ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARTINO BARDI

An asymptotic formula for the Green's function of an elliptic operator

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, nº 4 (1987), p. 569-586
<http://www.numdam.org/item?id=ASNSP_1987_4_14_4_569_0>

An Asymptotic Formula for the Green's Function of an Elliptic Operator

MARTINO BARDI (*)

1. Introduction

Let $G^\varepsilon(x,y)$ be the Green's function with pole at y of the Dirichlet problem for the uniformly elliptic operator L^ε, i.e., the weak solution of

$$\begin{cases} L^\varepsilon u := -\varepsilon a_{ij} u_{x_i x_j} + b_i u_{x_i} = \delta_y & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \varepsilon \leq 1$, δ_y is the Dirac measure at $y \in \Omega$, $\Omega \subseteq \mathbb{R}^N$ is bounded and open, and we adopt, here and in the following, the summation convention. In this paper we show that under certain conditions on the vector field b, as $\varepsilon \searrow 0$ $G^\varepsilon(\cdot, y)$ converges exponentially to 0, uniformly on compact subsets of $\Omega \backslash \{y\}$, with rate of decay $\dfrac{-I(x,y) + o(1)}{\varepsilon}$, $I(x,y) \geq 0$, and we give a representation formula for $I(x,y)$.

The exponential decay as $\varepsilon \searrow 0$ of the Green's functions of the parabolic operators $\dfrac{\partial}{\partial t} + L^\varepsilon$ was studied by Varadhan [17, 18] in the case $b \equiv 0$, and by Friedman [7, 8] in the general case. Friedman employed the Ventcel-Freidlin estimates from the theory of large deviations of stochastically perturbed dynamical systems and some rather delicate parabolic estimates due to Aronson. His result is the following. For any $x(\cdot) \in W^{1,2}_{\text{loc}}([0,\infty), \overline{\Omega})$ define

$$\|\dot{x}(s) + b(x(s))\|^2 := a^{ij}(x(s)(\dot{x}(s) + b(x(s)))_i \, (\dot{x}(s) + b(x(s)))_j, \tag{1.1}$$

where $((a^{ij})) = a^{-1}$ is the inverse matrix of a. If $\partial\Omega$, a_{ij} and b_i are smooth,

(*) This work was done while the author was visiting the Department of Mathematics of the University of Maryland, partially supported by the Consiglio Nazionale delle Ricerche.

Pervenuto alla Redazione il 28 Luglio 1986.

then the Green's function with pole in y, $q^\varepsilon(t,x,y)$, of $\frac{\partial}{\partial t}+L^\varepsilon$, satisfies

$$\lim_{\varepsilon\searrow 0} -\varepsilon \log q^\varepsilon(t,x,y) = \inf \{\frac{1}{4}\int_0^t \|\dot{x}(s)+b(x(s))\|^2 ds | x(0)=x, \quad x(t)=y,\ x(s)\in\Omega\ \forall\ 0\le s\le t\}. \tag{1.2}$$

The corresponding formula we propose for the elliptic case is the following:

$$\lim_{\varepsilon\searrow 0} -\varepsilon \log G^\varepsilon(x,y) = I(x,y) := \inf \{\frac{1}{4}\int_0^t \|\dot{x}(s)+b(x(s))\|^2 ds; \quad x(\cdot)\in W^{1,2}([0,t],\Omega),\ x(0)=x,\ x(t)=y,\ \text{for some } t\in[0,\infty)\}. \tag{1.3}$$

It is clear, however, that unlike the parabolic case, such a formula can be true only under some strong assumptions on the vector field b, since in the simple case $b\equiv 0$, $G^\varepsilon \to +\infty$ as $\varepsilon\searrow 0$ uniformly on compact subsets of $\Omega\backslash\{y\}$. The main result of this paper is the proof of formula (1.3) in the case that b satisfies the following condition:

$$\text{(B1)}\qquad \begin{cases} \text{if } x(\cdot)\in W^{1,2}_{\text{loc}}([0,\infty),\overline{\Omega}), & \text{then} \\ \displaystyle\int_0^\infty |\dot{x}(s)+b(x(s))|^2 ds = \infty. \end{cases}$$

Condition (B1) was first used by Fleming [5] in the study of a singular perturbation problem arising in stochastic control theory. Its physical meaning is that it takes an infinite amount of energy to resist the flow determined by $-b$ and stay forever in $\overline{\Omega}$. In particular, b is "regular", i.e., it has no zeroes in $\overline{\Omega}$. We remark that the definition of $I(x,y)$ coincides with that of "quasipotential" of the vector field b with respect to the point y in Freidlin-Wentzell [6, p. 108].

Our proof of (1.3) is completely independent of formula (1.2) and also of the probabilistic methods used by Friedman. Instead we follow the PDE approach to WKB-type results initiated in the recent paper by L.C. Evans and H. Ishii [4], where new, totally analytic and simpler proofs are given of three results due respectively to Varadhan, Fleming, and Ventcel-Freidlin. The idea of Evans-Ishii is basically the following: 1) apply a logarithmic transformation to the unknown function, in our case

$$v^\varepsilon(x,y) := -\varepsilon \log G^\varepsilon(x,y),$$

and find a PDE that v^ε solves; 2) prove estimates, independent of ε, on v^ε and its gradient; 3) show that a subsequence of v^ε converges as $\varepsilon\searrow 0$, to the *viscosity solution* of a Hamilton-Jacobi equation (see Crandall-Lions [2], Crandall-Evans-Lions [1] and P.L. Lions [14]); 4) by deterministic control theory methods find a representation formula for such a solution. .

The main difficulties in the implementation of this plan in our case are in the treatment of the two boundary layers that our problem exhibits, one at the boundary $\partial\Omega$, where v^ε goes to $+\infty$, and the other around the singularity y, where v^ε goes to $-\infty$: notice that the limit $I(x,y)$ is positive and bounded. To deal with these problems we shall establish in §2 suitable estimates of v^ε around y, and we shall introduce in §3 various approximating problems.

The pioneering work about singular perturbation of elliptic operators is due to Levinson [13]. We refer to Schuss [15] for an introduction to the physical motivations and an extensive bibliography. The theory of viscosity solutions has been utilized for problems of this type also by P.L. Lions [14, Ch. 6] and Kamin [12]. For results in the nonregular case, i.e., b having one or more zeroes in Ω, we refer to Freidlin-Wentzell [6], Friedman [8, Ch. 14], Kamin [11], Day [3], and the papers quoted therein. Kamin [19] has also treated recently a nonregular problem where the relevant Hamilton-Jacobi equation has more than one viscosity solution.

The paper is organized as follows: in §2 we list the hypotheses, recall a few definitions and basic facts about the Green's function, prove the estimates for v^ε and deduce from them a convergence result; in §3 we prove the representation formula for the limit.

Acknowledgements.

I wish to thank Professor L.C. Evans for suggesting the problem and for his advice and encouragement. I am very grateful to H. Ishii for suggesting a proof of the estimate in Proposition 2.20 that makes use of hypothesis (B1) instead of a different technical condition I had used in the first draft of this paper. I am also grateful to the members of the Department of Mathematics of the University of Maryland for their hospitality during the preparation of this work and to the C.N.R. for the financial support.

2. Estimates and Convergence

Throughout the paper we assume the summation convention and $0 < \varepsilon \leq 1$. We will write for brevity $H := W^{1,2}_{\text{loc}}([0,\infty),\mathbb{R}^N)$.

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, satisfy

(A1) Ω is open, bounded and connected, with smooth boundary $\partial\Omega$.

Let $S^{N\times N}$ be the space on $N\times N$ symmetric matrices and let $a:\Omega\to S^{N\times N}$ satisfy

(A2) $$\begin{cases} a\in C^{1,\alpha}(\Omega) & \text{for some } \alpha>0 \quad \text{and} \\ a_{ij}(x)\xi_i\xi_j\geq|\xi|^2 & \text{for every } \xi\in\mathbb{R}^N,\ x\in\Omega. \end{cases}$$

Let $b:\overline{\Omega}\to\mathbb{R}^N$ satisfy

(B0) $$b\in C^{1,\alpha}(\Omega) \quad \text{for some} \quad \alpha>0.$$

and define
$$d_i^\varepsilon := b_i+\varepsilon a_{ijx_j}, \quad D:=\sup_{\substack{0<\varepsilon\leq 1\\ i=1,\dots,N}}\|d_i^\varepsilon\|_{L^\infty(\Omega)}.$$

It is well known that in the above hypotheses, for every $f\in C^0(\overline{\Omega})$, the unique weak solution of

(2.1) $$\begin{cases} -(\varepsilon a_{ij}u_{x_i})_{x_j}+d_i^\varepsilon u_{x_i}=f & \text{in } \Omega, \\ u\in W_0^{1,2}(\Omega), \end{cases}$$

belongs to $W^{2,N}_{\mathrm{loc}}(\Omega)\cap C^0(\overline{\Omega})$ and it solves

(2.2) $$\begin{cases} -\varepsilon a_{ij}u_{x_ix_j}+b_iu_{x_i}=f \text{ a.e.} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

see for instance [9, Ch. 9]. The problem adjoint to (2.1) is

(2.3) $$\begin{cases} -(\varepsilon a_{ij}v_{x_i}+d_j^\varepsilon v)_{x_j}=\psi & \text{in } \Omega, \\ v\in W_0^{1,2}(\Omega), \end{cases}$$

which is also uniquely solvable in the weak sense.

DEFINITION [16]. A function $G^\varepsilon(x,y)$ defined for $x,y\in\Omega$, $x\neq y$ is a Green's function for the problem (2.1) if $G^\varepsilon(\cdot,y)\in L^1(\Omega)$, $\forall y\in\Omega$, and

$$\int_\Omega G^\varepsilon(x,y)\psi(x)\mathrm{d}x=v(y) \quad \text{for all } y\in\Omega,$$

for every $\psi\in C^0(\overline{\Omega})$ and v the corresponding weak solution of (2.3).

From the theory of Stampacchia [16] it follows that there exists a unique Green's function for the problem (2.1), and it satisfies the following properties:

(2.4) $$\begin{cases} G^\varepsilon(x,y)=\tilde{G}^\varepsilon(y,x) \\ \text{where } \tilde{G}^\varepsilon \text{ is the Green's function for problem (2.3), i.e.} \\ \int_\Omega \tilde{G}^\varepsilon(x,y)f(x)\mathrm{d}x=u(y) \quad \forall y\in\Omega, \\ \text{for all } f\in C^0(\overline{\Omega}) \text{ and } u \text{ the corresponding solution of (2.1);} \end{cases}$$

$$\tilde{G}^\varepsilon(x,y) \geq 0; \tag{2.5}$$

$$\begin{cases} \text{for each } y \in \Omega \; G^\varepsilon(\cdot,y),\; \tilde{G}^\varepsilon(\cdot,y) \in W_0^{1,q}(\Omega) \\ \text{for every } q < \dfrac{N}{N-1}; \end{cases} \tag{2.6}$$

$$\begin{cases} \tilde{G}^\varepsilon(\cdot,y) \in W^{1,2}_{\text{loc}}(\Omega\backslash\{y\}) \\ \text{and it is a weak solution in } \Omega\backslash\{y\} \text{ of} \\ -(\varepsilon a_{ij} v_{x_i} + d_j^\varepsilon v)_{x_j} = 0. \end{cases} \tag{2.7}$$

PROPOSITION 2.8. *Under assumptions* (A1), (A2), (B0) *we have* $G^\varepsilon(\cdot,y) \in C^{3,\alpha}(\Omega\backslash\{y\})$ *and it satisfies*

$$-\varepsilon a_{ij} G^\varepsilon_{x_i x_j} + b_i G^\varepsilon_{x_i} = 0 \text{ for all } x \in \Omega\backslash\{y\}.$$

PROOF. Fix ε and y and define $u(x) = G^\varepsilon(x,y)$. Let f_n be an approximation of the identity and let u^n be the weak solution of

$$\begin{cases} -(\varepsilon a_{ij} u^n_{x_i})_{x_j} + d_i^\varepsilon u^n_{x_i} = f_n & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega. \end{cases}$$

By [16, Thm. 9.1] we have

$$\|u^n\|_{W_0^{1,q}(\Omega)} \leq K \quad \text{for} \quad q < \frac{N}{N-1}.$$

Then a subsequence of u^n converges in $L^q(\Omega)$ and weakly in $W_0^{1,q}(\Omega)$ to u. Now fix $\Omega' \subset\subset \Omega\backslash\{y\}$ with smooth boundary. For n big enough u^n solves

$$-\varepsilon(a_{ij} u^n_{x_i})_{x_j} + d_i^\varepsilon u^n_{x_i} = 0 \quad \text{in } \Omega'.$$

Thus, by standard methods we have

$$\int_{\Omega'} |Du^n|^2 \, dx \leq C,$$

so that a subsequence of u_n converges in $L^2(\Omega')$ and weakly in $W^{1,2}(\Omega')$, necessarily to u. Thus u is a weak solution in Ω' of

$$-(\varepsilon a_{ij} u_{x_i})_{x_j} + d_i^\varepsilon u_{x_i} = 0.$$

Thus u is continuous and the proposition follows from the Schauder theory. □

By the above proposition and the strong maximum principle we have $G^\varepsilon(x,y) > 0$ for all $x \in \Omega,\ x \neq y$, so that we can define

$$v^\varepsilon(x,y) := -\varepsilon \log G^\varepsilon(x,y).$$

It is easy to check that $v^\varepsilon(\cdot,y)$ satisfies

$$\text{(2.9)} \qquad \begin{cases} -\varepsilon a_{ij} v^\varepsilon_{x_i x_j} + a_{ij} v^\varepsilon_{x_i} v^\varepsilon_{x_j} + b_i v^\varepsilon_{x_i} = 0 & \text{in} \quad \Omega\backslash\{y\}, \\ v^\varepsilon(x,y) \to -\infty & \text{as } x \to y, \\ v^\varepsilon(x,y) \to +\infty & \text{as } x \to \partial\Omega. \end{cases}$$

For the solution of the above PDE it is possible to obtain interior estimates for the gradient independent of ε, as shown by Evans-Ishii [4]:

LEMMA 2.10. *For each $\Omega' \subset\subset \Omega\backslash\{y\}$ there exists a constant $C(\Omega')$, independent of ε, such that every C^3 solution of the* PDE *in* (2.9) *satisfies*

$$\sup_{\Omega'} |D_x v^\varepsilon| \leq C(\Omega').$$

PROOF. See [4, Lemma 2.2]. □

We are now going to prove interior estimates, independent of ε, for $|v^\varepsilon|$. To do this we will estimate the Green's function of the adjoint problem $\tilde{G}^\varepsilon$ and exhibit the dependence of the constants on ε. The crucial exponential dependence on ε^{-1} of the bounds for $\tilde{G}^\varepsilon$ comes from the constant in the Harnack inequality:

LEMMA 2.11. *Let $\Omega' \subseteq \Omega$ be open and $u \in W^{1,2}(\Omega')$, $u \geq 0$, be a solution of*

$$-(\varepsilon a_{ij} u_{x_i} + d^\varepsilon_j u)_{x_j} = 0 \quad \text{in} \quad \Omega'.$$

Then, for any ball $B(z,4r) \subset \Omega'$, $r > \frac{4\varepsilon}{3}$, we have

$$\text{(2.12)} \qquad \sup_{B(z,r)} u \leq C^{r/\varepsilon} \inf_{B(z,r)} u,$$

where $C = C(N, D, \|a_{ij}\|_{L^\infty(\Omega)})$.

PROOF. Fix $x_0 \in B(z,r)$ and define

$$\tilde{a}_{ij}(x) := a_{ij}(x_0 + \varepsilon x),\ \tilde{d}_j(x) := d^\varepsilon_j(x_0 + \varepsilon x),\ \tilde{u}(x) := u(x_0 + \varepsilon x).$$

Then $\tilde{u}$ solves

$$-(\tilde{a}_{ij}\tilde{u}_{x_i} + \tilde{d}_j \tilde{u})_{x_j} = 0 \quad \text{in} \quad \tilde{\Omega} := \{x |\ x_0 + \varepsilon x \in \Omega'\},$$

$\tilde{a}$ satisfies (A2) and $\|\tilde{d}_j\|_{L^\infty(\tilde{\Omega})} \leq D$. Since $B(0,4) \subseteq \tilde{\Omega}$, by the Harnack inequality there exists C such that

$$\sup_{B(x_0,\varepsilon)} u = \sup_{B(0,1)} \tilde{u} \leq C \inf_{B(0,1)} \tilde{u} = C \inf_{B(x_0,\varepsilon)} u .$$

Since any two points in $B(z,r)$ can be connected by a chain of $\left[\frac{2r}{\varepsilon}\right]$ appropriately overlapping balls of radius ε, we obtain (2.12).

□

REMARK 2.13. The dependence on ε of the constant in the Harnack inequality displayed in (2.12) is sharp, as the following simple example shows:

$$-\varepsilon\Delta u + u_{x_i} = 0 \quad \text{in} \quad \mathbb{R}^N$$

has the positive solution $u(x) = e^{x_i/\varepsilon}$ that assumes the values $e^{r/\varepsilon}$ and $e^{-r/\varepsilon}$ on the boundary of $B(0,r)$.

□

PROPOSITION 2.14. *Assume* (A1), (A2), (B0). *The function* $\tilde{G}^\varepsilon(x,y)$ *defined in* (2.4) *satisfies the inequality*

$$\tilde{G}^\varepsilon(x,y) \geq \frac{C_1\, e^{-C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^{N-2}}, \quad \text{for } \frac{4\varepsilon}{3} \leq |x-y| \leq 1 \wedge \text{dist}\,(y,\partial\Omega)/2$$

where C_1 *and* C_2 *are constants independent of* ε.

PROOF. Fix $x \neq y$ and define $r = |x-y|$, $u(z) := \tilde{G}^\varepsilon(z,y)$. Define $S_1 := \{z \in \Omega |\ \frac{r}{2} \leq |z-y| \leq r\}$, $S_2 := \{z \in \Omega |\ \frac{r}{4} \leq |z-y| \leq \frac{3r}{2}\}$ and let $\varsigma \in C_0^\infty(\Omega)$ be such that $0 \leq \varsigma \leq 1$, $\varsigma \equiv 1$ in S_1, $\varsigma \equiv 0$ in $\Omega \backslash S_2$, $|D\varsigma| \leq \frac{C}{r}$. By (2.7), using $\phi = u\varsigma^2$ as a test function, we get

$$\varepsilon \int_\Omega |Du|^2\varsigma^2 dz \leq (\frac{\varepsilon C}{r} + D) \int_\Omega \sum_i |u_{x_i}| u\varsigma dz + \frac{C}{r} \int_\Omega u^2\varsigma dz$$

where we indicate by C any constant depending only on N, D and $\|a_{ij}\|_{L^\infty(\Omega)}$. Thus

$$(2.15) \qquad \int_{S_1} |Du|^2 dz \leq \frac{C}{\varepsilon}(\frac{C\varepsilon}{r^2} + \frac{C}{r} + \frac{C}{\varepsilon}) \int_{S_2} u^2 dz \leq (\frac{C}{r^2} + \frac{C}{\varepsilon^2}) r^N \sup_{S_2} u^2.$$

Now let $\phi \in C_0^\infty(\Omega)$ be such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B(y, \frac{r}{2})$, $\phi \equiv 0$ in $\Omega \backslash B(y,r)$, $|D\phi| \le \frac{C}{r}$. As a consequence of (2.4) and the regularity of the coefficients we have

$$\int_\Omega (\varepsilon a_{ij} u_{x_i} \phi_{x_j} + d_i^\varepsilon u \phi_{x_i}) \mathrm{d}x = \phi(y).$$

Then

$$\begin{aligned} 1 &\le \varepsilon \frac{C}{r} \int_{S_1} |Du| \mathrm{d}z + \frac{C}{r} \int_{S_1} u \mathrm{d}z \\ &\le \frac{\varepsilon C}{r} r^{N/2} (r^N (\frac{C}{r^2} + \frac{C}{\varepsilon^2}) \sup_{S_2} u^2)^{1/2} + C r^{N-1} \sup_{S_2} u \\ &\le C r^{N-2} \sup_{S_2} u \end{aligned}$$

for $r \le 1$, where we have got the second inequality from Schwarz inequality and (2.15). Now, in order to apply the Harnack inequality (Lemma 2.11), we observe that any ball $B(z,R)$ with $z \in S_2$ and $R = \frac{r}{20}$ is such that $B(z,4R) \subseteq \Omega \backslash \{y\}$, and that any two points in S_2 can be connected by a chain of appropriately overlapping such balls whose number depends only on N. Then we obtain

$$1 \le C r^{N-2} \, C^{r/\varepsilon} \, u(x),$$

which yields the conclusion.

□

REMARK 2.16. For this proof we borrowed some ideas from Grüter-Widman [10].

□

In order to get the estimate from above of G^ε around the pole y we shall use hypothesis (B1). Define

$$\Omega_\gamma := \{x \in \mathbb{R}^N : \ \mathrm{dist}(x,\Omega) < \gamma\}.$$

The main consequence of (B1) is the following Lemma, which is a slight extension of Lemma 4.2 in [4]:

LEMMA 2.17. *Assume* (B0) *and* (B1) *and let* $\tilde{b}$ *be a Lipschitz extension of* b *to a neighbourhood of* Ω. *Then there exist* $\alpha > 0$, $T > 0$, $\gamma > 0$ *such that*

$$\int_0^S |\dot{x}(s) - \tilde{b}(x(s))|^2 \mathrm{d}s \ge \alpha S$$

for all $S \ge T$ *and for all* $x(\cdot) \in W^{1,2}([0,S], \overline{\Omega}_\gamma)$.

PROOF. First observe that (B1) is equivalent to

$$\int_0^\infty |\dot{x}(s) - b(x(s))|^2 ds = \infty$$

for all $x(\cdot) \in W^{1,2}_{loc}([0,\infty),\overline{\Omega})$. Then the proof (by contradiction) is essentially the same as that of Lemma 4.2 in [4].

□

LEMMA 2.18. *Under the assumptions of Lemma* 2.17 *there exist* $\gamma > 0$ *and* $w \in W^{1,\infty}(\Omega_\gamma)$ *such that*

$$\tilde{b}_i w_{x_i} \geq 3, \qquad \text{a.e. in } \Omega_\gamma. \tag{2.19}$$

PROOF. For a given $x(\cdot) \in H$, $x(0) = x$, let t_x be the first exit time of $x(\cdot)$ from Ω_γ, i.e.

$$t_x := \inf \{t > 0 : x(t) \notin \Omega_\gamma\}.$$

Let α, T, γ be the constants provided by Lemma 2.17 and define for $x \in \overline{\Omega}_\gamma$

$$w(x) := \inf \{\int_0^{t_x} (\frac{3}{\alpha}|\dot{x}(s) - \tilde{b}(x(s))|^2 - 3)\ ds : x(\cdot) \in H,\ x(0) = x\}.$$

By standard arguments w is Lipschitz continuous in $\overline{\Omega}_\gamma$.

Now, observe that w is the value function of the control problem of minimizing

$$\int_0^{t_x} (\frac{3}{\alpha}|\beta(s) - \tilde{b}(x(s))|^2 - 3)\ ds$$

where $x(\cdot)$ satisfies $\dot{x}(s) = \beta(s)$, $x(0) = x$, and the control $\beta(\cdot) \in L^2_{loc}([0,\infty),\mathbb{R}^N)$. The Hamilton-Jacobi-Bellman equation associated to this problem is

$$\frac{\alpha}{12}|Dw|^2 - \tilde{b}_i w_{x_i} + 3 = 0.$$

Since w is continuous it is a viscosity solution of this equation (see [14, Thm. 1.10]), then by Rademacher's theorem it also satisfies the equation a.e., which implies (2.19).

□

PROPOSITION 2.20. *Assume* (A1), (A2), (B0), (B1). *Then the function* $\tilde{G}^\varepsilon(x,y)$ *defined in* (2.4) *satisfies*

$$\tilde{G}^\varepsilon(x,y) \leq \frac{C_1\ e^{C_2 \frac{|x-y|}{\varepsilon}}}{|x-y|^N}, \text{ for } 20\varepsilon/3 < |x-y| < \text{ dist}(y,\partial\Omega)/2 \text{ and } \varepsilon < C_3,$$

where the constants C_1, C_2, C_3 *are independent of* ε.

PROOF. We extend b to a Lipschitz vector field $\tilde{b}$ defined in a neighbourhood of Ω and consider the function w constructed in Lemma 2.18. We call

$$w^\eta(x) := (w * \rho_\eta)(x), \qquad \text{for } 0 < \eta < \gamma,\ x \in \Omega,$$

the convolution of w with a mollifier ρ_η. Then w^η is smooth and satisfies

$$\sup_\Omega |w^\eta| \le C, \text{ for } 0 < \eta < \gamma,$$
$$b_i w^\eta_{x_i} \ge 3 - C\eta \quad \text{in } \Omega,$$

and

$$|w^\eta_{x_i x_j}| \le C/\eta^2 \quad \text{in } \Omega.$$

Now we choose η_0 small enough so that $v := w^{\eta_0}$ satisfies

$$b_i v_{x_i} \ge 2,$$

and

$$-\varepsilon a_{ij} v_{x_i x_j} \ge -\varepsilon C/\eta_0^2.$$

Therefore v satisfies

$$-\varepsilon a_{ij} v_{x_i x_j} + b_i v_{x_i} \ge 1, \text{ for all } \varepsilon \le \eta_0^2/C =: C_3,\ x \in \Omega.$$

Now let u^ε be the solution of

$$-\varepsilon a_{ij} u^\varepsilon_{x_i x_j} + b_i u^\varepsilon_{x_i} = 1, \text{ a.e. in } \Omega,$$
$$u^\varepsilon = 0, \text{ on } \partial\Omega.$$

By the Alexandrov-Bakelman-Pucci maximum principle (see e.g. [9, Thm. 9.1]) we then have

$$\sup_\Omega u^\varepsilon \le \sup_\Omega v + \sup_{\partial\Omega} v^- \le C, \text{ for } \varepsilon \le C_3.$$

Using the definition of $\tilde{G}^\varepsilon$ we get

$$\int_\Omega \tilde{G}^\varepsilon(x,y)\mathrm{d}x = u^\varepsilon(y) \le C, \text{ for } \varepsilon \le C_3,\ y \in \Omega.$$

Now taking $\rho = |x-y|/5$, $4\varepsilon/3 < \rho < \operatorname{dist}(x,\partial\Omega)/4$, by the Harnack inequality (Lemma 2.11) we have

$$\tilde{G}^\varepsilon(x,y) \le C^{\rho/\varepsilon} \inf_{z \in B(x,\rho)} \tilde{G}^\varepsilon(z,y) \le \frac{C^{\rho/\varepsilon}}{\rho^N} \int_\Omega \tilde{G}^\varepsilon(x,y)\mathrm{d}x.$$

□

THEOREM 2.21. *Assume* (A1), (A2), (B0), (B1). *For each* $y \in \Omega$ *there exists a sequence* $\varepsilon_k \searrow 0$ *and a function* $v(\cdot, y) \in C^{0,1}(\overline{\Omega})$ *such that* $\lim_k v^{\varepsilon_k}(\cdot, y) = v(\cdot, y)$ *uniformly on compact subsets of* $\Omega \backslash \{y\}$ *and* $v(\cdot, y)$ *is a viscosity solution of the Hamilton-Jacobi equation*

$$a_{ij} v_{x_i} v_{x_j} + b_i v_{x_i} = 0 \quad \text{in } \Omega \backslash \{y\}. \tag{2.22}$$

Moreover there exists a positive constant C *such that*

$$|v(x, y)| \leq C|x - y|, \text{ for } |x - y| \leq 1 \wedge \operatorname{dist}(y, \partial\Omega)/2. \tag{2.23}$$

PROOF. Lemma 2.10 and Propositions 2.14 and 2.20 imply that $\{v^\varepsilon(\cdot, y),\ 0 < \varepsilon \leq 1\}$ is bounded in $W^{1,\infty}_{\text{loc}}(\Omega)$. Therefore a subsequence converges uniformly to $v(\cdot, y) \in W^{1,\infty}_{\text{loc}}(\Omega)$, which is a viscosity solution of (2.22) because $v^\varepsilon(\cdot, y)$ solves (2.9), see Crandall-Lions [2, §IV.1]. By the results of Crandall-Lions [2, §I.4], (2.22) implies $|Dv(\cdot, y)| \leq C$ in $\Omega \backslash \{y\}$ and then $v(\cdot, y)$ has a unique Lipschitz extension to $\overline{\Omega}$.

□

3. The Representation Formula for the Limit

We recall the definition:

$$I(x, y) := \inf\{\frac{1}{4} \int_0^t \|\dot{x}(s) + b(x(s))\|^2 ds \mid 0 \leq t < \infty,$$
$$x(\cdot) \in W^{1,2}([0, t], \Omega), x(0) = x,\ x(t) = y\}$$

where $\|\dot{x}(s) + b(x(s))\|^2$ is defined by (1.1). Our main result is the following:

THEOREM 3.1. *Assume* (A1), (A2), (B0), (B1). *Then*

$$\lim_{\varepsilon \searrow 0} v^\varepsilon(\cdot, y) = I(\cdot, y)$$

uniformly on compact subsets of $\Omega \backslash \{y\}$.

PROOF. Let $v(\cdot, y) = \lim_k v^{\varepsilon_k}(\cdot, y)$ uniformly on compact subsets of $\Omega \backslash \{y\}$ for some $\varepsilon_k \searrow 0$. Our goal is to prove that $v(x, y) = I(x, y)$ for all $x, y \in \Omega$. For $x \in \Omega$ let τ_x be the first exit time of $x(\cdot) \in H$, $x(0) = x$, from $\Omega \backslash \{y\}$. Since by Theorem 2.21 $v(\cdot, y)$ is a viscosity solution of (2.22) and we are assuming

(B1), the following representation formula of Evans-Ishii [4, Thm. 4.1] holds:

$$v(x,y) = \inf\{\frac{1}{4}\int_0^{\tau_x} \|\dot{x}(s)+b(x(s))\|^2 ds + v(x(\tau_x),y) | x(\cdot)\in H, x(0)=x\},$$

for all $x \in \Omega\backslash\{y\}$.

Since either $x(\tau_x) = y$ or $x(\tau_x) \in \partial\Omega$, we have

$$\begin{aligned} v(x,y) &\le \inf\{\tfrac{1}{4}\int_0^{\tau_x} \|\dot{x}(s)+b(x(s))\|^2 ds + v(x(\tau_x),y) | x(\cdot)\in H,\ x(0)=x, \\ &\qquad x(\tau_x)=y\} \\ &= \inf\{\tfrac{1}{4}\int_0^{\tau_x} \|\dot{x}(s)+b(x(s))\|^2 ds | x(\cdot)\in H,\ x(0)=x,\ x(\tau_x)=y\} \\ &= I(x,y) \end{aligned}$$

because (2.23) implies $v(y,y)=0$.

We are now going to prove that $v(x,y) \ge I(x,y)$ for all $x,y \in \Omega$. We fix $y \in \Omega$ and in order to simplify the notation we drop the second variable y in v^ϵ, v, I and in all the functions defined in the following. Define $\Omega' = \Omega \cup \{x \notin \Omega |\ \mathrm{dist}(x,\partial\Omega) < \beta\}$, for $\beta > 0$ small, let τ'_x be the exit time of $x(\cdot)\in H$ $x(0)=x\in\Omega'$ from $\Omega'\backslash\{y\}$, and extend a and b to be Lipschitz and bounded in all $\mathbb{R}^N$. For $\lambda \ge 0$, $x \in \Omega'$, define

$$I'_\lambda(x) := \inf\{\frac{1}{4}\int_0^{\tau'_x} e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds | x(\cdot)\in H,\ x(0)=x,$$
$$x(\tau'_x) = y \text{ if } \tau'_x < \infty\}.$$

For all $\lambda \ge 0$ I'_λ is locally Lipschitz with $|DI'_\lambda| \le \frac{1}{4}(1+\|b\|_{L^\infty(\Omega)})^2$ so that

(3.2) $$|I'_\lambda| \le C, \qquad C \text{ independent of } \lambda,$$

and it is not difficult to show, using the argument in [4, Lemma 2.4], that I'_λ is a viscosity solution of

$$\begin{cases} \lambda I'_\lambda + a_{ij} I'_{\lambda x_i} I'_{\lambda x_j} + b_i I'_{\lambda x_i} = 0 & \text{in } \Omega'\backslash\{y\} \\ I'_\lambda(y) = 0. \end{cases}$$

Now define for $0 < \gamma < \beta$ the mollification

$$I^\gamma_\lambda(x) := (I'_\lambda * \rho_\gamma)(x), \qquad x \in \overline{\Omega},$$

where ρ_γ is an approximation of the identity. It is easy to deduce from Jensen's inequality, (A2), (B0) and (3.2) that

$$\lambda I^\gamma_\lambda + a_{ij} I^\gamma_{\lambda x_i} I^\gamma_{\lambda x_j} + b_i I^\gamma_{\lambda x_i} \le C\gamma, \qquad x \in \Omega,$$

where C is independent of γ and λ. (3.2) implies also

$$-\varepsilon a_{ij} I^{\gamma}_{\lambda x_i x_j} \leq \frac{C\varepsilon}{\gamma^2}, \qquad \text{for all } x \in \Omega,$$

so that I^{γ}_{λ} satisfies

$$L^{\varepsilon}_{\lambda} I^{\gamma}_{\lambda} \leq C_0(\gamma + \frac{\varepsilon}{\gamma^2}) \qquad \text{in } \Omega \tag{3.3}$$

for a suitable constant C_0 independent of ε, γ and λ, where $L^{\varepsilon}_{\lambda}$ is the quasilinear elliptic operator

$$L^{\varepsilon}_{\lambda} w := -\varepsilon a_{ij} w_{x_i x_j} + a_{ij} w_{x_i} w_{x_j} + b_i w_{x_i} + \lambda w.$$

Furthermore, since $I'_{\lambda}(y) = 0$, we have $I^{\gamma}_{\lambda}(y) \leq C_1 \gamma$ and thus

$$I^{\gamma}_{\lambda}(x) \leq C_1 \gamma + C_2 R \text{ for all } x \in \partial B(y,R), \tag{3.4}$$

where the constants are independent of λ, γ and R, $0 < R < \text{dist}(y, \partial\Omega)$.

Now fix a constant M such that

$$I^{\gamma}_{\lambda} \leq M, \text{ on } \partial\Omega, \text{ for all } \lambda, \gamma, \tag{3.5}$$

and define $v^{\varepsilon}_{\lambda,R}$ to be the solution of

$$\begin{cases} L^{\varepsilon}_{\lambda} \ v^{\varepsilon}_{\lambda,R} = 0 & \text{in } \Omega \backslash B(y,R), \\ v^{\varepsilon}_{\lambda,R} = C_1\gamma + C_2 R & \text{on } \partial B(y,R), \\ v^{\varepsilon}_{\lambda,R} = M & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

(for the existence and regularity of $v^{\varepsilon}_{\lambda,R}$ see e.g. [9, Thm. 15.10]). By the comparison principle [9, Thm. 10.1] and (3.3-4-5) we have

$$I^{\gamma}_{\lambda} \leq v^{\varepsilon}_{\lambda,R} + \frac{C_0}{\lambda} \ (\gamma + \frac{\varepsilon}{\gamma^2}) \quad \text{in } \Omega \backslash B(y,R), \tag{3.7}$$

and

$$v^{\varepsilon}_{\lambda,R} \geq 0 \quad \text{in } \Omega \backslash B(y,R). \tag{3.8}$$

Again by the comparison principle, (3.6) (3.8) and (2.9) (2.20) we get

$$v^{\varepsilon}_{\lambda,R} \leq v^{\varepsilon} + C_1\gamma + C_2 R + CR + \varepsilon(C - N \log R) \quad \text{in } \Omega \backslash B(y,R).$$

Combining this last inequality with (3.7) and letting $\varepsilon \to 0$, $\gamma \to 0$ and $R \to 0$ in this order we get

$$I'_{\lambda}(x) \leq v(x) \quad \text{for all } x \in \Omega. \tag{3.9}$$

We are now going to show that

$$I_\lambda\ (x) := \ \inf\{\frac{1}{4}\int_0^{\tau_x} e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds | x(\cdot)\in H, x(0)=x, x(\tau_x)=y\}$$

satisfies

$$I_\lambda\ (x) \le \liminf_{\beta\searrow 0}\ I'_\lambda(x) \le v(x) \text{ for all } x\in\Omega. \tag{3.10}$$

Fix $\varepsilon > 0$. Let $\beta_n \searrow 0$ as $n\to\infty$, $\Omega_n := \Omega \cup \{x\notin\Omega | \text{dist}(x,\partial\Omega) < \beta_n\}$, let τ_x^n be the exit time from $\Omega_n\backslash\{y\}$ of $x(\cdot)\in H, x(0)=x$ and let $I'_{\lambda,n}$ be I'_λ for $\beta=\beta_n$. Now take $x_n(\cdot)\in H$ such that $x_n(0)=x$, $x_n(\tau_x^n)=y$ if $\tau_x^n<\infty$ and

$$\frac{1}{4}\int_0^{\tau_x^n} e^{-\lambda s}\|\dot{x}_n(s)+b(x_n(s))\|^2 ds \le I'_{\lambda,n}(x)+\varepsilon.$$

Define

$$z_n(s) := \begin{cases} x_n\ (s) & \text{for } s\le\tau_x^n \\ \text{the solution of} \begin{cases}\dot{z}=-b(z)\\ z(\tau_x^n)=y\end{cases} & \text{for } s>\tau_x^n,\ \text{if } \tau_x^n<\infty. \end{cases}$$

It is easy to see, using (3.9), that for all $T>0$

$$\frac{1}{4}\int_0^{T} e^{-\lambda s}|\dot{z}_n(s)|^2 ds \le Cv(x)+C\varepsilon+CT,$$

so that the sequence $\{z_n(\cdot)\}$ is bounded in $W^{1,2}([0,T],\ \Omega_0)$. Hence there exists $z(\cdot)\in H$ and a subsequence of $z_n(\cdot)$, still denoted by $z_n(\cdot)$, which converges to $z(\cdot)$ weakly in $W^{1,2}([0,T],\ \Omega_0)$ and uniformly on $[0,T]$.

Now for $x(\cdot)\in H$, $x(0)=x$ define

$$s_x := \begin{cases} \inf\{s : x(s)=y\} & \text{if} \quad \{s: x(s)=y\}\neq\emptyset \\ +\infty & \text{if} \quad \{s: x(s)=y\}=\emptyset. \end{cases}$$

We claim that $z(s)\in\overline{\Omega}$ for all $0\le s\le s_x$. To prove this assume $z(\tilde{s})\notin\overline{\Omega}$. Let $\alpha :=\ \text{dist}(z(\tilde{s}),\partial\Omega)$ and fix $\overline{n}$ such that $|z_n(s)-z(s)|<\frac{\alpha}{2}$ for $0\le s\le\tilde{s}$, $n>\overline{n}$. Let $\tilde{n}$ be such that $\beta_{\tilde{n}}<\frac{\alpha}{2}$ and define $\overline{\overline{n}} := \max\{\overline{n},\tilde{n}\}$. Then for all $n>\overline{\overline{n}}$ we have $z_n(\tilde{s})\notin\Omega_n$ and thus $x_n(\tau_x^n)=y$ with $\tau_n := \tau_x^n[x_n(\cdot)]<\tilde{s}$. Hence τ_n has a subsequence converging to $\overline{s}\le\tilde{s}$ and it is easy to see that $z(\overline{s})=y$, which implies $\tilde{s}>s_x$ and proves the claim.

Now define

$$J_\lambda(x) := \inf\{\frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds | x \in H, x(0)=x, \ x(s)\in\overline{\Omega} \quad \text{for } 0 \le s \le s_x\}.$$

We claim that

$$J_\lambda(x) \le \liminf_n \ I'_{\lambda,n}(x). \tag{3.11}$$

To prove this we recall that for each $T>0$ the functional

$$x(\cdot) \mapsto \int_0^T e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds$$

is sequentially weakly lower semicontinuous by a classical theorem of Tonelli. Then

$$\begin{aligned}\int_0^T e^{-\lambda s}\|\dot{z}(s) \ + \ b(z(s))\|^2 ds &\le \liminf_n \int_0^T e^{-\lambda s}\|\dot{z}_n(s)+b(z_n(s))\|^2 ds \\ &\le \liminf_n \int_0^{\tau_x^n} e^{-\lambda s}\|\dot{x}_n(s)+b(x_n(s))\|^2 ds \\ &\le \liminf_n \ 4I'_{\lambda,n}(x)+4\varepsilon.\end{aligned}$$

Thus

$$\begin{aligned}J_\lambda(x) &\le \frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{z}(s)+b(z(s))\|^2 ds \\ &\le \liminf_n \ I'_{\lambda,n}(x)+\varepsilon,\end{aligned}$$

and the claim is proved by the arbitrariness of ε.

Next we claim that

$$I_\lambda(x) \le J_\lambda(x). \tag{3.12}$$

We first observe that by a Lemma of Evans-Ishii (see [4, Remark 4.3]) hypothesis (B1) implies that there exist T_0, λ_0 such that

$$\frac{1}{4}\int_0^T e^{-\lambda s}\|\dot{x}(s)+b(x(s))\|^2 ds \ge v(x)+1 \ge J_\lambda(x)+1,$$

for all $T \geq T_0$, $0 \leq \lambda \leq \lambda_0$, $x(\cdot) \in H$ satisfying $x(s) \in \overline{\Omega}$ for all $0 \leq s \leq T_0$. Then, if we assume $\lambda \leq \lambda_0$ and fix $0 < \varepsilon < 1$, we can find $x(\cdot) \in H$ such that

$$(3.13) \qquad \begin{cases} x(0) = x, \ x(s_x) = y, \ s_x \leq T_0, \ x(s) \in \overline{\Omega} \text{ for } 0 < s < s_x, \\ \frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{x}(s) + b(x(s))\|^2 ds \leq J_\lambda(x) + \varepsilon. \end{cases}$$

Since $\partial\Omega$ is smooth there exists a smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{cases} \Omega = \{x \in \mathbb{R}^N | \phi(x) > 0\}, \ \partial\Omega = \{x \in \mathbb{R}^N : \phi(x) = 0\}, \\ |D\phi| = 1 \text{ on } \partial\Omega. \end{cases}$$

Define

$$(3.14) \qquad \overline{x}(s) := \begin{cases} x + sD\phi(x) & \text{for } 0 \leq s \leq \varepsilon \\ x(s-\varepsilon) + \varepsilon D\phi(x(s-\varepsilon)) & \text{for } \varepsilon \leq s < s_x + \varepsilon \\ y + (2\varepsilon + s_x - s)D\phi(y) & \text{for } s_x + \varepsilon \leq s \leq s_x + 2\varepsilon. \end{cases}$$

Clearly

$$I_\lambda(x) \leq \frac{1}{4}\int_0^{s_x+2\varepsilon} e^{-\lambda s}\|\dot{\overline{x}}(s) + b(\overline{x}(s))\|^2 ds,$$

and using the definitions (1.1) (3.13) (3.14) and the smoothness of a, b, and ϕ, it is not hard to show that

$$\int_0^{s_x} |\dot{x}(s)|^2 ds \leq C,$$

and to deduce from it that

$$I_\lambda(x) \leq \frac{1}{4}\int_0^{s_x} e^{-\lambda s}\|\dot{x}(s) + b(x(s))\|^2 ds + 0(\varepsilon) \leq J_\lambda(x) + 0(\varepsilon).$$

This proves the claim (3.12), and then, by (3.11) and the arbitrariness of $\beta_n \searrow 0$, the proof of (3.10) is complete. It remains to show that

$$(3.15) \qquad I_\lambda(x) \to I(x) \text{ for all } x \in \Omega.$$

Using [4, Remark 4.3] as above, we find λ_0, T_0 such that for all $\lambda \leq \lambda_0$ and fixed $\varepsilon > 0$ there exists $x_\lambda(\cdot) \in H$ such that

$$\begin{cases} x_\lambda(0) = x, \ x_\lambda(\tau_x) = y, \ \tau_x < T_0, \\ \frac{1}{4}\int_0^{\tau_x} e^{-\lambda s}\|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 ds \leq I_\lambda(x) + \varepsilon. \end{cases}$$

Then

$$I_\lambda(x) + \varepsilon \geq \frac{1}{4}\, e^{-\lambda T_0} \int_0^{\tau_x} \|\dot{x}_\lambda(s) + b(x_\lambda(s))\|^2 \mathrm{d}s$$
$$\geq e^{-\lambda T_0}\, I(x) \geq I(x) - \varepsilon$$

for λ small enough. This gives (3.15) and completes the proof.

□

REMARK 3.16. Several ideas in this proof are taken from Evans-Ishii [4, §2].

□

REFERENCES

[1] M.G. CRANDALL, L.C. EVANS, P.L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **282** (1984), pp. 487-502.

[2] M.G. CRANDALL, P.L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), pp. 1-42.

[3] M. DAY, *Exponential leveling for stochastically perturbed dynamical systems*, SIAM J. Math. Anal. **13** (1982), pp. 532-540.

[4] L.C. EVANS, H. ISHII, *A* PDE *approach to some asymptotic problems concerning random differential equations with small noise intensities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), pp. 1-20.

[5] W.H. FLEMING, *Exit probabilities and stochastic optimal control*, Appl. Math. Optim. **4** (1978), pp. 329-346.

[6] M.I. FREIDLIN, A.D. WENTZELL, *Random perturbations of dynamical systems*, Springer-Verlag, New York 1984.

[7] A. FRIEDMAN, *Small random perturbations of dynamical systems and applications to parabolic equations*, Indiana Univ. Math. J. **24** (1974), pp. 533-553; Erratum, ibid. **24** (1975), p. 903.

[8] A. FRIEDMAN, *Stochastic differential equations and applications*, Vol. 2, Academic Press, New York 1976.

[9] D. GILBARG, N.S. TRUDINGER, *Elliptic partial differential equations of second order*, 2nd edition, Springer-Verlag, Berlin 1983.

[10] M. GRÜTER, K.O. WIDMAN, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), pp. 303-342.

[11] S. KAMIN, *On elliptic singular perturbation problems with several turning points*, in: *Theory and applications of singular perturbations*, W. Eckhaus and E.M. de Jager eds., Lecture Notes in Math. **942**, Springer-Verlag 1982.

[12] S. KAMIN, *Exponential descent of solutions of elliptic singular perturbation problems*, Comm. Partial Differential Equations **9** (1984), pp. 197-213.

[13] N. LEVINSON, *The first boundary value problem for* $\varepsilon\,\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$ *for small* ε, Ann. of Math. **51** (1950), pp. 428-445.

[14] P.L. LIONS, *Generalized solutions of Hamilton-Jacobi equations*, Pitman, Boston 1982.

[15] Z. SCHUSS, *Singular perturbation methods in stochastic differential equations of mathematical physics*, SIAM Rev. **22** (1980), pp. 119-155.

[16] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), pp. 189-258.

[17] S.R.S. VARADHAN, *On the behavior of the fundamental solution of the heat equation with variable coefficients*, Comm. Pure Appl. Math. **20** (1967), pp. 431-455.

[18] S.R.S. VARADHAN, *Diffusion processes in a small time interval*, Comm. Pure Appl. Math. **20** (1967), pp. 659-685.

[19] S. KAMIN, *Singular perturbation problems and the Hamilton-Jacobi equation*, Integral Equations Operator Theory **9** (1986), pp. 95-105.

Seminario Matematico
Università di Padova
Via Belzoni, 7
35131 Padova