An asymptotic method in the theory of series

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PROEFSCHRIFT

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STANISLAUS, THOMAS, MARIA ACKERMANS
GEBOREN TE AMSTERDAM

TECHNISCHE HOGESCHOOL EINDHOVEN

DIT PROEFSCHRIFT IS GOEDGEKEURD

DOOR DE PROMOTOR

P R O F. D R. N. G. D E B R U IJ N

Voor mijn ouders;
voor José en de kinderen.

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Introduction and summary

This thesis gives a contribution to the theory of series with positive terms. Many of the classical results in this theory are inequalities involving two series, the terms of which stand in certain relation to each other. As an example we take a well-known theorem due to G.H.Hardy (see e.g. [8] theorem 326).

If $a_n \geqslant 0$ $(n=1,2,\dots)$, $0 < \Sigma_{n=1}^{\infty} a_n < \infty$, $0 < p < 1$, then

$$\Sigma_{n=1}^{\infty} n^{-1/p}(a_1^p + \dots + a_n^p)^{1/p} < (1-p)^{-1/p} \Sigma_{n=1}^{\infty} a_n; \qquad (0.1)$$

for each value of p is the constant $(1-p)^{-1/p}$ best possible in the sense that the theorem becomes false if $(1-p)^{-1/p}$ is replaced by a smaller constant.

We refer to (0.1) as to Hardy's inequality.
We will study the corresponding inequality for finite series; i.e. we will consider

$$\Sigma_{n=1}^{N} n^{-1/p}(a_1^p + \dots + a_n^p)^{1/p} \leqslant \lambda \Sigma_{n=1}^{N} a_n, \qquad (0.2)$$

for some natural number N. Formula (0.2) is usually referred to as a finite section of (0.1). Let $\lambda_N(p)$ denote for some value of $p \in (0,1)$ the smallest (or best possible) value of λ for which (0.2) holds for that value of p for all $a_1 \geqslant 0,\dots,a_N \geqslant 0$. That such a smallest value does exist follows from the fact that $\lambda_N(p)$ is the maximum of the continuous function $F(x_1,\dots,x_N) =$ $= \Sigma_{n=1}^{N} n^{-1/p}(x_1^p + \dots + x_N^p)^{1/p}$ on the compact set defined by $\Sigma_{n=1}^{N} x_n = 1$, $x_1 \geqslant 0,\dots,x_N \geqslant 0$. As (0.1) also holds for series with $a_n = 0$ for $n > N$, we see that $\lambda_N(p) < (1-p)^{-1/p}$. Moreover, considering only the sequences $a_1,\dots,a_N$ with $a_N = 0$, we see that $\lambda_N(p) \geqslant \lambda_{N-1}(p)$. From this it follows that $\lim_{N\to\infty} \lambda_N(p) \leqslant$ $\leqslant (1-p)^{-1/p}$. However, from the fact that the constant $(1-p)^{-1/p}$ in (0.1) is best possible, it follows that this limit cannot be smaller than $(1-p)^{-1/p}$. So we have

$$\lambda_N(p) = (1-p)^{-1/p} + o(1) \qquad (N\to\infty). \qquad (0.3)$$

For the meaning of the symbols o and $\mathcal{O}$ we refer to [1].

In this thesis we intend to obtain more information about the asymptotic behaviour of best possible constants in finite sections of classical inequalities, such as $\lambda_N(p)$, if the number of terms in the section tends to infinity. Actually, instead of (0.3) we shall find for $\lambda_N(p)$ the formula

$$\lambda_N(p) = (1-p)^{-1/p} - (1-p)^{-1-1/p}\ 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}). \tag{0.4}$$

Although there are many theorems of the same type as (0.1), only few results are known about the best possible constants occurring in the corresponding finite sections. Using eigenvalue theory of truncated integral equations, N.G.de Bruijn and H.S.Wilf [3] have derived an asymptotic formula for the best possible constant in Hilbert's inequality for finite series

$$\Sigma_{m,n=1}^{N} (m+n)^{-1} a_m a_n \leqslant \lambda\ \Sigma_{n=1}^{N} a_n^2 \qquad (a_1 \geqslant 0,\ldots,a_N \geqslant 0).$$

In a seperate note H.S.Wilf [12] remarked that the method of [3] can be extended to several other cases, some of which are also discussed in this thesis with slightly improved results (see Sec.8). N.G. de Bruijn [2] proved for the best possible constant λ_N of a finite section of N terms of Carleman's inequality (see [4])

$$\Sigma_{n=1}^{\infty} (a_1\cdots a_n)^{1/n} < e\ \Sigma_{n=1}^{\infty} a_n \qquad (a_1 \geqslant 0,\ldots,0 < \Sigma_{n=1}^{\infty} a_n < \infty), \tag{0.5}$$

the asymptotic formula

$$\lambda_N = e - 2\pi^2 e\,(\log N)^{-2} + \mathcal{O}((\log N)^{-3}). \tag{0.6}$$

The method employed in this thesis is essentially the one used in N.G. de Bruijn's paper [2] on Carleman's inequality.
Just as in Carleman's original proof of (0.5) the basic tool is the theory of Lagrange multipliers. By this the problem transforms into a question concerning an iteration process, and the study of that

iteration process produces proofs of formulas such as (0.4) and (0.6).

It may be mentioned that J.W.S.Cassels [5] also used Lagrange multiplier theory to prove inequalities such as (0.1) and (0.5), but he obtained no information about the corresponding finite sections.

This thesis consists of 13 Sections and an appendix.

Sec.1 contains some results and examples on iteration processes, and has no direct relation to series. The principal result in this Section is theorem 1.2, which states that if under certain conditions $\{z_n\}$ converges to a zero z_o of a continuous function $\varphi(z)$ and

$$z_{n+1} - z_n = n^{-1}\varphi(z_n) + n^{-2}\phi(z_n) + \mathcal{O}(n^{-3}\chi(z_n)) \quad (n > N_o), \qquad (0.7)$$

then either $a-z_n = \mathcal{O}(n^{-1})$ or $a-z_n = \mathcal{O}((\log n)^{-1})$. In both cases the theorem gives even more information about $a-z_n$. The arguments in the proof of theorem 1.2 may be regarded as typical of a way of arguing that will be applied in the following Sections.

A complete discussion of (0.2) starts in Sec.2. The proof of (0.4) is completed at the end of Sec.7. As $\lambda_N(p)$ is the maximum of a differentiable function in $(x_1,\dots,x_N)$ subject to certain restrictions, we make use of a Lagrange multiplier to determine it (in Sec.2). By our calculations we then obtain an iteration process (the iterates depending on a parameter λ) which is of the form

$$z_1 = 1, \quad z_{n+1} - z_n = n^{-1}(\varphi(\lambda,z_n) + R_n(\lambda,z_n)),$$

where $\varphi(\lambda,z)$ is a convex function of z attaining one minimum in $(1,\infty)$ which is positive for λ less than a certain number ω $(\omega > 1)$, zero for $\lambda = \omega$, and negative for $\lambda > \omega$; and where R_n is a nuisance term which for fixed λ and z is $\mathcal{O}(n^{-1})$ but which, nevertheless, tends to $+\infty$ if $z \uparrow \lambda^q n^q$, where $q = p(1-p)^{-1}$. If $z_m \geqslant \lambda^q m^q$, z_{m+1}

is not defined and the procedure stops (or breaks down). We define a breakdown index N_λ which is, roughly speaking, the last value of m for which z_{m+1} can still be defined. Determining N_λ for each value of p, we shall obtain information about $\lambda_N(p)$, which equals the only value of λ for which the N^{th} iterate equals $\lambda^q N^q$. In Sec. 3 we prove that for $\lambda \geqslant \omega$ no breakdown occurs, so $\lambda_N < \omega$. Some additional results in this Section constitute a proof of Hardy's theorem for infinite series. The important result that there exists a $C > 0$ such that $\varphi(\lambda, z_n(\lambda)) > C\, n^{-\frac{1}{2}}$ for $\lambda \in (1, \omega)$ is proved in Sec.4. By virtue of this result the $R_n(\lambda, z_n(\lambda))$ is small in comparison to $\varphi(\lambda, z_n(\lambda))$ when $z_n(\lambda)$ is small and n is large. In Sec.5 we introduce a number ρ which exceeds the zero of $\varphi(\omega, z)$ and define D_λ to be the largest index such that $z_n(\lambda)$ is less than ρ. We also define an auxiliary index E_λ with $D_\lambda < E_\lambda \leqslant N_\lambda \leqslant \infty$, and prove some results in preparation of Sec.6. In this Section it is proved that

$$\log N_\lambda = \log D_\lambda + \mathcal{O}(1) \qquad (\lambda \uparrow \omega),$$

and that the recurrence relation is so well approximated by the differential equation $dz/d(\log n) = \varphi(\lambda, z)$ that

$$\log D_\lambda = \int_1^\rho (\varphi(\lambda, z))^{-1} dz + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

A standard treatment of the latter integral is the topic of Sec.7. In Sec.8 an asymptotic formula for the best possible constant in a finite section of an inequality due to E.T. Copson (see e.g. [8] theorem 331) is derived from (0.4) by Hölder's inequality. As an additional result we obtain a formula for the largest eigenvalue of finite submatrices of a certain infinite matrix.

In Sec.9 a class of iteration problems is described which can be treated analogously to the one of the Hardy case. Theorem 9.1 generalizes the results of Secs.3 - 7. The proof of this theorem is omitted, since no essentially new arguments are needed.

Applying theorem 9.1 to the iteration problems arising from some other inequalities of E.T. Copson ([6]) we obtain asymptotic formulas for the best possible constants in finite sections. This may be found in Sec.10. The inequalities in this Section are general-

izations of those in Secs.2 and 8. Not only Lagrange multiplier theory but also application of Hölder's theorem is used in order to transform the finite section problems into iteration problems.
In Sec.11 we show that an inequality of K.Knopp gives rise to an iteration problem of the same nature, and we formulate an analogue of theorem 9.1. Application of this theorem produces a formula for the constant in the finite sections.
In Sec.12 the result of Sec.11 is extended to some other inequalities, some due to E.T. Copson, another originating from our systematic treatment.
For the results of the Secs.10, 11 and 12 we refer to the list of formulas on pages 78 and 79.
In Sec.13 we discuss some cases, at first sight seemingly of the same type as the previous ones, but in fact behaving quite differently.
In the appendix we throw some more light on theorem 1.2, discussing an example in detail. For an iteration problem of the form (0.7) in which it depends on the value of a continuous parameter, whether the solutions are $\mathcal{O}(n^{-1})$ or $\mathcal{O}((\log n)^{-1})$, we give a formula which is uniform in the parameter and thus illustrates how the different types of solutions are related. One of the tools will be a version of Banach's theorem on the fixed point of a contraction operator, by means of which we show the existence of small solutions of an auxiliary iteration problem.
A list of formulas and a list of references may be found after the appendix.
With respect to the notation Secs.2 - 7 are to be regarded as a unity. Notations introduced in the other Sections are valid only in the Section where they are introduced, with the exception of the T's and S's in Secs.9 - 12, which denote properties.

1. Preliminary results

This Section contains two theorems on iteration processes and some examples. The iteration processes we will study have the following form

$$z_1 = \alpha, \quad z_{n+1} - z_n = F_n(z_n) \qquad (n=1,2,\ldots). \tag{1.1}$$

Theorem 1.1. If the F_n in (1.1) satisfy

$$F_n(x) = n^{-1}(\varphi(x) + \mathcal{O}(n^{-1}\psi(x))) \qquad (n\to\infty, -\infty < x < \infty), \tag{1.2}$$

whereas φ and ψ are continuous functions and φ has a discrete set of zeros; and if, moreover, the sequence $\{z_n\}$ given by (1.1) and (1.2) is bounded, then $\lim_{n\to\infty} z_n$ exists and $\varphi(\lim_{n\to\infty} z_n) = 0$.

Proof. As the sequence $\{z_n\}$ is bounded, the z_n are in some compact interval J on which ψ is continuous. So the $\mathcal{O}$-term may be replaced by $\mathcal{O}(n^{-1})$ and we obtain

$$z_1 = \alpha, \quad z_{n+1} - z_n = n^{-1}[\varphi(z_n) + \mathcal{O}(n^{-1})].$$

As the maximum of $|\varphi(x)|$ on J exists, we have (with $A > 0$, $B > 0$) $|z_{n+1} - z_n| < n^{-1}A + n^{-2}B$ and so $|z_{n+1} - z_n| \to 0$. Consequently, every point x satisfying $\zeta_1 = \liminf_{n\to\infty} z_n \leqslant x \leqslant \limsup_{n\to\infty} z_n = \zeta_2$ is an accumulation point(*) of the sequence $\{z_n\}$.
If $\zeta_1 < \zeta_2$, there exists a δ_1 and a δ_2 in (ζ_1,ζ_2) with the property that $\varphi(x) \neq 0$ for $x \in [\delta_1,\delta_2]$. We shall prove that $\zeta_1 \geqslant \delta_1$ if $\varphi(x) > 0$ on $[\delta_1,\delta_2]$; the proof that $\zeta_2 \leqslant \delta_2$ if $\varphi(x) < 0$ on $[\delta_1,\delta_2]$ is analogous. Let $\min\{\varphi(x) |\ x \in [\delta_1,\delta_2]\} = \mu > 0$.
Let m be so large that the $\mathcal{O}(n^{-1})$-term is larger than $-\frac{1}{2}\mu$, and that $|z_{n+1} - z_n| < \delta_2 - \delta_1$ for $n \geqslant m$, whereas $z_m > \delta_1$. It is evident that such a number m can be determined, since $\zeta_2 > \delta_1$ implies that $z_n > \delta_1$ for infinitely many values of n.

(*) We call x an accumulation point of the sequence $\{z_n\}$ if each open interval containing x also contains z_n for infinitely many values of n. The possibility that all these z_n may be equal to x is not excluded.

Then $z_n > \delta_1$ for all $n \geqslant m$, which is proved by an induction argument. As a matter of fact, $z_n > \delta_1$ and $n \geqslant m$ imply $z_{n+1} > \delta_1$, as either $z_n \geqslant \delta_2$ and $z_{n+1} \geqslant z_n - |z_{n+1} - z_n| > z_n - (\delta_2 - \delta_1) \geqslant \delta_1$, or $z_n \in (\delta_1, \delta_2)$ and $z_{n+1} - z_n > \frac{1}{2}n^{-1}\mu > 0$. So $\zeta_1 \geqslant \delta_1$. As this constitutes a contradiction, we have proved that $\liminf_{n\to\infty} z_n = \limsup_{n\to\infty} z_n = \beta$. It only remains to be proved that $\varphi(\beta) = 0$. For this purpose we suppose that $\varphi(\beta) \neq 0$ and we can find an interval $J' = [\beta-\delta, \beta+\delta]$ such that $\varphi(x) \neq 0$ on J'. $z_n \in J'$ if $n \geqslant m_1$. Now $\varphi(x) > 0$ on J' implies $\min \{\varphi(x) \mid x \in J'\} = \mu_1 > 0$ and $\varphi(x) < 0$ on J' implies $\max \{\varphi(x) \mid x \in J'\} = \mu_2 < 0$. So we have in the case of $\varphi(\beta) > 0$, that $\beta + \delta \geqslant z_{n+1} = z_{m_1} + \Sigma_{\nu=m_1}^{n} (z_{\nu+1} - z_\nu) > z_{m_1} + \Sigma_{\nu=m_1}^{n} \nu^{-1}(\mu_1 - B\nu^{-1})$ for $n > m_1$. If $\varphi(\beta) < 0$, we have $\beta - \delta < z_{m_1} + \Sigma_{\nu=m_1}^{n} \nu^{-1}(\mu_2 + B\nu^{-1})$. If ν is sufficiently large, we have $\mu_1 - B\nu^{-1} > \frac{1}{2}\mu_1$ and $\mu_2 + B\nu^{-1} < \frac{1}{2}\mu_2$. The above inequalities are therefore in contradiction with the fact that $\Sigma_{\nu=k}^{n} \nu^{-1} \to \infty$ if $n \to \infty$. By this the proof of theorem 1.1 is completed.
We would make some remarks.

Remark 1. We can prove the convergence of z_n to a zero of φ in the same way as in theorem 1.1, if (1.2) is replaced by $F_n(x) = a_n(\varphi(x) + \mathcal{O}(b_n\phi(x)))$ where $a_n \geqslant 0$ (or $a_n \leqslant 0$) for all n, $a_n \to 0$, $b_n \to 0$ if $n \to \infty$, and $\Sigma_{n=1}^{\infty} a_n$ diverges.

Remark 2. A sequence z_n given by (1.1) with F_n satisfying (1.2) may diverge as is shown by the simple example

$$z_1 = 2, \quad z_{n+1} - z_n = -n^{-1} z_n^3.$$

In this case we have $z_1 = 2$, $z_2 = -6$ and it can be proved by induction that $z_{2n-1} \geqslant 2(2n-1)$, $z_{2n} \leqslant -4n$. In this example $\liminf_{n\to\infty} z_n = -\infty$, $\limsup_{n\to\infty} z_n = +\infty$ and the sequence $\{z_n\}$ has no finite accumulation points.

Remark 3. Even if a sequence $\{z_n\}$ given by an iteration procedure indicated by (1.1) and (1.2) has a finite accumulation point, the limit of z_n does not necessarily exist, as will be shown by the following example which we shall describe only roughly. We take $\varphi(x) \equiv 1$ and we try to find a continuous function Φ, such that the sequence $\{z_n\}$ given by

$$z_1 = 0, \quad z_{n+1} - z_n = n^{-1}(1 + n^{-1}\phi(z_n))$$

has zero as accumulation point, whereas a subsequence of $\{z_n\}$ tends to $+\infty$. The function ϕ will be zero, except for a set of disjoint negative peaks at relatively large distances. We shall construct a sequence $a_1, a_2, a_3, \ldots$ of real numbers, with the properties $0 < a_1 < a_2 < \ldots$ and $a_n \to \infty$ if $n \to \infty$, a sequence $n_1, n_2, n_3, \ldots$ of positive integers with $n_k \to \infty$ if $k \to \infty$, and a sequence $d_1, d_2, d_3, \ldots$ of positive real numbers. The function $\phi(x)$ is zero except for values of x in the intervals $(a_k - n_k^{-1}, a_k + n_k^{-1})$; ϕ is linear on the intervals $[a_k - n_k^{-1}, a_k]$ and $[a_k, a_k + n_k^{-1}]$, whereas $\phi(a_k) = -d_k$ $(k=1,2,\ldots)$.

If $a_1 = \Sigma_{\nu=1}^{100} \nu^{-1}$ and $n_1 = 100$, we have $z_1 = 0$ and $z_{n+1} = \Sigma_{\nu=1}^{n} \nu^{-1}$ for $n=1,\ldots,100$. So $\phi(z_{100}) = 0$, and $z_{101} = a_1$; we therefore take d_1 so large that $z_{102} = 0$; then we find $z_{103} = (102)^{-1}$, $z_{104} = \Sigma_{\nu=102}^{103} \nu^{-1}$ and so on, until z_m exceeds $a_1 - n_1^{-1}$. It is easy to show that $m_1 = \min\{n \mid z_n > a_1 + n_1^{-1}\}$ is finite, since $1 + n^{-1}\phi(x) > 1 - n^{-1}d_1$ for $x \in [0, a_1 + n_1^{-1}]$, and $1 - n^{-1}d_1 > \frac{1}{2}$ if n is sufficiently large. We take $a_2 = z_{m_1} + \Sigma_{\nu=m_1}^{m_1^2} \nu^{-1}$ and $n_2 = m_1^2$; then $\phi(z_n) = 0$ for $n = m_1, m_1+1, \ldots, m_1^2$; $z_{n_2+1} = a_2$. We take d_2 so large that $z_{n_2+2} = 0$. If $m_2 = \min\{n \mid z_n > a_2 + n_2^{-1}\}$, we take $a_3 = z_{m_2} + \Sigma_{\nu=m_2}^{m_2^2} \nu^{-1}$, $n_3 = m_2^2$, and d_3 so large that $z_{n_3+2} = 0$. When a_k, n_k and d_k have been constructed in this way, and if $m_k = \min\{n \mid z_n > a_k + n_k^{-1}\}$, we take $a_{k+1} = z_{m_k} + \Sigma_{\nu=m_k}^{m_k^2} \nu^{-1}$, $n_{k+1} = m_k^2$ and d_{k+1} so large that $z_{n_{k+1}} + 2 = 0$. It will be clear that the z_n obtained in this way are dense everywhere on the positive real axis.

It is easy to show that no example of this phenomenon can be found with $\phi(x) \equiv 0$ and consequently having the form $z_{n+1} - z_n = n^{-1}\varphi(z_n)$, where φ is a continuous function. The existence of two "forces" φ and $n^{-1}\phi$ working in opposite directions is essential for this effect.

We now come to the major result of this Section. It also concerns an iteration process of the form (1.1), but (1.2) is replaced by the more special formula

$$F_n(x) = \frac{1}{n}\left[\varphi(x) + \frac{\phi(x)}{n} + \mathcal{O}\left(\frac{\chi(x)}{n^2}\right)\right]. \qquad (1.3)$$

It is, however, this situation which we shall meet in the following Sections. As theorem 1.2 asserts something about the behaviour of the z_n if $n \to \infty$, no requirements are made upon the beginning of the iteration.

Theorem 1.2. If $b < c$, $\varphi \in C^{(4)}([b,c])$, $\phi \in C^{(1)}([b,c])$, $\chi \in C^{(0)}([b,c])$; $b < a < c$, $\varphi'(a) = 0$, $\varphi''(a) > 0$, $\phi(a) > 0$; and if, moreover, the sequence $\{z_n\}$ satisfies the two conditions

$$\lim_{n\to\infty} z_n = a$$

and

$$z_{n+1} - z_n = \frac{1}{n}\left(\varphi(z_n) + \frac{\phi(z_n)}{n} + \mathcal{O}\left(\frac{\chi(z_n)}{n^2}\right)\right) \qquad (n > N_0), \qquad (1.4)$$

then either $\lim_{n\to\infty} n(a - z_n) = \phi(a)$, or

$$z_n = a - \frac{1}{\alpha \log n} - \frac{\beta}{\alpha^3}\frac{\log\log n}{(\log n)^2} + \mathcal{O}\left(\frac{1}{(\log n)^2}\right) \quad (n\to\infty) \qquad (1.5)$$

where $\alpha = \frac{1}{2}\varphi''(a)$, $\beta = \frac{1}{6}\varphi'''(a)$.

($C^{(k)}([b,c])$ denotes the class of all functions which have continuous k^{th} derivatives $(k=0,1,2,\dots)$ on (b,c) and continuous right k^{th} derivatives in b and left k^{th} derivatives in c.)

Proof. The proof is divided into different parts.

(1) First, we make some trivial simplifications and conclusions. From $\lim_{n\to\infty} z_n = a$, it follows that $\varphi(a) = 0$. As $\{z_n\}$ is bounded and χ is continuous we may replace $\mathcal{O}(n^{-2}\chi(z_n))$ by $\mathcal{O}(n^{-2})$. It will be convenient to have

$$z_{n+1} - z_n = \frac{1}{n+1}\left(\varphi(z_n) + \frac{\phi_1(z_n)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \qquad (1.6)$$

instead of (1.4). If we write (1.4) in this form, then $\phi_1(x) = \phi(x) + \varphi(x)$; so $\phi_1(a) = \phi(a)$.

We restrict ourselves to a (possibly small) closed subinterval $J = [\delta_1, \delta_2] \subset [b,c]$ with the following properties: $a \in (\delta_1, \delta_2)$ and $0 \leqslant \varphi(x) < \phi(a)$; $|\varphi'(x)| < \frac{1}{4}$; $\varphi''(x) > 0$; and $\frac{1}{2}\phi(a) < \phi(x) < \frac{3}{2}\phi(a)$ (so $\frac{1}{2}\phi(a) < \phi_1(x) < \frac{5}{2}\phi(a)$) for $x \in J$. The justification of these restrictions of J will be apparent later on.

Let n_o be an integer which exceeds N_o, and which is so large that for $n > n_o$ $z_n \in J$ and that the $\mathcal{O}$-term in (1.6) is in absolute value $< \frac{1}{2}n^{-1}\phi(a)$. Then it will be clear that $z_{n+1} - z_n > 0$ for $n > n_o$, and this implies $a - z_n \downarrow 0$.
From now on we consider only values of n exceeding n_o. For $x \in J$ we have

$$\varphi(x) = \alpha(x-a)^2 + \beta(x-a)^3 + \mathcal{O}((x-a)^4)$$

and

$$\phi_1(x) = \phi(a) + \mathcal{O}(x-a).$$

Substituting $t_n = a - z_n$ we can write (1.6) as

$$t_{n+1} - t_n = \frac{-1}{n+1}\left\{\alpha t_n^2 - \beta t_n^3 + \mathcal{O}(t_n^4) + \frac{\phi(a)}{n} + \mathcal{O}\left(\frac{t_n}{n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}.$$

We put $nt_n = n(a - z_n) = s_n$. As we already know that $t_n = o(1)$ $(n \to \infty)$, we may write the recurrence relation for the s_n as

$$s_{n+1} - s_n = \frac{1}{n}\{s_n - \phi(a) + (s_n+1)\, o(1)\}. \tag{1.7}$$

We already have $s_n > 0$ for $n > n_o$ and $n^{-1}s_n \to 0$.

(2) Next, we prove that we have either $\lim_{n\to\infty} s_n = \phi(a)$ or $s_n \to \infty$ if $n \to \infty$. As $\{s_n\}$ is a solution of an iteration process which is of the form (1.1), (1.2), either we have $\lim_{n\to\infty} s_n = \phi(a)$, or $\{s_n\}$ is unbounded. The latter implies that a subsequence diverges to $+\infty$. But if a subsequence of $\{s_n\}$ tends to $+\infty$ then $\{s_n\}$ itself tends to $+\infty$. This may be seen if we write (1.7) as

$$s_{n+1} - s_n = \frac{1}{n}\left\{\frac{1}{2}(s_n-1) - \phi(a) + s_n\left(o(1)+\frac{1}{2}\right) + \left(\frac{1}{2}+o(1)\right)\right\}.$$

If $o(1) + \frac{1}{2} > 0$ (which is the case if n is sufficiently large) and $s_n > B > 2\phi(a) + 1$, we have $s_{n+1} - s_n > 0$ and hence $s_{n+1} > B$. As this holds for every B, $\limsup_{n\to\infty} s_n = +\infty$ implies $s_n \to +\infty$. We have thus proved that either $n(a - z_n) \to \phi(a)$ or $n(a - z_n) \to +\infty$. It remains to be proved that in the latter case $\{z_n\}$ satisfies (1.5).

(3) Very roughly speaking, (1.5) means that the influence of ϕ and of the $\mathcal{O}$-term is negligible in comparison with the influence of φ.

We shall prove that there exists a positive constant C such that $\varphi(z_n) > Cn^{-\frac{1}{2}}$ if $n > n_o$. We would remark that for $n > n_o$, $\varphi(z_n) > 0$, as $\delta_1 \leqslant z_n < a$. Let n_1 ($n_1 > n_o$) be chosen so large that $s_n > 5\phi(a)$ or $z_n < a - 5n^{-1}\phi(a)$ for $n \geqslant n_1$. Now J is so chosen that $\varphi(z) <$ $< \frac{1}{4}(a-z)$ if $z \in (\delta_1, a)$, and we have therefore

$$z_{n+1} - z_n < (n+1)^{-1} \left(\tfrac{1}{4}(a - z_n) + 3n^{-1} \phi(a)\right), \tag{1.8}$$

for $n \geqslant n_1$. If $w_n = a - z_n - 4n^{-1}\phi(a)$ then (1.8) becomes

$$w_n - w_{n+1} < \frac{1}{4}(n+1)^{-1} w_n.$$

Moreover, $w_{n_1} > 5n_1^{-1}\phi(a) - 4n_1^{-1}\phi(a) = n_1^{-1}\phi(a) > 0$. For $n \geqslant n_1$ we have $w_{n+1} > (1 - \frac{1}{4}(n+1)^{-1})w_n > (1 - (n+1)^{-1})^{\frac{1}{4}} w_n$; and so $w_n > w_{n_1} n_1^{\frac{1}{4}} n^{-\frac{1}{4}}$. If we determine a number C_1 ($C_1 > 0$) such that $\varphi(x) \geqslant C_1 (x-a)^2$ for $x \in [\delta_1, a]^{(**)}$ then we have

$$\varphi(z_n) \geqslant C_1 (z_n - a)^2 > C_1 w_n^2 > C_1 w_{n_1}^2 n_1^{\frac{1}{2}} n^{-\frac{1}{2}} > C_1 (\phi(a))^2 n_1^{-\frac{3}{2}} n^{-\frac{1}{2}} =$$
$$= C_2 n^{-\frac{1}{2}}$$

for $n \geqslant n_1$.
If $C = \min(C_2, C_3)$ where $C_3 = \frac{1}{2}\min\{n^{\frac{1}{2}} \varphi(z_n) \mid n = n_o+1, \ldots, n_1\}$ then we have $\varphi(z_n) > C\, n^{-\frac{1}{2}}$ for $n > n_o$.

(4) The next step in our proof is to show that, heuristically speaking, the differential equation

$$\frac{d(z(n))}{d(\log n)} = \varphi(z(n))$$

can be used as an approximation of (1.6). In fact we shall show that for $n > n_o$

$$\int_{\delta_1}^{z_n} (\varphi(t))^{-1} dt = \log n + \mathcal{O}(1). \tag{1.9}$$

If $K_n = \int_{\delta_1}^{z_n} (\varphi(t))^{-1} dt - \log n$ ($n > n_o$), then we shall obtain the

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(**) The existence of C_1 can be proved as follows:
$\varphi(x) = (x-a)^2 \varphi_1(x)$; $\varphi_1(x) = \alpha + \beta(x-a) + \mathcal{O}((x-a)^2)$. So $\varphi_1(a) = \alpha > 0$; $\varphi_1(x) \neq 0$ on $[\delta_1, a]$ and $\varphi_1(x)$ is continuous. So $\min\{\varphi_1(x) \mid x \in [\delta_1, a]\} = C_1 > 0$.

result $K_n = \mathcal{O}(1)$ $(n > n_o)$, showing that the sequence $\Sigma_{\nu=n_o+1}^{\infty} \varkappa_\nu$ with $\varkappa_\nu = K_{\nu+1} - K_\nu$ converges absolutely.
To this end we need an estimate for $\varphi(z_n) - \varphi(y)$ where $y \in (z_n, z_{n+1})$.
Such an estimate is

$$0 < \varphi(z_n) - \varphi(y) < \varphi(z_n) - \varphi(z_{n+1}) <$$
$$< \tfrac{1}{4}(z_{n+1} - z_n) < \tfrac{1}{4}(n+1)^{-1} \left[\varphi(z_n) + 3\phi(a)n^{-1}\right] <$$
$$< \tfrac{1}{4}(n+1)^{-1} \left(\max\{\varphi(x) \mid x \in [\delta_1, a]\} + 3\phi(a)\right) < Dn^{-1}.$$

Using the mean value theorem for integrals we obtain (with $y_1 \in (z_n, z_{n+1})$)

$$\left|K_{n+1} - K_n\right| = \left|\int_{z_n}^{z_{n+1}} \frac{dt}{\varphi(t)} + \log\frac{n}{n+1}\right| = \left|\frac{z_{n+1} - z_n}{\varphi(y_1)} + \log\frac{n}{n+1}\right| =$$
$$= \frac{1}{n+1}\left|\frac{\varphi(z_n) + n^{-1}\phi_1(z_n) + \mathcal{O}(n^{-2})}{\varphi(y_1)} + (n+1)\log\frac{n}{n+1}\right| =$$
$$= \frac{1}{n+1}\left|\frac{\varphi(z_n) - \varphi(y_1) + n^{-1}\phi_1(z_n) + \mathcal{O}(n^{-2})}{\varphi(y_1)} + 1 + (n+1)\log\frac{n}{n+1}\right| <$$
$$< \frac{1}{n}\,\frac{\varphi(z_n) - \varphi(y_1) + 3n^{-1}\phi(a)}{\varphi(y_1)} + \frac{1}{n}\left|1 + (n+1)\log\frac{n}{n+1}\right| <$$
$$< \frac{1}{n}\,\frac{n^{-1}D + 3n^{-1}\phi(a)}{\tfrac{1}{2}C\, n^{-\frac{1}{2}}} + \frac{1}{2n^2} < \frac{E}{n\sqrt{n}},$$

as $\varphi(y_1) > \varphi(z_{n+1}) > C(n+1)^{-\frac{1}{2}} > \tfrac{1}{2}C\, n^{-\frac{1}{2}}$ and

$$\left|1 + (n+1)\log\left(1 - (n+1)^{-1}\right)\right| = \Sigma_{\nu=2}^{\infty}\, \nu^{-1}\, (n+1)^{1-\nu} < (2n)^{-1}.$$

As $|K_{n+1} - K_n| < E\, n^{-\frac{3}{2}}$, with some constant E, and $\Sigma_{n=1}^{\infty} n^{-\frac{3}{2}}$ converges, we have $K_n = \mathcal{O}(1)$, and hence (1.9).

(5) It remains to be proved only that formula (1.9) implies (1.5).
As $\varphi(x) = \alpha(x-a)^2 + \beta(x-a)^3 + \mathcal{O}((x-a)^4)$ $(x \to a)$ we have

$$\frac{1}{\varphi(x)} = \frac{1}{\alpha(x-a)^2} - \frac{\beta}{\alpha^2(x-a)} + \mathcal{O}(1) \qquad (x \to a)$$

and thus

$$\int_{\delta_1}^{y} \frac{dt}{\varphi(t)} = \frac{1}{\alpha(a-y)} + \frac{\beta}{\alpha^2} \log \frac{1}{a-y} + \mathcal{O}(1) \qquad (y \uparrow a).$$

As $z_n \uparrow a$, the combination of this result with (1.9) yields

$$\frac{1}{\alpha(a-z_n)} + \frac{\beta}{\alpha^2} \log \frac{1}{a-z_n} = \log n + \mathcal{O}(1) \qquad (n \to \infty).$$

From this implicit asymptotic formula we obtain an expression for z_n by iteration (cf. [1] Ch.2).
If $(a-z_n)^{-1} = v_n$ we have $v_n \to \infty$ and $n^{-1} v_n = o(1)$ if $n \to \infty$. So we derive from

$$\alpha^{-1} v_n = \log n + \mathcal{O}(1) - \beta\alpha^{-2} \log v_n,$$

that $\alpha^{-1} v_n < (2 + |\beta|\, \alpha^{-2}) \log n \quad (n \to \infty)$; and the latter formula implies $\log v_n = \mathcal{O}(\log \log n)$. But this yields

$$\alpha^{-1} v_n = \log n + \mathcal{O}(\log \log n).$$

Taking logarithms again we obtain

$$-\log \alpha + \log v_n = \log \log n + \mathcal{O}\big((\log \log n)(\log n)^{-1}\big),$$

and therefore $\log v_n = \log \log n + \mathcal{O}(1)$. This result yields at once

$$\alpha^{-1} v_n = \log n - \beta\alpha^{-2} \log \log n + \mathcal{O}(1).$$

From the latter formula we obtain without difficulty

$$z_n = a - \frac{1}{\alpha \log n} - \frac{\beta}{\alpha^3} \frac{\log \log n}{(\log n)^2} + \mathcal{O}\Big(\frac{1}{(\log n)^2}\Big).$$

This completes the proof of theorem 1.2.

Remark 4. It will be clear that we could prove $\varphi(z_n) > C_\alpha\, n^{-\alpha}$ for each $\alpha > 0$ in part (3) of the preceding proof. To this end, it suffices to take instead of J a (possibly small) subinterval, on which $|\varphi'(x)| < \frac{1}{2}\alpha$. If $\phi_1(x) + \mathcal{O}(n^{-1}) < \beta$ on that interval for some $\beta > \phi(a)$, we take n_1 so large that z_n lies in the considered interval and $s_n > \rho$ for $n > n_1$, where ρ is so large that the substitution $w_n = (a-z_n) - (1-\frac{1}{2}\alpha)^{-1} n^{-1}\beta$ yields for $n = n_1$

$$w_{n_1} > n_1^{-1} \big(\rho - (1-\tfrac{1}{2}\alpha)^{-1}\beta\big) > 0.$$

As $C_\alpha n^{-\alpha} > C_\alpha n^{-\beta}$ for $\beta > \alpha$, it follows that it has to be proved only that $\varphi(z_n) > C_\alpha n^{-\alpha}$ for a small value of α, in order to obtain $\varphi(z_n) > C n^{-\frac{1}{2}}$. From this it follows that we have one degree of freedom more than we need. In the proof of theorem 1.2 we took ρ sufficiently large. In Sec.4 we have to prove an analogous result in a much more complicated situation. There we shall use the other degree of freedom taking α sufficiently small and ρ fixed but $> \beta$.

Remark 5. In Secs.11 and 12 we shall meet with a situation where all conditions of theorem 1.2 are satisfied except $\varphi''(a) > 0$, $\phi(a) > 0$ which are replaced by $\varphi''(a) < 0$, $\phi(a) < 0$.
A completely analogous proof shows that also in this case we have either $n(a-z_n) \to \phi(a)$ or

$$z_n = a - \frac{1}{\alpha \log n} - \frac{\beta}{\alpha^3}\,\frac{\log\log n}{(\log n)^2} + \mathcal{O}\Big(\frac{1}{(\log n)^2}\Big).$$

Remark 6. The sign of $\phi(a)$ is irrelevant if we know that for $n > N_0$ all z_n are on the same side of a. If we omit e.g. the requirement $\phi(a) > 0$ in theorem 1.2 and replace it by $z_n < a$, then we have almost at once $n(a-z_n) \to +\infty$ in the cases where $\phi(a) < 0$. Moreover, if $\phi(a) \leqslant 0$ and $n(a-z_n) \to +\infty$, then the proof of (1.5) does not meet with much difficulty, as we still can prove $\varphi(z_n) >$ $> C n^{-\frac{1}{2}}$, (e.g. instead of (1.8) we may take $z_{n+1} - z_n <$ $< \frac{1}{4}(n+1)^{-1}(a-z_n)$).

We conclude this Section with a more or less detailed discussion of two examples illustrating theorem 1.2. For a detailed discussion of another example, we refer to the appendix.

Example 1. Let $\{z_n(x)\}$ be the sequence of functions defined on [0,1] by the following iteration process

$$\begin{cases} z_1(x) \equiv 0, \\ z_{n+1}(x) - z_n(x) = (n+1)^{-1}\,\big((z_n(x)-1)^2 + n^{-1}x\big). \end{cases} \tag{1.10}$$

(1) $z_n(x)$ increases with n, for each fixed value of x. If for

some x' and m we have $z_m(x') \geqslant 1$, then $z_{m+1}(x') = 1 + \alpha$ $(\alpha > 0)$ and $z_{m+n}(x') > 1 + \alpha$ $(n=2,3,\ldots)$, and so $z_n(x') \to \infty$ if $n \to \infty$ by theorem 1.1. Let $\Phi_n(z)$ denote $z + (n+1)^{-1}(z-1)^2$, then for $n \geqslant 2$

$$\frac{d(\Phi_n(z_n))}{d\,z_n} = 1 + \frac{2(z_n-1)}{n+1} > 1 - \frac{2}{n+1} > 0.$$

As $z_2(x) = \frac{1}{2}(1+x)$ and $x\,n^{-1}(n+1)^{-1}$ are increasing functions of x, we have that $z_n(x)$ is an increasing function of x for $n \geqslant 2$.
For $x = 0$, it is proved that $0 \leqslant z_n \leqslant 1 - n^{-1}$ by induction, using $z_{n+1} = \Phi_n(z_n)$, and so $z_{n+1} \leqslant \Phi_n(1-n^{-1}) = 1-n^{-1} + n^{-2}(n+1)^{-1} \leqslant$ $\leqslant 1 - (n+1)^{-1}$ if $z_n \leqslant 1-n^{-1}$. So $\lim_{n\to\infty} z_n(0) = 1$. For $x = 1$ we have $z_2(1) = 1$ and the sequence $z_n(1)$ increases to $+\infty$.
Let x_n $(n \geqslant 2)$ denote the only solution of the equation $z_n(x) = 1$ then $x_n > 0$ and $x_{n+1} < x_n$ as $z_{n+1}(x_n) = 1 + x_n\,n^{-1}(n+1)^{-1} > 1$.
Let $\lim_{n\to\infty} x_n = c$ then $c < x_3 = -3 + \sqrt{14} < \frac{3}{4}$ and $c \geqslant \frac{1}{4}$, since it can be proved by induction that $z_n(\frac{1}{4}) < 1 - \frac{1}{2}n^{-1}$. So we have proved for the z_n defined by (1.10) the following proposition.

> There exists a number c, $0 < c < 1$, such that $z_n(x) \to +\infty$ if $n \to \infty$ for $x > c$, and $\lim_{n\to\infty} z_n(x) = 1$ for $x \leqslant c$.

Instead of the latter proposition we shall prove

$$z_n(c) = 1 - c\,n^{-1} + \mathcal{O}(n^{-2}),$$

and

$$z_n(x) = 1 - (\log n)^{-1} + \mathcal{O}((\log n)^{-2}) \qquad (0 \leqslant x < c)$$

(2). If $s_n(x) = n(1 - z_n(x))$ then (1.10) yields for the s_n:

$$\begin{cases} s_1(x) = 1, \\ s_{n+1}(x) - s_n(x) = n^{-1}\,(s_n(x) - x - n^{-1}(s_n(x))^2). \end{cases} \tag{1.11}$$

By the result already proved we have $s_n(x) \to -\infty$ if $n \to \infty$ for $x > c$, and $s_n(x) > 0$ for $0 \leqslant x \leqslant c$.
Applying theorem 1.2 we obtain that for $x \leqslant c$ either $s_n(x) \to x$ or $s_n(x) \to \infty$. We shall prove that the first possibility is realized only for $x = c$.
If $s_m(x') \leqslant x'$ for some m and x' then $s_n(x') \to -\infty$ if $n \to \infty$; so

$s_n(x) > x$ for $0 \leqslant x \leqslant c$. If $\Psi_n(s) = s + n^{-1}s - n^{-2}s^2$ then $s_{n+1}(x) = \Psi_n(s_n(x)) - x\,n^{-1}$. $\Psi_n(s)$ increases for $s \leqslant \frac{1}{2}(n+n^2)$. As $s_n = n(1-z_n) \leqslant n$, $\Psi_n(s_n)$ decreases if s_n decreases. Moreover, $s_2(x) = 1-x$ and $-x\,n^{-1}$ are decreasing functions, so s_n is a decreasing function of x on $[0,1]$.
If $u_2 = 0$, $v_2 = \frac{1}{2}$ then $s_2(u_2) = 1$; $s_2(v_2) = v_2$; $s_3(u_2) = \frac{5}{4} > 1$; $s_3(v_2) = v_2 - \frac{1}{4}v_2^2 < v_2$; so numbers u_3 and v_3 can be found with $u_2 < u_3 < v_3 < v_2$ and $s_3(u_3) = 1$, $s_3(v_3) = v_3$. Repeating this argument we have after the k^{th} $(k \geqslant 3)$ step an interval $[u_k, v_k]$ such that $0 < u_k < v_k < \frac{1}{2}$; $s_k(u_k) = 1$; $s_{k+1}(u_k) = 1+k^{-1}(\frac{1}{2}-k^{-1}) > 1$; $s_k(v_k) = v_k$; $s_{k+1}(v_k) = v_k - k^{-2}v_k^2 < v_k$; and then we construct a proper subinterval $[u_{k+1}, v_{k+1}] \subset [u_k, v_k]$.
As the sequence $\{u_n\}$ increases and $\{v_n\}$ decreases, we have

$$0 < \lim_{n\to\infty} u_n = c_\ell \leqslant \lim_{n\to\infty} v_n = c_r < \tfrac{1}{2}.$$

If $x \in [0, c_\ell)$ then $s_n(x) \to +\infty$; if $x \in [c_\ell, c_r]$ then $\lim_{n\to\infty} s_n(x) = x$, whereas $x \in (c_r, 1]$ implies $s_n(x) \to -\infty$. We shall prove $c_\ell = c_r$ by deriving a contradiction from $c_\ell < c_r$. If $c_\ell < c_r$, we choose n so large that $s_n(c_\ell) < c_r$; since s_n decreases this would imply $c_r > s_n(c_\ell) > s_n(c_r) > c_r$. From $c_\ell = c_r$ it follows that $c_\ell = c_r = c$.

(3) <u>The case: $x = c$</u>. Theorem 1.2 does not give any further information in this case. We already know that $s_n(c) = c + o(1)$ $(n\to\infty)$ (or $z_n(c) = 1 - c\,n^{-1} + o(1)\,n^{-1}$ $(n\to\infty)$) and we shall prove that

$$s_n(c) = c + \tfrac{1}{2}n^{-1}c^2 + n^{-1}o(1) \qquad (n\to\infty).$$

Using the substitutions $b_n = s_n(c) - c$; $d_n = n\,b_n$, we have $b_n > 0$, $b_n = o(1)$ $(n\to\infty)$, whereas b_n and d_n satisfy

$$b_1 = 1-c, \quad b_{n+1} - b_n = n^{-1}(b_n - n^{-1}(b_n+c)^2)$$

and

$$\begin{cases} d_1 = 1-c, \\ d_{n+1} - d_n = n^{-1}\{2d_n - c^2 + n^{-1}(d_n - 2d_nc - c^2) - n^{-2}(d_n^2 + 2d_nc) - n^{-3}d_n^2\}. \end{cases}$$

From the fact that $b_n > 0$, $b_n \to 0$, it follows that $b_{n+1} - b_n < 0$ infinitely often. So $0 < n\,b_n = d_n < b_n^2 + 2b_nc + c^2$ infinitely often, and the sequence $\{d_n\}$ has an accumulation point in $[0,c^2]$. On the other hand it follows from $d_n = n \cdot o(1)$ that $d_{n+1} - d_n = n^{-1}\{2d_n - c^2 + o(1)\}$; and this implies that either $\lim_{n\to\infty} d_n = \frac{1}{2}c^2$ or $d_n \to +\infty$, (we use $d_n > 0$). So $\lim_{n\to\infty} d_n = \frac{1}{2}c^2$ and this means

$$z_n(c) = 1 - n^{-1}c + n^{-2}(\tfrac{1}{2}c^2 + o(1)) = 1 - n^{-1}c + \mathcal{O}(n^{-2}).$$

We shall not give more terms of the expansion of $z_n(c)$.

(4) <u>The case: $0 \leqslant x < c$</u>. The fact that $s_n(x) \to \infty$ for $x \in [0,c)$ gives at once

$$z_n(x) = 1 - (\log n)^{-1} + \mathcal{O}((\log n)^{-2})$$

by application of theorem 1.2.

Of course the $\mathcal{O}$-term in this formula depends on x. In order to obtain a (possibly rough) estimate which holds uniformly in $0 \leqslant x < c$, we can refine the arguments used in the proof of theorem 1.2. In this way it can be proved that

$$z_n(x) = 1 - \frac{1}{\log n} + \mathcal{O}\left(\left(\frac{c+x}{c-x}\right)^{4\frac{c+x}{c-x}} \frac{1}{(\log n)^2}\right), \quad (n\to\infty,\ 0 \leqslant x < c)$$

but we shall not give the proof here.

<u>Example 2</u>. This deals with an iteration process of the form $z_{n+1} - z_n = n^{-1}(\varphi(z_n) + \mathcal{O}(n^{-1}\phi(z_n)))$, for which the z_n converge to a zero a of φ with $\varphi'(a) \neq 0$. We have formulated no general results other than the almost trivial theorem 1.1 for situations like this, nor shall we do so. With the exception of Sec.13 we shall not meet situations of this type. Nevertheless, the following example shows how some of our methods can be employed.

Let $\{z_n(x)\}$ be a sequence of functions defined for $x \geqslant 0$ by the following process:

$$\begin{cases} z_1(x) \equiv 0, \\ z_{n+1}(x) - z_n(x) = (n+1)^{-1}\left((z_n(x))^2 - 1 + n^{-1}x\right). \end{cases} \tag{1.12}$$

We shall prove that there exists a number c $(1 < c < \frac{5}{2})$ such that

$$0 \leqslant x < c \text{ implies } z_n(x) = -1 + n^{-1}x + \mathcal{O}(n^{-2}\log n), \tag{1.13}$$

$$z_n(c) = 1 - \tfrac{1}{3}n^{-1}c + \mathcal{O}(n^{-2}), \tag{1.14}$$

and $x > c$ implies $z_n(x) \to \infty$ if $n \to \infty$.

(1) All z_n are polynomials; $z_n(x)$ is an increasing function of x for $n \geqslant 2$. This may be seen as follows: $z_1 > -\frac{1}{2}$; $z_n > -\frac{1}{2}(n+1)$ implies $z_{n+1} > z_n + (n+1)^{-1}(z_n^2-1) > -\frac{1}{2}(n+2)$, thus $z_n > -\frac{1}{2}(n+1)$ for all n; $\frac{dz_{n+1}}{dx} = z'_{n+1}(x) = z'_n(x) + \frac{2z_n(x)\cdot z'_n(x)}{n+1} + \frac{1}{n(n+1)} > z'_n(x)\left(1 + \frac{2z_n}{n+1}\right)$;

so $z'_n(x) > 0$ implies $z'_{n+1}(x) > 0$; $z'_2(x) = \frac{1}{2}$ as $z_2(x) = \frac{1}{2}(x-1)$.

All $z_n > -1$, since $z_1 > -1$ and $z_n > -1$ implies $z_{n+1} > -1 + n^{-1}(n+1)^{-1}x > -1$. If $z_m(x) \geqslant 1$ for some m and x, then $z_n(x) \to \infty$ if $n \to \infty$ for that value of x.

From these considerations and from theorem 1.1 it follows that if $n \to \infty$ there are only three possibilities: $z_n(x) \downarrow -1$; $z_n(x) \uparrow +1$; $z_n(x) \to \infty$. For $x = 0$ we have $\lim_{n\to\infty} z_n(0) = -1$ as $z_n(0) \leqslant -1 + n^{-1}$ (with equality for $n = 1$ and $n = 2$ only). But even for $x = 1$ we can easily prove $z_n(1) \to -1$ by induction, as $z_n(1) < -1 + 3n^{-1}$. If $x_2 = 3$ then $z_2(x_2) = 1$ whereas $z_3(x_2) = 1 + x_2 2^{-1}(2+1)^{-1} > 1$. So, for the number x_3 with $z_3(x_3) = 1$, we have $x_3 < x_2$ ($x_3 = -3 + \sqrt{30} < \frac{5}{2}$). By repetition we find a decreasing sequence $\{x_n\}$ with $c = \lim_{n\to\infty} x_n \geqslant 1$. For $x > c$ we have $z_n(x) \to \infty$ if $n \to \infty$; for $0 \leqslant x \leqslant c$ we have $|z_n(x)| < 1$ and so either $\lim_{n\to\infty} z_n = +1$ or $\lim_{n\to\infty} z_n = -1$.

We will first prove that $z_n(x) \to -1$ for $0 \leqslant x < c$. Suppose $z_n(x_o) \to +1$ for some $x_o < c$, then there exists an m such that $n \geqslant m$ implies $z_n(x_o) \geqslant 0$. $z'_m(x)$ is positive and continuous on $[x_o, c]$, so $\min\{z'_m(x) \mid x \in [x_o, c]\} = \alpha > 0$. We then have $z'_{m+1}(x) = z'_m(x) + (m+1)^{-1}(2z_m z'_m + m^{-1}) > \alpha$; likewise $z'_{m+2}(x) > \alpha, \ldots$, $z'_{m+k}(x) > \alpha$, on $[x_o, c]$. If $n > m$ is so large that $z_n(x_o) > 1 - \frac{1}{2}\alpha(c-x_o)$ then $z_n(c) > z_n(x_o) + \alpha(c-x_o) = 1 + \frac{1}{2}\alpha(c-x_o) > 1$, which is impossible. So $z_n(x) \to -1$ for $0 \leqslant x < c$.

On the other hand let $\lim_{n\to\infty} z_n(c) = -1$, then we shall find a number $c_1 > c$ with $\lim_{n\to\infty} z_n(c_1) = -1$, in contradiction to the maximum property of c. For $x < 3$, and $m \geqslant 3$ it follows from $z_m(x) < 0$ that $z_{m+1}(x) < 0 + (m+1)^{-1}(-1 + 3m^{-1}) \leqslant 0$. If $\lim_{n\to\infty} z_n(c) = -1$, we can find an m $(m \geqslant 3)$ such that $z_m(c) < -\frac{1}{2}$, but as $z_m(x)$ is a continuous and increasing function of x, we can find c_1 $(c < c_1 < 3)$ such that $z_m(c_1) < 0$. But then $z_n(c_1) < 0$ for all $n \geqslant m$ and so $\lim_{n\to\infty} z_n(c_1) = -1$. So it follows that $\lim_{n\to\infty} z_n(c) = +1$.

We complete the discussion by proving the asymptotic formulas (1.13) and (1.14).

(2) <u>The case: x = c</u>. Now we have $z_n \to +1$, $z_n < 1$. The substitution $t_n = 1-z_n$ leads to $t_1 = 1$, $t_{n+1} - t_n = (n+1)^{-1}(2t_n - t_n^2 - cn^{-1})$. As $t_n \to 0$, $t_n > 0$ we conclude that $t_{n+1} - t_n < 0$ infinitely often. So $2t_n - t_n^2 - cn^{-1} < 0$ infinitely often. As $t_n \to 0$, this can occur only if $t_n < 1 - \sqrt{1-n^{-1}c}$ infinitely often. This means that the sequence $\{s_n\}$ with $s_n = nt_n$ has a finite accumulation point. Moreover, $\{s_n\}$ satisfies

$$s_1 = 1, \quad s_{n+1} - s_n = n^{-1}\,[3s_n - c - n^{-1}s_n^2].$$

In the same way as in example 1, we find that

$$z_n(c) = 1 - \frac{c}{3n} - \frac{c^2 + o(1)}{36n^2} \qquad (n \to \infty)$$

which proves (1.14).

(3) <u>The case: 0 ≤ x < c</u>. Of course formula (1.13) is not uniform in x. By the substitution $s_n(x) = n(z_n(x) + 1)$ we get

$$s_1(x) = 1, \quad s_{n+1}(x) - s_n(x) = n^{-1}(x - s_n + n^{-1}s_n^2).$$

As $z_n+1 = o(1)$ $(n\to\infty)$ we have $s_{n+1} - s_n = n^{-1}(x - s_n + s_n \cdot o(1))$. As $s_n > 0$, we have either $s_n \to x$ or $s_n \to \infty$. If $s_n \to \infty$ then $s_n > x$ and $x - s_n + n^{-1}s_n^2 > 0$ infinitely often; so $s_n > n - x - n^{-1}x^2 - \ldots$ infinitely often and thus $n^{-1}s_n > \frac{1}{2}$ infinitely often, in contradiction to the fact that $n^{-1}s_n = o(1)$. So we have proved $s_n(x) \to x$ if $n \to \infty$.

If $d_n(x) = n(s_n(x) - x)$, then $d_1 = 1 - x$,

$$d_{n+1} - d_n = n^{-1}[x^2 + n^{-1}(2xd_n - d_n + x^2) + n^{-2}(2xd_n + d_n^2) + n^{-3}d_n^3].$$

As we already know that $n^{-1}d_n(x) = o(1)$ $(n\to\infty)$, we have

$$d_{n+1}(x) - d_n(x) = n^{-1}(x^2 + o(1)) = n^{-1}\, \mathcal{O}(1).$$

From this it follows that $d_n = \mathcal{O}(\log n)$, and this implies

$$z_n(x) = -1 + \frac{x}{n} + \mathcal{O}\left(\frac{\log n}{n^2}\right).$$

2. Hardy's inequality(*)

In this Section we start a discussion about the finite sections of the inequality stated in the following theorem due to G.H.Hardy (see [8] theorem 326).

If $0 < p < 1$, then

$$\Sigma_{n=1}^{\infty} \{n^{-1} (a_1^p + \ldots + a_n^p)\}^{1/p} \leqslant (1-p)^{-1/p} \Sigma_{n=1}^{\infty} a_n, \qquad (2.1)$$

for all convergent series $\Sigma_{n=1}^{\infty} a_n$ with non-negative terms; unless all a_n are zero, there is strict inequality and the constant $(1-p)^{-1/p}$ is best possible.

If $\lambda_N(p)$ is the best possible constant in the following inequality for finite series with non-negative terms $a_1,\ldots,a_N$,

$$\Sigma_{n=1}^{N} \{n^{-1} (a_1^p + \ldots + a_n^p)\}^{1/p} \leqslant \lambda_N(p)\, \Sigma_{n=1}^{N} a_n,$$

then $\lambda_N(p)$ is the maximum of

$$F(\underline{x}) = F(x_1,\ldots,x_N) = \Sigma_{n=1}^{N} \{n^{-1} (x_1^p + \ldots + x_n^p)\}^{1/p}$$

under the restrictions $\Sigma_{n=1}^{N} x_n = 1$, $x_1 \geqslant 0,\ldots,x_N \geqslant 0$. (Throughout these Sections p has a fixed value in (0,1).) The points $\underline{x} = (x_1,\ldots,x_N)$ satisfying $\Sigma_{n=1}^{N} x_n = 1$, $x_1 \geqslant 0,\ldots,x_N \geqslant 0$ form a compact set S in R_N on which the function F is continuous; F

(*) Notations introduced in this Section are also valid in Secs. 3-7.

therefore attains a maximum on S. Moreover, the maximum of F is attained at a point $\underline{x}$ for which $x_1 > 0,\dots,x_N > 0$. In fact, let $\underline{y}\in S$ have $y_k = 0$, $y_\ell = \delta > 0$ $(1\leq k,\ell\leq N)$, then we consider $\underline{z}(t)$ defined by $z_n(t) = y_n$ $(1\leq n\leq N,\ n\neq k,\ n\neq\ell)$, $z_k(t) = \delta t$, $z_\ell(t) = \delta(1-t)$. So $\underline{z}(t)\in S$ for all $t\in[0,1]$. Now it suffices to observe that we have $f'(t)\to +\infty$ if $t\downarrow 0$, where $f(t) = F(\underline{z}(t))$.

We shall use the theory of Lagrange multipliers to determine the maximum of F on S. As $F(x_1,\dots,x_N)$ is homogeneous of degree 1 in $x_1,\dots,x_N$ we have

$$F(x_1,\dots,x_N) = \Sigma_{n=1}^N x_n \frac{\partial}{\partial x_n} (F(x_1,\dots,x_N)).$$

If therefore

$$\frac{\partial}{\partial x_k} (F(x_1,\dots,x_N) - \lambda\,\Sigma_{n=1}^N x_n) = 0 \qquad (k=1,\dots,N) \tag{2.2}$$

or

$$\frac{\partial}{\partial x_k} (F(x_1,\dots,x_N)) = \lambda$$

for a set $(x_1',\dots,x_N',\lambda')$, then we have $F(x_1',\dots,x_N') = \lambda'\,\Sigma_{n=1}^N x_n'$. Now (2.2) and $\Sigma_{n=1}^N x_n = 1$ are necessary conditions for an extremum of F on the subset of S with $x_1\neq 0,\dots,x_N\neq 0$. But the maximum of F is attained at a point of this subset; and so the maximum equals the largest stationary value of λ i.e. the largest value of λ for which there is a solution $x_1',\dots,x_N'$ of

$$\frac{\partial}{\partial x_k}[\Sigma_{n=1}^N \{n^{-1}(x_1^p + \dots + x_n^p)\}^{1/p} - \lambda\,\Sigma_{n=1}^N x_n] = 0 \qquad (k=1,\dots,N), \tag{2.3}$$

and

$$\Sigma_{n=1}^N x_n = 1.$$

Now (2.3) can be written as

$$\lambda x_k^{1-p} = \Sigma_{n=k}^N n^{-1}\{n^{-1}(x_1^p + \dots + x_n^p)\}^{-1+1/p} \qquad (k=1,\dots,N). \tag{2.4}$$

If we take differences, (2.4) is transformed into

$$\begin{cases} \lambda(x_k^{1-p} - x_{k+1}^{1-p}) = k^{-1}\{k^{-1}(x_1^p + \dots + x_k^p)\}^{-1+1/p} & (k=1,\dots,N-1), \\ \lambda x_N^{1-p} = N^{-1}\{N^{-1}(x_1^p + \dots + x_N^p)\}^{-1+1/p}. \end{cases} \tag{2.5}$$

So we have to solve (2.5) and $\Sigma_{n=1}^{N} x_n = 1$; if we omit the latter equation the $x_1,\ldots,x_N$ are determined except for a multiplicative constant. Because of the homogeneity of the problem, no information about λ is lost. This justifies our omitting $\Sigma_{n=1}^{N} x_n = 1$.
We write (2.5) in a more tractable form by the substitutions

$$p(1-p)^{-1} = q; \quad z_k = k^{-1}x_k^{-p}(x_1^p + \ldots + x_k^p) \qquad (k=1,\ldots,N). \quad (2.6)$$

From (2.6) it follows that $0 < q < \infty$ and further that

$$z_1 = 1, \; z_{k+1} - z_k \cdot kx_k^p \cdot (k+1)^{-1} \; x_{k+1}^{-p} = (k+1)^{-1} \qquad (k=1,\ldots,N). \quad (2.7)$$

By the substitutions (2.6) we transform (2.5) into

$$\begin{cases} 1 - x_k^{p-1} \; x_{k+1}^{1-p} = (\lambda k)^{-1} \; z_k^{1/q}, & (k=1,\ldots,N-1) \\ \lambda = N^{-1} \; z_N^{1/q}. \end{cases} \quad (2.8)$$

Combination of (2.8) with (2.7) gives

$$z_1 = 1, \; z_{k+1} = \frac{1}{k+1} + \frac{k}{k+1} \; z_k(1 - \lambda^{-1}k^{-1}z_k^{1/q})^{-q} \quad (k=1,\ldots,N-1), \quad (2.9)$$

$$z_N = (\lambda N)^q. \quad (2.10)$$

The fact that $\lambda_N = \lambda_N(p)$ is the largest value of λ for which (2.9) and (2.10) have as solution $(1,z_2',\ldots,z_N')$ trivially implies that we have $z_k < (\lambda_N k)^q$ for $k=1,\ldots,N-1$. It will become clear that there is exactly one value of λ for which (2.9) and (2.10) have a positive solution. We consider the either finite (if $z_m \geqslant (\lambda m)^q$ for some m) or infinite sequence $\{z_n\}$, $z_n = z_n(\lambda)$ depending on the parameter λ, given by

$$z_1 = 1, \; z_{n+1} = \frac{1}{n+1} + \frac{n}{n+1} \; z_n(1 - \lambda^{-1}n^{-1}z_n^{1/q})^{-q}. \quad (2.11)$$

The latter expression is of the form $z_{n+1} = \Phi_n(\lambda,z_n)$, where $\Phi_n(\lambda,x)$ is defined if $x > 0$, $\lambda > 0$ and $x < (\lambda n)^q$. For each n, $\Phi_n(\lambda,x)$ is a continuous function of λ and x (provided that $0 < x < (\lambda n)^q$); for fixed λ and n it is an increasing function of x; for fixed x and n it is a decreasing function of λ. If $x \geqslant 1$ and $\Phi_n(\lambda,x)$ is defined, then $\Phi_n(\lambda,x) \geqslant \Phi_n(\lambda,1) > 1$.
We shall prove that $1 = \lambda_1 < \lambda_2 < \ldots$, and that $z_{n+1}(\lambda)$ can be

calculated from (2.11) if $\lambda > \lambda_n$. $z_{n+1}(\lambda)$ will be seen to be a decreasing continuous function of λ for $\lambda > \lambda_n$.
The process starts from the constant function $z_1(\lambda) \equiv 1$. We observe that $\lambda = 1 = \lambda_1$ is the only solution of $z_1(\lambda) - \lambda^q = 0$. Because of the properties of Φ_1 we can calculate $z_2(\lambda)$ for $\lambda > 1$; $z_2(\lambda)$ is a continuous decreasing function of λ. Further, $z_2(\lambda) \to +\infty$ if $\lambda \downarrow 1$; $z_2(\lambda) \downarrow 1$ if $\lambda \to \infty$. So $z_2(\lambda) - (2\lambda)^q = 0$ has exactly one root in (λ_1, ∞); from what has been said above, it is obvious that this root equals λ_2. For $\lambda > \lambda_2$ we find the continuous decreasing function $z_3(\lambda)$, which tends to $+\infty$ if $\lambda \downarrow \lambda_2$ and to 1 if $\lambda \to \infty$. $z_3(\lambda) - (3\lambda)^q = 0$ has exactly one root in (λ_2, ∞); and this root equals λ_3. By repetition of this argument we obtain the desired proof. λ_n is the only solution of $z_n(\lambda) - (n\lambda)^q = 0$ $(n=1,2,\ldots)$. From now on we shall deal with (2.11) instead of (2.9) and (2.10).
For a fixed value of λ it may happen that we can calculate $z_1, z_2, \ldots, z_m, z_{m+1}$ but no more z's; this happens if $z_k < (k\lambda)^q$ for $k=1,\ldots,m$, and $z_{m+1} \geqslant (\lambda(m+1))^q$. We say then that the iteration process breaks down at m+1. We now define the breakdown index N_λ as follows: if there exists an m for which breakdown occurs, N_λ is defined by

$$N_\lambda = \max \{n \mid z_n(\lambda) < (n\lambda)^q\};$$

for values of λ for which the process does not break down we define $N_\lambda = \infty$. For any finite N_λ the process breaks down at $N_\lambda+1$. So $N_\lambda = m$ if and only if $\lambda_m < \lambda \leqslant \lambda_{m+1}$, and $N_\lambda = \infty$ if and only if λ exceeds all λ_m, although we have not yet established whether this ever may happen. Formula (2.11) is equivalent to

$$z_1 = 1,\; z_{n+1} - z_n = (n+1)^{-1}\{1 - (n+1)z_n + nz_n(1 - \lambda^{-1}n^{-1}z_n^{1/q})^{-q}\}. \tag{2.12}$$

We can write this as

$$z_1 = 1,\; z_{n+1} - z_n = (n+1)^{-1}[\varphi(\lambda, z_n) + R_n(\lambda, z_n)], \tag{2.13}$$

where

$$\varphi(\lambda, x) = 1 - x + q\lambda^{-1}\, x^{1+1/q} \tag{2.14}$$

and

$$R_n(\lambda, x) = \Sigma_{\nu=2}^{\infty}\, (-1)^\nu \binom{-q}{\nu}\, \lambda^{-\nu} n^{1-\nu} x^{1+\nu/q} =$$

$$= \frac{q(q+1)}{2!\,\lambda^2 n} x^{1+2/q} + \frac{q(q+1)(q+2)}{3!\,\lambda^3 n^2} x^{1+3/q} + \dots . \quad (2.15)$$

Henceforward we shall make use of the notations introduced below. $\varphi(\lambda,x) = -1 + (q+1)\lambda^{-1} x^{1/q}$; $\phi(\lambda,x)$ is the partial derivative of $\varphi(\lambda,x)$ with respect to x. $\chi(\lambda,x) = (q+1)(q\lambda)^{-1} x^{-1+1/q}$; so $\chi(\lambda,x)$ is the second partial derivative of $\varphi(\lambda,x)$ with respect to x. For every $\lambda > 0$, we have $\chi(\lambda,x) > 0$ for $x \in (0,\infty)$; this implies that $\varphi(\lambda,x)$ is a strongly convex function. For every $\lambda > 0$, $\phi(\lambda,x)$ has one zero, $\varphi(\lambda,x)$ has one minimum for $x \in (0,\infty)$. The minimum equals $1 - \lambda^q(q+1)^{-q-1}$, which we denote by $b(\lambda)$ and it is attained for $x = \lambda^q(q+1)^{-q} = c(\lambda)$. Further, $a(\lambda)$ denotes $\frac{1}{2}\chi(\lambda,c(\lambda)) = (2q)^{-1}\lambda^{-q}(q+1)^q$; ω denotes the value $(q+1)^{1+1/q}$. Notice, that $b(\lambda) \geqslant 0$ if $\lambda \leqslant \omega$, and $b(\lambda) < 0$ if $\lambda > \omega$. We have $b(\lambda) = 0$ only if $\lambda = \omega$; the value of x corresponding to this is $c(\omega) = q+1$.
We shall show that no breakdown occurs for $\lambda \geqslant \omega$; in fact we shall prove (in Sec.3) that for $\lambda \geqslant \omega$ $z_n(\lambda)$ is defined for all n, and $z_n(\lambda) < q+1$. For $0 < \lambda < \omega$ we shall find a breakdown situation, and we shall show that $N_\lambda \to \infty$ if $\lambda \uparrow \omega$. Our main interest will be in the asymptotic behaviour of N_λ if $\lambda \uparrow \omega$. We shall conclude this Section with some heuristic arguments which don't prove anything, it is true, but which may, nevertheless, show us the way of handling our problem.
If instead of (2.13) one considers the iteration process $z_1(\lambda) \equiv 1$, $z_{n+1}(\lambda) - z_n(\lambda) = (n+1)^{-1}\varphi(\lambda,z_n)$, one can prove that for this process $z_n(\lambda) < q+1$ if $\lambda \geqslant \omega$, and that $z_n(\lambda)$ increases to $+\infty$ if $\lambda < \omega$. One may expect that for $\lambda \geqslant \omega$, the sequence $\{z_n\}$ calculated from (2.13) does not behave very differently from the sequence calculated from the process simplified by omitting $R_n(\lambda,z_n)$, which is small if n is large. So one may try first to get some information about the behaviour of $z_n(\lambda)$ for $\lambda \geqslant \omega$. This will be done in Sec. 3. In that Section we shall prove not only that $\lim_{n\to\infty} z_n(\omega) = q+1$, but also that $n(1+q - z_n(\omega)) \to \infty$. This means that $z_n(\omega)$ satisfies an asymptotic formula, as stated in theorem 1.2. We obtained that formula on account of $\varphi(\omega,x)$ only, as will be remembered. One thus may get the impression that also for $\lambda < \omega$ but close to ω, the influence of $R_n(\lambda,z_n)$ is small if $z_n(\lambda)$ is close to $c(\lambda)$. Our method consists in fact in proving this. Different parts of the proof are prompted by arguments used in the proof of theorem 1.2.

The next part consists of the results of Sec.4, where it is proved that $\varphi(\lambda, z_n(\lambda)) > Cn^{-\frac{1}{2}}$ for $\lambda \leqslant \omega$, where C does not depend on λ. This is a generalization of part (3) of the proof of theorem 1.2. If we replaced (2.13) by the differential equation

$$\frac{d(\log n)}{dz} = (\varphi(\lambda, z))^{-1}, \tag{2.16}$$

we should obtain if $N_\lambda < \infty$

$$\log N_\lambda = \int_1^{z_{N_\lambda}} (\varphi(\lambda, z))^{-1}\, dz.$$

If $\lambda \uparrow \omega$ the peak of $(\varphi(\lambda, x))^{-1}$ tends to ∞. In Sec.5 we introduce a number ρ ($\rho > q+1$) and we conjecture an asymptotic formula, viz.

$$\log N_\lambda = \int_1^{\rho} (\varphi(\lambda, x))^{-1} dx + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{2.17}$$

(Notice that the maximum of $(\varphi(\lambda, x))^{-1}$ is attained for a value of $x < q+1$). The proof of (2.17) is given in Secs.5 and 6, of which Sec.5 contains only auxiliary results.

A standard proof of the formula

$$\int_1^{\rho} (\varphi(\lambda, x))^{-1} dx = \pi\, (a(\omega)\, b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega) \tag{2.18}$$

will be given in Sec.7.

Once (2.18) has been proved we may transform it into

$$\lambda_N = \omega - 2\pi^2 (q+1)\, \omega (\log N)^{-2} + \mathcal{O}((\log N)^{-3}) \tag{2.19}$$

(it will then be clear that $\lambda \uparrow \omega$ implies $N \to \infty$).

If we write (2.19) with the original parameter p, it gets the final form

$$\lambda_N = (1-p)^{-1/p} - 2\pi^2 (1-p)^{-1-1/p} (\log N)^{-2} + \mathcal{O}((\log N)^{-3}). \tag{2.20}$$

3. The case: $\lambda \geqslant \omega$

This Section is devoted to some fundamental results about the behaviour of $z_n(\lambda)$ for $\lambda \geqslant \omega$. We mentioned before that for $\lambda \geqslant \omega$ the process (2.11) does not break down and that $z_n(\lambda) < q+1$ for all n. In we fact shall prove the following slightly stronger lemma.

Lemma 3.1. If $\lambda \geqslant \omega$ then $z_n(\lambda)$ is defined for all n and

$$z_n(\lambda) \leqslant q+1 - qn^{-1}. \tag{3.1}$$

Proof. As $z_n(\lambda)$ is a decreasing function of λ for $\lambda > \lambda_{n-1}$ $(n>1)$ it suffices to prove that $\omega > \lambda_n$ and $z_n(\omega) \leqslant q+1 - qn^{-1}$ for all n; once this has been proved we shall have that $z_n(\lambda)$ exists for $\lambda \geqslant \omega$ and $z_n(\lambda) \leqslant z_n(\omega) \leqslant q+1 - qn^{-1}$. We supply the proof by induction with respect to n. $z_1(\omega) = 1 = q+1 - q$. As $1 < (q+1)^{q+1} = \omega^q$, $z_2(\omega)$ can be calculated. If we suppose that $z_n(\omega)$ is defined and $z_n(\omega) \leqslant q+1 - qn^{-1}$ then it follows that $z_n(\omega) < (\omega n)^q = (q+1)^{q+1}n^q$, and so $z_{n+1}(\omega)$ can be calculated. We then find that $z_{n+1}(\omega) = \Phi_n(\omega, z_n(\omega))$. $\Phi_n(\omega, q+1 - qn^{-1})$ is defined, and because of the monotonicity property of $\Phi_n(\lambda,x)$ with respect to x it is not less than $\Phi_n(\omega, z_n(\omega))$. So $z_{n+1}(\omega) \leqslant \Phi_n(\omega, q+1 - qn^{-1})$, and the proof will be completed if we show that

$$\Phi_n(\omega, q+1 - qn^{-1}) \leqslant q+1 - q(n+1)^{-1}. \tag{3.2}$$

But we have (with $\mu = q+1 - qn^{-1}$)

$$\Phi_n(\omega,\mu) = \frac{1}{n+1} + \frac{n}{n+1}\,\mu\left(1 - \mu^{1/q}(q+1)^{-1-1/q}\,n^{-1}\right)^{-q}. \tag{3.3}$$

We use the inequality

$$1 - \frac{q}{(q+1)n} \leqslant \left(1 + \frac{1}{(q+1)n}\right)^{-q} = 1 - \frac{q}{(q+1)n} + \Sigma_{k=2}^{\infty}\binom{-q}{k}(q+1)^{-k}\,n^{-k} \tag{3.4}$$

(the series in the right-hand side is an alternating series whose sum is positive).

The inequality $1 - x(q+1)^{-1}n^{-1} \geqslant x$ holds for $x \leqslant \left(1+(q+1)^{-1}n^{-1}\right)^{-1}$; so we may substitute $x = \left(1 - q(q+1)^{-1}n^{-1}\right)^{1/q}$ in it, on account of (3.4). The result of this substitution is

$$1 - \left(1 - \frac{q}{(q+1)n}\right)^{1/q} \frac{1}{(q+1)n} \geqslant \left(1 - \frac{q}{(q+1)n}\right)^{1/q}. \tag{3.5}$$

Combining (3.3) with (3.5) we obtain

$$\Phi_n(\omega, q+1 - qn^{-1}) \leqslant (n+1)^{-1} + n(n+1)^{-1}(q+1) = q+1 - q(n+1)^{-1}.$$

This completes the proof.

<u>Lemma 3.2</u>. $\lim_{n\to\infty} z_n(\omega) = q+1.$

<u>Proof</u>. We define $R_n^*(\lambda,x) = \begin{cases} R_n(\lambda,x) & 0 \leqslant x \leqslant q+1 \\ R_n(\lambda,q+1) & x > q+1 \\ 0 & x < 0 \end{cases}$

As $z_n(\omega) \in (0,q+1)$ we replace (2.13) (for $\lambda = \omega$) by $z_1 = 1$, $z_{n+1} - z_n = (n+1)^{-1} (\varphi(\omega,z_n) + R_n^*(\omega,z_n))$. Theorem 1.1 yields at once $\lim_{n\to\infty} z_n(\omega) = q+1$.

If $\lambda > \omega$, then $\varphi(\lambda,q+1) < \varphi(\omega,q+1) = 0$, $\varphi(\lambda,x)$ being still a strongly convex function of x with $\varphi(\lambda,1) > 0$ and $\varphi(\lambda,x) > 0$ for x large. This implies that $\varphi(\lambda,x)$ has two zeros $t(\lambda)$ and $u(\lambda)$, satisfying $1 < t(\lambda) < q+1 < u(\lambda)$, for each value of $\lambda > \omega$. As the minimum of $\varphi(\lambda,x)$ is attained for $x > q+1$, $\varphi(\lambda,x)$ is decreasing on $[1,q+1]$. For $\lambda > \omega$ we have the following analogue of lemma 3.2.

<u>Lemma 3.3</u>. If $\lambda > \omega$, then $\lim_{n\to\infty} z_n(\lambda) = t(\lambda)$.

<u>Proof</u>. As we have proved $z_n(\lambda) < q+1$ for $\lambda \geqslant \omega$, it suffices to show that every $z_n(\lambda)$ exceeds a fixed number, in order to apply theorem 1.1 in the same way as in the previous lemma. As $z_{n+1} - z_n$ can be negative only if $z_n > t(\lambda)$ we have that $z_n(\lambda) \geqslant$ $\geqslant \min\{t(\lambda) + b(\lambda), 1\}$.

<u>Remark</u>. As $\psi(\lambda,t(\lambda)) = -r(\lambda) > -2$ and $\chi(\lambda,x) > 0$, we have $\varphi(\lambda,x) > -r(\lambda)(x-t(\lambda))$ for $x > t(\lambda)$. If $z_n(\lambda) > t(\lambda)$, then $z_{n+1}(\lambda) > z_n(\lambda) + (n+1)^{-1} \varphi(\lambda,z_n(\lambda)) > (1 - r(\lambda)(n+1)^{-1}) z_n(\lambda) +$ $+ r(\lambda)(n+1)^{-1} t(\lambda) > t(\lambda)$. So $z_n(\lambda) - t(\lambda)$ does not change sign more than once.

Let

$$d(\lambda) = \frac{1}{6}\,\frac{1-q}{q^2}\cdot\left(\frac{q+1}{\lambda}\right)^{2q} = \frac{1}{3!}\left[\frac{\partial^3\ \varphi(\lambda,x)}{\partial x^3}\right]_{x=c(\lambda)}$$

then we have the following lemma.

Lemma 3.4.

$$z_n(\omega) = c(\omega) - \frac{1}{a(\omega)\log n} - \frac{d(\omega)}{(a(\omega))^3}\,\frac{\log\log n}{(\log n)^2} + \mathcal{O}\left(\frac{1}{(\log n)^3}\right). \quad (3.7)$$

Proof. Application of theorem 1.2 for the interval $[0,q+2]$ gives the result almost at once. We would only observe that lemma 3.1 means that $n(q+1 - z_n(\omega)) > q$, whereas $R_n(\omega,x)$ written as $n^{-1}\,\phi_0(x) + \mathcal{O}(n^{-2}\chi_0(x))$ on $[0,q+2]$ yields $\phi_0(q+1) = \frac{1}{2}q$. So $n(q+1 - z_n(\omega)) \to \infty$.

We conclude this Section with some remarks on the theorem that states the infinite inequality (2.1); it must be noticed, however, that the proof of (2.1) resulting from our remarks on finite sections, which we shall give below, is much more complicated than Elliott's proof given in [8] theorem 326. If we were interested in the infinite inequality only, our approach would be too complicated for the result. Since $q = p(1-p)^{-1}$, $\omega = (q+1)^{1+1/q}$ we have $\omega = (1-p)^{-1/p}$. Once (2.20) will have been proved, we shall have proved that for all convergent series with non-negative terms we have (2.1):

$$\Sigma_{n=1}^{\infty}\ \{n^{-1}(a_1^p + \ldots + a_n^p)\}^{1/p} \leqslant (1-p)^{-1/p}\ \Sigma_{n=1}^{\infty}\ a_n.$$

Indeed, if $\Sigma_{n=1}^{\infty}\ \{n^{-1}(b_1^p + \ldots + b_n^p)\}^{1/p} > (1-p)^{-1/p}\ \Sigma_{n=1}^{\infty}\ b_n$ for some sequence $\{b_n\}$, then there would exist a "finite" sequence $b_1,\ldots,b_m$, $0,\ldots$ with $\Sigma_{n=1}^{m}\ \{n^{-1}(b_1^p + \ldots + b_n^p)\}^{1/p} > (1-p)^{-1/p}\ \Sigma_{n=1}^{m}\ b_n$, and this contradicts formula (2.20). (Notice that $\lambda_1 < \lambda_2 < \ldots$, see Sec.2) Moreover, the constant $(1-p)^{-1/p}$ in (2.1) is best possible; for, if we replaced it by a smaller one, we could find, on account of (2.20), a "finite" series violating (2.1) with the new constant. The fact that there is strict inequality for all convergent series, is the only detail of the theorem quoted in Sec.2, which does not follow from (2.20). We shall revert to this question later. First we shall prove that for any sequence $\{x_n\}$, with $n^{-1}\ x_n^{-p}(x_1^p + \dots + x_n^p) = z_n(\omega)$ $(n=1,2,\dots)$, $\Sigma_{n=1}^{\infty} x_n$ diverges. (All such sequences differ only by a multiplicative constant.) Using lemma 3.1 we derive from $z_n(\omega) \leqslant q+1 - qn^{-1}$ $(n=1,2,\dots)$, that $(n(q+1) - q)\ x_n^p \geqslant x_1^p + \dots + x_n^p$; and this implies $x_n^p \geqslant$ $\geqslant (n-1)^{-1}\ (q+1)^{-1}(x_1^p + \dots + x_{n-1}^p)$ and thus

$$x_n^p \geqslant (n-1)^{-1}(q+1)^{-1}\ x_1^p\ \Pi_{\nu=1}^{n-2}\ (1 + \nu^{-1}(q+1)^{-1}) \quad (n \geqslant 2). \qquad (3.8)$$

Using the fact that there exists a positive constant C such that $\Pi_{\nu=1}^{n-2}\ (1 + \nu^{-1}(q+1)^{-1}) \geqslant C(n-1)^{1/(q+1)}$ we find that there exists a constant C^* (we may take for x_1 any positive number) such that $x_n \geqslant C^*\ (n-1)^{\sigma}$ $(n=2,\dots)$, with $\sigma = ((q+1)^{-1} - 1)\ p^{-1}$.
Because of the relation between p and q we have $\sigma = -1$ and so $\Sigma_{n=1}^{\infty} x_n$ diverges.
Using the result of lemma 3.4, we can obtain even much more information about the x_n. From (2.8), which reads for $\lambda = \omega$: $1 - (x_n^{-1}\ x_{n+1})^{1-p} = \omega^{-1}\ n^{-1}(z_n(\omega))^{1/q}$, (3.7), and the definitions of ω, $a(\omega)$, $c(\omega)$ and $d(\omega)$, we find by straightforward calculation:

$$\log x_{n+1} - \log x_n = -\frac{1}{n} + \frac{2}{n \log n} + \frac{\alpha \log \log n}{n\ (\log n)^2} + \mathcal{O}\left(\frac{1}{n(\log n)^2}\right),$$

where $\alpha = d(\omega)\left(q(a(\omega))^3\right)^{-1} = \frac{4}{3}(1-q^2)$. In order to obtain a formula for x_n we use a summation method (see [1] Ch.3). If $y_n =$ $= \log x_n + \log n - 2 \log \log n$ we have

$$y_n - y_{n+1} = -\frac{\alpha \log \log n}{n(\log n)^2} + \mathcal{O}\left(\frac{1}{n(\log n)^2}\right).$$

As $\Sigma_{\nu=n}^{\infty}\ (y_n - y_{n+1})$ converges we have $y_n = \Sigma_{\nu=n}^{\infty}\ (y_\nu - y_{\nu+1}) +$ $+ \lim_{n\to\infty} y_n$. If $\lim_{n\to\infty} y_n = \log A$ $(A > 0)$ we find

$$y_n = \log A - \alpha(\log n)^{-1} \log \log n + \mathcal{O}((\log n)^{-1}).$$

From this it is derived without difficulty that

$$x_n = An^{-1}\{(\log n)^2 - \alpha(\log n)(\log \log n) + \mathcal{O}(\log n)\}. \qquad (3.9)$$

The divergence of the special series $\Sigma_{n=1}^{\infty} x_n$ is not yet sufficient to prove the strict inequality in (2.1) for all convergent series (except $\Sigma_{n=1}^{\infty} 0$). In order to prove this, we can argue as follows. Let $\Sigma_{n=1}^{\infty} c_n$ be a convergent series giving equality, i.e.

$$\Sigma_{n=1}^{\infty} \{n^{-1}(c_1^p + \dots + c_n^p)\}^{1/p} = (1-p)^{-1/p} \Sigma_{n=1}^{\infty} c_n.$$

Now $c_1 \geqslant c_2 \geqslant \dots$, otherwise we should obtain a series violating (2.1) by rearrangement of the c's; moreover $c_n > 0$ for all n, and we may assume $c_1 \leqslant 1$. We take $y_n = n^{-1} c_n^{-p} (c_1^p + \dots + c_n^p)$ and prove that $y_n = z_n(\omega)$. As $y_1 = 1$, $y_{n+1} = (n+1)^{-1} + nc_n^p(n+1)^{-1} c_{n+1}^{-p} y_n$, we have to prove that $c_n^p c_{n+1}^{-p} = (1 - \omega^{-1} n^{-1} y_n^{1/q})^{-q}$, or that

$$\omega(c_n^{1-p} - c_{n+1}^{1-p}) = n^{-1} \{n^{-1}(c_1^p + \dots + c_n^p)\}^{-1+1/p}. \qquad (3.10)$$

In order to prove (3.10) for a fixed value of n, we consider for $m \geqslant n+2$ the function

$$H(x_1,\dots,x_m) = F(x_1,\dots,x_m) + \\ + \Sigma_{\nu=1}^{\infty} \{(m+\nu)^{-1}(x_1^p+\dots+x_m^p + c_{m+1}^p+\dots+c_{m+\nu}^p)\}^{1/p},$$

(F is the same function as in Sec.2). We know that H attains the maximum $\omega\Sigma_{\nu=1}^{\infty} c_\nu$ on the set, defined by $\Sigma_{\nu=1}^{m} x_\nu = \Sigma_{\nu=1}^{m} c_\nu$, $x_1 \geqslant 0,\dots,x_m \geqslant 0$, at the point $(c_1,\dots,c_m)$. On the compact set V defined by $c_{m+1} \leqslant x_1,\dots,x_m \leqslant 1$ the partial derivatives of H are obtained by differentiating term by term, and hence

$$\frac{\partial}{\partial x_k} H(x_1,\dots,x_m) = \frac{\partial}{\partial x_k} F(x_1,\dots,x_m) + \\ + x_k^{p-1} \Sigma_{\nu=1}^{\infty}(m+\nu)^{-1}\{(m+\nu)^{-1}(x_1^p+\dots+x_m^p+c_{m+1}^p+\dots+c_{m+\nu}^p)\}^{-1+1/p}$$

(k=1,...,m), as the latter series converges uniformely on V; (it has $\Sigma_{\nu=1}^{\infty} c_\nu'$ as a majorant, where $c_n' = 1$ if n=1,...,m, $c_n' = c_n$, if $n > m$). $H(x_1,\dots,x_m)$ is a homogeneous function of degree 1. We infer that

$$\left(\frac{\partial}{\partial x_k} H(x_1,\dots,x_m)\right)_{x_1=c_1,\dots,x_m=c_m} = \omega \qquad (k=1,\dots,m).$$

The latter relation is the same as

$$c_k^{p-1}\ \Sigma_{n=k}^{\infty}\ n^{-1}(n^{-1}(c_1^p + \ldots + c_n^p))^{-1+1/p} = \omega \quad (k=1,\ldots,m). \quad (3.11)$$

Multiplying both sides of (3.11) by c_k^{1-p}, and subtracting the formulas for k=n and k=n+1, we obtain (3.10).

By this we have proved that there is strict inequality in (2.1) for all non-trivial convergent series.

4. A fundamental result about $\varphi(\lambda,z_n(\lambda))$

As we are interested in what happens if $\lambda\uparrow\omega$, we may confine ourselves to values of λ for which $\omega-\lambda$ is small. So, from now on we shall only consider values of λ in $(\lambda_0,\omega]$ where $\lambda_0 = (q+1)^{1/q}$, but further restrictions will be made in the following Sections. If $\lambda > \lambda_0$ and $z_n(\lambda) < q+1$ then certainly $n \leqslant N_\lambda$, so the restriction guarantees that no breakdown occurs, when $z_n(\lambda) < q+1$.

We have the following monotonicity property: if $\lambda \leqslant \omega$ then $z_1(\lambda) < z_2(\lambda) < \ldots < z_{N_\lambda}(\lambda)$ if $N_\lambda < \infty$ or $z_1(\lambda) < z_2(\lambda) < \ldots$, if $N_\lambda = \infty$. This follows at once from (2.13), as $z_{n+1}(\lambda) - z_n(\lambda) > 0$.

The present Section is almost completely devoted to the fundamental lemma 4.1. The version in which we prove it, is, however, somewhat stronger than the one we actually need. The usefulness of this lemma was already suggested by the proof of theorem 1.2. It is analogous to part (3) of that proof. However, it should be remarked that the situation is now essentially more difficult, as we state a result which is uniform in λ. We refer to the methodological observations made in Sec.1 (Remark 4).

Lemma 4.1. For every positive number α there exists a constant C_α ($C_\alpha > 0$) such that for all $\lambda \in (\lambda_0,\omega]$, and all n $(1 \leqslant n < N_\lambda + 1$, if $N_\lambda < \infty$ or $1 \leqslant n < \infty$, if $N_\lambda = \infty)$, $\varphi(\lambda,z_n(\lambda))$ satisfies

$$\varphi(\lambda,z_n(\lambda)) > C_\alpha\ n^{-\alpha}. \quad (4.1)$$

Proof. It obviously suffices to prove that for every α ($\alpha > 0$) there exist C_α ($C_\alpha > 0$) and $\zeta_\alpha \in [\lambda_0, \omega)$ such that (4.1) holds for all $\lambda \in (\zeta_\alpha, \omega]$, as a matter of fact, if we take $C^*_\alpha = \min\{C_\alpha, b(\zeta_\alpha)\}$ we have $\varphi(\lambda, z_n(\lambda)) > C^*_\alpha\, n^{-\alpha}$ for all $\lambda \in (\lambda_0, \omega]$. Moreover, it will be clear that once we have found ζ_{α_1} and C_{α_1}, we may take $\zeta_\alpha = \zeta_{\alpha_1}$ and $C_\alpha = C_{\alpha_1}$ for every $\alpha > \alpha_1$. So it suffices to prove lemma 4.1 for $\alpha < \frac{1}{2}$. Further, the result $\varphi(\omega, z_n(\omega)) > C_{\alpha,\omega}\, n^{-\alpha}$ is a consequence of lemma 3.4; so we confine ourselves to values of $\lambda < \omega$.

$\varphi(\lambda,x)$, $\phi(\lambda,x)$ and $\chi(\lambda,x)$ are continuous functions for $\lambda_0 < \lambda < \infty$, $0 < x < \infty$. (For the notations we refer to Sec.2.) In the point $(\omega, q+1)$ we have $\varphi(\omega,q+1) = 0$, $\phi(\omega,q+1) = 0$, $\chi(\omega,q+1) > 0$, so there exist b_1, η_1 and γ_1 with $b_1 > 0$, $\lambda_0 \leqslant \eta_1 < \omega$, $1 \leqslant \gamma_1 < q+1$ such that for the set G defined by $\eta_1 < \lambda < \omega$, $\gamma_1 < x < q+1$ the following propositions hold:

(i) if $(\lambda,x) \in G$, then $|\varphi(\lambda,x)| = \varphi(\lambda,x) < 1$; (4.2)

(ii) if $(\lambda,x) \in G$, then $|\phi(\lambda,x)| < \frac{1}{2}\alpha$; (4.3)

(iii) $\inf\{\chi(\lambda,x) \mid (\lambda,x) \in G\} = 2b_1$. (4.4)

From the results of Sec.3 we use that $\lim_{n\to\infty} z_n(\omega) = q+1$, and that $z_n(\omega) \leqslant q+1 - qn^{-1} = c(\omega) - qn^{-1} < c(\omega) - (9/10)qn^{-1}$. This enables us to choose an index m, which we take $\geqslant 6$, for reasons that will be explained below, with the property

$$\gamma_1 < z_m(\omega) < c(\omega) - \frac{9}{10}\, qm^{-1}. \tag{4.5}$$

Using the continuity of $z_m(\lambda)$ in $\lambda = \omega$, and the continuity of $c(\lambda)$ we are able to determine η_2 with $\eta_1 \leqslant \eta_2 < \omega$ such that for $\lambda \in (\eta_2, \omega]$ we still have

$$\gamma_1 < z_m(\lambda) < c(\lambda) - \frac{9}{10}\, qm^{-1}. \tag{4.6}$$

We define an index A_λ depending on λ as

$$A_\lambda = \max\{n \mid z_n(\lambda) < c(\lambda)\}$$

if the maximum exists and $A_\lambda = \infty$ if $z_n(\lambda) < c(\lambda)$ for all n. As we speak only about values of λ exceeding λ_0, it will be obvious that we have $A_\lambda \leqslant N_\lambda$. For $\lambda \in (\eta_2, \omega)$ it is trivial that $A_\lambda \geqslant m$, whereas $A_\lambda < \infty$ for those values of λ. In order to prove the latter proposition it suffices to remark that for fixed $\lambda \in (\lambda_0, \omega)$ we have $z_{n+1}(\lambda) - z_n(\lambda) > (n+1)^{-1}\, b(\lambda)$, so the sequence $z_2(\lambda), z_3(\lambda), \ldots$ increases faster than $\{B_\lambda \log n\}$ $(B_\lambda > 0)$, possibly until the process breaks down, but it will certainly not break down for $z_n(\lambda) < q+1$.

From now on we consider only values of $\lambda \in (\eta_2, \omega)$. We have to estimate $R_n(\lambda, z_n(\lambda))$. If $z_n(\lambda) < c(\lambda)$ then $R_n(\lambda, z_n(\lambda)) < R_n(\lambda, c(\lambda))$; and $R_n(\lambda, c(\lambda)) < R_n(\omega, c(\omega)) = R_n(\omega, q+1)$ for $\lambda \in (\eta_2, \omega)$. Using the trivial fact that $(q+\nu) < (q+1)(\nu+1)$ for all natural ν, we have that $R_n(\omega, q+1) < \frac{1}{2}q\, \Sigma_{k=1}^{\infty}\, n^{-k} = \frac{1}{2}q(n-1)^{-1}$; and this is not larger than $(3/5)\,qn^{-1}$ for $n \geqslant m$. Combination of these estimates yields for $\lambda \in (\eta_2, \omega)$ and $m \leqslant n \leqslant A_\lambda$

$$R_n(\lambda, z_n(\lambda)) \leqslant \frac{3}{5}\, qn^{-1}. \tag{4.7}$$

For a fixed λ in (η_2, ω) we have that $\varphi(\lambda, x)$ is a decreasing, strongly convex function of x in $(\gamma_1, c(\lambda)]$; therefore, for $x \in (\gamma_1, c(\lambda))$ we have

$$\varphi(\lambda, x) < \frac{\varphi(\lambda, \gamma_1) - \varphi(\lambda, c(\lambda))}{c(\lambda) - \gamma_1}\, (c(\lambda) - x) + b(\lambda). \tag{4.8}$$

Application of (4.7) and (4.8) to (2.13) gives

$$z_{n+1}(\lambda) - z_n(\lambda) < \frac{1}{n+1}\left[\frac{\varphi(\lambda, \gamma_1) - \varphi(\lambda, c(\lambda))}{c(\lambda) - \gamma_1}(c(\lambda) - z_n(\lambda)) + b(\lambda) + \frac{3}{5}\, qn^{-1}\right], \tag{4.9}$$

for $\lambda \in (\eta_2, \omega)$, $m \leqslant n \leqslant A_\lambda$.

The mean value theorem allows us to write

$$0 < \frac{\varphi(\lambda, \gamma_1) - \varphi(\lambda, c(\lambda))}{c(\lambda) - \gamma_1} = |\phi(\lambda, x_1)|, \tag{4.10}$$

where $x_1 \in (\gamma_1, c(\lambda))$, so $(\lambda, x_1) \in G$.

Combination of (4.9), (4.10) and (4.3) gives

$$z_{n+1}(\lambda) - z_n(\lambda) < \frac{1}{n+1}\left\{\tfrac{1}{2}\alpha(c(\lambda) - z_n(\lambda)) + b(\lambda) + \frac{3}{5}\,\frac{q}{n}\right\}, \tag{4.11}$$

for $\lambda \in (\eta_2, \omega)$, $m \leqslant n \leqslant A_\lambda$.

So we have approximated (2.13) for $m \leqslant n \leqslant A_\lambda$ by a linear recurrence relation. By the substitution

$$w_n(\lambda) = c(\lambda) - z_n(\lambda) + 2\alpha^{-1} b(\lambda) - (1 - \tfrac{1}{2}\alpha)^{-1}\, \frac{3}{5}\, qn^{-1} \tag{4.12}$$

(4.11) is reduced to the homogeneous relation $w_n(\lambda) - w_{n+1}(\lambda) < \frac{1}{2}(n+1)^{-1} \alpha\, w_n(\lambda)$, which can be written as

$$w_{n+1}(\lambda) > (1 - \tfrac{1}{2}\alpha(n+1)^{-1})\, w_n(\lambda). \tag{4.13}$$

For $\lambda \in (\eta_2, \omega)$, $m \leqslant n \leqslant A_\lambda$, we derive, from (4.13) and $\frac{1}{2}\alpha < 1$, that

$$w_{n+1}(\lambda) > w_n(\lambda)\ (1 - (n+1)^{-1})^{\frac{1}{2}\alpha}. \tag{4.14}$$

From (4.6) and $(1-\frac{1}{2}\alpha) > \frac{1}{2}$ it follows that for $\lambda \in (\eta_2, \omega]$, $c(\lambda) - z_m(\lambda) - (1-\frac{1}{2}\alpha)^{-1}(3/5)qm^{-1} > (1/10)qm^{-1}$ and so $w_m(\lambda) > \frac{1}{10}\, qm^{-1}$.
Now (4.14) implies $w_{m+1}(\lambda) > 0, \ldots, w_{A_\lambda+1}(\lambda) > 0$, and $w_n(\lambda) > w_m(\lambda)\, m^{\frac{1}{2}\alpha}\, n^{-\frac{1}{2}\alpha}$ for $m < n \leqslant A_\lambda+1$. From the definition of A_λ and (4.12) we infer that $w_{A_\lambda+1}(\lambda) < 2\alpha^{-1}\, b(\lambda)$. So $w_{A_\lambda+1}(\lambda) > w_m(\lambda)\, m^{\frac{1}{2}\alpha}\, (A_\lambda+1)^{-\frac{1}{2}\alpha}$ gives

$$2\alpha^{-1}\, b(\lambda) > m^{\frac{1}{2}\alpha}\, (A_\lambda+1)^{-\frac{1}{2}\alpha}\, w_m(\lambda). \tag{4.15}$$

Therefore, if $n > m$ and

$$n < \{2\alpha^{-1} b(\lambda)\}^{-2/\alpha}\, m(w_m(\lambda))^{2/\alpha}, \tag{4.16}$$

then $2\alpha^{-1} b(\lambda) < m^{\frac{1}{2}\alpha}\, n^{-\frac{1}{2}\alpha}\, w_m(\lambda)$ and so by (4.15) $n < A_\lambda+1$, $n \leqslant A_\lambda$ which yields

$$c(\lambda) - z_n(\lambda) + 2\alpha^{-1} b(\lambda) > w_n(\lambda) > m^{\frac{1}{2}\alpha}\, n^{-\frac{1}{2}\alpha}\, w_m(\lambda). \tag{4.17}$$

If $m < n \leqslant A_\lambda$, $\lambda \in (\eta_2, \omega)$, then $z_n(\lambda) < c(\lambda)$ and $(\lambda, z_n(\lambda)) \in G$; and by application of (4.4) and Taylor's theorem we get

$$\varphi(\lambda, z_n(\lambda)) > b(\lambda) + b_1 \cdot (c(\lambda) - z_n(\lambda))^2. \tag{4.18}$$

As (4.2) implies $b(\lambda) < 1$ if $\lambda \in (\eta_2, \omega)$, we derive from (4.18)

$$\begin{aligned}\varphi(\lambda, z_n(\lambda)) &> b(\lambda))^2 + b_1 \cdot (c(\lambda) - z_n(\lambda))^2 \geqslant \\ &\geqslant \tfrac{1}{4}\alpha^2 b_2 \{4\alpha^{-2} (b(\lambda))^2 + (c(\lambda) - z_n(\lambda))^2\},\end{aligned} \tag{4.19}$$

where $b_2 = \min\{1,\ 4\alpha^{-2} b_1\}$.
Using $d^2 + e^2 \geqslant \frac{1}{2}(d+e)^2$ which holds for all d and e, we get from (4.19)

$$\varphi(\lambda, z_n(\lambda)) > \tfrac{1}{8}\alpha^2 b_2 \{2\alpha^{-1} b(\lambda) + c(\lambda) - z_n\}^2. \tag{4.20}$$

For values of n satisfying $n > m$ and (4.16), we obtain when combining (4.17) with (4.20)

$$\varphi(\lambda, z_n(\lambda)) > \frac{1}{8}\alpha^2 b_2\ m^\alpha\ w_m^2(\lambda)\ n^{-\alpha}. \tag{4.21}$$

If we denote $\frac{1}{8}\alpha^2 b_2\ m^{\alpha-2}\ q^2 \cdot 10^{-2}$ by b_3, then $b_3 > 0$ and for values of n satisfying $n > m$ and (4.16) we infer from (4.21) that $\varphi(\lambda, z_n(\lambda)) > b_3\ n^{-\alpha}$. For values of n for which (4.16) does not hold we have

$$\varphi(\lambda, z_n(\lambda)) \geqslant b(\lambda) > (b(\lambda))^2 \geqslant \frac{1}{4}\alpha^2 m^\alpha (w_m(\lambda))^2 n^{-\alpha}. \tag{4.22}$$

With $b_4 = \frac{1}{4}\alpha^2\ m^{\alpha-2}\ q^2 \cdot 10^{-2}$, (4.22) implies that $\varphi(\lambda, z_n(\lambda)) > b_4 n^{-\alpha}$. Finally for $1 \leqslant n \leqslant m$ we have

$$\varphi(\lambda, z_n(\lambda)) \geqslant \inf \{\varphi(\lambda, x) \mid (\lambda, x) \in G^*\} = b_5 > 0, \tag{4.23}$$

where G^* denotes the set defined by $\eta_2 < \lambda < \omega$, $1 \leqslant x < q+1 - \frac{9}{10}\ qm^{-1}$. So $\varphi(\lambda, z_n(\lambda)) > b_5 n^{-\alpha}$ for $1 < n \leqslant m$. If $b_6 = q\omega^{-1}$ then $\varphi(\lambda, z_1) = \varphi(\lambda, 1) = q\lambda^{-1} > b_6$.

If we take $\zeta_\alpha = \eta_2$; $C_\alpha = \min\{b_3, b_4, b_5, b_6, C_{\alpha,\omega}\}$ then for all $\lambda \in (\zeta_\alpha, \omega]$ and all n with $1 \leqslant n < N_\lambda + 1$ (or $1 \leqslant n < \infty$), we have $\varphi(\lambda, z_n(\lambda)) > C_\alpha\ n^{-\alpha}$. This completes the proof of lemma 4.1.

In the following Sections we shall use lemma 4.1 only for $\alpha = \frac{1}{2}$ and write C_1 instead of $C_{\frac{1}{2}}$. (Later, we shall use constants which will be denoted $C_2, C_3, \ldots$, but this notation has nothing to do with the C_α of this lemma.)

Lemma 4.2. For each number A there exists a number ζ_A, $\zeta_A \in [\lambda_0, \omega)$ such that $\lambda \in (\zeta_A, \omega)$ implies $A_\lambda > A$.

Proof. Let $A > 0$. Using $z_{[A]+1}(\omega) < c(\omega)$ and the continuity of $c(\lambda)$ and of $z_{[A]+1}(\lambda)$ in $\lambda = \omega$ we choose $\zeta_A \in [\lambda_0, \omega)$ such that for $\zeta_A < \lambda \leqslant \omega$ we still have $z_{[A]+1}(\lambda) < c(\lambda)$ which implies $A_\lambda \geqslant [A]+1$.

We will conclude this Section with a discussion of the proof of lemma 4.1. Among the properties used, the convexity of $\varphi(\lambda, x)$ should be mentioned first. The continuity of $z_n(\lambda)$ and $c(\lambda)$ and the results of Sec.3 [viz. $\lim_{n\to\infty} z_n(\omega) = c(\omega)$ and the estimate given by lemma 3.1 which is of the form $z_n(\omega) < c(\omega) - \beta_1 n^{-1}$ $(\beta_1 > 0)$] enable us to choose m and η_2 such that (4.6) holds for m and $\lambda \in (\eta_2, \omega)$. (4.6) is of the form

$$\gamma_1 < z_m(\lambda) < c(\lambda) - \beta_1 m^{-1}.$$

Moreover, we have the estimate of R_n given by (4.7) which can be written as $R_n(\lambda, z_n(\lambda)) \leqslant \beta_2 n^{-1}$, and which holds for $\lambda \in (\eta_2, \omega)$, $1 \leqslant z_n \leqslant c(\lambda)$. The substitution (4.12) which can be written as

$$w_n(\lambda) = c(\lambda) - z_n(\lambda) + 2\alpha^{-1} b(\lambda) - \beta_3 n^{-1}.$$

gives the homogeneous expression (4.13) because $\frac{1}{2}\alpha\beta_3 + \beta_2 = \beta_3$. Further we have used $w_m(\lambda) > 0$ which followed from $\beta_1 - \beta_3 > 0$. At the beginning of the proof we observed that it is sufficient to consider only small values of α. As we have to take $\beta_3 = (1-\frac{1}{2}\alpha)^{-1}\beta_2$ it follows that we can restrict ourselves to small values of α, obtaining $\beta_3 < \beta_1$ if and only if $\beta_2 < \beta_1$.
We remarked already (see Sec.1 Remark 4) that we could give a different proof of the existence of a constant C with $\varphi(\lambda, z_n(\lambda)) > C\, n^{-\frac{1}{2}}$, starting from the fact that $z_n(\omega) < (q+1 - qn^{-1})$ implies $n(q+1 - z_n(\omega)) \to \infty$ (see lemma 3.4).

5. Some auxiliary results

In this Section we shall introduce some new notations and discuss some results in preparation of Sec.6. The usefulness of these results will be clear in that Section. We start, introducing the numbers ρ and σ defined by

$$\rho = (2\omega)^q q^{-q}$$
$$\sigma = \max\{2^{1+1/q}\omega q^{-1},\ 4(q+1)\omega q^{-1}\}. \tag{5.1}$$

We confine ourselves to values of $\lambda \in (\lambda_1, \omega)$, where $\lambda_1 = \max(\frac{1}{2}q+1, \lambda_0)$.

<u>Lemma 5.1</u>. If $\lambda \in (\lambda_1, \omega)$, and $x > \rho$ then $\varphi(\lambda, x) > q(2\lambda)^{-1} x^{1+1/q}$.

<u>Proof</u>. First we observe that $\varphi(\lambda,\rho) - q(2\lambda)^{-1} \rho^{1+1/q} =$ $= 1 - \rho + q(2\lambda)^{-1} \rho^{1+1/q} > \rho(-1 + q(2\lambda)^{-1} \rho^{1/q}) = \rho(-1 + \omega\lambda^{-1}) \geqslant 0$. Further, $\varphi(\lambda,x) - q(2\lambda)^{-1} x^{1+1/q}$ is an increasing function for $x > \rho$, its derivative with respect to x, $-1 + (q+1)(2\lambda)^{-1} x^{1/q}$ being positive for $x > \rho > (2\lambda)^q(q+1)^{-q}$.

<u>Lemma 5.2</u>. If $A_\lambda \geqslant \sigma$, and $N_\lambda < \infty$ then $z_{N_\lambda+1}(\lambda) > 2\rho$.

<u>Proof</u>. We remember $A_\lambda \leqslant N_\lambda$. By the definition of N_λ we have $z_{N_\lambda+1} \geqslant \lambda^q(N_\lambda+1)^q$, and as $\lambda > \lambda_1 \geqslant \lambda_0 > 1$, $N_\lambda+1 > A_\lambda \geqslant \sigma$ we have $z_{N_\lambda+1} > \sigma^q > 2\rho$.

In virtue of lemma 4.2 there exists a $\lambda_2 \in [\lambda_1, \omega)$ such that $\lambda \in (\lambda_2, \omega]$ implies $A_\lambda \geqslant \sigma$; to show this we only have to take $\lambda_2 = \max\{\lambda_1, \zeta_{\sigma-1}\}$. From now on we shall consider only values of λ in (λ_2, ω). We give the following definitions of D_λ and E_λ

$$D_\lambda = \max\{n \mid z_n(\lambda) < \rho\}, \tag{5.2}$$

if the maximum exists, if the maximum does not exist, and so $z_n(\lambda) < \rho$ for all n, we define $D_\lambda = \infty$.

$$E_\lambda = \max\{n \mid (z_n(\lambda))^{1/q} < \tfrac{1}{2} n\lambda(q+1)^{-1},\ n \geqslant A_\lambda\} \tag{5.3}$$

if the maximum exists; if $z_n(\lambda) < (\frac{1}{2}n\lambda)^q(q+1)^{-q}$ for arbitrarily large n, we define $E_\lambda = \infty$. We observe that the index-set W_λ, of which E_λ is the maximum is not empty if $\lambda \in (\lambda_2, \omega)$; as $c(\lambda) < q+1 <$ $< \{\frac{1}{2}\sigma(\frac{1}{2}q+1)(q+1)^{-1}\}^q$, we have $A_\lambda \in W_\lambda$. Further, $D_\lambda \geqslant A_\lambda$, as $\rho >$ $> c(\lambda)$. If $\lambda \in (\lambda_2, \omega)$, then $D_\lambda \leqslant N_\lambda$, since $z_{N_\lambda+1}(\lambda) > 2\rho$ if the process should break down; and $D_\lambda < \infty$ as the sequence $\{z_n(\lambda)\}$ cannot be bounded, since $z_{n+1}(\lambda) - z_n(\lambda) > (n+1)^{-1}b(\lambda)$.

The following lemma states a property of E_λ.

<u>Lemma 5.3</u>. There exists $\lambda_3 \in [\lambda_2, \omega)$, such that $\lambda \in (\lambda_3, \omega)$ and $n \leqslant E_\lambda$ (or $n < \infty$ if $E_\lambda = \infty$) imply that

$$[z_n(\lambda)]^{1/q} < \tfrac{1}{2} n\lambda(q+1)^{-1}.$$

Proof. If $z_n(\lambda) \geq \{\frac{1}{2} n\lambda(q+1)^{-1}\}^q$ for some $n \leq N_\lambda$, then we have (see (2.11))

$$z_{n+1}(\lambda) \geq \Phi_n \left(\lambda, \{ \frac{n\lambda}{2(q+1)}\}^q\right) = \frac{1}{n+1} + \frac{n}{n+1} \cdot \frac{\lambda^q n^q}{(2q+1)^q},$$

and the latter quantity is certain to exceed $\{\frac{1}{2}(n+1)(q+1)^{-1}\}^q$, if $n^{q+1}(n+1)^{-(q+1)} > (2q+1)^q(2q+2)^{-q}$, which is the case if $n > A^*$. So we take $\lambda_3 \in [\lambda_2, \omega)$ such that $A_\lambda > A^*$ for $\lambda \in (\lambda_3, \omega)$, (thus $\lambda_3 \geq \max(\lambda_2, \zeta_{A^*})$).

Lemma 5.3 has the consequence that $E_\lambda < \infty$ implies

$$\max\{n \mid (z_n(\lambda))^{1/q} < \tfrac{1}{2} n\lambda(q+1)^{-1},\ n \geq A_\lambda\} =$$
$$= \min\{n \mid (z_n(\lambda))^{1/q} \geq \tfrac{1}{2} n\lambda(q+1)^{-1},\ n \geq A_\lambda\} - 1,$$

for $\lambda \in (\lambda_3, \omega)$.

We shall use the following lemma, in order to obtain a relation between D_λ and E_λ.

Lemma 5.4. There exists a number $\lambda_4 \in [\lambda_3, \omega)$ such that
$z_{D_\lambda+1}(\lambda) < 2\rho$ for $\lambda \in (\lambda_4, \omega)$.

Proof. If $z_{D_\lambda+1}(\lambda) \geq 2\rho$, then we conclude from $z_{D_\lambda}(\lambda) < \rho$ that

$$\rho < z_{D_\lambda+1}(\lambda) - z_{D_\lambda}(\lambda) = (D_\lambda+1)^{-1} \{\varphi(\lambda, z_{D_\lambda}(\lambda)) + R_{D_\lambda}(\lambda, z_{D_\lambda}(\lambda))\};$$

and this implies

$$\rho < D_\lambda^{-1} (p(1) + R_{D_\lambda}(1, \rho)), \tag{5.4}$$

where $p(\lambda) = \max\{\varphi(\lambda, 1), \varphi(\lambda, \rho)\}$.

But (5.4) is impossible if D_λ is large enough, say $D_\lambda > B$. Thus it suffices to take $\lambda_4 = \max\{\lambda_3, \zeta_B\}$ (see lemma 4.2), as we have then $D_\lambda \geq A_\lambda > B$ for $\lambda \in (\lambda_4, \omega)$. This completes the proof.

Lemma 5.5. If $\lambda \in (\lambda_4, \omega)$, then $D_\lambda < E_\lambda$.

Proof. This lemma is an immediate consequence of the preceding one. As $z_{D_\lambda+1}(\lambda) < 2\rho$ we have $(z_{D_\lambda+1}(\lambda))^{1/q} < (2\rho)^{1/q} \leq \frac{1}{2}\sigma(q+1)^{-1} <$ $< \frac{1}{2}(D_\lambda+1)\lambda(q+1)^{-1}$; and this inequality (for which we use

$D_\lambda \geqslant A_\lambda \geqslant \sigma$, $\lambda \geqslant \lambda_0 > 1$) implies $D_\lambda + 1 \leqslant E_\lambda$.

We shall conclude this Section with two lemmas concerning the parts of the recursion $1 - D_\lambda$ and $D_\lambda - E_\lambda$ (or $D_\lambda - \infty$) respectively. Lemma 5.6 is analogous to a part of the proof of theorem 1.2, and it will be employed in the same way as its analogue in theorem 1.2.

Lemma 5.6. There exists a constant C_2, such that for $\lambda \in (\lambda_4, \omega)$, $1 \leqslant n \leqslant D_\lambda$ and $x \in (z_n(\lambda), z_{n+1}(\lambda))$ the following inequality holds:

$$|\varphi(\lambda,x) - \varphi(\lambda, z_n(\lambda))| < C_2 n^{-1}. \tag{5.5}$$

Proof. From lemma 5.4 it follows that $z_n(\lambda) < 2\rho$ for all n with $1 \leqslant n \leqslant D_\lambda + 1$, (notice that $z_{n+1}(\lambda) > z_n(\lambda)$ if $n < N_\lambda$). First, we suppose $2(2\rho)^{1/q} \leqslant n \leqslant D_\lambda$, (as $D_\lambda \geqslant \sigma \geqslant 2(2\rho)^{1/q}$ there are values of n satisfying this condition). As $\lambda > \lambda_4$ implies $\lambda > \lambda_1 \geqslant \frac{1}{2}q+1$ (and thus $\lambda > 1$) we have $2\lambda > q+2$, $3\lambda > q+3$, ... For $\lambda \in (\lambda_4, \omega)$ and $2(2\rho)^{1/q} \leqslant n \leqslant D_\lambda$ we have therefore the following estimates:

$$\begin{aligned} R_n(\lambda, z_n(\lambda)) &< R_n(\lambda, 2\rho) = \\ &= \Sigma_{\nu=2}^{\infty}\, q(q+1)\cdots(q+\nu-1)(\nu!)^{-1}\lambda^{-\nu} n^{1-\nu}(2\rho)^{1+\nu/q} < \\ &< q(q+1)\lambda^{-2}(2\rho)^{1+1/q}\, \Sigma_{\nu=2}^{\infty}\, \nu^{-1}(n^{-1}(2\rho)^{1/q})^{\nu-1} < \\ &< q(q+1)\lambda_4^{-2}(2\rho)^{1+2/q}\, n^{-1} = d_1 n^{-1} < \tfrac{1}{2} d_1 (2\rho)^{-1/q} = d_2 . \end{aligned}$$

If $\max\{ R_n(\lambda_4, z_n(\lambda_4)) \mid 1 \leqslant n < 2(2\rho)^{1/q}\} = d_3$, and $\max(d_2, d_3) = d_4$, then we have for all $\lambda \in (\lambda_4, \omega)$ and $1 \leqslant n \leqslant D_\lambda$ the estimate(*)

$$R_n(\lambda, z_n(\lambda)) \leqslant d_4 . \tag{5.6}$$

Let T denote the compact set $\{(\lambda,x) \mid \lambda \in [\lambda_4, \omega],\ x \in [1, 2\rho]\}$.
Let $\max\{\varphi(\lambda,x) \mid (\lambda,x) \in T\} = c_1$; $\max\{|\phi(\lambda,x)| \mid (\lambda,x) \in T\} = c_2$.

(*) For the present lemma the estimate (5.6) will do; for further use (lemma 6.2), we observe that we have in fact

$$R_n(\lambda, z_n(\lambda)) \leqslant d_5\, n^{-1}, \tag{5.7}$$

for $\lambda \in (\lambda_4, \omega)$ and $1 \leqslant n \leqslant D_\lambda$, if $d_5 = \max(d_1,\ d_3 \cdot 2(2\rho)^{1/q})$.

If $\lambda \in (\lambda_4, \omega)$, $1 \leqslant n \leqslant D_\lambda$ and $x \in (z_n(\lambda), z_{n+1}(\lambda))$, then we have by the mean value theorem and the estimate (5.6)

$$|\varphi(\lambda,x) - \varphi(\lambda,z_n)| \leqslant c_2 \ (x-z_n) < c_2 \ (z_{n+1}-z_n) =$$
$$= \frac{c_2}{n+1}\{\varphi(\lambda,z_n) + R_n(\lambda,z_n)\} \leqslant \frac{c_2}{n+1} \ (c_1 + d_4) \leqslant \frac{2c_2\,(c_1+d_4)}{n} = \frac{C_2}{n}.$$

Lemma 5.7. There exists a constant C_3 ($C_3 > 0$) such that for $\lambda \in (\lambda_4, \omega)$, $D_\lambda < n < E_\lambda$ (or $D_\lambda < n < \infty$) and $x \in (z_n(\lambda), z_{n+1}(\lambda))$:

$$\varphi(\lambda,x) < C_3 \ \varphi(\lambda,z_n(\lambda)).$$

Proof. If $E_\lambda = D_\lambda + 1$, there is nothing to prove. We start proving that $R_n(\lambda,z_n(\lambda)) < \varphi(\lambda,z_n(\lambda))$ if $\rho < z_n(\lambda) < 2^{-q}n^q\lambda^q(q+1)^{-q}$. In that case we have

$$R_n(\lambda,z_n(\lambda)) < \frac{q}{2\lambda}(z_n(\lambda))^{1+1/q}\left\{\frac{1}{2} + \frac{q+2}{3(q+1)}\cdot\frac{1}{4} + \frac{(q+2)(q+3)}{3\cdot 4(q+1)^2}\cdot\frac{1}{8}+\ldots\right\} <$$
$$< \frac{q}{2\lambda}(z_n(\lambda))^{1+1/q}$$

(because $(q+\nu) < (\nu+1)(q+1)$ for all natural ν). Lemma 5.1 gives immediately that $R_n(\lambda,z_n(\lambda)) < \varphi(\lambda,z_n(\lambda))$. So

$$z_{n+1}(\lambda) - z_n(\lambda) < 2(n+1)^{-1} \ \varphi(\lambda,z_n(\lambda)). \tag{5.8}$$

Because $z_{D_\lambda+1}(\lambda) > \rho > c(\lambda)$ it follows that for $x \in (z_n(\lambda), z_{n+1}(\lambda))$ we have

$$\varphi(\lambda,x) < \varphi(\lambda,z_{n+1}(\lambda)) = \varphi(\lambda,z_n(\lambda)) + (z_{n+1}(\lambda)-z_n(\lambda))\psi(\lambda,y)$$

with $y \in (z_n(\lambda), z_{n+1}(\lambda))$. For $y > c(\lambda)$, however, we have

$$0 < \psi(\lambda,y) = -1 + (q+1)\lambda^{-1}y^{1/q} < (q+1)\lambda^{-1}y^{1/q}.$$

From this, by applying the formulas (5.8) and (5.3), we obtain without difficulty $\varphi(\lambda,x) < \varphi(\lambda,z_n(\lambda)) + (z_{n+1}(\lambda) - z_n(\lambda)) \cdot$ $\cdot\ (q+1)\lambda^{-1}(z_{n+1}(\lambda))^{1/q} < \varphi(\lambda,z_n(\lambda))\{1+2(n+1)^{-1}(q+1)\lambda^{-1}(z_{n+1}(\lambda))^{1/q}\} <$ $< 2\varphi(\lambda,z_n(\lambda))$, which holds for $\lambda \in (\lambda_4, \omega)$, $D_\lambda < n < E_\lambda$ (or $D_\lambda < n < \infty$ if $E_\lambda = \infty$) and $x \in (z_n(\lambda), z_{n+1}(\lambda))$. So we have proved lemma 5.7, and we have found at the same time that $C_3 = 2$ meets the requirements.

6. The behaviour of E_λ and N_λ

The aim of this Section is to prove formula (2.17). We define for $\lambda \in (\lambda_4, \omega)$ and $y \in [1, \infty)$

$$\theta_\lambda(y) = \int_1^y (\varphi(\lambda,x))^{-1}\, dx.$$

Notice, that $\int_1^\infty (\varphi(\lambda,x))^{-1}\, dx < \infty$ for $\lambda \in (\lambda_4, \omega)$, as $1 + \frac{1}{q} > 1$. With this notation (2.17) reads

$$\log N_\lambda = \theta_\lambda(\rho) + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{6.1}$$

The validness of this formula is stated in the following lemma.

<u>Lemma 6.1</u>. If $\lambda < \omega$, then $N_\lambda < \infty$ and (6.1) holds.

The proof of lemma 6.1 consists of three different parts related to the parts of the recursion $1 - D_\lambda$, $D_\lambda - E_\lambda$ and $E_\lambda - N_\lambda$ respectively. We shall formulate the partial results in the following lemmas.

<u>Lemma 6.2</u>.
$$\log D_\lambda = \theta_\lambda(\rho) + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{6.2}$$

<u>Lemma 6.3</u>. If $\lambda < \omega$, then $E_\lambda < \infty$ and
$$\log E_\lambda = \log D_\lambda + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{6.3}$$

<u>Lemma 6.4</u>. If $\lambda < \omega$, then $N_\lambda < \infty$ and
$$\log N_\lambda = \log E_\lambda + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{6.4}$$

Combination of the results (6.2), (6.3) and (6.4) constitutes a proof of lemma 6.1. The proof of lemma 6.2 is analogous to part (4) of the proof of theorem 1.2.

<u>Proof of lemma 6.2</u>. If $S_n(\lambda,x)$ denotes $\varphi(\lambda,x) - \varphi(\lambda,z_n(\lambda))$ then lemma 5.6 states that for $\lambda \in (\lambda_4, \omega)$, $1 \leq n \leq D_\lambda$ and $x \in (z_n(\lambda), z_{n+1}(\lambda))$, we have $|S_n(\lambda,x)| < C_2 n^{-1}$. Using the mean value theorem we may write

$$\theta_\lambda(z_{n+1}(\lambda)) - \theta_\lambda(z_n(\lambda)) = (z_{n+1}(\lambda) - z_n(\lambda))\, \theta_\lambda'(x),$$

with $x \in (z_n(\lambda), z_{n+1}(\lambda))$. So, we have

$$\theta_\lambda(z_{n+1}(\lambda)) - \theta_\lambda(z_n(\lambda)) = \frac{1}{n+1}\,\frac{\varphi(\lambda,z_n(\lambda)) + R_n(\lambda,z_n(\lambda))}{\varphi(\lambda,z_n(\lambda)) + S_n(\lambda,x)}, \tag{6.5}$$

as $\theta_\lambda'(x) = (\varphi(\lambda,x))^{-1}$
Moreover, we have the estimate (5.7) $R_n(\lambda,z_n(\lambda)) \leqslant d_3 n^{-1}$, for $\lambda \in (\lambda_4,\omega)$, $1 \leqslant n \leqslant D_\lambda$. If $K_n(\lambda) = \theta_\lambda(z_n(\lambda)) - \log n$, then we get in (6.5)

$$K_{n+1}(\lambda) - K_n(\lambda) = \frac{1}{n+1}\left[\frac{\varphi(\lambda,z_n(\lambda)) + R_n(\lambda,z_n(\lambda))}{\varphi(\lambda,z_n(\lambda)) + S_n(\lambda,x)}\right] + \log\left(1-\frac{1}{n+1}\right).$$

If n_1 is chosen so that $C_1 - C_2\, n_1^{-\frac{1}{2}} > \frac{1}{2}C_1$, then we have for $n_1 \leqslant n \leqslant D_\lambda$ that

$$\left|K_{n+1}(\lambda) - K_n(\lambda)\right| < \frac{1}{n}\left|\frac{S_n(\lambda,x) - R_n(\lambda,z_n(\lambda))}{\varphi(\lambda,z_n(\lambda)) + S_n(\lambda,x)}\right| +$$

$$+ \frac{1}{n}\left|1 + (n+1)\log\left(1-\frac{1}{n+1}\right)\right| < \frac{1}{n}\,\frac{d_3 n^{-1} + C_2 n^{-1}}{C_1 n^{-\frac{1}{2}} - C_2 n^{-1}} + \frac{1}{2n^2} <$$

$$< \frac{1}{n\sqrt{n}}\left[\frac{2}{C_1}(C_2 + d_3) + \frac{1}{2}\right] = \frac{C_4}{n\sqrt{n}}. \tag{6.6}$$

Relation (6.6) gives $K_{D_\lambda+1}(\lambda) = K_{n_1}(\lambda) + \Sigma_{\nu=n_1}^{D_\lambda}\left(K_{\nu+1}(\lambda) - K_\nu(\lambda)\right) < K_{n_1}(\lambda) + \Sigma_{\nu=1}^{\infty} C_4\, \nu^{-\frac{3}{2}} = K_{n_1}(\lambda) + C_5$. In order to obtain $K_{D_\lambda+1}(\lambda) = \mathcal{O}(1)$ uniformly in an interval $(\lambda^*.\omega)$ we make a further restriction for λ. We take $\lambda_5 \in [\lambda_4,\omega)$ so large that $z_{n_1}(\lambda) < c(\lambda)$ for $\lambda \in (\lambda_5,\omega)$; so $\lambda_5 \geqslant \max\{\lambda_4, \zeta_{n_1}\}$ (see lemma 4.2). We shall show that

$$|K_{n_1}(\lambda)| < C_6, \tag{6.7}$$

for $\lambda \in (\lambda_5,\omega)$, where C_6 does not, of course, depend on λ. In virtue of lemma 4.1, we have $\varphi(\lambda,z_{n_1}(\lambda)) > C_1\, n_1^{-\frac{1}{2}}$ for $\lambda \in (\lambda_5,\omega)$. By the mean value theorem we have $\theta_\lambda(z_{n_1}(\lambda)) = (z_{n_1}(\lambda)-1)(\varphi(\lambda,x))^{-1}$, with $x \in (1, z_{n_1}(\lambda))$. But $\varphi(\lambda,x)$ is decreasing on $(1, z_{n_1}(\lambda))$, as $z_{n_1}(\lambda) < c(\lambda)$. So $\theta_\lambda(z_{n_1}(\lambda)) < (z_{n_1}(\lambda)-1)(\varphi(\lambda,z_{n_1}(\lambda)))^{-1} < q\, C_1^{-1}\, n_1^{\frac{1}{2}}$. If $C_6 = q\, C_1^{-1}\, n_1^{\frac{1}{2}} + \log n_1$, then we have proved that $|K_{n_1}(\lambda)| < \theta_\lambda(z_{n_1}(\lambda)) + \log n_1 < C_6$ for $\lambda \in (\lambda_5,\omega)$.
For $\lambda \in (\lambda_5,\omega)$ we have therefore

$$|\theta_\lambda(z_{D_\lambda+1}(\lambda)) - \log(D_\lambda+1)| < C_5 + C_6 = C_7. \tag{6.8}$$

Formula (6.8) implies that $\log D_\lambda = \theta_\lambda(z_{D_\lambda+1}(\lambda)) + \mathcal{O}(1) \quad (\lambda \uparrow \omega)$. Further it is obvious that for $\lambda \in (\lambda_5, \omega)$

$$\theta_\lambda(z_{D_\lambda+1}(\lambda)) - \theta_\lambda(\rho) < \int_\rho^{2\rho} (\varphi(\omega,x))^{-1}\, dx = C_8.$$

So we have proved

$$\log D_\lambda = \theta_\lambda(\rho) + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

<u>Proof of lemma 6.3</u>. For $\lambda \in (\lambda_5, \omega)$, $D_\lambda < n < E_\lambda$ and $x \in (z_n(\lambda), z_{n+1}(\lambda))$ we have that $\varphi(\lambda,x) < 2\varphi(\lambda, z_n(\lambda))$ (by lemma 5.7). From this it is evident that

$$\begin{aligned}\theta_\lambda(z_{n+1}(\lambda)) - \theta_\lambda(z_n(\lambda)) &> \tfrac{1}{2}\big(z_{n+1}(\lambda) - z_n(\lambda)\big)\big(\varphi(\lambda, z_n(\lambda))\big)^{-1} > \\ &> \tfrac{1}{2}(n+1)^{-1}.\end{aligned} \tag{6.9}$$

As $\lim_{y\to\infty} \int_\rho^y (\varphi(\lambda,x))^{-1} dx$ exists, it is obvious that $E_\lambda < \infty$, for, otherwise we should obtain (by (6.9))

$$\int_\rho^\infty (\varphi(\lambda,x))^{-1} dx > \int_{z_{D_\lambda+1}(\lambda)}^\infty (\varphi(\lambda,x))^{-1} dx > \tfrac{1}{2} \Sigma_{n=D_\lambda+1}^\infty (n+1)^{-1},$$

which is a contradiction. If $C_9 = \int_\rho^\infty (\varphi(\omega,x))^{-1} dx$, we find by summation of (6.9)

$$\begin{aligned}C_9 &> \theta_\lambda(z_{E_\lambda}(\lambda)) - \theta_\lambda(z_{D_\lambda+1}(\lambda)) > \tfrac{1}{2} \Sigma_{n=D_\lambda+1}^{E_\lambda-1} (n+1)^{-1} > \\ &> \tfrac{1}{2} \Sigma_{n=D_\lambda+1}^{E_\lambda} \log\big((n+2)(n+1)^{-1}\big) = \tfrac{1}{2}\{\log(E_\lambda+2) - \log(D_\lambda+2)\}.\end{aligned}$$

So we have found $\log(E_\lambda+2) - \log(D_\lambda+2) < 2C_9$. But as $1 \leqslant D_\lambda < E_\lambda$ we have

$$\log E_\lambda - \log D_\lambda < \log 3 + \log(E_\lambda+2) - \log(D_\lambda+2).$$

So we have $\log E_\lambda < \log D_\lambda + \log 3 + 2C_9$, and thus

$$\log E_\lambda = \log D_\lambda + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

Proof of lemma 6.4. In fact we shall prove the stronger proposition that there exists a number M such that $N_\lambda - E_\lambda < M$ for all $\lambda \in (\lambda_5, \omega)$. The proof of the assertions in the lemma then follows immediately, since $N_\lambda < E_\lambda + M < \infty$, and $\log N_\lambda < \log E_\lambda + \log(1+ME_{\lambda_5}^{-1})$. We consider only the first term of $R_n(\lambda, z_n(\lambda))$. For $n > E_\lambda$, $n \leqslant N_\lambda$ (or $n < \infty$ if $N_\lambda = \infty$) we have $(z_n(\lambda))^{1/q} \geqslant \frac{1}{2}(q+1)^{-1} \lambda n$, and so

$$z_{n+1}(\lambda) - z_n(\lambda) > \frac{1}{n+1}\,\frac{q(q+1)}{2\lambda^2 n}\,(z_n(\lambda))^{1+2/q} \geqslant z_n(\lambda)\,\frac{n\,q}{8(n+1)(q+1)} >$$

$$> z_n(\lambda)\,\frac{q}{16(q+1)}.$$

If we denote $1 + \frac{1}{16}\,q(q+1)^{-1}$ by δ, then $\delta > 1$ and for $n > E_\lambda$ we have $z_{n+1}(\lambda) > \delta z_n(\lambda)$ until the process breaks down. So we have for $n > E_\lambda$ but $n \leqslant N_\lambda$ if $N_\lambda < \infty$

$$z_{n+1}(\lambda) > \delta^{\,n-E_\lambda}\, z_{E_\lambda+1}(\lambda). \tag{6.10}$$

Clearly $E_\lambda > 1$, for $\lambda \in (\lambda_5, \omega)$. We introduce the following abbreviations $\delta_1 = \delta^{1/q}$ (so $\delta_1 > 1$), $\sigma_1 = 2(q+1)$. σ_2 is defined as follows: if $\delta_1 \leqslant e^{2/e}$ then σ_2 is the largest root of $x^{-1}\log x = \frac{1}{2}\log\delta_1$, otherwise $\sigma_2 = 2$.
Let $M_1 = \max\{2(\log\sigma_1)(\log\delta_1)^{-1}, 2, \sigma_2\}$ and $M = M_1+1$, then we shall prove that for $N = E_\lambda + M$ we have

$$\delta_1^{M_1}\,(E_\lambda+1)\,\sigma_1^{-1} \geqslant N. \tag{6.11}$$

Once (6.11) will have been proved, we shall have $N_\lambda < N$ since otherwise $z_N(\lambda)$ would be defined and both $z_N(\lambda) < \lambda^q N^q$ and, in virtue of (6.10),

$$z_N(\lambda) > \delta^{\,N-E_\lambda-1}\cdot z_{E_\lambda+1}(\lambda) \geqslant \delta_1^{qM_1}\cdot(E_\lambda+1)^q\,\sigma_1^{-q}\,\lambda^q \geqslant N^q\lambda^q,$$

which is a contradiction. So $N_\lambda < E_\lambda + M$.
For the proof of (6.11) we use the following three facts.

(i) $M_1^{-1}\log\sigma_1 \leqslant \frac{1}{2}\log\delta_1$. (6.12)

(ii) From $M_1 \geqslant 2$, $E_\lambda+1 \geqslant 2$ it follows that $M_1(E_\lambda+1) \geqslant$ $\geqslant M_1+E_\lambda+1 = N$; this implies that $\log M_1 \geqslant \log N - \log(E_\lambda+1)$.

(iii) $M_1^{-1}\log M_1 \leqslant \frac{1}{2}\log\delta_1$.

Combination of (ii) and (iii) gives

$$M_1^{-1}\ (\log N - \log(E_\lambda+1)) \leqslant \tfrac{1}{2}\log \delta_1 . \tag{6.13}$$

Addition of (6.12) and (6.13) yields

$$\log \sigma_1 + \log N \leqslant M_1 \log \delta_1 + \log(E_\lambda+1), \tag{6.14}$$

which is equivalent to (6.11). This completes the proof of lemma 6.4, so that the proof of lemma 6.1 is now also completed.

We conclude this Section with a remark about an alternative method. N.G. de Bruijn in his paper [2] used one strong and elegant argument to obtain the result equivalent to $\log N_\lambda = \log D_\lambda + \mathcal{O}(1)$ $(\lambda \uparrow \omega)$. This argument enables him to avoid many of the troublesome arguments we needed in Secs.5 and 6. We shall explain how, for the case $q \leqslant 1$, the procedure can be simplified by a generalization of this argument. If $n > D_\lambda$ (we confine ourselves to values of $\lambda \in (\lambda_3, \omega)$), then $z_n(\lambda) \geqslant \rho$ and therefore $\varphi(\lambda, z_n(\lambda)) >$ $> \frac{1}{4} q\ \lambda^{-1}(z_n(\lambda))^{1+1/q} > \frac{1}{4} q\ \omega^{-1}(z_n(\lambda))^{1+1/q}$, by lemma 5.1. If $n > D_\lambda$ we have $z_{n+1}(\lambda) - z_n(\lambda) > \frac{1}{2}(n+1)^{-1}\ q\ \omega^{-1}(z_n(\lambda))^{1+1/q}$. For $n > D_\lambda$ we compare $z_n(\lambda)$ with y_n defined by

$$y_{D_\lambda+1} = z_{D_\lambda+1}(\lambda), \quad y_{n+1} - y_n = \tfrac{1}{2}(n+1)^{-1}\ q\omega^{-1}\ y_n^{1+1/q} \qquad (n > D_\lambda).$$

By induction we can see $y_n \leqslant z_n(\lambda)$ for $D_\lambda < n \leqslant N_\lambda$ (or $D_\lambda < n < \infty$ if $N_\lambda = \infty$). From this it follows that $y_n^{1/q} < \lambda n$ for $D_\lambda < n \leqslant N_\lambda$ $(D_\lambda < n < \infty)$. So $y_{n+1} - y_n < \frac{1}{2}(n+1)^{-1}\ q\ \omega^{-1}\lambda\, n y_n < \frac{1}{2} q\ y_n \leqslant \frac{1}{2} y_n$, and therefore $y_{n+1} < \frac{3}{2}\ y_n$. From this we derive $y_n^{-1} - y_{n+1}^{-1} =$ $= \frac{1}{2}(n+1)^{-1}\ q\ \omega^{-1}\ y_{n+1}^{-1}\ y_n^{1/q} > \frac{1}{2}\cdot\frac{2}{3}\ (n+1)^{-1}\ q\ \omega^{-1}\ y_n^{-1+1/q} > (n+1)^{-1} C_{10}$, for $D_\lambda < n \leqslant N_\lambda$ (or $< \infty$ if $N_\lambda = \infty$) and some constant $C_{10} > 0$. For the latter inequality sign it is essential that $-1+1/q \geqslant 0$; so $y_n^{-1+1/q} \geqslant \rho^{-1+1/q}$.
Now we can see that $N_\lambda < \infty$, since otherwise the inequality $y_n^{-1} - y_{n+1}^{-1} > C_{10}\ (n+1)^{-1}$ would hold for all $n > D_\lambda$ and so, as $\lim_{n\to\infty} y_n^{-1} = 0$,

$$y^{-1}_{D_\lambda+1} = \Sigma^{\infty}_{n=D_\lambda+1} (y^{-1}_n - y^{-1}_{n+1}) > C_{10} \quad \Sigma^{\infty}_{n=D_\lambda+1} (n+1)^{-1},$$

which is impossible as the latter series diverges. Moreover, from $y^{-1}_{D\lambda+1} > C_{10}\, \Sigma^{N\lambda}_{n=D\lambda+1} (n+1)^{-1}$ we find immediately that there exists a constant C_{11} such that $\log N_\lambda - \log D_\lambda < C_{11}$, for it is easily seen that $y^{-1}_{D_\lambda+1} = (z_{D_\lambda+1}(\lambda))^{-1} \leqslant \rho^{-1}$.

7. The integral $\theta_\lambda(\rho)$

In order to complete the proof of the formula

$$\log N_\lambda = \pi(a(\omega)\ b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega), \tag{7.1}$$

we have to prove the following lemma.

Lemma 7.1. $$\int_1^\rho (\varphi(\lambda,x))^{-1}dx = \pi(a(\omega)\ b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \quad (\lambda \uparrow \omega). \tag{7.2}$$

Proof. If λ is close to ω, the integrand $(\varphi(\lambda,x))^{-1}$ has a sharp peak at $x = c(\lambda)$. If $J = (\alpha,\beta)$ is a given interval, with $q+1 \in (\alpha,\beta)$ then we find (on account of the continuity of $c(\lambda)$) a number $\lambda_J \in [1,\omega)$, such that for $\lambda \in (\lambda_J,\omega)$ we have $c(\lambda) \in (\alpha,\beta)$. Then we have for $\lambda \in (\lambda_J,\omega)$

$$\left|\int_1^\rho \frac{dx}{\varphi(\lambda,x)} - \int_\alpha^\beta \frac{dx}{\varphi(\lambda,x)}\right| \leqslant \left|\int_1^\alpha \frac{dx}{\varphi(\omega,x)}\right| + \left|\int_\beta^\rho \frac{dx}{\varphi(\omega,x)}\right|,$$

and so we find

$$\int_1^\rho (\varphi(\lambda,x))^{-1}dx = \int_\alpha^\beta (\varphi(\lambda,x))^{-1}dx + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{7.3}$$

This means that the contribution of a fixed neighbourhood of q+1 is almost equal to the whole integral when $\lambda \uparrow \omega$. In a neighbourhood of $c(\lambda)$ we approximate the integrand by a function which is simpler (for q=1 it will be the same function as $(\varphi(\lambda,x))^{-1}$). This procedure is known as Laplace's method (see [1] ch.4).

If we take $y = x - c(\lambda)$, then J is transformed into an interval containing 0 if $\lambda \in (\lambda_J, \omega)$; instead of $\varphi(\lambda,x)$ we obtain $\varphi^*(\lambda,y)$, which can be written in a closed interval, containing zero, as

$$\varphi^*(\lambda,y) = b(\lambda) + a(\lambda)\, y^2 + d(\lambda)\, y^3 + \mathcal{O}(y^4), \tag{7.4}$$

where $d(\lambda) = \frac{1}{6}(1-q)\, q^{-2}\, (q+1)^{2q}\, \lambda^{-2q}$; and the $\mathcal{O}$-term is also uniform in λ in a closed neighbourhood of $\lambda = \omega$. In view of (7.3) it suffices to evaluate for some fixed δ $\int_{-\delta}^{\delta} (\varphi^*(\lambda,y))^{-1} dy$ for $\lambda \uparrow \omega$. On such an interval $(-\delta,\delta)$ we shall approximate $(\varphi^*(\lambda,y))^{-1}$ by $(b(\lambda) + a(\lambda)\, y^2)^{-1}$. Moreover, we take δ so small, and $\lambda_6 \in (1,\omega)$ so close to ω, that for $y \in (-\delta,\delta)$ and $\lambda \in (\lambda_6, \omega)$ we have $b(\lambda) \leqslant 1$, and

$$\varphi^*(\lambda,y) > \tfrac{1}{2}(b(\lambda) + a(\lambda)\, y^2). \tag{7.5}$$

The possibility of such a choice follows from

$$\varphi^*(\lambda,y) - \tfrac{1}{2}(b(\lambda) + a(\lambda)y^2) > y^2(\tfrac{1}{2}a(\lambda) + d(\lambda)y + \mathcal{O}(y^2)).$$

We shall further evaluate $\int_{-\delta}^{\delta} (\varphi^*(\lambda,y))^{-1} dy$ for $\lambda \uparrow \omega$. If $e(\lambda) = -d(\lambda)(a(\lambda))^{-1} = \frac{1}{3}\, q^{-1}(q-1)(q+1)^q\, \lambda^{-q}$ we have

$$|(1+e(\lambda)y)\cdot\varphi^*(\lambda,y) - (b(\lambda) + a(\lambda)y^2)| = \mathcal{O}(b(\lambda)\,|y|) + \mathcal{O}(y^4) \tag{7.6}$$

for $y \in (-\delta,\delta)$ and $\lambda \in (\lambda_6, \omega)$. As $y^4 < (a(\omega))^{-2}(b(\lambda) + a(\lambda)y^2)^2$, combination of (7.5) and (7.6) involves that there exist positive constants k_1 and k_2 such that for $y \in (-\delta,\delta)$ and $\lambda \in (\lambda_6, \omega)$

$$\left|\frac{1}{\varphi^*(\lambda,y)} - \frac{1+e(\lambda)y}{b(\lambda) + a(\lambda)y^2}\right| \leqslant k_1 \frac{b(\lambda)\ |y|}{(b(\lambda) + a(\lambda)y^2)^2} + k_2. \tag{7.7}$$

As $\int_{-\delta}^{\delta} e(\lambda)y\ (b(\lambda) + a(\lambda)y^2)^{-1} dy = 0$, (the intergand is an odd function), and

$$\int_{-\delta}^{\delta} \frac{b(\lambda)\ |y|}{(b(\lambda) + a(\lambda)y^2)^2}\, dy < b(\lambda)\int_0^{\infty} \frac{dy^2}{(b(\lambda) + a(\lambda)y^2)^2} = \frac{1}{a(\lambda)} < \frac{1}{a(\omega)},$$

we find from (7.7)

$$\int_{-\delta}^{\delta} \frac{dy}{\varphi^*(\lambda,y)} = \int_{-\delta}^{\delta} \frac{dy}{b(\lambda) + a(\lambda)y^2} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{7.8}$$

Now

$$\int_{-\delta}^{\delta} (b(\lambda)+ a(\lambda)y^2)^{-1}dy = \int_{-\infty}^{\infty} (b(\lambda)+a(\lambda)y^2)^{-1}dy + \mathcal{O}(1) \qquad (\lambda \uparrow \omega)$$

and the latter integral equals $\pi(a(\lambda)\ b(\lambda))^{-\frac{1}{2}}$.

As $a(\lambda) - a(\omega) = \frac{1}{2}q^{-1}(q+1)^q\ \lambda^{-q}\ b(\lambda)$ we have

$$\pi(a(\lambda)\ b(\lambda))^{-\frac{1}{2}} = \pi(a(\omega)\ b(\lambda))^{-\frac{1}{2}} + o(1) \qquad (\lambda \uparrow \omega),$$

which completes the proof of lemma 7.1.

Bearing in mind the relation between λ_N and N_λ we derive from formula (7.1) that

$$b(\lambda_N) = \frac{\pi^2}{a(\omega)(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right), \qquad (N \to \infty)$$

as $\lambda \uparrow \omega$ implies $N_\lambda \to \infty$.

From the latter formula we derive formula (2.20) without difficulty.

So

$$\lambda_N = (1-p)^{-1/p} - 2\pi^2(1-p)^{-1-1/p}(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

8. An inequality due to E.T.Copson

We write (2.1) in an equivalent form; taking $a_n^* = a_n^p$, and $p^* = p^{-1}$ in (2.1), and then omitting the asterisks we obtain

$$\Sigma_{n=1}^{\infty}\ n^{-p}\ (a_1 + \ldots + a_n)^p \leqslant p^p(p-1)^{-p}\ \Sigma_{n=1}^{\infty}\ a_n^p. \qquad (8.1)$$

Now $p > 1$, and (8.1) holds for all convergent series $\Sigma_{n=1}^{\infty}\ a_n^p$ with $a_1 \geqslant 0$, $a_2 \geqslant 0$, ...; unless all the a_n are zero, there is strict inequality; the constant $p^p(p-1)^{-p}$ is best possible.

Translating the result of the previous Sections for finite sections of (8.1) we obtain an asymptotic formula for the best possible constant in these finite sections.

If $\lambda_N(p)$ is the smallest value of λ such that

$$\Sigma_{n=1}^{N} n^{-p} (a_1 + \ldots + a_n)^p \leqslant \lambda \Sigma_{n=1}^{N} a_n^p \qquad (p > 1), \qquad (8.2)$$

then we have for $\lambda_N(p)$ the formula

$$\lambda_N(p) = \left(\frac{p}{p-1}\right)^p - \left(\frac{p}{p-1}\right)^{p+1} \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \qquad (8.3)$$

We consider the following inequality due to E.T.Copson (see e.g. [8] theorem 331)

$$\Sigma_{n=1}^{\infty} (b_n + b_{n+1} + \ldots)^p \leqslant p^p \Sigma_{n=1}^{\infty} n^p b_n^p \qquad (p > 1) \qquad (8.4)$$

which holds for all sequences $b_1, b_2, \ldots$, with $b_1 \geqslant 0$, $b_2 \geqslant 0$, ..., such that $\Sigma_{n=1}^{\infty} n^p b_n^p$ converges; unless all the b_n are zero, there is strict inequality; the constant p^p is best possible.
Now the inequalities (8.1) and (8.4) are reciprocal in the sense that either of them can be derived from the other by applying the "converse of Hölder's theorem" (see [8] theorem 15).
In this Section we shall derive from (8.3) an asymptotic formula for $\mu_N(p)$, the best possible constant in the finite sections of (8.4); i.e. $\mu_N(p)$ is the smallest value of μ, for which

$$\Sigma_{n=1}^{N} (b_n + b_{n+1} + \ldots + b_N)^p \leqslant \mu \Sigma_{n=1}^{N} n^p b_n^p \qquad (p > 1) \qquad (8.5)$$

for all $b_1 \geqslant 0$, ..., $b_N \geqslant 0$.
We shall use a finite version of a well-known device, which is often used as a proof of the fact that (8.1) and (8.4) are reciprocal.

We define $c_{m,n}$ for $m=1,\ldots,N$; $n=1,\ldots,N$ by $c_{m,n} = n^{-1}$ if $m \leqslant n$; $c_{m,n} = 0$ if $m > n$. We write the double sum $F(\underline{x},\underline{y}) = \Sigma_{m,n=1}^{N} c_{m,n} x_m y_n$ in two different ways obtaining

$$\Sigma_{n=1}^{N} n^{-1}(x_1+\ldots+x_n)y_n = \Sigma_{m=1}^{N} x_m(m^{-1}y_m+\ldots+N^{-1}y_N). \qquad (8.6)$$

We only consider $x_1 \geqslant 0,\ldots,x_N \geqslant 0$; $y_1 \geqslant 0,\ldots,y_N \geqslant 0$ and apply Hölder's inequality to the left-hand side of (8.6). If q is the conjugate of p, i.e. $p^{-1} + q^{-1} = 1$ and therefore $q > 1$, we find

$$\Sigma_{n=1}^{N} n^{-1}(x_1+\ldots+x_n)y_n \leqslant \{\Sigma_{n=1}^{N} n^{-q}(x_1+\ldots+x_n)^q\}^{1/q} \cdot \{\Sigma_{n=1}^{N} y_n^p\}^{1/p}. \tag{8.7}$$

Applying (8.2) (written with q instead of p) to the right-hand side of (8.7) and replacing the left-hand side of (8.7) by the right-hand side of (8.6) we obtain

$$\Sigma_{m=1}^{N} x_m(m^{-1}y_m+\ldots+N^{-1}y_N) \leqslant \{\lambda_N(q)\ \Sigma_{n=1}^{N} x_n^q\}^{1/q} \cdot \{\Sigma_{n=1}^{N} y_n^p\}^{1/p}, \tag{8.8}$$

which holds for all $x_1 \geqslant 0,\ldots,x_N \geqslant 0$, $y_1 \geqslant 0,\ldots,y_N \geqslant 0$.
Now it can be seen easily that the maximum of the linear form $\Sigma_{m=1}^{N} d_m x_m$ under the restrictions $\Sigma_{n=1}^{N} x_n^q = 1$, $x_1 \geqslant 0,\ldots,x_N \geqslant 0$, equals $\{\Sigma_{n=1}^{N} d_n^p\}^{1/p}$. This is a consequence of Hölder's theorem but we can also prove it by using a Lagrange multiplier. We shall apply this result in (8.8) with $d_n = n^{-1} y_n + \ldots + N^{-1} y_N$. As (8.8) holds for all $x_1 \geqslant 0,\ldots,x_N \geqslant 0$, $y_1 \geqslant 0,\ldots,y_N \geqslant 0$, we can for each $y_1 \geqslant 0,\ldots,y_N \geqslant 0$ replace the left-hand side by the maximum it attains on $\Sigma_{n=1}^{N} x_n^q = 1$, and still have inequality. So

$$\{\Sigma_{n=1}^{N} (n^{-1}y_n+\ldots+N^{-1}y_N)^p\}^{1/p} \leqslant \{\lambda_N(q)\}^{1/q} \cdot \{\Sigma_{n=1}^{N} y_n^p\}^{1/p}, \tag{8.9}$$

which holds for all $y_1 \geqslant 0,\ldots,y_N \geqslant 0$. If we replace y_n by nb_n $(n=1,\ldots,N)$ (8.9) yields

$$\Sigma_{n=1}^{N} (b_n + \ldots + b_N)^p \leqslant \{\lambda_N(q)\}^{p/q}\ \Sigma_{n=1}^{N} n^p b_n^p, \tag{8.10}$$

which holds for all $b_1 \geqslant 0,\ldots,b_N \geqslant 0$.
Formula (8.10) implies $\mu_N(p) \leqslant \{\lambda_N(q)\}^{p/q}$.

In order to prove the opposite inequality, we reverse the arguments. Applying Hölder's inequality to the right-hand side of (8.6) we obtain

$$F(\underline{x},\underline{y}) \leqslant \{\Sigma_{n=1}^{N} x_n^q\}^{1/q} \cdot \{\Sigma_{n=1}^{N} (n^{-1}y_n+\ldots+N^{-1}y_N)^p\}^{1/p}. \tag{8.11}$$

If we apply (8.5) with $n^{-1} y_n = b_n$ $(n=1,\ldots,N)$ to the right-hand side of (8.11), we obtain that for all $x_1 \geqslant 0,\ldots,x_N \geqslant 0$, and all

$y_1 \geq 0,\ldots,y_N \geq 0$

$$F(\underline{x},\underline{y}) \leq \{\mu_N(p)\,\Sigma_{n=1}^N\, y_n^p\}^{1/p} \cdot \{\Sigma_{n=1}^N\, x_n^q\}^{1/q}. \tag{8.12}$$

For each set $x_1 \geq 0,\ldots,x_N \geq 0$ the maximum of $F(\underline{x},\underline{y}) = \Sigma_{n=1}^N\, n^{-1}(x_1 + \ldots + x_n)y_n$ under the restrictions $\Sigma_{n=1}^N\, y_n^p = 1$ $y_1 \geq 0,\ldots,y_N \geq 0$ equals $\{\Sigma_{n=1}^N\, n^{-q}(x_1 + \ldots + x_n)^q\}^{1/q}$. Combination of this result and (8.12) gives for all $x_1 \geq 0,\ldots,x_N \geq 0$

$$\Sigma_{n=1}^N\, n^{-q}\,(x_1 + \ldots + x_n)^q \leq \{\mu_N(p)\}^{q/p}\,\Sigma_{n=1}^N\, x_n^q. \tag{8.13}$$

But formula (8.13) implies $\lambda_N(q) \leq \{\mu_N(p)\}^{q/p}$.

So we have proved $\mu_N(p) = \{\lambda_N(q)\}^{p/q}$, and we derive an asymptotic formula for $\mu_N(p)$ from (8.3). It reads

$$\mu_N(p) = p^p - \frac{2\pi^2(p-1)p^{p+1}}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \tag{8.14}$$

If we take $p=2$ in (8.2) and (8.5) we find in (8.3) and (8.14)

$$\lambda_N(2) = \mu_N(2) = 4 - \frac{16\,\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \tag{8.15}$$

As $\mu_N(2)$ is the maximum $\Sigma_{n=1}^N\,(n^{-1}y_n + \ldots + N^{-1}y_N)^2$ if $\Sigma_{n=1}^N\, y_n^2 = 1$, it will be seen that $\mu_N(2)$ is the largest eigenvalue of the $N \times N$ matrix A of which the element in the i^{th} row and j^{th} column equals $(ij)^{-1}\,\Sigma_{n=1}^{\min(i,j)}\, 1 = (\max\,(i,j))^{-1}$. By a method using truncated integral equations H.S.Wilf [12] obtained for this eigenvalue $4 - 16\pi^2(\log N)^{-2} + \mathcal{O}((\log\log N)\cdot(\log N)^{-3})$. By means of matrix transformations he derived $\lambda_N(2) = \mu_N(2)$. As our $\mathcal{O}$-term is smaller, our result is slightly better.

9. A certain class of iteration problems

One can rightly wonder in virtue of which properties we came to a successful result in the iteration problem belonging to Hardy's inequality. In this Section we shall formulate a list of properties which allow a treatment as in Secs.2 - 7. This means that we shall describe a class of iteration problems, which have almost all the properties used in Secs.2 - 7 in common with the Hardy case. We divide these properties into three groups T_0, T_1 and T_2; T_0 contains the trivial properties characterizing the situation; T_1 concerns the behaviour of the iterates for the boundary value of the parameter; T_2 concerns the occurrence of breakdown. It will be clear that many of these properties can be replaced by other ones; e.g. the fact that in T_0 the $\Phi_n(\lambda,x)$ are decreasing with respect to λ and that $\lambda \uparrow \omega$ (the boundary value) can be replaced by $\Phi_n(\lambda,x)$ increasing and $\lambda \downarrow \omega$. An equivalent alternative of T_0, T_1, T_2 will be given in Sec.11. It should be emphasized that it is not our aim to give a set of conditions which are all necessary for an asymptotic behaviour of the same type as in the Hardy case; e.g. many of the strong monotonicity conditions can be omitted, but this causes a lot of unessential complications in the proofs, and in the cases in which we shall apply the results of this Section, these conditions are always met. Nor did we take much trouble to make the set minimal in the sense that no properties are mentioned that follow from other ones.

The z_n are given as functions of a parameter λ by the procedure

$$\begin{cases} z_1(\lambda) = 1 \\ z_{n+1}(\lambda)-z_n(\lambda) = \Phi_n(\lambda,z_n(\lambda))-z_n(\lambda) = \frac{1}{n+1}\left(\varphi(\lambda,z_n(\lambda))+R_n(\lambda,z_n(\lambda))\right) \end{cases} \tag{9.1}$$

Moreover, there is a breakdown condition i.e. there is a sequence of functions $\{f_n(\lambda)\}$; and roughly speaking, z_n is defined only for those values of λ which satisfy $z_1(\lambda) < f_1(\lambda),\ldots,z_{n-1}(\lambda) < f_{n-1}(\lambda)$.

$\underline{T_0}$ f_n, Φ_n, φ and R_n satisfy the following requirements.

(1) $f_n(\lambda)$ is continuous and increasing for $\lambda \in [\omega_1,\omega]$; for each value of λ we have $f_n(\lambda) \to +\infty$ monotonically if $n \to \infty$; there

exist constants Γ and σ such that $0 < f_n(\lambda) < \Gamma n^\sigma$ for $\lambda \in [\omega_1, \omega]$.

(2) $\Phi_n(\lambda,x)$ is for each value of n defined and continuous for (λ,x) satisfying $\lambda \in [\omega_1,\omega]$, $x \in [1, f_n(\lambda))$; it is decreasing with respect to λ, increasing with respect to x; $\Phi_n(\omega,1) \geqslant 1$ and $\Phi_n(\lambda,x) \to +\infty$ if $x \uparrow f_n(\lambda)$ for fixed n and λ.

(3) $\varphi(\lambda,x)$ is continuous on $D = \{(\lambda,x) \mid \lambda \in [\omega_1,\omega],\ 1 \leqslant x < \infty\}$; $\varphi(\lambda,x)$ is decreasing with respect to λ;

$$\frac{\partial}{\partial x}\varphi(\lambda,x) = \phi(\lambda,x),\ \frac{\partial^2}{\partial x^2}\varphi(\lambda,x) = \chi(\lambda,x),\ \frac{\partial^3}{\partial x^3}\varphi(\lambda,x),\ \frac{\partial^4}{\partial x^4}\varphi(\lambda,x)$$

exist and are continuous on D; $\chi(\lambda,x) > 0$ on D; $\varphi(\lambda,x)$ attains one strong minimum $b(\lambda)$ for $x = c(\lambda)$; $b(\lambda)$ and $c(\lambda)$ are continuous on $[\omega_1,\omega]$, $b(\lambda)$ is decreasing, $c(\lambda)$ is increasing; $b(\omega) = 0$; $c(\omega) > 1$. [We use the notation $a(\lambda) = \frac{1}{2}\chi(\lambda, c(\lambda))$.]

(4) On every compact set $G = \{(\lambda,x) \mid \lambda \in [\omega_1,\omega],\ x \in [1,d]\}$ we have $R_n(\lambda,x) = n^{-1}r(\lambda,x) + \mathcal{O}(n^{-2})$ for $n \geqslant N_G$; $r(\lambda,x)$ is continuous on D and decreasing with respect to λ.

<u>T_1</u> $z_n(\omega)$ satisfies the inequalities

$$z_n(\omega) < f_n(\omega); \quad z_n(\omega) < c(\omega) - \beta n^{-1}$$

for all n, whereas $\beta > 0$, $\beta > r(\omega, c(\omega))$.

<u>T_2</u> There exist ρ, τ, M, μ, L and L' with $\rho > c(\omega)$, $0 < \tau < 1$, $M > 0$, $\mu \in [\omega_1,\omega)$, $L > 0$, $L' > 0$ such that the following propositions hold for $n > M$ and $\lambda \in (\mu,\omega)$:

(1) $\Phi_n(\lambda, \tau f_n(\lambda)) \geqslant \tau f_{n+1}(\lambda)$;

(2) $\tau f_n(\lambda) > \rho$; if $x \in [\rho, \tau f_n(\lambda)]$ then $0 \leqslant R_n(\lambda,x) < L\,\varphi(\lambda,x)$ and $\phi(\lambda,x) < L\,n$;

(3) if $x \in [\tau f_n(\lambda), f(\lambda))$ then $R_n(\lambda,x) > L'nx$;

(4) $\int_\rho^\infty (\varphi(\lambda,x))^{-1}dx < \infty$.

We define $N_\lambda = \max\{n \mid z_1(\lambda) < f_1(\lambda),\ z_2(\lambda) < f_2(\lambda), \ldots, z_n(\lambda) < f_n(\lambda)\}$, if this maximum exists, and take $N_\lambda = \infty$ if it does not. We have $N_\omega = \infty$. The question is, how the z_n behave with respect to breakdown. The following theorem supplies the answer.

Theorem 9.1. If the iteration procedure (9.1) satisfies T_0, T_1 and T_2 then $N_\lambda < \infty$ for $\lambda \in [\omega_1, \omega)$ and

$$\log N_\lambda = \pi(a(\lambda)\ b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

This theorem can be proved by a series of arguments which are analogous to parts of Secs.2 - 7. Only some extra care is needed, as $R_n(\lambda,x)$ may be negative for some λ and x and, therefore, $z_{n+1}(\lambda)$ may be less than $z_n(\lambda)$. We shall not give the proof here.

10. Generalizations of Hardy's inequality

In this Section we consider finite sections of two inequalities concerning series with positive terms, both due to Copson (see [6]). We shall give an asymptotic formula for $\lambda_N(p,s)$ and $\mu_N(p,s)$ if $N \to \infty$, where $\lambda_N(p,s)$ is defined if $p \geqslant s > 1$ as the best possible constant, such that for all sequences $a_1 \geqslant 0, \ldots, a_N \geqslant 0$ the following inequality hold

$$\Sigma_{n=1}^{N}\ n^{-s}(a_1+\ldots+a_n)^p \leqslant \lambda_N(p,s)\ \Sigma_{n=1}^{N}\ n^{-s}(n\ a_n)^p \qquad (p \geqslant s > 1) \qquad (10.1)$$

and $\mu_N(p,s)$ is defined if $p > 1 > s \geqslant 0$, as the best possible constant such that for $b_1 \geqslant 0, \ldots, b_N \geqslant 0$

$$\Sigma_{n=1}^{N}\ n^{-s}(b_n+\ldots+b_N)^p \leqslant \mu_N(p,s)\ \Sigma_{n=1}^{N}\ n^{-s}(n\ b_n)^p \qquad (p > 1 > s \geqslant 0). \qquad (10.2)$$

One might notice that (10.1) and (10.2) are generalizations of the inequalities considered in Sec.8. Formula (10.1) reduces to (8.2) if $p = s$; (10.2) reduces to (8.5) if $s = 0$.
From Copson's paper we know that $\lambda_N(p,s) \to p^p(s-1)^{-p}$ if $N \to \infty$, $\mu_N(p,s) \to p^p(1-s)^{-p}$ if $N \to \infty$. In this Section we will show that (10.1) and (10.2) both give rise to the same iteration problem,

which will turn out to be of the type described in the previous Section.

For the determination of $\lambda_N(p,s)$ we use a Lagrange multiplier, analogous to the discussion in Sec.2. First we would observe that $\lambda_N(p,s)$ is the maximum of $F(x_1,\dots,x_N) = \Sigma_{n=1}^N n^{-s}(x_1+\dots+x_n)^p$ subject to the restrictions $\Sigma_{n=1}^N n^{-s}(n\,x_n)^p = 1$, $x_1 \geqslant 0,\dots,x_N \geqslant 0$. The maximum is attained at a point $\underline{x} = (x_1,\dots,x_N)$ for which $x_1 > 0,\dots,x_N > 0$, which can be proved in the same way as in Sec.2. From the homogeneity of degree p of $F(x_1,\dots,x_N)$ it follows that

$$F(x_1,\dots,x_N) = p^{-1}\ \Sigma_{n=1}^N\ x_n \frac{\partial}{\partial x_n} F(x_1,\dots,x_N).$$

From this it follows that, if

$$\begin{cases} \dfrac{\partial}{\partial x_k}\left(F(x_1,\dots,x_N)-\lambda\Sigma_{n=1}^N\ n^{-s}(nx_n)^p\right) = 0 \quad (k=1,\dots,N), \\ \Sigma_{n=1}^N\ n^{-s}(nx_n)^p = 1 \end{cases} \tag{10.3}$$

for a set $(x_1',\dots,x_N',\lambda')$, then we have $F(x_1',\dots,x_N') = \lambda'$. So we find that λ_N is the largest value of λ for which (10.3) has a solution $x_1,\dots,x_N$ with $x_1 > 0,\dots,x_N > 0$. As before (10.3) has only one solution with $x_1 > 0,\dots,x_N > 0$. The system (10.3) can be written as

$$\begin{cases} \Sigma_{n=k}^N\ n^{-s}(x_1+\dots+x_n)^{p-1} = \lambda\, k^{p-s}\, x_k^{p-1} \quad (k=1,\dots,N), \\ \Sigma_{n=1}^N\ n^{-s}(nx_n)^p = 1. \end{cases} \tag{10.4}$$

In analogy with Sec.2 we find that this is equivalent to

$$\begin{cases} z_1 = 1, \quad z_{k+1} = \dfrac{1}{k+1} + \left(\dfrac{k}{k+1}\right)^{1-s'}\ z_k\left(1-\lambda^{-1}k^{-1}z_k^{1/q}\right)^{-q} \\ \qquad\qquad\qquad\qquad\qquad\qquad (k=1,\dots,N-1), \\ z_N = \lambda^q\ N^q. \end{cases} \tag{10.5}$$

To obtain (10.5) we have taken differences in (10.4) and used the substitutions $(kx_k)^{-1}\ (x_1+\dots+x_k) = z_k$ $(k=1,\dots,N)$; $(p-1)^{-1} = q$; $(p-1)^{-1}(p-s) = s'$. So q and s' satisfy the conditions $q > 0$, $0 \leqslant s' < 1$. If $s' = 0$ (which corresponds to $s = p$), then the system (10.5) is the same as (2.9) and (2.10).

Now we shall discuss (10.2). We remark that (10.1) and (10.2) are reciprocal in the sense that either of them can be derived from the other by the so-called converse of Hölder's inequality (see Sec.8). The latter fact has been proved by Hardy in [7]. We shall use the finite version of a well-known device in order to apply Hölder's inequality (see e.g. [1], Sec.8,9); this device can be considered a refinement of the one used in Sec.8, its principal tool being a change of order of summation. With $x_n = a_n^p$ $(n=1,\dots,N)$, $\mu_N^* = \mu_N^{-1}$ (10.2) gets

$$\Sigma_{n=1}^{N} n^{p-s} x_n \geqslant \mu_N^* \, \Sigma_{n=1}^{N} n^{-s} (x_n^{1/p}+\dots+x_N^{1/p})^p. \tag{10.6}$$

For any set of positive numbers p_{kn} $(1 \leqslant n \leqslant k \leqslant N)$ we get by changing the order of summation and applying Hölder's inequality $(q' = 1-p)$

$$\Sigma_{k=1}^{N} x_k \, \Sigma_{n=1}^{k} p_{kn} = \Sigma_{n=1}^{N} \Sigma_{k=n}^{N} x_k \, p_{kn} \geqslant$$
$$\geqslant \Sigma_{n=1}^{N} (x_n^{1/p}+\dots+x_N^{1/p})^p \cdot (\Sigma_{k=n}^{N} p_{kn}^{1/q'})^{q'}. \tag{10.7}$$

The equality sign in (10.7) holds if and only if $p_{kn}^{1/q'}$ is proportional to $x_k^{1/p}$ and so $p_{kn}^{1/q'} = y_n^{1/p} x_k^{1/p}$ $(1 \leqslant n \leqslant k \leqslant N)$. This implies that μ_N^* in (10.6) is the smallest (and as it will become clear the only) value of μ for which $y_1,\dots,y_N$, $x_1,\dots,x_N$ can be found satisfying

$$\left\{\begin{array}{l} x_k^{q'/p} \, \Sigma_{n=1}^{k} y_n^{q'/p} = k^{p-s} \\ y_n^{1/p} \, \Sigma_{k=n}^{N} x_k^{1/p} = \mu^{1/q'} n^{-s/q'} \end{array}\right. \quad (1 \leqslant n \leqslant k \leqslant N). \tag{10.8}$$

Eliminating $y_1,\dots,y_N$ from (10.8) and putting $\Sigma_{k=n}^{N} x_k^{1/p} = t_n$ $(n=1,\dots,N)$ we obtain

$$\left\{\begin{array}{ll} \mu \, \Sigma_{n=1}^{k} n^{-s} t_n^{-q'} = (t_k - t_{k+1})^{-q'} k^{p-s} & (k=1,\dots,N-1) \\ \mu \, \Sigma_{n=1}^{N} n^{-s} t_n^{-q'} = t_n^{-q'} N^{p-s}. & \end{array}\right. \tag{10.9}$$

Taking differences we get from (10.9)

$$\begin{cases} \mu\, t_1^{-q'} = (t_1 - t_2)^{-q'} \\ \mu\, k^{-s}\, t_k^{-q'} = (t_k - t_{k+1})^{-q'} k^{p-s} - (t_{k-1} - t_k)^{-q'} (k-1)^{p-s} \quad (k=2,\dots,N-1) \\ \mu\, N^{-s}\, t_N^{-q'} = t_N^{-q'}\, N^{p-s} - (t_{N-1} - t_N)^{-q'} (N-1)^{p-s}. \end{cases} \tag{10.10}$$

We employ the following substitutions $t_k^{-1}\, t_{k+1} = u_k$ $(k=1,\dots,N-1)$; $u_N = 0$ (an elegant way of formulating this is introducing a new variable t_{N+1} in (10.9) and taking $t_{N+1} = 0$); $(1-u_k)^{-q'} = \mu v_k$ $(k=1,\dots,N)$; $k^{p-1}\, v_k = z_k$ $(k=1,\dots,N)$; $\mu^{1/q'} = \lambda$; $q' = -q$. As $q' < 0$, we have $q > 0$, and to the smallest value of μ corresponds the largest value of λ. Carrying out the substitutions listed above we obtain from (10.10) the same process as (10.5) with s instead of s'. So we shall study an iteration process given by

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1-s} z_n(\lambda)\left(1-\lambda^{-1} n^{-1} (z_n(\lambda))^{1/q}\right)^{-q} \tag{10.11}$$

where $0 \leqslant s < 1$, $q > 0$. The process breaks down at N if $z_N(\lambda) \geqslant \lambda^q N^q$, but $z_1(\lambda) < \lambda^q, \dots, z_{N-1}(\lambda) < \lambda^q (N-1)^q$.

We shall show that the process given by (10.11) is of the type described in the previous Section (for $s = 0$, we know already that it is). It then follows that there is exactly one value of λ (denoted by λ_N^*) such that the process breaks down at N and $z_N(\lambda) = \lambda^q N^q$. Moreover, the theory yet developed will give us an asymptotic formula for λ_N^* if $N \to \infty$. Let this asymptotic formula be written symbolically as

$$\lambda_N^* = \mathrm{Asf}\ (q,\ s,\ N), \tag{10.12}$$

then in view of the substitutions made we have for the original problems the results that $\lambda_N(p,s)$ from (10.1) satisfies

$$\lambda_N(p,s) = \mathrm{Asf}\ ((p-1)^{-1},\ (p-s)(p-1)^{-1},\ N), \tag{10.13}$$

and that $\mu_N(p,s)$ from (10.2) satisfies

$$\{\mu_N(p,s)\}^{1/(p-1)} = \mathrm{Asf}\ (p-1,\ s,\ N). \tag{10.14}$$

We shall now show that the process (10.11) satisfies T_0, T_1, T_2

from Sec.9. To this end we write it in the forms

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \Phi_n(\lambda, z_n(\lambda))$$

and

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) - z_n(\lambda) = (n+1)^{-1}\big(\varphi(\lambda, z_n(\lambda)) + R_n(\lambda, z_n(\lambda))\big),$$

where
$$\Phi_n(\lambda,x) = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1-s} x\left(1-\lambda^{-1}n^{-1}x^{1/q}\right)^{-q},$$
$$\varphi(\lambda,x) = 1-(1-s)\,x + q\,\lambda^{-1}\,x^{1+1/q},$$
$$R_n(\lambda,x) = nx\,\{(1+n^{-1})^{s}(1-\lambda^{-1}n^{-1}x^{1/q})^{-q} - 1 - sn^{-1} - \lambda^{-1}n^{-1}qx^{1/q}\}.$$

The breakdown condition is given by $z_n \geqslant n^q\lambda^q$ so $f_n(\lambda) = n^q\lambda^q$.

As to T_0 we take $\omega_1 = 1$; $\omega = \left((q+1)(1-s)^{-1}\right)^{1+1/q}$; (from the properties of $\varphi(\lambda,x)$ it follows that we have to take this value for ω in order to get T_0 satisfied). By straightforward verification one can show that T_0 is satisfied. We confine ourselves to the following short remarks, $\psi(\lambda,x) = -(1-s) + (q+1)\lambda^{-1}\,x^{1/q}$;
$\chi(\lambda,x) = (q+1)q^{-1}\lambda^{-1}x^{-1+1/q}$; $a(\lambda) = \frac{1}{2}q^{-1}(1-s)^{1-q}\lambda^{-q}(q+1)^q$;
$b(\lambda) = 1-(1-s)^{q+1}(q+1)^{-q-1}\lambda^q$; $c(\lambda) = (1-s)^q(q+1)^{-q}\,\lambda^q$;
$a(\omega) = \frac{1}{2}q^{-1}(q+1)^{-1}(1-s)^2$; $c(\omega) = (1-s)^{-1}(q+1) > 1$;
$r(\lambda,x) = x\,\{\frac{1}{2}\,s(s-1) + q\,s\,\lambda^{-1}x^{1/q} + \frac{1}{2}\,q(q+1)\,\lambda^{-2}\,x^{2/q}\}$;
$r(\omega,c(\omega)) = \frac{1}{2}(q-c) < c(\omega)-1$. The presentation of $R_n(\lambda,x)$ on compact sets G is proved by observing that $n^2R_n(\lambda,x) - n\,r(\lambda,x)$ converges uniformely on G to a continuous function.

The following lemma proves that the process under consideration satisfies T_1.

<u>Lemma 10.1</u>. $z_n(\omega) < f_n(\omega)$, $z_n(\omega) \leqslant c(\omega) - n^{-1}(c(\omega)-1)$ $(n=1,2,\ldots)$.

<u>Proof</u>. We use induction with respect to n. $z_1(\omega) = 1 < \omega^q$. We remark that $c(\omega) < \omega^q \leqslant f_n(\omega)$. So, if for some n $z_n(\omega) \leqslant c(\omega) - n^{-1}(c(\omega)-1)$, then $z_{n+1}(\omega)$ is defined and not larger than $\Phi_n(\omega,\, c(\omega) - n^{-1}(c(\omega)-1))$. Thus, we have to prove only that the latter quantity is not larger than $c(\omega) - (n+1)^{-1}(c(\omega)-1)$. Simple calculation shows that the inequality $\Phi_n(\omega,\, c(\omega) - n^{-1}(c(\omega)-1)) \leqslant$

$\leqslant c(\omega) - (n+1)^{-1}(c(\omega)-1)$ is equivalent to

$$(1-n^{-1}(1-(c(\omega))^{-1}))^{1/q}((nc(\omega))^{-1}+(1+n^{-1})^{s/q})\leqslant 1. \qquad (10.15)$$

In order to prove (10.15) we replace n^{-1} by x, obtaining a function, say $g(x)$, in the left-hand side of (10.15), and we shall show that $g(x)\leqslant 1$ for $0\leqslant x\leqslant 1$. Throughout the rest of this proof we write $c(\omega) = t^{-1}$; so $t(q+1) = 1-s$. $g(x) = (1-(1-t)x)^{1/q}(tx+(1+x)^{s/q})$. As $g(0) = 1$, it suffices to prove $g'(x)\leqslant 0$ for $0\leqslant x\leqslant 1$. $g'(x)\leqslant$ $\leqslant 0$ is equivalent to $h(x)\leqslant 0$ if
$h(x) = qt-(1-t)t(1+q)x+(1+x)^{-1+s/q}\,(s-1+t-x(1-t)(s+1))$.
Again $h(0) = qt+s-1+t = 0$, so it suffices to prove that $h'(x)\leqslant 0$ for $0\leqslant x\leqslant 1$. Bearing in mind the relation between s, q and t we have to prove that $(1+x)^{-2+s/q}(-2qs-(1-t)(s+1)sx)\leqslant(1-t)(1-s)q$, which is trivial, as the left-hand side is negative, and the right-hand side is positive. This completes the proof.
One can show that equality in (10.15) only occurs for $n=1$, but we shall not need this.

Instead of T_2 one can prove much stronger propositions but we shall not do this. One has to be aware, however, of the freedom of choice we have in determining ρ, τ, M, μ, L and L'. First, we choose $\rho > c(\omega)$ such that for $x>\rho$ and $\lambda\in[1,\omega]$ $r(\lambda,x)>0$, and $\varphi(\lambda,x)>\frac{1}{2}q\,\lambda^{-1}\,x^{1+1/q}$. As $R_n(\lambda,x) > n^{-1}\,r(\lambda,x)$, we have $R_n(\lambda,x)\geqslant$ $\geqslant 0$, if $r(\lambda,x)\geqslant 0$. For τ we may take any number in $(0,1)$. We take $\tau = 2^{-q}$. If we now denote $x^{1/q}(\lambda n)^{-1}$ by t, we have
$\lambda x^{-1-1/q}\,R_n(\lambda,x) = t^{-1}((1+n^{-1})^{s}(1-t)^{-q}-1-sn^{-1}-qt) = t^{-1}\,\ell(t)$,
and since $(1+n^{-1})^{s}\leqslant 1+sn^{-1}$ and $(1-t)^{-q} = 1+qt+[(1-t)^{-q}-1-qt]$
we find that $t^{-1}\,\ell(t)\leqslant sq+(1+s)\,t^{-1}((1-t)^{-q}-1-qt)$. So we find that for $x<\tau\lambda^{q}\,n^{q}$ we have $\lambda x^{-1-1/q}\,R_n(\lambda,x)\leqslant$
$\leqslant\max\{sq+(1+s)\,t^{-1}((1-t)^{-q}-1-qt)\mid 0\leqslant t\leqslant\frac{1}{2}\} = L_1$. So $L_1 =$ $= sq+2(1+s)(2^{q}-1-\frac{1}{2}q)$. For $\rho<x<\tau\lambda^{q}n^{q}$ we have $0<R_n(\lambda,x)\leqslant$ $\leqslant L_1\lambda^{-1}\,x^{1+1/q} < 2q^{-1}L_1\,\varphi(\lambda,x)$; $\phi(\lambda,x)<(q+1)\tau\,n$. In order to answer $T_2(2)$ it suffices to take $L = \max(2q^{-1}L_1,\,(q+1)\tau)$, and $n\geqslant M_0$ where $\tau M_0^{q}>\rho$. For $\tau\lambda^{q}n^{q}<x<\lambda^{q}n^{q}$ we use $n^{-1}x^{-1}R_n(\lambda,x) =$

$= \ell(t)$ and so $n^{-1}x^{-1} R_n(\lambda,x) > 2^q - 1 - \frac{1}{2}q - n^{-1}$, which exceeds L' if $n > M_1$ for appropriate L' and M_1. This yields $T_2(3)$. In order to arrange for $T_2(1)$ to be satisfied it is sufficient to take $n > M_2$, where M_2 is chosen such that for $n > M_2$ $((n+1)^{-1}n)^{1-s+q} \geqslant$ $\geqslant (1-\tau^{1/q})^q = 2^{-q}$. We then have to take $M = \max(M_0, M_1, M_2)$. For μ any number in $[1,\omega)$ can be used. As $q > 1$, $T_2(4)$ is satisfied.

By theorem 9.1 we find for the N_λ from problem (10.11) the asymptotic formula

$$\log N_\lambda = \pi(a(\lambda)\ b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

As $a(\lambda) - a(\omega) = \frac{1}{2}q^{-1}(q+1)^q(1-s)^{1-q}\lambda^{-q}\ b(\lambda)$ we have $\pi(a(\lambda)\ b(\lambda))^{-\frac{1}{2}} =$ $= \pi(a(\omega)\ b(\lambda))^{-\frac{1}{2}} + o(1) \quad (\lambda \uparrow \omega)$ and so

$$\log N_\lambda = \pi(a(\omega)\ b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \qquad (10.16)$$

In the same way as in the Hardy case we obtain from formula (10.16) the formula for λ_N^*. So we find for (10.12)

$$\lambda_N^* = \left(\frac{q+1}{1-s}\right)^{1+1/q} - \frac{2}{1-s}\left(\frac{q+1}{1-s}\right)^{2+1/q} \frac{\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \qquad (10.17)$$

For the original $\lambda_N(p,s)$ from (10.1) we find from (10.17) the formula

$$\lambda_N(p,s) = \left(\frac{p}{s-1}\right)^p - \frac{2(p-1)}{(s-1)}\left(\frac{p}{s-1}\right)^{p+1} \frac{\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right), \qquad (10.18)$$

which is written in the original parameters according to (10.13). For the best possible constant $\mu_N(p,s)$ from (10.2) we find by combining (10.17) and (10.14) the formula

$$\mu_N(p,s) = \left(\frac{p}{1-s}\right)^p - \frac{2(p-1)}{(1-s)}\left(\frac{p}{1-s}\right)^{p+1} \frac{\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \qquad (10.19)$$

If we take $s=p$ in (10.18) and $s=0$ in (10.19) we find again the results of Sec.8, (8.3) and (8.14) respectively.

One can specialize (10.2) for $p=2$, in order to obtain a formula for the largest eigenvalue of a certain matrix. Doing so, we find that the largest eigenvalue λ_N' of a finite section of N rows and columns of the symmetric matrix A with $A_{ij} = (ij)^{\frac{1}{2}s-1} \sum_{\nu=1}^{\min(i,j)} \nu^{-s}$ $0 \leqslant s < 1$ satisfies

$$\lambda_N' = 4(1-s)^{-2} - 16\pi^2(1-s)^{-4}(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

11. An inequality due to K.Knopp

If one compares N.G. de Bruijn's result (see [2]) for Carleman's inequality (quoted as formula (0.6)) with formula (2.20), one may notice that if $p \downarrow 0$ in (2.20) one obtains (0.6) with the exception of the $\mathcal{O}$-term for which we have not proved that it is bounded if $p \downarrow 0$. Moreover, it is known that $\lim_{p \downarrow 0}[n^{-1}(a_1^p + \ldots + a_n^p)]^{1/p} = (a_1 \cdots a_n)^{1/n}$ (see [11], II No.82). In this Section we shall study an inequality due to K.Knopp (see [9]) which can be considered to be the analogue of (2.1) for $p < 0$. If we use the letter t instead of $-p$, this inequality reads

$$\Sigma'^{\infty}_{n=1} \; (n^{-1}(a_1^{-t} + \ldots + a_n^{-t}))^{-1/t} < (1+t)^{1/t} \; \Sigma^{\infty}_{n=1} \; a_n \tag{11.1}$$

where $a_1 > 0$, $a_2 \geqslant 0$, ..., $t > 0$ and $\Sigma^{\infty}_{n=1} a_n$ converges; the prime in Σ' means that if one has $a_1 > 0, \ldots, a_m > 0$, $a_{m+1} = 0$, $a_{m+2} \geqslant 0$, ..., one must read $\Sigma^{m}_{n=1}$ instead of $\Sigma^{\infty}_{n=1}$. The constant $(1+t)^{1/t}$ is best possible, whereas there is strict inequality. The problem to be discussed in this Section is to derive an asymptotic formula for $\lambda_N(t)$ if $N \to \infty$. $\lambda_N(t)$ is the best possible constant such that for all sequences $a_1 \geqslant 0, \ldots, a_N \geqslant 0$

$$\Sigma'^{N}_{n=1} \; (n^{-1}(a_1^{-t} + \ldots + a_n^{-t}))^{-1/t} \leqslant \lambda_N(t) \; \Sigma^{N}_{n=1} \; a_n. \tag{11.2}$$

It will be clear that now the prime means that one must read $\Sigma^{m}_{n=1}$ instead of $\Sigma^{N}_{n=1}$ if $a_1 > 0, \ldots, a_m > 0$, $a_{m+1} = 0$, $a_{m+2} \geqslant 0, \ldots, a_N \geqslant 0$. We shall characterize $\lambda_N(t)$ as the maximum of the function $F'(\underline{x}) = F'(x_1,\ldots,x_N) = \Sigma'^{N}_{n=1} \; (n^{-1}(x_1^{-t} + \ldots + x_n^{-t}))^{-1/t}$ on the compact set S in R_N given by $\Sigma^{N}_{n=1} x_n = 1$, $x_1 \geqslant 0, \ldots, x_N \geqslant 0$. First we shall make clear that $\max\{F'(\underline{x}) \mid \underline{x} \in S\}$ exists and is attained at a point $\underline{x}'$ for which $x_1' > 0, \ldots, x_N' > 0$. Let the b's denote non-negative numbers, the c's positive numbers. If for some k, $1 \leqslant k < N$, we have $b_{k+1} \geqslant b_k$ then it is obvious that

$$F'(b_1,\ldots,b_{k-1},b_k,b_{k+1},\ldots,b_N) \leqslant F'(b_1,\ldots,b_{k-1},b_{k+1},b_k,b_{k+2},\ldots,b_N).$$

So we have at once that $\sup\{F'(\underline{x}) \mid \underline{x} \in S\} \leqslant \sup\{F'(\underline{x}) \mid \underline{x} \in T\}$ where T is the compact set defined by $\Sigma^{N}_{n=1} x_n = 1$, $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_N \geqslant$

$\geqslant 0$. As $T \subset S$ we see that $\sup \{F'(\underline{x}) \mid \underline{x} \in S\} = \sup \{F'(\underline{x}) \mid \underline{x} \in T\}$. As $F'(\underline{x})$ is continuous on the compact set T, the supremum is a maximum. $T^* = \{(x_1,\dots,x_N) \mid \Sigma_{n=1}^N x_n = 1,\ \ x_1 \geqslant x_2 \geqslant \dots \geqslant x_N > 0\}$, so $T^* \subset T$. For each point of $T \setminus T^*$ we can find a point of T^* where the value of F' is larger. This follows from the fact that for a point $(c_1,c_2,\dots,c_k,0,\dots,0) \in T \setminus T^*$ $(k < N)$ we have :

$$\lim_{y \downarrow 0} F'(c_1,\dots,c_{k-1},(1-y)c_k,yc_k,0,\dots,0) =$$
$$= F'(c_1,\dots,c_k,0,\dots,0),$$

and

$$\lim_{y \downarrow 0} \frac{d}{dy} F'(c_1,\dots,c_{k-1},(1-y)c_k,yc_k,0,\dots,0) =$$
$$= c_k^{-t}\{-k^{1/t}(c_1^{-t}+\dots+c_k^{-t})^{-1-1/t}+(k+1)^{1/t}\ c_k^{t+1}\} > 0.$$

So $\max \{F'(\underline{x}) \mid \underline{x} \in T\}$ is attained at a point of T^*. On T^* however, we have $F'(x_1,\dots,x_N) = F(x_1,\dots,x_N) = \Sigma_{n=1}^N (n^{-1}(x_1^{-t}+\dots+x_n^{-t}))^{-1/t}$. So $\lambda_N(t)$ is the maximum of $F(x_1,\dots,x_N)$ under the restrictions $\Sigma_{n=1}^N x_n = 1,\ \ x_1 \geqslant x_2 \geqslant \dots \geqslant x_N > 0$. $F(x_1,\dots,x_N)$ is homogeneous of degree one in $x_1,\dots,x_N$. By the same arguments as used in Sec.2 we have therefore that $\lambda_N(t)$ is the uniquely determined largest value of λ for which

$$\begin{cases} \dfrac{\partial}{\partial x_k} \left[\Sigma_{n=1}^N \{n^{-1}(x_1^{-t}+\dots+x_n^{-t})\}^{-1/t} - \lambda \Sigma_{n=1}^N x_n \right] = 0, & (k=1,\dots,N), \quad (11.3) \\ \Sigma_{n=1}^N x_n = 1 & (11.4) \end{cases}$$

have a positive solution. If we subsequently take differences, omit (11.4) and employ the substitutions

$$\begin{cases} k^{-1}\ x_k^t\ (x_1^{-t} + \dots + x_k^{-t}) = z_k & (k=1,\dots,N), \\ t(t+1)^{-1} = s & (\text{so}\ \ 0 < s < 1), \end{cases} \qquad (11.5)$$

we obtain in the same way as in Sec.2 that λ_N is the only value of λ for which

$$\begin{cases} z_1 = 1, \quad z_{k+1} = \dfrac{1}{k+1} + \dfrac{k}{k+1}\ z_k(1-\lambda^{-1}k^{-1}z_k^{-1/s})^s & (k=1,\dots,N-1), \\ z_N^{1/s} = (\lambda N)^{-1} \end{cases} \qquad (11.6)$$

has a solution $(1,\dots,z_N)$.

Again, we consider z_n as functions of λ defined by

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \frac{1}{n+1} + \frac{n}{n+1} z_n(\lambda)(1 - \lambda^{-1}n^{-1}(z_n(\lambda))^{-1/s})^s. \tag{11.7}$$

Now $z_n(\lambda)$ is defined for $\lambda > \lambda_{n-1}$; and λ_n is the value of λ such that $z_n(\lambda) = n^{-s}\lambda^{-s} = f_n(\lambda)$. If we write (11.7) in the forms

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \Phi_n(\lambda, z_n(\lambda)) \tag{11.8}$$

and

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) - z_n(\lambda) = (n+1)^{-1}\big(\varphi(\lambda, z_n(\lambda)) + R_n(\lambda, z_n(\lambda))\big) \tag{11.9}$$

with breakdown functions $f_n(\lambda)$, then we have

$$\Phi_n(\lambda,x) = (n+1)^{-1} + n(n+1)^{-1}\, x(1 - \lambda^{-1}n^{-1}x^{-1/s})^s,$$

$$\varphi(\lambda,x) = 1 - x - s\lambda^{-1}\, x^{1-1/s},$$

$$R_n(\lambda,x) = -x\{\tfrac{1}{2!}s(1-s)\lambda^{-2}n^{-1}x^{-2/s} + \tfrac{1}{3!}s(1-s)(2-s)\lambda^{-3}n^{-2}x^{-3/s} + \dots\}.$$

Now $\varphi(\lambda,x)$ is a negative concave function if $\lambda < (1-s)^{1-1/s} = \omega$, and the maximum of $\varphi(\lambda,x)$ tends to zero if $\lambda \uparrow \omega$.
The problem turns out to be analogous to the previous cases, the only difference being that in this case $\{z_n(\lambda)\}$ decreases to breakdown. We shall list the analogous properties S_0, S_1, S_2 of the properties T_0, T_1, T_2 from Sec.9. The properties S_0, S_1, S_2 describe a situation in which the analogy of Secs.2 - 7 can be carried out with decreasing $\{z_n(\lambda)\}$.

For an iteration problem, given in the forms (11.8) and (11.9), with breakdown functions $f_n(\lambda)$, the properties S_0, S_1 and S_2 read.

$\underline{S_0}$

(1) $f_n(\lambda)$ is continuous and decreasing for $\lambda \in [\omega_1, \omega]$; for each λ we have $f_n(\lambda) \downarrow 0$ if $n \to \infty$; there exist constants $\Gamma > 0$ and $\sigma > 0$ such that $f_n(\lambda) > \Gamma n^{-\sigma}$ for $\lambda \in [\omega_1, \omega]$;

(2) $\Phi_n(\lambda,x)$ is a continuous function on the set in the (λ,x)-plane $\{(\lambda,x) \mid \lambda \in [\omega_1, \omega],\ x \in (f_n(\lambda), 1]\}$; it increases with λ as well as with x; $\Phi_n(\omega,1) \leqslant 1$; there exists an M_0 such that for $\lambda \in [\omega_1, \omega]$ and $n \geqslant M_0$

$$\lim_{x \downarrow f_n(\lambda)} \Phi_n(\lambda,x) < f_{n+1}(\lambda);$$

(3) $\varphi(\lambda,x)$ is continuous on $D = \{(\lambda,x) \mid \lambda \in [\omega_1,\omega],\ x \in (0,1]\}$. $\varphi(\lambda,x)$ is increasing with respect to λ;

$$\frac{\partial}{\partial x}\varphi(\lambda,x)=\psi(\lambda,x),\quad \frac{\partial^2}{\partial x^2}\varphi(\lambda,x)=\chi(\lambda,x),\quad \frac{\partial^3}{\partial x^3}\varphi(\lambda,x),\quad \frac{\partial^4}{\partial x^4}\varphi(\lambda,x)$$

exist and are continuous on D; $\chi(\lambda,x) \leq 0$ on D; $\varphi(\lambda,x)$ attains one strong maximum $b(\lambda)$ for $x = c(\lambda)$; $b(\lambda)$ and $c(\lambda)$ are continuous on $[\omega_1,\omega]$, $b(\lambda)$ is increasing; $c(\lambda)$ is decreasing; $b(\omega) = 0$; $0 < c(\omega) < 1$. [We use the notation $a(\lambda) = \frac{1}{2}\chi(\lambda,c(\lambda))$.]

(4) On every compact set $G = \{(\lambda,x) \mid \lambda \in [\omega_1,\omega],\ x \in [d,1];\ d > 0\}$ we have $R_n(\lambda,x) = n^{-1}r(\lambda,x) + \mathcal{O}(n^{-2})$ for $n \geqslant N_G$; $r(\lambda,x)$ is continuous on D and increasing with respect to λ.

$\underline{S_1}$ $z_n(\omega)$ satisfies the inequalities

$$z_n(\omega) > f_n(\omega);\quad z_n(\omega) > c(\omega) + \beta n^{-1}$$

for all n, whereas $\beta > 0$, $\beta > -r(\omega,c(\omega))$.

$\underline{S_2}$ There exist ρ, τ, M, μ, L and L' with $0 < \rho < c(\omega)$, $\tau > 1$, $M > 0$, $\mu \in [\omega_1,\omega)$, $L > 0$, $0 < L' < 1$ such that the following propositions hold for $n > M$ and $\lambda \in (\mu,\omega)$.

(1) $\Phi_n(\lambda, \tau f_n(\lambda)) \leqslant \tau f_{n+1}(\lambda)$;

(2) $\tau f_n(\lambda) < \rho$; if $x \in [\tau f_n(\lambda),\rho]$, then $L\varphi(\lambda,x) < R_n(\lambda,x) \leqslant 0$ and $\psi(\lambda,x) < Ln$;

(3) if $x \in (f_n(\lambda), \tau f_n(\lambda)]$, then $R_n(\lambda,x) < -L'nx$;

(4) $\int_0^\rho (\varphi(\lambda,x))^{-1}dx > -\infty$.

If $N_\lambda = \max\{n \mid z_1(\lambda) > f_1(\lambda),\dots,z_{n-1}(\lambda) > f_{n-1}(\lambda), z_n(\lambda) > f_n(\lambda)\}$, then $N_\omega = \infty$ and we can prove the following theorem.

<u>Theorem 11.1</u>. If an iteration procedure written in the forms (11.8) and (11.9), satisfies S_0, S_1 and S_2 then $N_\lambda < \infty$ for $\lambda \in [\omega_1,\omega)$ and

$$\log N_\lambda = \pi(a(\lambda)\, b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega).$$

In the present case of Knopp's inequality we take $\omega_1 = 1$; ω equals $(1-s)^{1-1/s}$; Φ_n, φ, R_n and f_n are already given. Moreover, $\phi(\lambda,x) = -1 + (1-s)\lambda^{-1}x^{-1/s}$; $\chi(\lambda,x) = -s^{-1}(1-s)\lambda^{-1}\,x^{-1-1/s}$; $a(\lambda) = -\tfrac{1}{2}s^{-1}(1-s)^{-s}\,\lambda^{s}$; $b(\lambda) = 1-(1-s)^{s-1}\,\lambda^{-s}$; $c(\lambda) = (1-s)^{s}\lambda^{-s}$; $a(\omega) = -\tfrac{1}{2}s^{-1}(1-s)^{-1}$; $c(\omega) = 1-s$; $r(\lambda,x) = -\tfrac{1}{2}s(1-s)\lambda^{-2}\,x^{1-2/s}$; $r(\omega,c(\omega)) = -\tfrac{1}{2}s$. The verification of S_0 presents no difficulties; S_1 is a consequence of lemma 11.1 given below; by arguments of the same type as in Sec.10 we can prove that S_2 is satisfied.

<u>Lemma 11.1.</u> $z_n(\omega) > n^{-s}\,\omega^{-s}$, $z_n(\omega) \geqslant 1-s+sn^{-1}$ $\qquad (n=1,2,\ldots)$.

<u>Proof.</u> As $\omega > 1$ both inequalities hold for $z_1(\omega) = 1$. We shall proceed by induction with respect to n. First we prove that $1-s+sn^{-1} > \omega^{-s}n^{-s}$ $(n=1,2,\ldots)$. If $(1-s)^{-1}n^{-1}$ is denoted by η then $\omega n(1-s+sn^{-1})^{1/s} > 1$ is equivalent to $1+s\eta > \eta^{s}$. But this is proved by $1+s\eta > (1+\eta)^{s}$, $(0 < s < 1)$.
Suppose we have proved $z_n(\omega) \geqslant 1-s+sn^{-1}$ then it follows that $z_n(\omega) > n^{-s}\,\omega^{-s}$, so $z_{n+1}(\omega)$ is defined. If we show that $z_{n+1}(\omega) \geqslant 1-s+s(n+1)^{-1}$ then the proof by induction can be completed. Now $z_{n+1}(\omega) \geqslant \Phi_n(\omega,\ 1-s+sn^{-1})$ and we shall prove that $\Phi_n(\omega,\ 1-s+sn^{-1}) > 1-s+s(n+1)^{-1}$. But, as $\eta = (1-s)^{-1}\,n^{-1}$ we have to prove that (with $\xi = (1+s\eta)^{1/s}$) $\xi(1-\xi^{-1}\eta) > 1$; and this is equivalent to $1+s\eta > (1+\eta)^{s}$.

As the iteration process (11.7) satisfies S_0, S_1 and S_2 and as $\pi(a(\lambda)\,b(\lambda))^{-\frac{1}{2}} = \pi(a(\omega)\,b(\lambda))^{-\frac{1}{2}} + o(1)$ $\quad(\lambda \uparrow \omega)$ [notice that $a(\lambda) - a(\omega) = -\tfrac{1}{2}s^{-1}(1-s)^{-s}\,\lambda^{s}\,b(\lambda)$], we find by applying theorem 11.1

$$\log N_\lambda = \pi(a(\omega)\,b(\lambda))^{-\frac{1}{2}} + \mathcal{O}(1) \qquad (\lambda \uparrow \omega). \tag{11.10}$$

By simple calculation we obtain from (11.10) a formula for λ_N

$$\lambda_N = (1-s)^{1-1/s} - 2(1-s)^{2-1/s}\,\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

With the original parameter this formula reads

$$\lambda_N(t) = (t+1)^{1/t} - 2(t+1)^{-1+1/t}\,\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}). \tag{11.11}$$

We observe that if t is replaced by -p formula (11.11) reads the same as (2.20) which is valid for $0 < p < 1$.

The analogues of the formulas (3.7) and (3.9) hold, since it can be proved that

$$\int_1^{z_n(\omega)} (\varphi(\omega,x))^{-1} dx = \log n + \mathcal{O}(1) \qquad (n \to \infty).$$

12. Extensions of Section 11

In this Section we shall discuss some results which stand in the same relation to Sec.11 as the results of Secs.8 and 10 to Secs. 2 - 7. First, we shall derive from the result (11.11) the analogous result for the Hölder reciprocal of Knopp's inequality. Thereafter we shall discuss a pair of reciprocal inequalities which are generalizations of Knopp's inequality and its reciprocal.

It is appropriate to start with a slightly different form of the result of the previous Section. If $\lambda_N(p)$ is the best possible constant such that for all $a_1 > 0, \ldots, a_N > 0$ the inequality

$$\Sigma_{n=1}^{N} \{n^{-1}(a_1 + \ldots + a_n)\}^p \leqslant \lambda_N(p)\, \Sigma_{n=1}^{N} a_n^p \tag{12.1}$$

holds, whereas $p < 0$, then $\lambda_N(p)$ satisfies

$$\lambda_N(p) = \left(\frac{p}{p-1}\right)^p - \left(\frac{p}{p-1}\right)^{p+1} \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right). \tag{12.2}$$

If a_n^{-t} in (11.2) is replaced by a_n $(n=1,\ldots,N)$ [notice that for attaining the maximum, all a_n in (11.2) are positive] and t is replaced by $-p^{-1}$ then (11.2) transforms in (12.1) and (11.11) in (12.2).

Starting again from formula (8.6) and writing down the analogues for $p < 0$ (hence $0 < q < 1$ as $p^{-1} + q^{-1} = 1$) of all the arguments in Sec.8 ((12.2) is the analogue of (8.3)) we produce a proof of the formula

$$\mu_N(q) = q^q + 2q^{q+1}(1-q)\ \pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}) \qquad (0<q<1) \tag{12.3}$$

for the best possible constant such that

$$\Sigma_{n=1}^{N}\ (z_1 + \dots + z_N)^q \geqslant \mu_N(q)\ \Sigma_{n=1}^{N}\ (n\ z_n)^q, \tag{12.4}$$

for $0 < q < 1$ and all $z_1 \geqslant 0, \dots, z_N \geqslant 0$.
Formula (12.4) is a finite section of a well-known inequality due to Copson (see [6], or [8] theorem 344).

We start the discussion of the second topic of this Section with a generalization of this inequality due to Copson. In [6] it is proved that for $a_1 \geqslant 0, a_2 \geqslant 0, \dots,\quad 0 < p < 1,\quad s \leqslant 0$ and $\Sigma_{n=1}^{\infty} n^{-s}\ (n\ a_n)^p < \infty$ we have

$$\Sigma_{n=1}^{\infty}\ n^{-s}(a_n + a_{n+1} + \dots)^p > p^p(1-s)^{-p}\ \Sigma_{n=1}^{\infty}\ n^{-s}(n\ a_n)^p, \tag{12.5}$$

unless all the a_n are zero; the constant $p^p(1-s)^{-p}$ is the best possible. If $s = 0$ (12.5) reduces to the inequality mentioned above. Now we consider $\mu_N(p,s)$, the best possible constant such that for all $x_1 \geqslant 0, \dots, x_N \geqslant 0;\quad 0 < p < 1,\quad s \leqslant 0$

$$\Sigma_{n=1}^{N}\ n^{p-s}x_n \leqslant \mu_N(p,s)\ \Sigma_{n=1}^{N}\ n^{-s}(x_n^{1/p} + \dots + x_N^{1/p})^p. \tag{12.6}$$

Writing down the analogues for $0 < p < 1$ (hence $q = 1-p > 0$) of the arguments in Sec.10, starting from the analogue of formula (10.7) we obtain the system

$$\begin{cases} z_1 = 1, \quad z_{k+1} = \dfrac{1}{k+1} + \left(\dfrac{k}{k+1}\right)^{1-s} z_k(1-\lambda^{-1}k^{-1}z_k^{-1/q})^q, \quad (k=1,\dots,N-1) \\ z_N^{1/q} = \lambda N, \end{cases} \tag{12.7}$$

and if λ_N^*, the value of λ for which (12.7) has a solution, satisfies an asymptotic formula (if $N \to \infty$) written symbolically as

$$\lambda_N^* = \mathrm{Asf}(q,\ s,\ N), \tag{12.8}$$

then $\mu_N(p,s)$ satisfies

$$(\mu_N(p,s))^{1/(1-p)} = \mathrm{Asf}(1-p,\ s,\ N). \tag{12.9}$$

With the experience of Sec.10, we expect that an inequality can be formulated, finite sections of which, by direct use of the method of the Lagrange multiplier, give rise to an iteration problem, which is identical to (12.7). Moreover, this inequality has to be a generalization of (11.1) and reciprocal to (12.5). We start with the finite series problem, formulating the infinite inequality afterwards. So we consider the following generalization of (12.1); (we again take $-p = t$ as parameter, the condition imposed on s will later appear to be in accordance with the analogy)

$$\Sigma_{n=1}^{N} n^{-s}(a_1 + \dots + a_n)^{-t} \leqslant \lambda_N(t,s)\ \Sigma_{n=1}^{N} n^{-s}(n\ a_n)^{-t}, \qquad (12.10)$$

$a_1 > 0,\dots,a_N > 0;\quad t > 0,\quad s \leqslant -t$. For $s = -t = p$ (12.10) reduces to (12.1). Using Lagrange multiplier theory to calculate $\lambda_N(t,s)$ just as before, we find that $\lambda_N(t,s)$ is the only value of λ for which there exist positive $x_1,\dots,x_N$ with

$$\begin{cases} \lambda\{x_k^{-(t+1)}k^{-(s+t)} - x_{k+1}^{-(t+1)}(k+1)^{-(s+t)}\} = k^{-s}(x_1+\dots+x_k)^{-(t+1)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k=1,\dots,N-1) \\ \lambda\ x_N^{-(t+1)}\ N^{-(s+t)} = N^{-s}(x_1 + \dots + x_N)^{-(t+1)}, \\ \Sigma_{n=1}^{N} n^{-s}(n\ x_n)^{-t} = 1. \end{cases}$$

The reduction of this system does not differ from previous cases so we only mention the substitutions used, omitting all calculations. By $z_k = k^{-1}\ x_k^{-1}(x_1 + \dots + x_k)\quad (k=1,\dots,N);\quad q = (t+1)^{-1}$ (so $0 < q < 1$); and $s' = (t+s)(t+1)^{-1}$ (so $s' \leqslant 0$), we again obtain (12.7) with s' instead of s. So $\lambda_N(t,s)$ from (12.10) satisfies

$$\lambda_N(t,s) = \mathrm{Asf}((t+1)^{-1},\ (t+s)(t+1)^{-1},\ N). \qquad (12.11)$$

We would observe that (12.10) is a finite section of the inequality

$$\Sigma_{n=1}^{\infty} n^{-s}(a_1 + \dots + a_n)^{-t} \leqslant (1-s)^t\ t^{-t}\ \Sigma_{n=1}^{\infty} n^{-s}(n\ a_n)^{-t}, \quad (12.12)$$

which holds for $a_1 > 0, a_2 > 0,\dots;\quad t > 0,\quad s \leqslant -t$; $\Sigma_{n=1}^{\infty} n^{-s}(n\ a_n)^{-t} < \infty$. The proof of this inequality follows from our asymptotic considerations; as in the previous cases there is strict inequality (cf.Sec.3).

In order to obtain an explicit expression for formula (12.8) by applying theorem 11.1, we would prove that the iteration procedure

$$z_1(\lambda) = 1; \quad z_{n+1}(\lambda) = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1-s} z_n(\lambda)\left(1-\lambda^{-1}n^{-1}(z_n(\lambda))^{-1/q}\right)^q, \tag{12.13}$$

satisfies S_0, S_1 and S_2. As this proof does not differ essentially from previous cases in Secs.10 and 11, we shall omit it, giving only some calculations. If we write (12.13) in the form (11.9) we find $\varphi(\lambda,x) = 1 - (1-s)x - q\lambda^{-1} x^{1-1/q}$ and
$R_n(\lambda,x) = nx\{(1+n^{-1})^s(1-\lambda^{-1}n^{-1}x^{-1/q})^q - 1 - sn^{-1} + \lambda^{-1}n^{-1}q\, x^{-1/q}\}$.
Moreover, we have

$f_n(\lambda) = n^{-q}\lambda^{-q}$; $\quad \omega = ((1-s)(1-q)^{-1})^{-1+1/q}$;

$a(\lambda) = -\frac{1}{2}q^{-1}(1-q)^{-q}(1-s)^{q+1}\lambda^q$; $\quad a(\omega) = -\frac{1}{2}q^{-1}(1-q)^{-1}(1-s)^2$;

$b(\lambda) = 1 - (1-s)^{1-q}(1-q)^{q-1}\lambda^{-q}$; $\quad c(\lambda) = (1-q)^q(1-s)^{-q}\lambda^{-q}$;

$r(\lambda,x) = \frac{1}{2}x\{s(s-1) - 2sq\lambda^{-1}x^{-1/q} + q(q-1)\lambda^{-2}x^{-2/q}\}$;

$c(\omega) = (1-q)(1-s)^{-1}$; $\quad r(\omega,c(\omega)) = -\frac{1}{2}(q+s)$.

For (12.8) we find by theorem 11.1

$$\lambda_N^* = \left(\frac{1-s}{1-q}\right)^{-1+1/q} - \left(\frac{1-s}{1-q}\right)^{-3+1/q} \frac{2\pi^2}{(1-q)(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right).$$

For the original problems, discussed in this Section, we find from this result the asymptotic formulas wanted.
For $\mu_N(p,s)$ from (12.6) we have by application of (12.9)

$$\mu_N(p,s) = \left(\frac{1-s}{p}\right)^p - \left(\frac{1-s}{p}\right)^{p-1} \frac{1-p}{1-s}\,\frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right).$$

For $\lambda_N(t,s)$ from (12.10) we obtain by (12.11)

$$\lambda_N(t,s) = \left(\frac{1-s}{t}\right)^t - \left(\frac{1-s}{t}\right)^{t-1} \frac{t+1}{1-s}\,\frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right).$$

If we take $s = 0$ and $s = -t$ respectively these formulas agree with the results already known for the inequalities (12.4) and (12.1).

13. Instability

All iteration problems we have studied, can be obtained by a choice of the parameters s and q in the following problem

$$z_1 = 1, \quad z_{n+1} = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1-s} z_n \left(1-\lambda^{-1}n^{-1}z_n^{-1/q}\right)^{-q},$$

with breakdown functions $\lambda^q n^q$. With $s = 0$ and $q > 0$ this problem becomes the one arising from Hardy's inequality. The generalization considered in Sec.10 had $0 \leqslant s < 1$, $q > 0$. The problem of Sec.11 had $s = 0$, $-1 < q < 0$; in the generalization of this problem in Sec.12 we had $s < 0$, $-1 < q < 0$.

In his book [1], N.G. de Bruijn discusses an inequality due to Copson, which gives rise to an iteration problem, which can be written in the form given above, with the parameter values $s = \frac{1}{2}$, $q = -\frac{1}{2}$ (see [1] Sec.8.9 and Sec.8.10). The behaviour with respect to breakdown is then different from the cases studied so far in this thesis. N.G. de Bruijn discovered a remarkable discontinuity in the asymptotic behaviour of the z_n's. We shall discuss an example which shows the same behaviour. This example has $s < 0$, $q > 0$, but it can be shown that for $1 < q < 0$, and $s < 0$, one can find equivalent examples.

The behaviour, we shall observe is related to example 2 of Sec.1, as the preceding cases are to example 1 of that Section. Most of the proofs will be omitted as they are more or less analogous to previous cases or to arguments in N.G. de Bruijn's discussion in [1] (Sec.8.10). Writing $t = -s$ we have the iteration process

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1+t} z_n(\lambda)\left(1-\lambda^{-1}n^{-1}(z_n(\lambda))^{1/q}\right)^{-q}, \tag{13.1}$$

with breakdown at N if $z_n(\lambda) < \lambda^q n^q$ for $n < N$, and $z_N(\lambda) \geqslant \lambda^q N^q$. The value of λ such that $z_N(\lambda) = \lambda^q N^q$ is denoted by $\lambda_N(q,t)$. That $\lambda_N(q,t)$ is uniquely determined is proved in the same way as in Secs.2 and 9. If we write $z_n(\lambda,q,t)$ to express the functionality with respect to q and t then we see that for fixed λ and q, $z_n(\lambda,q,t)$, when defined, is a decreasing function of t. As $z_n((q+1)^{1+1/q},q,0)$ is defined for all n, we have $z_n((q+1)^{1+1/q},q,t) <$ $< \infty$ for $t > 0$ and all n and this implies $\lambda_n(q,t) < (q+1)^{1+1/q}$ for

all n. $\lim_{n\to\infty} \lambda_n(q,t)$ exists, as $\{\lambda_n(q,t)\}$ is a bounded increasing sequence; we denote this limit by $\eta(q,t)$. $\eta(q,t) \leqslant (q+1)^{1+1/q}$.
In the same way as in Sec.10 we now have that for $s > p > 1$ and $a_1 \geqslant 0,\ a_2 \geqslant 0, \ldots$

$$\Sigma_{n=1}^{N}\ n^{-s}(a_1 + \ldots + a_n)^p \leqslant \lambda_N\left(\frac{1}{p-1},\ \frac{s-p}{p-1}\right)\ \Sigma_{n=1}^{N}\ n^{-s}(n\ a_n)^p$$

and for $p > 1$, $s < 0$ and $b_1 \geqslant 0,\ b_2 \geqslant 0, \ldots$

$$\Sigma_{n=1}^{N}\ n^{-s}(b_n + \ldots + b_N)^p \leqslant \{\lambda_N(p-1,\ -s)\}^{p-1}\ \Sigma_{n=1}^{N}\ n^{-s}(n\ b_n)^p.$$

Thus we are speaking about the inequalities concerning series with non-negative terms (cf. Copson [6])

$$\Sigma_{n=1}^{\infty}\ n^{-s}(a_1 + \ldots + a_n)^p \leqslant \eta\left(\frac{1}{p-1},\ \frac{s-p}{p-1}\right)\ \Sigma_{n=1}^{\infty}\ n^{-s}(n\ a_n)^p \qquad (13.2)$$

$$(s > p > 1),$$

and

$$\Sigma_{n=1}^{\infty}\ n^{-s}(b_n+b_{n+1}+\ldots)^p \leqslant\ (\eta(p-1,\ -s))^{p-1}\ \Sigma_{n=1}^{\infty}\ n^{-s}(n\ b_n)^p$$

$$(p > 1,\ s < 0). \qquad (13.3)$$

If we write (13.1) in the usual form

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = (n+1)^{-1}(\varphi(\lambda, z_n(\lambda)) + R_n(\lambda, z_n(\lambda))),$$

we denote by $\omega(q,t)$ --in accordance with our previous notations-- the value of λ for which the strongly convex function $\varphi(\lambda,x) = 1 - (1+t)x + q\lambda^{-1}\ x^{1+1/q}$ has a minimum equal to zero. It is found that $\omega(q,t) = ((t+1)^{-1}(q+1))^{1+1/q}$.
Without proof we mention the following results:

Lemma 13.1. If q is fixed, then $\eta(q,t)$ is a decreasing function of t.

Lemma 13.2. The process (13.1) satisfies T_2 (see Sec.9).

Lemma 13.3. $\eta(q,t) \geqslant \omega(q,t)$ for each positive t and q.

A discontinuity of the type N.G. de Bruijn discovered occurs only if $\eta(q,t) > \omega(q,t)$.

In order to give a clearer description of the different behaviour we take $t = q$ and study the best possible constant η of the inequality

$$\Sigma_{n=1}^{\infty} n^{-1-p}(x_1+\ldots+x_n)^p \leqslant \eta\Sigma_{n=1}^{\infty} n^{-1} x_n^p, \quad (p>1). \tag{13.4}$$

From the above discussion we know that $1 < \eta \leqslant (q+1)^{1+1/q}$, and that for $\lambda \geqslant \eta$ the iteration process

$$z_1(\lambda) \equiv 1, \quad z_{n+1}(\lambda) = \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1+q} z_n(\lambda)\left(1-\lambda^{-1}n^{-1}(z_n(\lambda))^{1/q}\right)^{-q} \tag{13.5}$$

does not break down. Moreover, from lemma (13.2) we know that $\eta \geqslant \omega(q,q)$ and as $\omega(q,q) = 1$ we have $\eta > \omega(q,q)$. Writing (13.5) in the usual form we have $\varphi(\lambda,x) = 1-(1+q)x+q\lambda^{-1}x^{1+1/q}$. If $\lambda > 1$, $\varphi(\lambda,1) < 0$, whereas $\chi(\lambda,x) = (\lambda q)^{-1}(q+1)x^{-1+1/q}$ is still positive if $x > 0$; so $\varphi(\lambda,x)$ is still convex. As $\varphi(\lambda,0) = 1$, and $\varphi(\lambda,x) \to +\infty$ if $x \to \infty$, we have that $\varphi(\lambda,x)$ has two zeros $s(\lambda)$ and $t(\lambda)$ with $0 < s(\lambda) < 1 < t(\lambda) < \infty$. We observe that if $1 < \lambda_1 < \lambda_2$ then $s(\lambda_1) > s(\lambda_2)$ and $t(\lambda_1) < t(\lambda_2)$. The discontinuity announced above is given by the following results.

I. If $\lambda > \eta$, then $\lim_{n\to\infty} z_n(\lambda) = s(\lambda)$, (lemma 13.8)

II. for $\lambda = \eta$ we have $\lim_{n\to\infty} z_n(\eta) = t(\eta)$. (lemma 13.9)

The effect II is called instability.

We can give the proof in the following steps:

<u>Lemma 13.4</u>. If $z_m(\lambda) \geqslant t(\lambda)$ for some value of m and $\lambda > 1$, then $\lambda < \eta$.

<u>Lemma 13.5</u>. If $\lambda \geqslant \eta$, then $\liminf_{n\to\infty} z_n(\lambda) \geqslant s(\lambda)$.

<u>Lemma 13.6</u>. If $\liminf_{n\to\infty} z_n(\lambda) = s(\lambda)$ for $\lambda \geqslant \eta$, then $\lim_{n\to\infty} z_n(\lambda) = s(\lambda)$.

<u>Lemma 13.7</u>. The sequence $\{z_n(\lambda)\}$ has for $\lambda \geqslant \eta$, no accumulation point in $(s(\lambda), t(\lambda))$.

<u>Lemma 13.8</u>. If $\lambda > \eta$ then $\lim_{n\to\infty} z_n(\lambda) = s(\lambda)$.

<u>Lemma 13.9</u>. $\lim_{n\to\infty} z_n(\eta) = t(\eta)$.

Extending the analogy with the results of Sec.9, we can determine a number ρ (sufficiently exceeding $t(\eta)$), and define

$$D_\lambda = \max\{n \mid z_n(\lambda) < \rho\}.$$

As before N_λ denotes the breakdown-index. The number ρ can be chosen in such a way that we can prove

<u>Lemma 13.10</u>. If $\lambda < \eta$, then $D_\lambda < \infty$, $N_\lambda < \infty$ and

$$\log N_\lambda = \log D_\lambda + \mathcal{O}(1) \qquad (\lambda \uparrow \eta).$$

There is another remarkable difference between the present problem and the problems discussed in the previous Sections. When, in the previous Sections, we considered infinite series, we had always strict inequality. We shall show that in the present case we have a sequence $x_1, x_2, \ldots$ such that $\Sigma_{n=1}^{\infty} n^{-1} x_n^p$ converges and

$$\Sigma_{n=1}^{\infty} n^{-1-p}(x_1 + \ldots + x_n)^p = \eta\, \Sigma_{n=1}^{\infty} n^{-1} x_n^p.$$

Remembering some results in Sec.3, we shall show that for the sequence $x_1, x_2, \ldots$ determined as follows

$$x_1 = 1, \quad n^{-1} x_n^{-1}(x_1 + \ldots + x_n) = z_n(\eta) \qquad (n=1,2,\ldots), \qquad (13.6)$$

the series $\Sigma_{n=1}^{\infty} n^{-1} x_n^p$ converges. We observe that the $x_1, x_2, \ldots$ are uniquely determined by (13.6). As $\lim_{n\to\infty} z_n(\eta) = t(\eta) > 1$, we may determine α and M such that $\alpha > 1$, $M \geqslant 2$, and that $n \geqslant M$ implies $z_n(\eta) > \alpha$. Now $n^{-1} x_n^{-1}(x_1 + \ldots + x_n) > \alpha$ for $n \geqslant M$, and thus

$$x_n < (\alpha n-1)^{-1}(x_1 + \ldots + x_{n-1}) < \alpha^{-1}(n-1)^{-1}(x_1 + \ldots + x_{n-1}) <$$
$$< \alpha^{-1}(n-1)^{-1}(x_1 + \ldots + x_M)\, \Pi_{\nu=M}^{n-2} (1+\alpha^{-1}\nu^{-1}).$$

As $(1+\alpha^{-1}\nu^{-1}) < ((\nu-1)^{-1}\nu)^{1/\alpha}$, we find $\Pi_{\nu=M}^{n-2} (1+\alpha^{-1}\nu^{-1}) <$ $< ((M-1)^{-1}(n-2))^{1/\alpha}$ and, therefore, $x_n < A\, n^{-1+1/\alpha}$ for all n and some appropriately chosen positive constant A. As $-1+1/\alpha < 0$, $\Sigma_{n=1}^{\infty} n^{-1} x_n^p$ converges.

We conclude this Section with some remarks about arbitrary values of q, and t. So from now on we have no longer $t = q$.
First, we remark that if $\eta(q,t) = \omega(q,t)$ we can prove in the same way as in lemma 13.2, ..., 13.7 that $\lim_{n\to\infty} z_n(\omega) = c(\omega)$, and $\lambda >$ $> \omega$ implies $\lim_{n\to\infty} z_n(\lambda) = s(\lambda)$. In this case, however, there is no discontinuity as $s(\eta) = t(\eta) = c(\omega)$.

Lemma 13.11. If $t \geqslant q$ then

$$\lim_{\lambda \downarrow \eta(q,t)} \lim_{n\to\infty} z_n(\lambda) < \lim_{n\to\infty} \lim_{\lambda \downarrow \eta(q,t)} z_n(\lambda).$$

Lemma 13.12. If $t \geqslant q$, then $\eta(q,t) > \omega(q,t)$ and $\lambda > \eta$ implies $\lim_{n\to\infty} z_n(\lambda) = s(\lambda)$, whereas $\lim_{n\to\infty} z_n(\eta) = t(\eta)$.

In Secs.1 - 12 we developed a method by which under certain conditions the influence of $R_n(\lambda,x)$ can be neglected. One of the conditions was $\eta(q,t) = \omega(q,t)$. It will be clear that if $\eta(q,t) >$ $> \omega(q,t)$ no such method exists as the $R_n(\lambda,x)$ play an essential part in that case.

We have proved that for $t \geqslant q$ we have $\eta(q,t) > \omega(q,t)$, but from lemma 13.1 it follows that then also for some $t < q$ we have $\eta(q,t) > \omega(q,t)$. As for such t we have $\omega(q,t) > 1$, the phenomenon is much more complicated, as we can not longer see from $\varphi(\lambda,x)$ alone, whether it occurs or not.

For the cases that $\eta(q,t) > \omega(q,t)$, we have no explicit formula for $\eta(q,t)$; one can, however, make numerical estimates for each value of q and t, but we shall not do so.

Appendix

For iteration problems satisfying the requirements of theorem 1.2, two different types of convergent solutions are possible. In example 1 of Sec.1 the type of solution depends on the value of a continuous parameter. In this appendix we intend to throw some more light on this phenomenon by another typical example to be discussed in detail. In fact, we shall have sequences $\{z_n\}$ depending on a continuous parameter x, with $z_n = \mathcal{O}(n^{-1})$ at a certain point $x = c$, $z_n = \mathcal{O}((\log n)^{-1})$ for all values x in an open interval having c as left end point. We shall find an asymptotic expression for z_n which holds uniformly for x in a right neighbourhood of c, $[c,c+h)$. By this it will become clearer how the two types of convergent solutions of the iteration problem are related. As in Sec.2 comparison of the recurrence relation with a differential equation will suggest the desired asymptotic formula (see (6)).

(1) We consider a sequence $\{z_n\}$ satisfying

$$z_{n+1} - z_n = -n^{-1}(z_n^2 + n^{-1}). \tag{A.1}$$

The $\{z_n\}$ depend on a parameter x which runs through $[0,1]$ (therefore we sometimes write $z_n(x)$). In order to avoid difficulties arising from non-monotonicity in the beginning of the iteration, we start from z_{100}, and we prescribe its value by requiring $z_{100} = x$. The number 100 is certainly large enough. It is easy to see that $z_n(x_o) \to -\infty$ for $n\to\infty$ if $z_m(x_o) \leqslant 0$ for some m and x_o. Moreover, if $z_n > \delta > 0$ for all n, then $\Sigma_{\nu=100}^{\infty}(z_{\nu+1} - z_\nu)$ diverges, hence a contradiction. So there are only two possibilities $z_n \to -\infty$ or $z_n \to 0$. If $\lim_{n\to\infty} z_n = 0$, then either $z_n = (\log n)^{-1} + \mathcal{O}((\log n)^{-2})$ or $z_n = n^{-1}(1 + o(1))$ if $n\to\infty$, (by theorem 1.2 applied for the sequence $\{-z_n\}$).

(2) We shall first prove that there exists a number c $(0 < c < 1)$ with $z_n(x) \to 0$ for $x \in [c,1]$ and $z_n(x) \to -\infty$ for $x \in [0,c)$.
By induction we can prove $z_n(1) \geqslant 2n^{-1}$. To this end we remark that $z_{100}(1) = 1$, and that $dz_{n+1}/dz_n = 1 - 2z_n n^{-1} > 0$ for $n \geqslant 100$ and $z_n \leqslant 1$. So, $2n^{-1} \leqslant z_n \leqslant 1$ implies $2(n+1)^{-1} < 2n^{-1} - 4n^{-3} - n^{-2} \leqslant z_{n+1} < 1$.

Moreover, $z_n(x)$ increases with x for $x \in [0,1]$ and $n \geqslant 100$, since z_{100} increases and $dz_{n+1}/dz_n > 0$. If x_n is the root of $z_n(x) = 0$, then $x_{100} = 0$; $x_n < 1$ for all n; and $x_n < x_{n+1}$, since $z_{n+1}(x_n) = z_n(x_n) - n^{-1}z_n^2(x_n) - n^{-2} = -n^{-2} < 0$. Therefore, $\lim_{n\to\infty} x_n$ exists; $\lim_{n\to\infty} x_n = c$ say; and $0 < c \leqslant 1$ (we even have proved $0 < c \leqslant 1/50$). It is easy to see that the number c has the desired properties.

(3) We can prove that $z_n(x) = (\log n)^{-1} + \mathcal{O}((\log n)^{-2})$ for $x \in (c,1]$, and that $z_n(c) = n^{-1} + \mathcal{O}(n^{-2})$.
If $s_n(x)$ is defined by $nz_n = s_n$, then we have $s_{100} = 100x$ and $s_{n+1} - s_n = n^{-1}(s_n - 1 - n^{-1}(s_n^2 + 1) - n^{-2}s_n^{-2})$. We know already that $s_n(x) \to -\infty$ for $x \in [0,c)$; $s_n(1) \geqslant 2$; and $s_n(x) = o(n)$ $(n\to\infty)$ for $x \in [c,1]$. In the same way as for example 1 of Sec.1, we can prove that $x \in (c,1]$ implies $s_n(x) \to +\infty$ and hence $z_n(x) = (\log n)^{-1} + \mathcal{O}((\log n)^{-2})$, and that $s_n(c) \downarrow 1$ and $z_n(c) = n^{-1} + \mathcal{O}(n^{-2})$.

(4) We denote $(\log n)(z_n(x) \log n - 1)$ by $t_n(x)$. Then we already know that $t_n(c) \to -\infty$ if $n\to\infty$, and $t_n = \mathcal{O}(1)$ for $x \in (c,1]$. Straightforward evaluation yields for $t_n = \mathcal{O}(1)$

$$t_{n+1} - t_n = -n^{-2}(\log n)^2 + \mathcal{O}(n^{-2}) - n^{-1}(\log n)^{-2}\, t_n^2,$$

so $\lim_{n\to\infty} t_n(x)$ exists for $x \in (c,1]$; we denote this limit by $t(x)$. We shall now prove that $t(x) < 0$ in a right neighbourhood of c. As $c \leqslant 1/50$, we have $c < (\log 100)^{-1}$. If we restrict ourselves to values of x in $(c, (\log 100)^{-1})$, then we have $z_{100} < (\log 100)^{-1}$. For $x \in (c, (\log 100)^{-1})$ the assumption $t(x) \geqslant 0$ leads to a contradiction, since $t(x) \geqslant 0$ implies that $N = \max\{n \mid t_n < 0\}$ exists and exceeds 99; but for $n > N$ we then have

$$z_{n+1} = \Sigma_{\nu=N}^{n} (z_{\nu+1} - z_\nu) + z_N < z_N - \Sigma_{\nu=N+1}^{n} \nu^{-1}(z_\nu^2 + \nu^{-1}) <$$
$$< (\log N)^{-1} - \Sigma_{\nu=N+1}^{n} (\nu^{-1}(\log \nu)^{-2} + \nu^{-2}).$$

(Notice that $t_n < 0$ means $z_n < (\log n)^{-1}$). As

$$\Sigma_{\nu=N+1}^{n}(\nu^{-1}(\log \nu)^{-2} + \nu^{2}) > (\log(N+1))^{-1} + (N+1)^{-1} +$$

$$- (\log(n+1))^{-1} - (n+1)^{-1}$$

and $(\log N)^{-1} - (\log(N+1))^{-1} - (N+1)^{-1} < 0$ for $N \geqslant 100$, we have $z_{n+1} < 0$ for sufficiently large n, which is impossible for $x \in (c,1]$.

(5) We shall prove that $c < x_1 < x_2 < (\log 100)^{-1}$ implies $-\infty < t(x_1) < t(x_2) < 0$. Let each of the sequences $\{z_n\}$ and $\{z'_n\}$ satisfy (A.1) for $n \geqslant 100$, whence $z_n \to 0$, $z'_n \to 0$ for $n \to \infty$, $z_n < (\log n)^{-1}$, $z'_n < (\log n)^{-1}$ for $n \geqslant m \geqslant 100$. Moreover, assume $z_{100} < z'_{100}$; (in our above notation we may also write $z_n = z_n(z_{100})$, $z'_n = z_n(z'_{100})$). If $\delta_n = z'_n - z_n$ then it follows from (A.1) that $\delta_{n+1} = (1-\alpha_n)\delta_n$ with $\alpha_n = n^{-1}(z'_n + z_n)$. As the z_n's are strictly monotonic functions of the initial value z_{100}, we have $\delta_m = z_m(z'_{100}) - z_m(z_{100}) > 0$. For $n > m$ we then have $\delta_m^{-1}\delta_{n+1} = \Pi_{\nu=m}^{n}(1-\alpha_\nu)$, and as $\alpha_\nu < 2\nu^{-1}(\log \nu)^{-1}$ this implies $\delta_m^{-1}\delta_{n+1} > \Pi_{\nu=m}^{n}(1-2\nu^{-1}(\log \nu)^{-1}) > C(\log n)^{-2}$ (*) for an appropriate positive C. So $(\log n)(z'_n \log n - 1) - (\log n)(z_n \log n - 1) > C\delta_m$, and this proves the strict monotonicity of $t(x)$ on $(c, (\log 100)^{-1})$. We make the trivial remark that solutions of (A.1) which differ asymptotically, have different values of z_{100}.

(6) <u>Comparison with a differential equation</u>. If $z_n \to 0$ then $z_{n+1} - z_n \to 0$, and it will be reasonable to compare the difference equation (A.1) with the differential equation

$$dz/dn = -n^{-1}(z^2 + n^{-1}),$$

where z is regarded as a function of the continuous variable n. If we take $w = z^{-1}$ as the dependent and $x = n^{-1}$ as the independent variable, then this differential equation is transformed into

$$dw/dx + w^2 + x^{-1} = 0,$$

(*) $\Pi_{\nu=m}^{n}(1-2\nu^{-1}(\log \nu)^{-1}) = \exp\{ \Sigma_{\nu=m}^{n} \log(1-2\nu^{-1}(\log \nu)^{-1})\} >$

$>\exp\{-\Sigma_{\nu=m}^{n} 2\nu^{-1}(\log \nu)^{-1} - 2(1-(1/50)(\log 100)^{-1})^{-2}\Sigma_{\nu=100}^{\infty} \nu^{-2}(\log \nu)^{-2}\} >$

$>\exp\{-2\int_{m-1}^{n} x^{-1}(\log x)^{-1}dx + C_1\} = \exp\{C_2 - 2\log\log n\}$.

a so-called "special Riccati equation". Applying the usual device for this type of equation, i.e. substituting $w = y' y^{-1}$, we obtain $xy'' + y = 0$. It should be kept in mind that we are interested in the behaviour of the solutions for $x \to 0$ only. In a neighbourhood of $x = 0$, the solutions of $xy'' + y = 0$ can be written as

$$y(x) = (c_1 + c_2 \log x)(x + \dots) + c_2(-1 + 0x + \dots).$$

It follows that

$$w(x) = \frac{(c_1 + c_2 \log x)(1 + \dots) + c_2(1 + \dots)}{(c_1 + c_2 \log x)(x + \dots) + c_2(-1 + 0x + \dots)}.$$

Since it is only the ratio of c_1 and c_2 that is significant, we put $\varepsilon = -c_2 c_1^{-1}$.

$$w = \frac{(1 - \varepsilon \log x)(1 + \dots) - \varepsilon(1 + \dots)}{x(1 - \varepsilon \log x)(1 + \dots) - \varepsilon(-1 + 0x + \dots)} =$$

$$= \frac{(1 + \varepsilon \log n)(1 + \dots) - \varepsilon(1 + \dots)}{n^{-1}(1 + \varepsilon \log n)(1 + \dots) + \varepsilon(1 - 0n^{-1} + \dots)}.$$

If $\varepsilon = 0$ we have $w(n) = n + \mathcal{O}(1)$ and $z(n) = n^{-1} + \mathcal{O}(n^{-2})$. For small $|\varepsilon|$ an approximation of w is $(n + \varepsilon n \log n)(1 + \varepsilon n + \varepsilon \log n)^{-1}$. First, we observe that for $\varepsilon < 0$ we have $w \to \infty$ for some finite value of n. This can be regarded as an analogue of the situation that the discrete sequence $\{z_n\}$ produces negative values of z_n from a certain value of n onward.
We shall direct our attention to $\varepsilon \geqslant 0$ only.
As $z = w^{-1}$ we can approximate $z(n)$ by $n^{-1} + \varepsilon(1 + \varepsilon \log n)^{-1}$. We shall show that these expressions are good approximations of solutions of (A.1), uniformly in ε for ε in some interval $0 \leqslant \varepsilon \leqslant \varepsilon_0$.

(7) Let $v_n(\varepsilon) = n^{-1} + \varepsilon(1 + \varepsilon \log n)^{-1}$ and

$$v_{n+1}(\varepsilon) - v_n(\varepsilon) + n^{-1}(v_n(\varepsilon))^2 + n^{-2} = b_n(\varepsilon).$$

We shall derive some estimates for $b_n(\varepsilon)$ which hold uniformly in ε. If $\varepsilon(1 + \varepsilon \log n)^{-1} = e_n(\varepsilon)$ then

$$e_{n+1}(\varepsilon) = e_n(\varepsilon) \left((e_n(\varepsilon)) \log (1+n^{-1}) + 1\right)^{-1}.$$

Now we have, if $n \geqslant 100$

$$b_n(\varepsilon) = (n+1)^{-1}-n^{-1}+n^{-2}+n^{-3}+2n^{-2}e_n+n^{-1}e_n^2-e_{n+1}e_n \log(1+n^{-1}) <$$

$$< 2n^{-3}+e_n(2n^{-2}+n^{-1}e_n-(e_n \log(1+n^{-1}))(1+e_n\log(1+n^{-1}))^{-1})<$$

$$< 2n^{-3}+e_n(2n^{-2}+n^{-1}e_n-e_n \log(1+n^{-1})+e_n^2(\log(1+n^{-1}))^2) <$$

$$< 2n^{-3}+e_n(2n^{-2}+(3/2)n^{-2}e_n^2)< 2n^{-3}+4n^{-2}e_n < n^{-2}(4\varepsilon+1/50).$$

Moreover, $b_n(\varepsilon) > 0$ for $\varepsilon \geqslant 0$, since $n^{-1}e_n^2 > e_{n+1}e_n \log(1+n^{-1})$.

(8) In $v_n(\varepsilon) + p_n(\varepsilon)$ is a solution of (A.1) then

$$p_{n+1} - p_n + 2n^{-1}p_nv_n + n^{-1}p_n^2 = -b_n. \tag{A.2}$$

We shall show that there exist solutions $\{p_n\}$ of (A.2) which are $\mathcal{O}((n \log n)^{-1})$ uniformly in ε in a right neighbourhood of 0. We now use some Banach space terminology. Let X be the space of all sequences $\underline{x} = \{x_{100}, x_{101}, \ldots\}$ with $x_n = \mathcal{O}(n^{-1})$. If we define a norm by $\|\underline{x}\| = \sup\{ n|x_n| \mid n=100,101,\ldots \}$, then X becomes a real Banach space. We shall only consider values of ε with $\varepsilon \leqslant 1/5$. Then we have $2v_n < \frac{1}{2}$ $(n \geqslant 100)$. (A further restriction on ε will be made below.) We shall show that for $\varepsilon \leqslant 1/5$ $p_{n+1} - p_n + 2n^{-1}p_nv_n(\varepsilon) = -n^{-1}a_n$ has a solution $\underline{p} \in X$ if $\underline{a} \in X$. If we consider $q_{n+1} - p_n + 2n^{-1}p_nv_n = -n^{-1}a_n$, or

$$p_n = (1 - 2n^{-1}v_n)^{-1} (n^{-1}a_n + q_{n+1}),$$

we see that this formula defines a mapping of $X \times X$ into X; we denote this mapping by $\underline{p} = A(\underline{a},\underline{q})$. We shall prove that if $\underline{a} \in X$, and if $\underline{p}(\underline{a})$ is defined by

$$(\underline{p}(\underline{a}))_n = p_n = \Sigma_{\nu=n}^{\infty}\{ \Pi_{k=n}^{\nu} (1-2k^{-1}v_k)^{-1}\} \nu^{-1}a_\nu,$$

then $\underline{p}(\underline{a}) = A(\underline{a}, \underline{p}(\underline{a}))$. First we prove that $\underline{p}(\underline{a}) \in X$. As

$$\Pi_{k=n}^{\nu} (1-2k^{-1}v_k)^{-1} \leqslant \Pi_{k=n}^{\nu} (1-\tfrac{1}{2}k^{-1})^{-1} \leqslant \nu^{\frac{1}{2}} (n-1)^{-\frac{1}{2}}$$

we have $|p_n| \leqslant (n-1)^{-\frac{1}{2}} \|\underline{a}\| \Sigma_{\nu=n}^{\infty} \nu^{-\frac{1}{2}} < 2(n-1)^{-1}\|\underline{a}\| < 3n^{-1} \|\underline{a}\|$.

So $p_n = \mathcal{O}(n^{-1})$ and $\underline{p} \in X$. Moreover, it is clear that

$$p_n = (1 - 2n^{-1}v_n)^{-1}(n^{-1}a_n + p_{n+1}).$$

So $\underline{p} = A(\underline{a}, \underline{p})$.
Let B be the mapping of X into X defined by $(B(\underline{r}))_n = r_n^2$. Moreover, we put $b_n = n^{-1}d_n$, whence $\underline{d} - \{d_n\} \in X$ for all ε in $[0,1/5]$. We now have to study the mapping A* of X × X into X defined by

$$A^*(\underline{r}, \underline{q}) = A(\underline{d} + B(\underline{r}), \underline{q}).$$

We must show that there exists a $\underline{p} \in X$ with $\underline{p} = A^*(\underline{p}, \underline{p})$. It will then be clear that $\{p_n\}$ is a solution of (A.2), since $\underline{p} = A(\underline{x}, \underline{p})$ means $p_n = (1 - 2n^{-1}v_n)^{-1}(n^{-1}x_n + p_{n+1})$ and therefore, with $\underline{x} = \underline{d} + B(\underline{p})$, it means,

$$p_n = (1 - 2n^{-1}v_n)^{-1}(b_n + n^{-1}p_n^2 + p_{n+1}).$$

Defining $\underline{s} = T(\underline{r})$ by

$$s_n = \Sigma_{\nu=n}^{\infty} \{\Pi_{k=n}^{\nu} (1 - 2k^{-1}v_k)^{-1}\} \nu^{-1}(d_\nu + r_\nu^2) \tag{A.3}$$

we have $\underline{s} = A(\underline{d} + B(\underline{r}), \underline{s})$, and we must prove the existence of an element $\underline{p} \in X$ with $\underline{p} = T(\underline{p})$.
To this end we use a theorem which is an extension of Banach's theorem on the fixed point of a contraction operator. With the notations

$$S(\underline{x}_o, \rho) = \{\underline{z} \mid \underline{z} \in X, \|\underline{z} - \underline{x}_o\| < \rho\}$$
$$\overline{S}(\underline{x}_o, \rho) = \{\underline{z} \mid \underline{z} \in X, \|\underline{z} - \underline{x}_o\| \leq \rho\}$$

this theorem is as follows (see [10], кap.I, §.7).

<u>Theorem</u>. If X is a Banach space and T is a mapping of X into X satisfying

$$\|T(\underline{x}) - T(\underline{y})\| \leq \alpha \|\underline{x} - \underline{y}\| \tag{A.4}$$

with a fixed $\alpha < 1$ for $\underline{x}, \underline{y} \in S(\underline{x}_o, \rho)$, and if

$$\|\underline{x}_o - S(\underline{x}_o)\| < (1-\alpha)\rho,$$

then the operator T has exactly one fixed point in $\overline{S}(\underline{x}_o, \rho)$.

We shall apply this theorem with $\underline{x}_0 = \underline{0}$. We have
$\|T(\underline{0})\| = \sup \{ n|(T(\underline{0}))_n| \mid n \geq 100 \} =$
$= \sup \{ n\Sigma_{\nu=n}^{\infty} \{\Pi_{k=n}^{\nu} (1-2k^{-1}v_k)^{-1}\} b_\nu \mid n \geq 100 \} \leq$
$\leq \sup \{ n(n-1)^{-\frac{1}{2}} (\Sigma_{\nu=n}^{\infty} \nu^{-\frac{3}{2}})(4\varepsilon+1/50) \mid n \geq 100 \} \leq (8\varepsilon+2/50)\cdot 100/99.$

We restrict ourselves to values of ε which are so small that $(8\varepsilon+2/50)\cdot 100/99 < 1/21$. In order to apply the above theorem we further must show that the operator T given by (A.3) satisfies a condition of the form (A.4) on a sphere around $\underline{0}$. In fact $\|T(\underline{x}) - T(\underline{y})\| < \frac{1}{2}\|\underline{x} - \underline{y}\|$ if $\underline{x}, \underline{y} \in S(0,1/10)$. This can be seen as follows
$|(T(\underline{x}) - T(\underline{y}))_n| = \Sigma_{\nu=n}^{\infty} \{\Pi_{k=n}^{\nu}(1-2k^{-1}v_k)^{-1}\} \nu^{-1}|x_\nu^2 - y_\nu^2| <$
$< (n-1)^{-\frac{1}{2}} \Sigma_{\nu=n}^{\infty} \nu^{-\frac{1}{2}} |x_\nu - y_\nu|(|x_\nu|+|y_\nu|) \leq$
$\leq (2/5)(n-1)^{-1} \sup \{\nu|x_\nu - y_\nu| \mid \nu \geq n\} < \frac{1}{2}n^{-1} \sup\{\nu|x_\nu - y_\nu| \mid \nu \geq n\}.$

Hence $\|T(\underline{x}) - T(\underline{y})\| = \sup\{n|(T(\underline{x}) - T(\underline{y}))_n| \mid n \geq 100\} <$
$< \frac{1}{2}\sup\{ n|x_n - y_n| \mid n \geq 100\} = \frac{1}{2}\|\underline{x} - \underline{y}\|$. We notice that, with $\alpha = \frac{1}{2}$, $\rho = 1/10$, $\|T(\underline{0})\| < 1/21 < (1-\alpha)\rho$. By use of the theorem we have the existence of a solution $\{p_n\}$ of (A.2) with $|p_n| \leq n^{-1}/10$ for all considered values of ε.
A simple observation suffices to obtain the desired result. As $p_n = \mathcal{O}(n^{-1})$, $v_n = \mathcal{O}((\log n)^{-1})$ and $b_n = \mathcal{O}(n^{-2}(\log n)^{-1})$ hold uniformly in ε, we have $p_{n+1} - p_n = \mathcal{O}(n^{-2}(\log n)^{-1})$ and hence $p_n =$ $= \mathcal{O}(n^{-1}(\log n)^{-1})$ uniformly in ε in an interval $0 \leq \varepsilon \leq \varepsilon_0$ $(\varepsilon_0 > 0)$.

(9) Our solution $v_n(\varepsilon) + p_n(\varepsilon)$ of (A.1) satisfies

$$\lim_{n\to\infty} (\log n)((v_n(\varepsilon) + p_n(\varepsilon))\log n - 1) = -\varepsilon^{-1},$$

(or $-\infty$ if $\varepsilon = 0$). We have already observed (see (5)) that if $\{z_n\}$ and $\{z_n'\}$ are solutions of (A.1) (for $n \geq 100$) with $z_n \to 0$, $z_n' \to 0$, then

$$\lim_{n\to\infty} (\log n)((z_n - z_n') \log n - 1) = 0,$$

if and only if $z_{100} = z_{100}'$. Roughly speaking, this means that all solutions of (A.1) for which $\lim_{n\to\infty} (\log n)(z_n \log n - 1)$ is large

negative, can be written as $v_n(\varepsilon) + p_n(\varepsilon)$. It is now also clear that $t(x)$ attains all values in an interval $(-\infty, \eta)$.
Thus, for those values of x for which $t(x) < -\varepsilon_0^{-1}$, [i.e. the values of x in $(c, v_{100}(\varepsilon_0) + p_{100}(\varepsilon_0))$] we have

$$z_n(x) = v_n(-(t(x))^{-1}) + p_n(-(t(x))^{-1}),$$

(notice the uniqueness of $\underline{p}(\varepsilon)$).
We further observe that $t(x)$ is continuous in a right neighbourhood of c (e.g. $(c, v_{100}(\varepsilon_0) + p_{100}(\varepsilon_0))$), since it is monotonic and attains all values in an interval $(-\infty, \eta)$. Moreover, $t(x) \to -\infty$, if $x \downarrow c$.
So we have the following picture of the solutions of (A.1). There exists an $h > 0$ and a continuous increasing function $\varepsilon(x)$ on $[c, c+h)$, such that $\varepsilon(c) = 0$ (hence $\varepsilon(x) > 0$ for $x > c$), and such that

$$z_n(x) = \frac{1}{n} + \frac{\varepsilon(x)}{1 + (\varepsilon(x))\log n} + \mathcal{O}\left(\frac{1}{n \log n}\right), \qquad \text{(A.5)}$$

uniformly in $x \in [c, c+h)$.
This result indicates, how the two types of converging solutions of (A.1) are related. As a matter of fact, for $z_n(c)$ (A.5) yields $z_n(c) = n^{-1}(1 + o(1))$, whereas for $x > c$ we obtain $z_n(x) = (\log n)^{-1} + \mathcal{O}((\log n)^{-2})$.

<u>A list of asymptotic formulas proved in this thesis</u>

This list contains only results concerning inequalities. References to the literature (given by means of [ ]) relate to the corresponding infinite inequalities. Secs. are Sections of this thesis. In all cases $a_1 \geqslant 0, \ldots, a_N \geqslant 0$.

(1) $\Sigma_{n=1}^{N} \{ n^{-1}(a_1^p + \ldots + a_n^p)\}^{1/p} \leqslant \lambda_N(p)\ \Sigma_{n=1}^{N} a_n \qquad (0<p<1)$

$\lambda_N(p) = (1-p)^{-1/p} - (1-p)^{-1-1/p}\ 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3})$.

[8] theorem 326; Secs.2 - 7. Equivalent to a special case of (3).

(2) $\Sigma_{n=1}^{N} (a_n + a_{n+1} + \ldots + a_N)^p \leqslant \lambda_N(p)\ \Sigma_{n=1}^{N} n^p a_n^p \qquad (p>1)$

$\lambda_N(p) = p^p - (p-1)p^{p+1}\ 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3})$.

[8] theorem 331; Sec.8. Special case of (4).

(3) $\Sigma_{n=1}^{N} n^{-s}(a_1 + \ldots + a_n)^p \leqslant \lambda_N(p,s)\ \Sigma_{n=1}^{N} n^{-s}(n\ a_n)^p \qquad (p \geqslant s > 1)$

$\lambda_N(p,s) = \left(\frac{p}{s-1}\right)^p - \frac{(p-1)}{(s-1)}\left(\frac{p}{s-1}\right)^{p+1} \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right)$.

[6]; Sec.10. Equivalent to (1) if p= s.

(4) $\Sigma_{n=1}^{N} n^{-s}(a_n + \ldots + a_N)^p \leqslant \lambda_N(p,s)\ \Sigma_{n=1}^{N} n^{-s}(n\ a_n)^p \quad (p>1>s\geqslant 0)$

$\lambda_N(p,s) = \left(\frac{p}{1-s}\right)^p - \frac{(p-1)}{(1-s)}\left(\frac{p}{1-s}\right)^{p+1} \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right)$.

[6]; Sec.10. Identical to (2) if s = 0.

(5) $\Sigma'^{N}_{n=1}\ \{n^{-1}(a_1^{-p}+\ldots+a_n^{-p})\}^{-1/p} \leqslant \lambda_N(p)\ \Sigma^{N}_{n=1}\ a_n$ $(p>0)$

$$\lambda_N(p) = (1+p)^{1/p} - (1+p)^{-1+1/p}\ 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

[9]; Sec.11. For the meaning of the prime see Sec.11. Equivalent to a special case of (8).

(6) $\Sigma^{N}_{n=1}\ (a_n+\ldots+a_N)^p \geqslant \lambda_N(p)\ \Sigma^{N}_{n=1}\ (n\ a_n)^p$ $(0<p<1)$

$$\lambda_N(p) = p^p + p^{p+1}(1-p)\ 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

[6], or [8] theorem 344; Sec.12. Equivalent to a special case of (7).

(7) $\Sigma^{N}_{n=1}\ n^{p-s}a_n \leqslant \lambda_N(p,s)\ \Sigma^{N}_{n=1}\ n^{-s}(a_n^{1/p}+\ldots+a_N^{1/p})^p$ $(0<p<1,\ s\leqslant 0)$

$$\lambda_N(p,s) = \left(\frac{1-s}{p}\right)^p - \left(\frac{1-s}{p}\right)^{p-1}\frac{1-p}{1-s}\ \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right).$$

[6]; Sec.12. Equivalent to (6) if $s=0$.

(8) $\Sigma^{N}_{n=1}\ n^{-s}(a_1+\ldots+a_n)^{-p} \leqslant \lambda_N(p,s)\Sigma^{N}_{n=1}\ n^{-s}(n\ a_n)^{-p}$ $(p>0,\ s\leqslant -p)$

$$\lambda_N(p,s) = \left(\frac{1-s}{p}\right)^p - \left(\frac{1-s}{p}\right)^{p-1}\frac{p+1}{1-s}\ \frac{2\pi^2}{(\log N)^2} + \mathcal{O}\left(\frac{1}{(\log N)^3}\right).$$

Sec.12. Equivalent to (5) if $s = -p$.

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Samenvatting

In dit proefschrift wordt voor een aantal ongelijkheden tussen eindige reeksen met positieve termen onderzocht, wat het asymptotische gedrag is van de in die ongelijkheden voorkomende constanten, indien het aantal termen van de reeks naar oneindig nadert.
In § 1 wordt een tweetal algemene stellingen over iteratieprocessen bewezen en met voorbeelden toegelicht.
Daarna begint in § 2 de tot in alle details uitgevoerde bepaling van een asymptotische formule voor de best mogelijke constante die optreedt in een eindige versie van de ongelijkheid van Hardy. Voor een vaste waarde van p in (0,1) trachten we een formule te vinden voor $\lambda_N(p)$ als $N\to\infty$, waarbij $\lambda_N(p)$ het kleinste getal is zodat voor alle $a_1 \geqslant 0,\dots,a_N \geqslant 0$ geldt :

$$\Sigma_{n=1}^{N} \{n^{-1}(a_1^p + \dots + a_n^p)\}^{1/p} \leqslant \lambda_N(p)\ \Sigma_{n=1}^{N} a_n.$$

In § 2 wordt aangetoond dat de vraag naar $\lambda_N(p)$ gelijkwaardig is met een vraag over een iteratieproces. De bestudering van dit iteratieproces in de §§ 3 t/m 7 geeft ons, behalve enige nevenresultaten in § 3, de gewenste asymptotische formule voor $\lambda_N(p)$:

$$\lambda_N(p) = (1-p)^{-1/p} - (1-p)^{-1-1/p}\, 2\pi^2(\log N)^{-2} + \mathcal{O}((\log N)^{-3}).$$

De gevolgde methode is in wezen die van N.G. de Bruijn's [2] artikel over de ongelijkheid van Carleman.
In § 8 wordt het eindige analogon van een bekende kunstgreep uit de theorie der reeksen aangewend om uit het bovenstaande resultaat een soortgelijke formule af te leiden voor de constante in een eindige versie van een ongelijkheid van Copson. Als nevenresultaat vinden we tevens een formule voor de grootste eigenwaarden van eindige deelmatrices van een zekere oneindige matrix.
Een klasse van iteratieproblemen, die een soortgelijke behandeling toelaten als in het geval van de ongelijkheid van Hardy, wordt beschreven in § 9.
De §§ 10, 11 en 12 zijn gewijd aan ongelijkheden die aanleiding geven tot soortgelijke iteratieproblemen, waarbij in § 10 direct het resultaat van § 9 wordt toegepast, terwijl we in de §§ 11 en 12

gebruik maken van een alternatieve vorm die in § 11 geformuleerd wordt.

In de laatste paragraaf worden ongelijkheden besproken, die bij eerste beschouwing analoog schijnen te zijn aan de voorgaande gevallen, maar die toch een geheel ander gedrag vertonen.

Een opmerkelijke eigenschap van iteratieprocessen, die in § 1 reeds naar voren komt, maar voor de gegeven beschouwingen over reeksen van weinig belang is, wordt in het aanhangsel door een uitgewerkt voorbeeld geillustreerd.

Curriculum vitae

De samensteller van dit proefschrift werd op 26 juli 1936 te Amsterdam geboren. Hij doorliep het gymnasium van het St.Ignatius-college in zijn geboortestad en studeerde van 1954 tot 1960 aan de Gemeentelijke Universiteit van Amsterdam in de faculteit van de wiskunde en natuurwetenschappen. Hier volgde hij onder andere colleges van de hoogleraren dr.E.W.Beth, dr.N.G.de Bruijn, dr.J.de Groot, dr.A.Heyting, dr.F.Loonstra en dr.J.Popken. Tevens was hij van 1958 tot 1960 als assistent werkzaam bij prof.dr.N.G.de Bruijn. Nadat hij het doctoraal examen met hoofdvak wiskunde en bijvak logica had afgelegd, werd hij benoemd tot wetenschappelijk ambtenaar aan de Technische Hogeschool te Eindhoven.

Typografie :
Mej.A.N.M.van de Griendt

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S.T.M.Ackermans
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S T E L L I N G E N

I

Zij A een Banach algebra met eenheidselement; a en b onderling verwisselbare elementen van A; $b^m = 0$; zij de complexe functie f locaal analytisch in een omgeving van het spectrum van a; zij F de "principal extension" van f; F_n de "principal extension" van $f^{(n)}$. Dan is

$$F(a+b) = \Sigma_{n=0}^{m-1} (n!)^{-1} F_n(a)\, b^n.$$

Litteratuur. E.Hille, R.S.Philips, Functional analysis and semigroups, blz. 164 e.v.

II

Bekend is dat in een ring de verzameling W der wortelgrootheden bevat is in de verzameling J der eigenlijk nilpotente elementen. J kan wezenlijk groter zijn dan W, zelfs in een commutatieve Banach algebra met eenheidselement.

Litteratuur. B.L.van der Waerden, Algebra II, blz. 143.
M.A.Naimark, Normed Rings, blz. 162.

III

Verscheidene stellingen over lineaire functionalen op een Riesz ruimte kunnen gegeneraliseerd worden tot stellingen over lineaire afbeeldingen van een Riesz ruimte in een complete Riesz ruimte.

Litteratuur. N.Bourbaki, Eléments de Mathématique, Livre VI Intégration, Chap.II, §.2.

IV

Het door Coxeter geuite vermoeden over afhankelijkheid in Artin's axioma's van de affiene meetkunde is juist, mits het goed geinterpreteerd wordt.

Litteratuur. H.S.M.Coxeter,
Introduction to geometry, blz. 191.
E.Artin, Geometric Algebra, blz.51 e.v.

V

In een Banach algebra met eenheidselement zij $M(\mathcal{O})$ de aanduiding voor de verzameling der elementen waarvan het spectrum in de open deelverzameling $\mathcal{O}$ van het complexe vlak ligt. Indien $M(\mathcal{O})$ samenhangend is, dan is $\mathcal{O}$ samenhangend.

VI

Zij A een commutatieve B*-algebra met eenheidselement; $\mathbb{C}$ het complexe vlak. Indien er een niet lege open deelverzameling O van $\mathbb{C}$ bestaat, zó dat $M(O)$ relatief compact in A is, dan is A volledig isomorf met een eindig dimensionale algebra $\mathbb{C}^n$.

VII

Zij A een Banach algebra met eenheidselement; $\mathbb{C}$ het complexe vlak; zij F de verzameling van de complex-waardige functies, f, gedefiniëerd op open deelverzamelingen van $\mathbb{C} \times A$, waarvoor $f(\zeta,a)$ locaal analytisch en $f(\alpha,z)$ locaal constant is (a, α vast, ζ, z variabel); zij Θ de deelverzameling van $\mathbb{C} \times A$ bestaande uit punten (α,a) waarvoor α tot het spectrum van a behoort, de zgn. spectrale graph. Dan is de natuurlijke afbeelding van de ruimte van de kiemen van functies van F in de punten van Θ op de ruimte van de analytische kiemen in $\mathbb{C}$ continu. Het is te verwachten dat deze afbeelding een bruikbaar hulpmiddel zal zijn in de functietheorie in Banach algebra's.

VIII

In wiskundige uiteenzettingen worden lacunes vaak overbrugd door zegswijzen als: "men ziet zonder veel moeite dat ..."; "het is onmiddellijk duidelijk dat ...". Het is mogelijk een genormaliseerde lijst op te stellen, die in de behoefte aan dergelijke uitdrukkingen voorziet. Beperking van de gebruikte uitdrukkingswijzen tot een dergelijke lijst zou de leesbaarheid van wiskundige geschriften ten goede komen.

IX

De laatste jaren is op verschillende wijzen een grootheid "studierendement" gedefiniëerd en gemeten. Uit individuele, kwalitatieve gegevens is waarschijnlijk informatie te verkrijgen die voor onderwijsverbetering waardevoller is.

X

Om de toekomstige ingenieurs vertrouwd te maken met de maatschappelijke verantwoordelijkheid die zij zullen dragen, is het nuttig hen tijdens hun studie te confronteren met de geschiedenis van de techniek en de invloed daarvan op de samenleving.