5
Total demand	25.37	26.49	27.79	29.10	30.41
Mean variance	25.00	25.00	25.00	25.00	25.00
IR-hedge	2.14	4.12	5.95	7.63	9.19
MPR-hedge	−1.77	−2.63	−3.16	−3.53	−3.78

Table IV. Monte Carlo simulation (%, $T = 1.0$)

$R\,(\equiv 1-\delta)$	0.5	1	1.5	4	5
Total demand	113.07	50.00	33.18	14.35	12.22
Mean variance	100.00	50.00	33.33	12.50	10.00
IR-hedge	−4.26	0.00	1.43	3.22	3.43
MPR-hedge	17.33	0.00	−1.58	−1.37	−1.22

Monte Carlo simulation are so close for IR-hedge while there is some difference for MPR-hedge, but the difference is small relative to the total demand. We also notice that the second-order scheme gives substantial improvement comparing with the first-order scheme which is equivalent to the case that we ignore the hedging components. (Note that $O(1)$ for MPR-hedge and IR-hedge components are *zero*.) Thus, we have confirmed that our method is not only computational efficient, but also gives sufficient approximations to the evaluation of the optimal portfolios.

Appendix A.

A.1. PROOFS OF LEMMAS

Proof of Lemma 1. Using (13), (14) and solutions of the set of stochastic differential equations (15)–(17), we obtain the result. □

Proof of Lemma 3. We first see that

$$g^{\alpha,0} = 0 \quad (\alpha = 0, 1, \ldots, r) \tag{A.1}$$

since $V(x, 0) \equiv 0$ from $[A1]$. Then, we compute explicitly $g^{\alpha,[j]}$, $j = 1, 2$, and by tedious routine work, we obtain the result. □

Proof of Lemma 4. Replacing $f(\cdot)$ in Equation (20) by $r(\cdot)$, $\theta_\alpha(\cdot)$, $(1/2)\theta_\alpha^2(\cdot)$ and applying the result in Lemma 3, we obtain the expansions of g_r^ϵ, $g_\theta^{\alpha,\epsilon}$, and $g_{\theta^2}^{\alpha,\epsilon}$. □

Proof of Lemma 5. Applying the expansions (27) and (28) in Lemma 4 directly, we obtain

$$\begin{aligned}
&\left(\int_t^T \partial r(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}w^\alpha(u) + \right.\\
&\left. + \sum_{\alpha=1}^r \int_t^T \theta_\alpha(X_u^\epsilon)\partial\theta_\alpha(X_u^\epsilon) Y_{t,u}^\epsilon V(x,\epsilon)\,\mathrm{d}u \right)\\
&= \epsilon\left(g_r^{[1]} + \sum_{\alpha=1}^r g_\theta^{\alpha,[1]} + \sum_{\alpha=1}^r g_{\theta^2}^{\alpha,[1]}\right) + o(\epsilon)
\end{aligned}$$

$$= \epsilon\bigg(\int_t^T \partial r^{[0]}(u) Y_{t,u}\,\mathrm{d}u + \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha^{[0]}(u) Y_{t,u} \times$$
$$\times\, \mathrm{d}w^\alpha(u) + \sum_{\alpha=1}^r \int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u) Y_{t,u}\,\mathrm{d}u\bigg)\partial_\epsilon V(x,0) + o(\epsilon).$$

Moreover, applying the expansion (19) in Lemma 2, we can obtain

$$\begin{aligned}
\zeta_{t,u}^\epsilon \;&\equiv\; (H_{0,t,T})^{-\delta/(1-\delta)} \\
&=\; \mathrm{e}^{\delta/(1-\delta)\int_t^T r^{[0]}(u)\,\mathrm{d}u}\, \mathrm{e}^{\delta/[2(1-\delta)^2]\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u} \times \\
&\quad\times \mathrm{e}^{-(1/2)(\delta/(1-\delta))^2\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u + [\delta/(1-\delta)]\int_t^T \theta^{[0]}(u)\,\mathrm{d}w(u)} \times \\
&\quad\times \bigg(1 + \epsilon\Big(\frac{\delta}{1-\delta}\Big)\int_t^T \partial r^{[0]}(u) D(t;u)\,\mathrm{d}u + \epsilon\Big(\frac{\delta}{1-\delta}\Big) \times \\
&\quad\times \sum_{\alpha=1}^r \int_t^T \partial\theta_\alpha^{[0]}(u) D(t;u)\,\mathrm{d}w^\alpha(u) + \\
&\quad+ \epsilon\Big(\frac{\delta}{1-\delta}\Big)\sum_{\alpha=1}^r \int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u) D(t;u)\,\mathrm{d}u\bigg) + o(\epsilon).
\end{aligned}$$

Then, by change of the measure technique with

$$\begin{aligned}
\hat{P}(A) \;&\equiv\; \mathbf{E}\bigg[1_A \exp\bigg(-\frac{1}{2}\Big(\frac{\delta}{1-\delta}\Big)^2 \int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u + \\
&\quad+ \Big(\frac{\delta}{1-\delta}\Big)\int_t^T \theta^{[0]}(u)\,\mathrm{d}w(u)\bigg)\bigg], \quad A \in \mathcal{F}_T
\end{aligned}$$

simple calculation yields the result. □

Proof of Lemma 6. Using (19) again, we can obtain the expansion:

$$\begin{aligned}
&\mathbf{E}[(H_{0,t,T})^{-\delta/(1-\delta)}] \\
&\quad= \mathrm{e}^{\delta/(1-\delta)\int_t^T r^{[0]}(u)\,\mathrm{d}u}\, \mathrm{e}^{[\delta/2(1-\delta)^2]\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u} \times \\
&\quad\times \bigg(1 + \epsilon\Big(\frac{\delta}{1-\delta}\Big)\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u + \epsilon\frac{\delta}{(1-\delta)^2} \times \\
&\quad\times \sum_{\alpha=1}^r \int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\bigg) + o(\epsilon),
\end{aligned} \tag{A.2}$$

where

$$\hat{D}_1(t;u) \equiv Y_{t,u}\int_t^u Y_{t,s}^{-1}\bigg\{\partial_\epsilon V_0^{[0]}(s)\,\mathrm{d}s + \Big(\frac{\delta}{1-\delta}\Big)\partial_\epsilon V^{[0]}(s)\theta^{[0]}(s)\,\mathrm{d}s\bigg\}.$$

Then, gathering (30) in Lemma 5 and (A.2), we obtain the result. □

Proof of Lemma 8. Note first that the asymptotic expansion of $E(2)$ is expressed as

$$\begin{aligned} E(2) \;=\; & \mathbf{E}[H^{[0]}_{0,t,T}\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] + \epsilon\mathbf{E}[H^{[0]}_{0,t,T}H^{[1]}_{0,t,T}\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] + \\ & + \epsilon\mathcal{Y}\mathbf{E}[(H^{[0]}_{0,t,T})^2 H^{[1]}_{0,t,T}\partial\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] + o(\epsilon). \end{aligned} \tag{A.3}$$

Then, by using Lemma 7, we can easily obtain the expression for the first term of (A.3).

$$\begin{aligned} \mathbf{E}[H^{[0]}_{0,t,T}\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] \;&=\; \mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\mathbf{E}^{(1)}[\phi'(\xi^{(1)}_{t,T,\mathcal{Y}}\mathrm{e}^{-\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_1(u)})] \\ &=\; \mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\bigg(\int_{-\infty}^{\infty}\phi'(\xi^{(1)}_{t,T,\mathcal{Y}}\mathrm{e}^{z})p(z)\,\mathrm{d}z\bigg), \end{aligned}$$

where

$$\xi^{(1)}_{t,T,\mathcal{Y}} = \mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\,\mathrm{e}^{(1/2)\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u},$$
$$z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_1(u),$$

and

$$p(z) = \frac{1}{\sqrt{2\pi\,\Sigma_z}}\,\mathrm{e}^{-z^2/2\Sigma_z}, \qquad \Sigma_z \equiv \int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u.$$

Here, $\mathbf{E}^{(1)}[\cdot]$ denotes the expectation operator under

$$P_1(A) \equiv \mathbf{E}\bigg[1_A \exp\bigg(-\int_t^T \theta^{[0]}(u)^*\,\mathrm{d}w(u) - \frac{1}{2}\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u\bigg)\bigg]$$

for all $A \in \mathcal{F}_T$, and $w_1(u) = w(u) + \int_t^u \theta^{[0]}(s)\,\mathrm{d}s$ denotes the standard Brownian motion under P_1.

We can also obtain expressions for the second and third terms of (A.3) after tedious calculation with Lemma 7 as follows:

$$\begin{aligned} &\epsilon\mathbf{E}[H^{[0]}_{0,t,T}H^{[1]}_{0,t,T}\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] \\ &\quad= \epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\mathbf{E}^{(1)}[(R_1 + \Theta_{21} + \Theta_1)\phi'(\mathcal{Y}H^{[0]}_{0,t,T})] \\ &\quad= \epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\mathbf{E}^{(1)}\bigg[\phi'(\mathcal{Y}H^{[0]}_{0,t,T})\mathbf{E}^{(1)}\bigg[(R_1 + \Theta_{21} + \Theta_1)|z \\ &\quad= -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_1(u)\bigg]\bigg] \\ &\quad= \epsilon\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\bigg(\int_{-\infty}^{\infty}(c_{12}z^2 + c_{11}z + c_{10})\phi'(\xi^{(1)}_{t,T,\mathcal{Y}}\,\mathrm{e}^{z})p(z)\,\mathrm{d}z\bigg), \end{aligned}$$

$$
\begin{aligned}
&\epsilon\mathcal{Y}\mathbf{E}[(H^{[0]}_{0,t,T})^2 H^{[1]}_{0,t,T}\partial\phi'(\mathcal{Y}H^{[0]}_{0,t,T})]\\
&= \epsilon\mathcal{Y}\,\mathrm{e}^{-2\int_t^T r^{[0]}(u)\,\mathrm{d}u}\mathbf{E}^{(2)}[(R_1+\Theta_{21}+\Theta_1)\partial\phi'(\mathcal{Y}H^{[0]}_{0,t,T})]\\
&= \epsilon\mathcal{Y}\,\mathrm{e}^{-2\int_t^T r^{[0]}(u)\,\mathrm{d}u}\mathbf{E}^{(2)}\bigg[\partial\phi'(\mathcal{Y}H^{[0]}_{0,t,T})\mathbf{E}^{(2)}\bigg[(R_1+\Theta_{21}+\Theta_1)|z\\
&= -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_2(u)\bigg]\bigg]\\
&= -\epsilon\mathcal{Y}\,\mathrm{e}^{-2\int_t^T r^{[0]}(u)\,\mathrm{d}u}\bigg(\int_{-\infty}^{\infty}\phi'(\xi^{(2)}_{t,T,\mathcal{Y}}\mathrm{e}^z)\partial_z\{(c_{22}z^2+c_{21}z+c_{20})p(z)\}\,\mathrm{d}z\bigg),
\end{aligned}
$$

where c_{i2}, c_{i1}, c_{i0} for $i = 1, 2$ are some constants which are explicitly given later. Here, we use the notations:

$$
\begin{aligned}
&\xi^{(2)}_{t,T,\mathcal{Y}} = \mathcal{Y}\,\mathrm{e}^{-\int_t^T r^{[0]}(u)\,\mathrm{d}u}\,\mathrm{e}^{(3/2)\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u},\\
&z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_2(u)
\end{aligned}
$$

and

$$
p(z) = \frac{1}{\sqrt{2\pi\,\Sigma_z}}\,\mathrm{e}^{-z^2/2\Sigma_z}, \qquad \Sigma_z \equiv \int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u.
$$

Moreover, $\mathbf{E}^{(2)}[\cdot]$ denotes the expectation operator under

$$
P_2(A) \equiv \mathbf{E}\bigg[1_A \exp\bigg(-2\int_t^T \theta^{[0]}(u)^*\,\mathrm{d}w(u) - 2\int_t^T |\theta^{[0]}(u)|^2\,\mathrm{d}u\bigg)\bigg]
$$

for all $A \in \mathcal{F}_T$, and $w_2(u) = w(u) + 2\int_t^u \theta^{[0]}(s)\,\mathrm{d}s$ denotes the standard Brownian motion under P_2.

Finally, in order to obtain $c_{i,j}$ for $i = 1, 2$ and $j = 0, 1, 2$ explicitly, we will evaluate conditional expectations, $\mathbf{E}^{(i)}\big[(R_1+\Theta_{21}+\Theta_1)|z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_i(u)\big]$ for $i = 1, 2$ by utilizing Gaussianity:

$$
\begin{aligned}
&\mathbf{E}^{(i)}\bigg[\Theta_1|z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_i(u)\bigg]\\
&= i\sum_{\alpha=1}^{r}\int_t^T \theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha\tilde{D}_i(t;u)\,\mathrm{d}u - \bigg(\frac{1}{\Sigma_z^2}z^2 - \frac{1}{\Sigma_z}\bigg)\times\\
&\quad\times\bigg\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha(u)\partial\theta_\alpha(u)Y_{t,u}\bigg(\int_t^u Y^{-1}_{t,u}\sum_{\alpha=1}^{r}\partial_\epsilon V_s\theta^{[0]}(s)\,\mathrm{d}s\bigg)\,\mathrm{d}u\bigg\} +\\
&\quad + \frac{1}{\Sigma_z}z\sum_{\alpha=1}^{r}\int_t^T \theta^{[0]}_\alpha\partial\theta^{[0]}_\alpha Y_{t,u}\int_t^u Y^{-1}_{t,s}\partial_\epsilon V_{0s}\,\mathrm{d}s\,\mathrm{d}u,
\end{aligned}
$$

$$\begin{aligned}
&\mathbf{E}^{(i)}\left[\mathbf{R}_1 | z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_i(u)\right] \\
&\quad = -\int_t^T \partial r^{[0]}(u)\tilde{D}_i(t;u)\,\mathrm{d}u + \frac{1}{\Sigma_z} z \int_t^T \partial r^{[0]}(u) Y_{t,u} \times \\
&\qquad \times \int_t^u Y_{t,s}^{-1} \partial_\epsilon V_s \theta^{[0]}(s)\,\mathrm{d}s\,\mathrm{d}u,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}^{(i)}\left[\Theta_{21} | z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_i(u)\right] \\
&\quad = -\sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]} \partial \theta_\alpha^{[0]} \tilde{D}_i(t;u)\,\mathrm{d}u + \frac{1}{\Sigma_z} z \sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]} \partial \theta_\alpha^{[0]} Y_{t,u} \times \\
&\qquad \times \int_t^u Y_{t,s}^{-1} \partial_\epsilon V_s \theta^{[0]}(s)\,\mathrm{d}s\,\mathrm{d}u,
\end{aligned}$$

where

$$\tilde{D}_i(t;u) \equiv Y_{t,u} \int_t^u Y_{t,s}^{-1} \partial_\epsilon V_{0s}\,\mathrm{d}s - \mathrm{i} Y_{t,u} \int_t^u Y_{t,s}^{-1} \partial_\epsilon V_s \theta^{[0]}(s)\,\mathrm{d}s.$$

Hence, we obtain the expressions of c_{i2}, c_{i1}, c_{i0} for $i = 1, 2$ by (45) and (46). □

Proof of Lemma 9. Using assumption [A1] $V(\cdot, 0) \equiv 0$, and computing conditional expectation of a Gaussian random variable;

$$\mathbf{E}^{(1)}\left[\sum_{\alpha=1}^{r} \int_t^T \partial \theta_\alpha^{[0]}(u) Y_{t,u}\,\mathrm{d}w^\alpha(u) | z = -\int_t^T \theta^{[0]}(u)\,\mathrm{d}w_1(u)\right] = c_{31} z + c_{30},$$

where c_{30} and c_{31} are constants defined by (47), we obtain the result. □

A.2. THE THIRD-ORDER SCHEME FOR A POWER UTILITY FUNCTION

We show the result of the third-order scheme, that is, an asymptotic expansion upto to the ϵ^2-order of the optimal portfolio for a power utility function:

$$\pi^*(t) = \frac{W}{1-\delta}[\theta^*(x) + \delta\{\epsilon A + \epsilon^2 B - \epsilon^2 AC\}]\sigma^{-1}(x) + o(\epsilon^2),$$

where

$$\begin{aligned}
A \equiv &\left(\int_t^T \partial r^{[0]}(u) Y_{t,u}\,\mathrm{d}u + \frac{1}{1-\delta} \times \right. \\
&\left. \times \sum_{\alpha=1}^{r} \int_t^T \theta_\alpha^{[0]}(u) \partial \theta_\alpha^{[0]}(u) Y_{t,u}\,\mathrm{d}u \right) \partial_\epsilon V(x, 0),
\end{aligned}$$

$$C \equiv \left(\frac{\delta}{1-\delta}\right)\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u + \\ + \frac{\delta}{(1-\delta)^2}\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u,$$

and B is the sum of the following terms:

1.
$$\left(\frac{\delta}{1-\delta}\right)\left\{\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\}\left\{\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

2.
$$\left(\frac{\delta}{1-\delta}\right)^2\left\{\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\} \times \\ \times \left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0) + \\ + \left(\frac{\delta}{1-\delta}\right)\left\{\sum_{\alpha=1}^{r}\int_t^T \partial\theta_\alpha^{[0]}(u)Y_{t,u}\left(\int_u^T \partial r^{[0]}(s)Y_{t,s}\,\mathrm{d}s\right) \times \\ \times Y_{t,u}^{-1}\partial_\epsilon V_u^{[0],(\cdot,\alpha)}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

3.
$$\left(\frac{\delta}{1-\delta}\right)\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0) \times \\ \times \left\{\int_t^T \partial r^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\};$$

4.
$$\left(\frac{\delta}{1-\delta}\right)\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\} \times \\ \times \left\{\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

5.
$$\left(\frac{\delta}{1-\delta}\right)^2\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\} \times \\ \times \left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0) + \\ + \left(\frac{\delta}{1-\delta}\right)\sum_{\alpha=1}^{r}\left\{\int_t^T \left(\sum_{\alpha'=1}^{r}\int_u^T \theta_{\alpha'}^{[0]}(s)\partial\theta_{\alpha'}^{[0]}(s)\,\mathrm{d}s\right) \times \\ \times Y_{t,u}^{-1}\partial_\epsilon V_u^{[0],(\cdot,\alpha)}\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

6.

$$\left(\frac{\delta}{1-\delta}\right)\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0)\,\times$$
$$\times\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\};$$

7.

$$\left(\frac{\delta}{1-\delta}\right)^2\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\}\times$$
$$\times\left\{\int_t^T \partial r^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

8.

$$\left(\frac{\delta}{1-\delta}\right)^2\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\}\times$$
$$\times\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

9.

$$\left(\frac{\delta}{1-\delta}\right)^3\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\,\mathrm{d}u\right\}\times$$
$$\times\left\{\sum_{\alpha=1}^{r}\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0)\,+$$
$$+\left(\frac{\delta}{1-\delta}\right)\left\{\sum_{\alpha=1}^{r}\int_t^T \partial\theta_\alpha^{[0]}(u)\hat{D}_1(t;u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0)\,+$$
$$+\left(\frac{\delta}{1-\delta}\right)^2\left\{\sum_{\alpha=1}^{r}\int_t^T\left(\sum_{\alpha'=1}^{r}\int_u^T \theta_{\alpha'}^{[0]}(s)\partial\theta_{\alpha'}^{[0]}(s)\,\mathrm{d}s\right)\times\right.$$
$$\left.\times\,Y_{t,u}^{-1}\partial_\epsilon V_u^{[0],(\cdot,\alpha)}\,\mathrm{d}u\right\}\partial_\epsilon V(x,0);$$

10.

$$\sum_{\alpha=1}^{r}\left(\int_t^T (\partial\theta_\alpha^{[0]}(u))^2[\hat{D}_1(t;u)]Y_{t,u}\,\mathrm{d}u\right)\partial_\epsilon V(x,0)\,+$$
$$+\sum_{\alpha=1}^{r}\left(\frac{1}{1-\delta}\right)\left(\int_t^T \theta_\alpha^{[0]}(u)\partial^2\theta_\alpha^{[0]}(u)[\hat{D}_1(t;u)]Y_{t,u}\,\mathrm{d}u+\right.$$
$$\left.+\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)[\hat{Y}_{t,u}^{[1]}]\,\mathrm{d}u\right)\partial_\epsilon V(x,0)\,+$$
$$+\sum_{\alpha=1}^{r}\left(\frac{1}{1-\delta}\right)\left(\int_t^T \theta_\alpha^{[0]}(u)\partial\theta_\alpha^{[0]}(u)Y_{t,u}\,\mathrm{d}u\right)\partial_\epsilon^2 V(x,0),$$

where

$$\hat{Y}^{[1]}_{t,u} \equiv \int_t^u Y_{t,u}Y^{-1}_{t,s}\bigg\{\partial_x^2 V_{0s}[\hat{D}_1(t;s)]Y_{t,s}\,\mathrm{d}s + \partial_\epsilon\partial_x V_{0s}Y_{t,s}\,\mathrm{d}s + + \left(\frac{\delta}{1-\delta}\right)\sum_{\alpha=1}^{r}\theta_\alpha(s)\partial_\epsilon\partial_x V_{\alpha s}Y_{t,s}\,\mathrm{d}s\bigg\};$$

11.

$$\left(\int_t^T \partial^2 r^{[0]}(u)[\hat{D}_1(t;u)]Y_{t,u}\,\mathrm{d}u + \int_t^T \partial r^{[0]}(u)\hat{Y}^{[1]}_{t,u}\,\mathrm{d}u\right)\partial_\epsilon V(x,0) + + \left(\int_t^T \partial r^{[0]}(u)Y_{t,u}du\right)\partial_\epsilon^2 V(x,0).$$

A.3. THE VALIDITY OF THE ASYMPTOTIC EXPANSIONS IN SECTION 6

We start with the following more or less well-known lemma.

LEMMA 10. *Let* $\theta, \lambda \in \mathbf{R} - \{0\}$. *Suppose that* $(\xi_t^\theta)_{t\in[0,T]}$ *is a linear diffusion process satisfying the stochastic differential equation*:

$$\mathrm{d}\xi_t^\theta = \theta\xi_t^\theta\,\mathrm{d}t + \mathrm{d}w_t, \quad \xi_0^\theta = x.$$

Let $\alpha = (\theta^2 - \lambda^2)/2$.

(1) *If* $x = 0$ *and if*

$$(\theta - \lambda)\left[\frac{\mathrm{e}^{2\lambda T} - 1}{2\lambda}\right] < 1,$$

then

$$\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] = \exp\left(-\frac{\theta-\lambda}{2}T\right)\left(1 - (\theta-\lambda)\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right]\right)^{-1/2} = \exp\left(-\frac{\theta-\lambda}{2}T\right)\left[\frac{2\lambda}{(\lambda-\theta)(\mathrm{e}^{2\lambda T}-1)+2\lambda}\right]^{1/2}. \tag{A.4}$$

(2) *Let x be arbitrary*.

(i) *If* $\theta < 0$, *then for any* $\lambda \in (\theta, -\theta)$,

$$\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \leqslant \exp\left(\frac{\lambda-\theta}{2}(x^2+T)\right).$$

(ii) *If* $\theta > 0$, *then for any* $\lambda \in (-\theta, \theta)$, *and if*

$$(\theta-\lambda)\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right] < \frac{1}{2},$$

$$\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \leqslant \exp\left(-\frac{\theta-\lambda}{2}(x^2+T) + + (\theta-\lambda)(x\,\mathrm{e}^{\lambda T})^2\right)\left[\frac{\lambda}{(\lambda-\theta)(\mathrm{e}^{2\lambda T}-1)+\lambda}\right]^{1/2}. \tag{A.5}$$

Proof. Let $\theta, \lambda \in \mathbf{R}$ (any real numbers). Denote by μ_{ξ^θ} and μ_{ξ^λ} the measures corresponding to the processes ξ_t^θ and ξ_t^λ:

$$\mathrm{d}\xi_t^\theta = \theta\xi_t^\theta\,\mathrm{d}t + \mathrm{d}w_t, \quad \xi_0^\theta = x, \qquad \mathrm{d}\xi_t^\lambda = \lambda\xi_t^\lambda\,\mathrm{d}t + \mathrm{d}w_t, \quad \xi_0^\lambda = x,$$

the measures μ_{ξ^θ} and μ_{ξ^λ} are equivalent and

$$\frac{\mathrm{d}\mu_{\xi^\theta}}{\mathrm{d}\mu_{\xi^\lambda}}(\xi_t^\lambda) = \exp\left\{(\theta-\lambda)\int_0^T \xi_t^\lambda\,\mathrm{d}\xi_t^\lambda - \frac{\theta^2-\lambda^2}{2}\int_0^T (\xi_t^\lambda)^2\,\mathrm{d}t\right\}.$$

Put $C = C([0, T])$. Then

$$\begin{aligned}\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] &= \int_C \exp\left(\alpha\int_0^T (x_t)^2\,\mathrm{d}t\right)\mu_{\xi^\theta}(\mathrm{d}x)\\ &= \int_C \exp\left(\alpha\int_0^T (x_t)^2\,\mathrm{d}t\right)\frac{\mathrm{d}\mu_{\xi^\theta}}{\mathrm{d}\mu_{\xi^\lambda}}(x)\mu_{\xi^\lambda}(\mathrm{d}x)\\ &= \mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\lambda)^2\,\mathrm{d}t\right)\exp\left\{(\theta-\lambda)\times\right.\right.\\ &\qquad \left.\left.\times\int_0^T \xi_t^\lambda\,\mathrm{d}\xi_t^\lambda - \frac{\theta^2-\lambda^2}{2}\int_0^T (\xi_t^\lambda)^2\,\mathrm{d}t\right\}\right].\end{aligned}$$

Let us take

$$\alpha = \frac{\theta^2-\lambda^2}{2}. \tag{A.6}$$

(In particular, if $\alpha \geqslant 0$, then $|\theta| \geqslant |\lambda|$.) Then

$$\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] = \mathbf{E}\left[\exp\left\{(\theta-\lambda)\int_0^T \xi_t^\lambda\,\mathrm{d}\xi_t^\lambda\right\}\right].$$

Using Itô's formula

$$(\xi_T^\lambda)^2 = (\xi_0^\lambda)^2 + 2\int_0^T \xi_t^\lambda\,\mathrm{d}\xi_t^\lambda + T,$$

we after all obtain

$$\begin{aligned}\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] &= \exp\left(-\frac{\theta-\lambda}{2}(x^2+T)\right)\cdot\\ &\quad\cdot\mathbf{E}\left[\exp\left\{\frac{\theta-\lambda}{2}(\xi_T^\lambda)^2\right\}\right].\end{aligned} \tag{A.7}$$

Noting that[2] ξ_T^λ is Gaussian:

$$\xi_T^\lambda \sim \mathbf{N}\left(x\,\mathrm{e}^{\lambda T}, \frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right).$$

(1) The case $x = 0$. We have

$$(\xi_T^\lambda)^2\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right]^{-1} \sim \chi_1^2,$$

where χ_1^2 is the chi-square distribution of degree one. It is known that

$$E[\mathrm{e}^{t\chi_1^2}] = (1-2t)^{-1/2} \quad \left(t < \tfrac{1}{2}\right).$$

Thus, if

$$\frac{\theta-\lambda}{2}\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right] < \frac{1}{2},$$

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \\
&\quad = \exp\left(-\frac{\theta-\lambda}{2}T\right)\left(1-2\frac{\theta-\lambda}{2}\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right]\right)^{-1/2} \\
&\quad = \exp\left(-\frac{\theta-\lambda}{2}T\right)\left[\frac{2\lambda}{(\lambda-\theta)(\mathrm{e}^{2\lambda T}-1)+2\lambda}\right]^{1/2},
\end{aligned} \tag{A.8}$$

where α, θ and λ must satisfy (A.6) and (A.8). The most simple case may be the one where $\lambda = 0$, $\alpha = \theta^2/2$ for given $\theta < 0$. In that case, (A.8) becomes $\theta T < 1$, which is automatically satisfied for $\theta < 0$.

(2) The case arbitrary $x \in \mathbf{R}$.[3]

(i) $\theta < 0$: It follows from (55) that

$$\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \leqslant \exp\left(\frac{\lambda-\theta}{2}(x^2+T)\right)$$

for $\lambda \in (\theta, -\theta)$.

(ii) $\theta > 0$: From (55), we have

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\alpha\int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \\
&\quad \leqslant \exp\left(-\frac{\theta-\lambda}{2}(x^2+T) + (\theta-\lambda)(x\,\mathrm{e}^{\lambda T})^2\right)\cdot \\
&\qquad \cdot\,\mathbf{E}\left[\exp\left\{(\theta-\lambda)(\xi_T^\lambda - x\,\mathrm{e}^{\lambda T})^2\right\}\right]
\end{aligned}$$

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[2] λ may still be positive or negative. If $\lambda = 0$, then the variance is T.

[3] When $x \neq 0$, ξ_T^λ has the noncentral χ^2 distribution with degree one. It is possible to express the moment generating function, but it is not clever for the present purpose.

for $\lambda \in (-\theta, \theta)$. Moreover, if (A.5) then

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\alpha \int_0^T (\xi_t^\theta)^2\,\mathrm{d}t\right)\right] \\
&\quad \leqslant \exp\left(-\frac{\theta-\lambda}{2}(x^2+T)+(\theta-\lambda)(x\,\mathrm{e}^{\lambda T})^2\right)\cdot \\
&\qquad \cdot\left(1-2(\theta-\lambda)\left[\frac{\mathrm{e}^{2\lambda T}-1}{2\lambda}\right]\right)^{-1/2} \\
&\quad = \exp\left(-\frac{\theta-\lambda}{2}(x^2+T)+(\theta-\lambda)(x\,\mathrm{e}^{\lambda T})^2\right)\cdot \\
&\qquad \cdot\left[\frac{\lambda}{(\lambda-\theta)(\mathrm{e}^{2\lambda T}-1)+\lambda}\right]^{1/2}.
\end{aligned}$$

Let us return to the example in Section 6 and verify the uniform L^p integrability of $\zeta^\epsilon_{t,T}$. Since r^ϵ_u is bounded, it suffices for the validity of the asymptotic expansion to show that the uniform (in ϵ) L^p-integrability of the functional

$$\mathcal{Z} = \exp\left(a\int_0^T X_u^{\epsilon(2)}\,\mathrm{d}w_u + b\int_0^T (X_u^{\epsilon(2)})^2\,\mathrm{d}u\right),$$

where a, b are constants. Let $p, q \in (1,\infty)$ and let $q' = q/(q-1)$. Taking $c = q(ap)^2/2$ and using Hölder's inequality, we see that

$$\begin{aligned}
\mathbf{E}[\mathcal{Z}^p] \;\leqslant\; &\mathbf{E}\left[\exp\left(qap\int_0^T X_u^{\epsilon(2)}\,\mathrm{d}w_u - qc\int_0^T (X_u^{\epsilon(2)})^2\,\mathrm{d}u\right)\right]^{1/q}\cdot \\
&\cdot\mathbf{E}\left[\exp\left(q'(c+bp)\int_0^T (X_u^{\epsilon(2)})^2\,\mathrm{d}u\right)\right]^{1/q'}.
\end{aligned}$$

Since $qc = (qap)^2/2$, the first factor on the right-hand side is not larger than one, hence it is sufficient to show that for any $L > 0$, there exists a constant $\epsilon(L) > 0$ such that

$$\sup_{\epsilon\in(0,\epsilon(L)]}\mathbf{E}\left[\exp\left(L\int_0^T (X_u^{\epsilon(2)})^2\,\mathrm{d}u\right)\right] < \infty. \tag{A.9}$$

If we put $x_t = (\epsilon\sigma_2)^{-1}(X_t^{\epsilon(2)} - \bar{X}^{(2)})$, then (x_t) satisfies the stochastic differential equation

$$\begin{aligned}
\mathrm{d}x_t &= -\kappa_2 x_t\,\mathrm{d}t + \mathrm{d}w_t, \\
x_0 &= \epsilon^{-1}c_0,
\end{aligned}$$

where $c_0 = (\sigma_2)^{-1}(X_0^{\epsilon(2)} - \bar{X}^{(2)})$. Therefore, in order to obtain (A.9), it suffices to show that for any $L > 0$, there exists a constant $\epsilon(L) > 0$ such that

$$\sup_{\epsilon\in(0,\epsilon(L)]}\mathbf{E}\left[\exp\left(L\epsilon^2\int_0^T (x_u)^2\,\mathrm{d}u\right)\right] < \infty. \tag{A.10}$$

Here x_u depends on ϵ. Applying Lemma 10 (2i) to the case that $\theta = -\kappa_2$, $x = c_0\epsilon^{-1}$ and $\lambda = -(\kappa^2 - 2L\epsilon^2)^{1/2}$, we obtain (58). □

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