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AN ASYMPTOTIC FORMULA FOR THE PERIOD OF A VOLTERRA-LOTKA SYSTEM

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An asymptotic formula for the period of
a Volterra-Lotka system

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Abstract

The periodic solution of a Volterra-Lotka system is considered as a relaxation oscillation. With perturbation techniques four local expansions are constructed from the implicitly given solution. Integration over the four regions leads to an asymptotic formula for the period.

1. Introduction

The system of differential equations

$$\frac{dh^*}{dt^*} = h^* (a-\alpha p^*), \tag{1a}$$

$$\frac{dp^*}{dt^*} = p^* (-b+\beta h^*) \tag{1b}$$

will have a one parameter family of periodic solutions with the equilibrium $(h^*,p^*) = (b/\beta, a/\alpha)$ as center point. Volterra [2] computed that for small disturbances of the equilibrium the period of such a solution is

$$T^* \approx \frac{2\pi}{\sqrt{ab}} . \tag{2}$$

We will construct an asymptotic formula for the period which holds for large disturbances. For that purpose we consider the periodic solution as a relaxation oscillation. The substitutions

$$h^* = \frac{b}{\beta}h, \qquad p^* = \frac{a}{\alpha}p, \tag{3a}$$

$$t^* = \frac{t}{a}, \qquad a = \varepsilon b, \tag{3b}$$

transform the system (1) into

$$\frac{dh}{dt} = h(1-p), \tag{4a}$$

$$\varepsilon\frac{dp}{dt} = p(-1+h). \tag{4b}$$

Let the initial values of (4) be

$$h(0) = h_1, \quad p(0) = 1 \tag{5}$$

where $0 < h_1 < 1$. In the (h,p)-plane (phase plane) we have the solution in an implicit form

$$h - \log h + \varepsilon\,(p - \log p) + \log h_1 - h_1 - \varepsilon = 0. \tag{6}$$

A system having a relaxation oscillation as solution contains a small parameter that multiplies one of the derivatives. In the following it is assumed that ε is such a small, positive parameter.

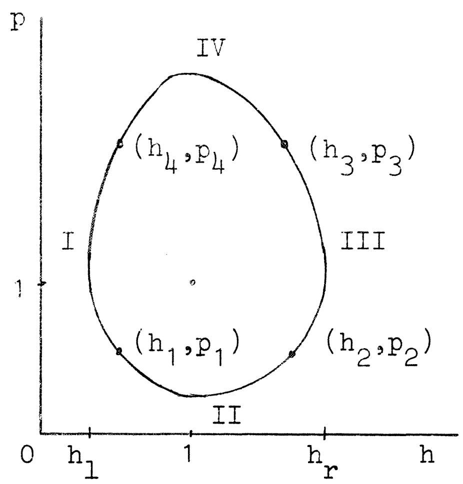

fig. 1

2. Local asymptotic approximations

In figure 1 the periodic solution is given by a closed curve in the phase plane. This curve is divided into four segments separated by the points (h_i, p_i), $i = 1,2,3,4$:

$$h_1 = h_l\ (1 + \frac{1}{h_l - 1}\ \varepsilon \log \varepsilon + \xi_1 \varepsilon),\ p_1 = \eta_1 \varepsilon, \qquad (7a)$$

$$h_2 = h_r\ (1 + \frac{1}{h_r - 1}\ \varepsilon \log \varepsilon + \xi_2 \varepsilon),\ p_2 = \eta_2 \varepsilon, \qquad (7b)$$

$$h_3 = h_r\ (1 - \xi_3), \qquad p_3 = \eta_3 \varepsilon^{-1}, \qquad (7c)$$

$$h_4 = h_l\ (1 + \xi_4), \qquad p_4 = \eta_4 \varepsilon^{-1}. \qquad (7d)$$

The constant h_r satisfies

$$h_r - \log h_r = h_l - \log h_l \qquad (8)$$

such that $h_l < 1 < h_r$. Since we have only one degree of freedom in choosing each of the points, there has to exist some relation between the positive constants ξ_i and η_i. In the four regions a convergent series expansion of the solution can be given in an explicit form. For the complete expansions the reader is referred to Veling [1] , where the proof of the validity of the following asymptotic approximations is given.

$$\text{I:}\ h = h_l + \frac{h_l}{1-h_l}\ (p - \log p - 1)\varepsilon + O\ \{(p - \log p - 1)^2 \varepsilon^2\}, \quad (9a)$$

$$\text{II:}\ p = \exp\ (\frac{h - \log h - h_l + \log h_l - \varepsilon}{\varepsilon}) + $$

$$+ O\ [\exp\ \{\frac{2(h - \log h - h_l + \log h_l - \varepsilon)}{\varepsilon}\}], \qquad (9b)$$

$$\text{III:}\ h = h_r + \frac{h_r}{1-h_r}\ (p - \log p - 1)\varepsilon + O\{(p - \log p - 1)^2 \varepsilon^2\}, \quad (9c)$$

$$\text{IV:}\ p = (\log h - h - \log h_l + h_l + \varepsilon)\ \varepsilon^{-1} - \log \varepsilon + O(1). \quad (9d)$$

For any ξ_1 independent of ε the expansions (9a) and (9b) are identical in (h_1, p_1) which can be verified by substituting one expansion into the other. This is also true for the other points. The constants ξ_3 and ξ_4 have upperbounds independent of ε in order to have converging expansions.

3. The period

The period is composed of the following integrals

$$T = T_I + T_{II} + T_{III} + T_{IV}, \tag{10}$$

$$T_I = \int_{p_4}^{p_1} \frac{dp}{(\frac{dp}{dt})}, \quad T_{II} = \int_{h_1}^{h_2} \frac{dh}{(\frac{dh}{dt})},$$

$$T_{III} = \int_{p_2}^{p_3} \frac{dp}{(\frac{dp}{dt})}, \quad T_{IV} = \int_{h_3}^{h_4} \frac{dh}{(\frac{dh}{dt})}.$$

Using (4) and (9) we obtain

$$T_I = \frac{2}{h_1 - 1} \varepsilon \log \varepsilon + \{\xi_1 - \frac{1 + \log(1-h_l) + \log \xi_4}{h_l - 1} + f(\xi_4)\}\varepsilon + O(\varepsilon^2 \log^2 \varepsilon),$$

$$T_{II} = (\log h_l - \log h_r) + (\frac{1}{h_l - 1} - \frac{1}{h_r - 1})\, \varepsilon \log \varepsilon + (\xi_2 - \xi_1)\, \varepsilon + O(\varepsilon^2 \log^2 \varepsilon),$$

$$T_{III} = \frac{-2}{h_r - 1} \varepsilon \log \varepsilon + \{-\xi_2 + \frac{1 + \log(h_r - 1) + \log \xi_3}{h_r - 1} + g(\xi_3)\}\varepsilon + O(\varepsilon^2 \log^2 \varepsilon),$$

$$T_{IV} = \{\frac{-1}{h_l-1} \log\log\frac{1}{h_l} + \frac{1}{h_r-1}\log\log h_r + \frac{1}{h_l-1}\log\xi_4 - \frac{1}{h_r-1}\log\xi_3 - f(\xi_4) - g(\xi_3) + I_l(h_l) + I_r(h_r)\}\varepsilon + O(\varepsilon^2\log^2\varepsilon),$$

where the functions $f(\xi_4)$ and $g(\xi_3)$ are contributions from higher order terms of (9) which will cancel out. The terms I_l and I_r denote the integrals

$$I_l(h_l) = \int_0^{-\log h_l} \{\frac{1}{x+h_l(1-e^x)} - \frac{1}{(1-h_l)x}\}\, dx,$$

$$I_r(h_r) = \int_{-\log h_r}^{0} \{\frac{1}{x+h_r(1-e^x)} - \frac{1}{(1-h_r)x}\}\, dx.$$

$$T = (h_r-h_l) + (\frac{-1}{1-h_l} + \frac{1}{1-h_r})\varepsilon\log\varepsilon + [\frac{1}{1-h_l} - \frac{1}{1-h_r} + \frac{1}{1-h_l}\log\{(1-h_l)\log\frac{1}{h_l}\} - \frac{1}{1-h_r}\log\{(h_r-1)\log h_r\} + I_l(h_l) + I_r(h_r)]\varepsilon + O(\varepsilon^2\log^2\varepsilon). \quad (11)$$

Notice that the artitrary constants ξ_i and η_i have canceled out.

In table I we compare the asymptotic formule (11) with the results obtained by numerical integration. Two different numerical methods were applied: a Runge-Kutta scheme (Rk4na, see Zonneveld [3]) for integrating system (4) and a method using the implicit formula (6). These numerical methods yield the same results in the required accuracy.

Table I

ε	h_1=.50		h_1=.25		h_1=.10	
	T_{as}	T_{num}	T_{as}	T_{num}	T_{as}	T_{num}
.5	3.5359	4.6599	4.7247	5.1734	5.8567	6.0920
.1	2.2470	2.3480	3.1303	3.1433	4.3014	4.3061
.05	1.8668	1.8875	2.8015	2.8009	4.00945	4.00939
.01	1.4320	1.4303	2.4612	2.4606	3.71766	3.71747
.005	1.3557	1.3548	2.4058	2.4055	3.67143	3.67136
.001	1.2816	1.2815	2.35364	2.35362	3.628628	3.628622

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