AN ASYMPTOTIC SAMPLING RECOMPOSITION THEOREM FOR GAUSSIAN SIGNALS

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Abstract. For any Gaussian signal and every given sampling frequency we prove an asymptotic property of type Shannon's sampling theorem, based on normalized cardinal sines, which keeps constant the sampling frequency. We generalize the Shannon's sampling theorem for a class of non band–limited signals which plays a central role in the signal theory, the Gaussian map is the unique function which reachs the minimum of the product of the temporal and frecuential width. This solve a conjecture stated in [1] and suggested by [3].

1. Introduction and statement of the main result

A key point for people who work on signal theory is the well–known Shannon–Whittaker–Kotel'nikov's theorem (see for instance [11] or [13]) working for band–limited maps of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener signals), and based on the normalized cardinal sinus map $\operatorname{sinc}(t)$ defined by

$$\operatorname{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0. \end{cases}$$

Another main result of the signal processing theory is the Middleton's sampling theorem for band step functions (see [10]). This result was one of the first modifications of the classical Sampling Theorem which only works for band–limited maps, see [12]. After this starting point many different extensions and generalizations of this theorem appeared in the literature trying to obtain approximations of non band–limited signals (see for instance [4] or [7]). Good surveys on these extensions are [5] or [13].

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[1] follows the spirit of the previous results in the sense of trying to obtain approximations of non band–limited signals by using band–limited ones by means of the increasing of the band size. But its approach is completely different to the previous ones in the sense that [1] keeps constant the sampling frequency generalizing in the limit the results of Marvasti et al. [9] and Agud et al. [2].

In this setting, [1] states the following asymptotic property of type sampling Shannon's theorem where the convergence of the series is considered in the Cauchy's principal value.

Property 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a map and $\tau\in\mathbb{R}^+$. We say that f holds the property $\mathcal{P}$ for τ if*

$$f(t)=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}}\left(\frac{k}{\tau}\right)\operatorname{sinc}(\tau t-k)\right)^n. \tag{1}$$

[1] proves that every constant signal holds property $\mathcal{P}$ for every given $\tau\in\mathbb{R}^+$ and conjectures that the Gaussian maps, i.e. maps of the form $e^{-\lambda t^2}$, $\lambda\in\mathbb{R}^+$ hold property $\mathcal{P}$ for every given $\tau\in\mathbb{R}^+$. To support the conjecture [1] proves that the Gaussian map e^{-t^2} holds expression (1) for the three first coefficients of the power series representation of e^{-t^2}.

Note that the veracity of the conjecture is also suggested by the Boas's estimation [3]. In [3] is stated that if f has an integrable Fourier transform, the pointwise error between f and its sampling series $\sum f(k)\operatorname{sinc}(t-k)$ is controlled by $\int_{|\xi|>\frac{1}{2}}\left|\hat{f}(\xi)\right|d\xi$. Since $(e^{-\pi\lambda t^2})^{\frac{1}{n}}=e^{-\pi(t\sqrt{\frac{\lambda}{n}})^2}$ its Fourier trasnform is $\sqrt{\frac{n}{\lambda}}e^{-\pi(\xi\sqrt{\frac{n}{\lambda}})^2}$ and

$$\sqrt{\frac{n}{\lambda}}\int_{|\xi|>\frac{1}{2}}e^{-\pi(\xi\sqrt{\frac{n}{\lambda}})^2}d\xi=\int_{|\xi|>\sqrt{\frac{n}{4\lambda}}}e^{-\pi\xi^2}d\xi\to 0$$

as $n\to\infty$. Thus Boas's estimate proves that the integer samples of the n–th root of the Gaussian maps converge to the n–th root, in a sense that is, presumably, consistent with (1).

The aim of this paper is to answer the conjecture in a positive way by proving that any Gaussian map $e^{-\lambda t^2}$ can be reconstructed as a limit of band–limited maps obtained from uniformly distributed samples at the points $\{\frac{k}{\tau}:k\in\mathbb{Z}\}$ where $\tau>0$ through the following formula:

$$e^{-\lambda t^2}=\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}}e^{-\lambda\frac{k^2}{n\tau^2}}\operatorname{sinc}(\tau t-k)\right)^n$$

being the convergence uniformly on compact sets. The statement of our main result is the following:

Theorem 1. *The Gaussian maps hold property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$.*

Note that since any Gaussian map $e^{-\lambda t^2}$ is analytical, for proving Theorem 1 is enough to show the equality between the coefficients of the power series representation of $e^{-\lambda t^2}$ and the coefficients of the expression $\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{-\lambda\frac{k^2}{n\tau^2}}\operatorname{sinc}(\tau t-k)\right)^n$ if such expression defines an analytical map.

We remark that the Gaussian map, which is mathematically important in itself, plays an important role in the signal theory because the Gaussian map is the unique function which reachs the minimum of the product of the temporal and frecuential width. This minimum is given by the Uncertainty Principle, see [8]. Therefore, to have recomposition results for these type of signal is interesting from the point of view of applications.

The paper is divided into three sections. In Section 2 we introduce some auxiliary results. The aim of Section 3 is to prove that expression $\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{-\frac{k^2}{n\tau^2}}\operatorname{sinc}(\tau t-k)\right)^n$ is convergent and generates an analytic map. Section 4 is devoted to the proof of Theorem 1.

2. Auxiliary results

In this section we present some auxiliary results which will play a key role in the proof of Theorem 1.

Lemma 2. *For every $r \in \mathbb{N}$ is held*

$$\sum_{j=1}^{r}\frac{(-1)^j}{\pi^{2j}\,(2r-2j+1)!}\sum_{k\in\mathbb{N}}\frac{(-1)^{k+1}}{k^{2j}}=\frac{-1}{2\,(2r+1)!}.$$

Proof. Let $f_j(z)=\dfrac{1}{z^{2j}\sin(\pi z)}$ with $z\in\mathbb{C}$ and $j\in\mathbb{N}\cup\{0\}$.
Computing the Laurent series representation of $f_0(z)$ is directly obtained that

$$\operatorname{Res}(f_0,0)=\frac{1}{\pi}. \tag{2}$$

For the maps $f_j(z)$ with $j>0$, let C_k the square of vertex $\left(k+\dfrac{1}{2}\right)(\pm 1\pm \mathrm{i})$.
By the Residual Theorem is

$$2\pi\mathrm{i}\left(\operatorname{Res}(f_j,0)+\sum_{\substack{r=-k\\ r\neq 0}}^{k}\operatorname{Res}(f_j,r)\right)=\int_{C_k} f_j(z)dz. \tag{3}$$

With simple calculations is obtained that

$$\sum_{\substack{r=-k \\ r\neq 0}}^{k} \mathrm{Res}(f_j, r) = \frac{1}{\pi} \sum_{\substack{r=-k \\ r\neq 0}}^{k} \frac{(-1)^r}{r^{2j}} = \frac{2}{\pi} \sum_{r=1}^{k} \frac{(-1)^r}{r^{2j}}. \tag{4}$$

Since $\lim_{k\to\infty} \int_{C_k} f_j(z)dz = 0$, taking limits when k goes to infinity in expression (3), is obtained

$$\mathrm{Res}(f_j, 0) = -\sum_{\substack{r\in\mathbb{Z} \\ r\neq 0}} \mathrm{Res}(f_j, r).$$

Therefore, by (2) and introducing (4) into the previous expression we can writte

$$\mathrm{Res}(f_j, 0) = \begin{cases} \dfrac{1}{\pi} & \text{if } j = 0, \\ \\ \dfrac{2}{\pi} \displaystyle\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^{2j}} & \text{if } j > 0. \end{cases} \tag{5}$$

On the other hand, by the Laurent series representation of the map $\dfrac{1}{\sin(\pi z)}$ around the point $z = 0$ is

$$\frac{1}{\sin(\pi z)} = \sum_{p=0}^{\infty} \beta_{2p-1}\, z^{2p-1} \tag{6}$$

and thus

$$f_j(z) = \frac{1}{z^{2j}\sin(\pi z)} = \sum_{p=0}^{\infty} \beta_{2p-1}\, z^{2(p-j)-1}.$$

From the previous expression we deduced for every $j \geq 0$

$$\beta_{2j-1} = \mathrm{Res}(f_j, 0). \tag{7}$$

Moreover, from the power series representation of the map $\sin(\pi z)$, (6) can be written in the form

$$\left(\sum_{q=0}^{\infty} \frac{(-1)^q}{(2q+1)!} (\pi z)^{2q+1} \right) \left(\sum_{p=0}^{\infty} \beta_{2p-1}\, z^{2p-1} \right) = 1$$

and making equal coefficients is

$$\sum_{j=0}^{r} \frac{(-1)^{r-j}}{(2r-2j+1)!} \pi^{2r-2j+1} \beta_{2j-1} = 0.$$

Now, changing β_{2j-1} for (7) and separating the term corresponding to $j = 0$ we have

$$\frac{1}{(2r+1)!}\operatorname{Res}(f_0, 0) + \sum_{j=1}^{r} \frac{(-1)^j}{\pi^{2j}\,(2r-2j+1)!}\operatorname{Res}(f_j, 0) = 0.$$

Finally, using (5) in the previous equality, the proof is over. ∎

The aim of the next section is to prove, for every $\tau > 0$, that

$$\lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}} \operatorname{sinc}(\tau t - k) \right)^n \tag{8}$$

exists and defines an analytic map. For proving it, we shall use the following result which is a simplified version of two well–known theorems on convergence of analytic maps (see [6, p. 241–242]).

Theorem 3. *Let $\mathcal{A}$ be an open connected set in $\mathbb{C}^p$. Let Φ be a set of analytic maps from $\mathcal{A}$ into a complex Banach space E. Let M be a oneness set in $\mathcal{A}$. If the following conditions are held:*

- i) *for every compact set L in $\mathcal{A}$ there exists $m_L > 0$ such that $\|f(z)\| \leq m_L$ for every $f \in \Phi$ and every $z \in L$,*
- ii) *$(f_n)_n \subseteq \Phi$ pointwise converges in M,*

then $(f_n)_n$ uniformly converges on compact sets of $\mathcal{A}$ to an analytic map.

3. Uniform convergence and analyticity

From now on we use the following notation

$$g(z,n) = \sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}} \operatorname{sinc}(\tau z - k). \tag{9}$$

In this section we shall prove that there exists the pointwise limit of $(g(z,n))^n$ and which it converges to an analytic map. For doing it we shall prove in the next two propositions that are verified the hypothesis of Theorem 3 taking:

$$\begin{aligned}
\mathcal{A} &= \mathbb{C} \setminus \frac{\mathbb{Z}}{\tau} \quad \text{open and connected set on } \mathbb{C},\\
M &= \mathcal{A} \cap \mathbb{R} \text{ oneness set for the analytic maps},\\
\Phi &= \left\{ (g(z,n))^n, n \in \mathbb{N} \right\},\\
E &= \mathbb{C}.
\end{aligned}$$

To prove it, we shall use notation from [1, Lemma 6] and [1, Proposition 7]:

$$\begin{aligned} l_k(x) &= \frac{1-e^{-k^2x}}{k^2x} \\ L(x) &= \sum_{k\in\mathbb{N}}(-1)^{k+1}l_k(x). \end{aligned} \tag{10}$$

Proposition 4. *For every L compact set of $\mathbb{C}\setminus\dfrac{\mathbb{Z}}{\tau}$ there exists $m_L>0$ such that $|\,(g(z,n))^n\,|\le m_L$ for every $z\in L$ and all $n\in\mathbb{N}$.*

Proof. Let $n\in\mathbb{N}$ fixed and consider the following complex map

$$G(z,n) = n\,\left(g(z,n)-1\right). \tag{11}$$

Using the definition of $g(z,n)$ given by (9) and $\sum_{k\in\mathbb{Z}}\operatorname{sinc}(z-k)=1$ for every $z\in\mathbb{C}$ (see [1, Lemma 3]), is

$$\begin{aligned} G(z,n) &= n\left(\sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}}\operatorname{sinc}(\tau z-k)-1\right) \\ &= n\sum_{k\in\mathbb{Z}}\left(e^{\frac{-k^2}{n\tau^2}}-1\right)\operatorname{sinc}(\tau z-k) \\ &= n\sum_{k\in\mathbb{N}}\left(e^{\frac{-k^2}{n\tau^2}}-1\right)\frac{(-1)^k\sin(\pi\tau z)}{\pi}\left(\frac{1}{\tau z-k}+\frac{1}{\tau z+k}\right) \\ &= -n\,\frac{2\tau\, z\sin(\pi\tau z)}{\pi}\sum_{k\in\mathbb{N}}\left(1-e^{\frac{-k^2}{n\tau^2}}\right)\frac{(-1)^k}{(\tau z)^2-k^2}. \end{aligned}$$

Separating the sum into even and odd terms we have

$$G(z,n) = B(z)a(z,n), \tag{12}$$

where

$$\begin{aligned} B(z) &= -\frac{2\tau\, z\sin(\pi\tau z)}{\pi}, \\ a(z,n) &= n\sum_{k\in\mathbb{N}}\left(\frac{1-e^{\frac{-(2k-1)^2}{n\tau^2}}}{(2k-1)^2-(\tau z)^2}-\frac{1-e^{\frac{-(2k)^2}{n\tau^2}}}{(2k)^2-(\tau z)^2}\right). \end{aligned} \tag{13}$$

Since L is a compact set, there exists a constant $\beta_L>0$ such that

$$|B(z)|\le\beta_L \tag{14}$$

for all $z\in L$.

For $z \in \mathbb{C} \setminus \dfrac{\mathbb{Z}}{\tau}$, let

$$A_p(z,n) = \frac{n\left(1 - e^{\frac{-p^2}{n\tau^2}}\right)}{p^2 - (\tau z)^2}.$$

Note that using notation given by (10)

$$A_p(z,n) = \frac{1}{1-(\frac{\tau z}{p})^2}\,\frac{1-e^{\frac{-p^2}{n\tau^2}}}{\frac{p^2}{n}} = \frac{1}{1-(\frac{\tau z}{p})^2}\,\frac{1}{\tau^2}\,l_p\left(\frac{1}{n\tau^2}\right) \tag{15}$$

and, by (13), $a(z,n)$ can be written in the form

$$a(z,n) = \sum_{k\in\mathbb{N}} \left(A_{2k-1}(z,n) - A_{2k}(z,n)\right). \tag{16}$$

After some calculations is

$$\begin{aligned}
&A_{2k-1}(z,n) - A_{2k}(z,n)\\
&\quad= \frac{1}{\tau^2}\left[l_{2k-1}\left(\frac{1}{n\tau^2}\right)\frac{1}{1-(\frac{\tau z}{2k-1})^2} - l_{2k}\left(\frac{1}{n\tau^2}\right)\frac{1}{1-(\frac{\tau z}{2k})^2}\right]\\
&\quad= \frac{1}{\tau^2}\left[l_{2k-1}\left(\frac{1}{n\tau^2}\right)\left(\frac{1}{1-(\frac{\tau z}{2k-1})^2} - \frac{1}{1-(\frac{\tau z}{2k})^2}\right)\right.\\
&\qquad\left.+\frac{1}{1-(\frac{\tau z}{2k})^2}\left(l_{2k-1}\left(\frac{1}{n\tau^2}\right) - l_{2k}\left(\frac{1}{n\tau^2}\right)\right)\right]\\
&\quad= \frac{1}{\tau^2}\left[l_{2k-1}\left(\frac{1}{n\tau^2}\right)(\tau z)^2\frac{4k-1}{((2k-1)^2-(\tau z)^2)((2k)^2-(\tau z)^2)}\right.\\
&\qquad\left.+\frac{(2k)^2}{(2k)^2-(\tau z)^2}\left(l_{2k-1}\left(\frac{1}{n\tau^2}\right) - l_{2k}\left(\frac{1}{n\tau^2}\right)\right)\right].
\end{aligned} \tag{17}$$

Thus, using the triangular inequality in (16)

$$\begin{aligned}
|a(z,n)| &= \left|\sum_{k\in\mathbb{N}}(A_{2k-1}(z,n) - A_{2k}(z,n))\right|\\
&\le \sum_{k\in\mathbb{N}} |A_{2k-1}(z,n) - A_{2k}(z,n)|\\
&\le \sum_{k\in\mathbb{N}}\left|l_{2k-1}\left(\frac{1}{n\tau^2}\right)z^2\frac{4k-1}{((2k-1)^2-(\tau z)^2)((2k)^2-(\tau z)^2)}\right|\\
&\qquad+\frac{1}{\tau^2}\sum_{k\in\mathbb{N}}\left|\frac{(2k)^2}{(2k)^2-(\tau z)^2}\left(l_{2k-1}\left(\frac{1}{n\tau^2}\right) - l_{2k}\left(\frac{1}{n\tau^2}\right)\right)\right|.
\end{aligned} \tag{18}$$

Now, on the one hand since the series

$$\sum_{k\in\mathbb{N}} \frac{4k-1}{|(2k-1)^2-(\tau z)^2|\,|(2k)^2-(\tau z)^2|}$$

converges for all $z \in L$ it is boundness and since $0 < l_{2k-1}\left(\frac{1}{n\tau^2}\right) \leq 1$, we have and upper bound α_L for the first part of the sum (18) because

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \left| l_{2k-1}\left(\frac{1}{n\tau^2}\right) z^2 \frac{4k-1}{((2k-1)^2-(\tau z)^2)\,((2k)^2-(\tau z)^2)} \right| \\
&\leq \sum_{k\in\mathbb{N}} |z|^2 \frac{4k-1}{|(2k-1)^2-(\tau z)^2|\,|(2k)^2-(\tau z)^2|} < \alpha_L.
\end{aligned} \tag{19}$$

On the other hand, since $z \in L$ and L is compact, there exists $\delta_L > 0$ such that for every $z \in L$, $|m^2-(\tau z)^2| \geq \delta_L$ for all $m \in \mathbb{N}$. Since

$$\lim_{k\to\infty} \frac{(2k)^2}{\min\limits_{z\in L}\left|(2k)^2-(\tau z)^2\right|} = 1,$$

by the limit definition given $\varepsilon > 0$ there is $k_0(L)$ such that for every $k \geq k_0$

$$1-\varepsilon < \frac{(2k)^2}{\min\limits_{z\in L}\left|(2k)^2-(\tau z)^2\right|} < 1+\varepsilon.$$

For $k < k_0$ is

$$\frac{(2k)^2}{\min\limits_{z\in L}\left|(2k)^2-(\tau z)^2\right|} \leq \frac{(2k_0)^2}{\delta_L}.$$

Therefore, taking $\lambda_L = \max\left\{1+\varepsilon, \frac{(2k_0)^2}{\delta_L}\right\}$ is

$$\left|\frac{(2k)^2}{(2k)^2-(\tau z)^2}\right| \leq \frac{(2k)^2}{\min\limits_{z\in L}\left|(2k)^2-(\tau z)^2\right|} \leq \lambda_L$$

uniformly on k.

From the previous expression using (10) and $L(x) \leq \frac{\pi}{2}$ for $x \in \mathbb{R}^+$ (see [1, Proposition 7]), we obtain the boundness of the second part of the sum (18) since

$$
\begin{aligned}
&\frac{1}{\tau^2}\sum_{k\in\mathbb{N}}\left|\frac{(2k)^2}{(2k)^2-(\tau z)^2}\right|\left|l_{2k-1}\left(\frac{1}{n\tau^2}\right)-l_{2k}\left(\frac{1}{n\tau^2}\right)\right| \\
&\leq \frac{\lambda_L}{\tau^2}\sum_{k\in\mathbb{N}}\left(l_{2k-1}\left(\frac{1}{n\tau^2}\right)-l_{2k}\left(\frac{1}{n\tau^2}\right)\right) \qquad (20)\\
&\leq \frac{\lambda_L}{\tau^2}\, L\left(\frac{1}{n\tau^2}\right) \leq \frac{\lambda_L}{\tau^2}\,\frac{\pi}{2}.
\end{aligned}
$$

So, by (19) and (20) we obtain from (18)

$$|a(z,n)| \leq \alpha_L + \frac{\lambda_L}{\tau^2}\,\frac{\pi}{2} = \gamma_L.$$

From the previous expression and by (14) we have for every $z \in L$

$$|G(z,n)| = |B(z)|\,|a(z,n)| \leq \beta_L\,\gamma_L < \infty. \tag{21}$$

Now, from the equation (11) the map $g(z,n)$ has the form

$$g(z,n) = 1 + \frac{G(z,n)}{n}. \tag{22}$$

Thus, using (21) is

$$|g(z,n)| \leq 1 + \frac{|G(z,n)|}{n} \leq 1 + \frac{\beta_L\,\gamma_L}{n},$$

and therefore

$$|\,(g(z,n))^n\,| \leq \left(1+\frac{\beta_L\,\gamma_L}{n}\right)^n \leq e^{\beta_L\,\gamma_L} = m_L,$$

ending the proof. ∎

Proposition 5. *$\{(g(z,n))^n\}_{n\in\mathbb{N}}$ pointwise converges in $\mathbb{R}$.*

Proof. Since $\operatorname{sinc}(k) = \delta_{k,0}$ for all $k \in \mathbb{Z}$, then $\{(g(z,n))^n\}_{n\in\mathbb{N}}$ converges in $\dfrac{\mathbb{Z}}{\tau}$.

Let $x \in \mathbb{R}\setminus\dfrac{\mathbb{Z}}{\tau}$ and $G(z,n)$ be the auxiliar map introduced in the previous proposition by (11). By (22)

$$(g(x,n))^n = \left(1+\frac{G(x,n)}{n}\right)^n$$

and consequently

$$\lim_{n\to\infty}(g(x,n))^n = e^{\lim\limits_{n\to\infty} G(x,n)}. \tag{23}$$

The previous equality implies that if $G(x,n)$ converges in n to a real number the proof is over.
By (12) is $G(x,n) = B(x)\,a(x,n)$, and therefore the convergence of $G(x,n)$ depends only on the convergence of $a(x,n)$ which by (16) is

$$a(x,n) = \sum_{k\in\mathbb{N}} \left(A_{2k-1}(x,n) - A_{2k}(x,n)\right).$$

As this series verifies the conditions of the Weierstrass's M–criterion, we can writte

$$\lim_{n\to\infty} a(x,n) = \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \left(A_{2k-1}(x,n) - A_{2k}(x,n)\right).$$

Now, using the expression of $A_p(x,n)$ given by (15) and $\lim_{x\to 0^+} l_k(x) = 1$, is

$$\lim_{n\to\infty} A_p(x,n) = \frac{1}{\tau^2}\frac{p^2}{p^2-(\tau x)^2}\lim_{n\to\infty} l_p\left(\frac{1}{n\tau^2}\right) = \frac{1}{\tau^2}\frac{p^2}{p^2-(\tau x)^2}$$

and so

$$\begin{aligned}\lim_{n\to\infty} a(x,n) &= \frac{1}{\tau^2}\sum_{k\in\mathbb{N}}\left(\frac{(2k-1)^2}{(2k-1)^2-(\tau x)^2} - \frac{(2k)^2}{(2k)^2-(\tau x)^2}\right)\\ &= x^2\sum_{k\in\mathbb{N}}\frac{4k-1}{((2k-1)^2-(\tau x)^2)((2k)^2-(\tau x)^2)}\end{aligned}$$

which converges for all $x \in \mathbb{R}\setminus\dfrac{\mathbb{Z}}{\tau}$.
Thus, is proved the existence of $\lim_{n\to\infty} a(x,n)$ which implies the existence of $\lim_{n\to\infty} G(x,n)$ and by (21) we know that this limit is finite. Finally, by (23) the proof finishs. ∎

Theorem 6. *For every $\tau > 0$, $\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}}\operatorname{sinc}(\tau t - k)\right)^n$ defines an analytic map.*

Proof. The proof is a direct consequence of the application of Theorem 3. The use of such result is possible by Propositions 4 and 5 where state the validity of the the hypothesis of Theorem 3. ∎

4. Converging to the Gaussian map: Proof of Theorem 1

The aim of this section is to prove our main result Theorem 1 by showing that the analytic map (8) obtained in the previous section is the

Gaussian map, i.e., for every $t \in \mathbb{R}$ is held the following equality

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}}\operatorname{sinc}(\tau t-k)\right)^n = e^{-t^2}$$

for every $\tau > 0$.

The methodology that we shall use will be to compute the coefficients of the power series representation of $(g(x,n))^n$ and to show that their limits are the coefficients of the power series representation of the Gaussian map.

We introduce the following notation which will simplify the computation of the coefficients of the power series representation in the sequel.

Definition 7. *For every $m \in \mathbb{N}\cup\{0\}$ and $n \in \mathbb{N}$ we define*

$$B^\tau_m = \frac{(-1)^m(\pi\tau)^{2m}}{(2m+1)!}; \tag{24}$$

$$C^\tau_{m,n} = \begin{cases} \dfrac{1}{2} & \text{if } m=0,\\ \tau^{2m}\displaystyle\sum_{k\in\mathbb{N}}\frac{(-1)^{k+1}}{k^{2m}}\,e^{\frac{-k^2}{n\tau^2}} & \text{if } m\geq 1; \end{cases} \tag{25}$$

$$D^\tau_{m,n} = \sum_{p=0}^{m} B^\tau_p C^\tau_{m-p,n}; \tag{26}$$

$$d^\tau_{m,n} = \begin{cases} 1 & \text{if } m=0,\\ \dfrac{(-1)^m}{(n\tau^2)^m m!} - \displaystyle\sum_{j=0}^{m-1} d^\tau_{j,n}\,\frac{B^\tau_{m-j}}{\tau^{2(m-j)}} & \text{if } m\geq 1. \end{cases} \tag{27}$$

The following two results on (25) and (26) will be key points for taking limits on the coefficients that we shall obtain.

Proposition 8. *For every $m \in \mathbb{N}\cup\{0\}$, $C^\tau_{m,n} = \dfrac{\tau^{2m}\,d^\tau_{m,n}}{2} + o\left(\dfrac{1}{n}\right)$.*

Proof. We shall prove

$$d^\tau_{m,n} = \frac{2\,C^\tau_{m,n}}{\tau^{2m}} + o\left(\frac{1}{n}\right).$$

The previous equality holds for $m=0$ since

$$n\,(d^\tau_{0,n} - 2C^\tau_{0,n}) = n\left(1-2\,\frac{1}{2}\right) = 0.$$

Let now $m=1$. Using that $\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$ is

$$\begin{aligned} n\left(d^{\tau}_{1,n} - \frac{2C^{\tau}_{1,n}}{\tau^2}\right) &= n\left(-\frac{1}{n\tau^2} + \frac{\pi^2}{6} - 2\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2} e^{\frac{-k^2}{n\tau^2}}\right) \\ &= -\frac{1}{\tau^2} + 2n\sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^2}\left(1 - e^{\frac{-k^2}{n\tau^2}}\right) \\ &= -\frac{1}{\tau^2} + \frac{2}{\tau^2}\sum_{k\in\mathbb{N}} (-1)^{k+1} l_k\left(\frac{1}{n\tau^2}\right) \\ &= -\frac{1}{\tau^2} + \frac{2}{\tau^2} L\left(\frac{1}{n\tau^2}\right). \end{aligned}$$

Therefore, taking limits when n goes to infinity and using $\lim_{x\to 0^+} L(x) = \frac{1}{2}$ (see [1, Proposition 7]) is

$$\lim_{n\to\infty} n\left(d^{\tau}_{1,n} - \frac{2\,C^{\tau}_{1,n}}{\tau^2}\right) = 0.$$

Let m be grater or equal than 2 and we assume that for every $j \le m$ is $d^{\tau}_{j,n} = \frac{2C^{\tau}_{j,n}}{\tau^{2j}} + o\left(\frac{1}{n}\right)$. Then, using (24) and by the induction hypothesis, (27) can be written in the form

$$\begin{aligned} d^{\tau}_{m+1,n} &= \frac{(-1)^{m+1}}{n^{m+1}\,\tau^{2m+2}\,(m+1)!} - \sum_{j=0}^{m} \frac{B^{\tau}_{m+1-j}}{\tau^{2m+2-2j}}\left(\frac{2\,C^{\tau}_{j,n}}{\tau^{2j}} + o\left(\frac{1}{n}\right)\right) \\ &= \frac{(-1)^{m+1}}{n^{m+1}\,\tau^{2m+2}\,(m+1)!} \\ &\quad + 2(-1)^m\,\pi^{2m+2}\sum_{j=0}^{m} \frac{(-1)^j\,C^{\tau}_{j,n}}{(2m-2j+3)!\,(\pi\tau)^{2j}} + o\left(\frac{1}{n}\right) \\ &= 2(-1)^m\,\pi^{2m+2}\sum_{j=0}^{m} \frac{(-1)^j\,C^{\tau}_{j,n}}{(2m-2j+3)!\,(\pi\tau)^{2j}} + o\left(\frac{1}{n}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{d^{\tau}_{m+1,n}}{2} - \frac{C^{\tau}_{m+1,n}}{\tau^{2m+2}} \\ &\quad = (-1)^m\,\pi^{2m+2}\sum_{j=0}^{m} \frac{(-1)^j\,C^{\tau}_{j,n}}{(2m-2j+3)!\,(\pi\tau)^{2j}} - \frac{C^{\tau}_{m+1,n}}{\tau^{2m+2}} + o\left(\frac{1}{n}\right) \\ &\quad = (-1)^m\,\pi^{2m+2}\left(\sum_{j=0}^{m+1} \frac{(-1)^j\,C^{\tau}_{j,n}}{(2m-2j+3)!\,(\pi\tau)^{2j}}\right) + o\left(\frac{1}{n}\right). \end{aligned}$$

Changing $C^{\tau}_{j,n}$ by the expression of (25) and using Lemma 2 with $r = m+1$, the previous expression has the form

$$\begin{aligned}
&\frac{d^{\tau}_{m+1,n}}{2} - \frac{C^{\tau}_{m+1,n}}{\tau^{2m+2}} \\
&\quad = (-1)^m \pi^{2m+2}\Bigg[\frac{1}{2\,(2m+3)!} + \sum_{j=1}^{m+1} \frac{(-1)^j}{\pi^{2j}\,(2m-2j+3)!} \sum_{k\in\mathbb{N}} \frac{(-1)^{k+1}}{k^{2j}} \\
&\qquad - \sum_{j=1}^{m+1} \frac{(-1)^j}{\pi^{2j}\,(2m-2j+3)!} \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\left(1 - e^{\frac{-k^2}{n\tau^2}}\right)}{k^{2j}}\Bigg] + o\left(\frac{1}{n}\right) \\
&\quad = (-1)^{m+1}\pi^{2m+2} \sum_{j=1}^{m+1} \left(\frac{(-1)^j}{\pi^{2j}(2m-2j+3)!} \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\left(1 - e^{\frac{-k^2}{n\tau^2}}\right)}{k^{2j}} \right) \\
&\qquad + o\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore, separating the term $j = 1$ from the others

$$n\left(\frac{d^{\tau}_{m+1,n}}{2} - \frac{C^{\tau}_{m+1,n}}{\tau^{2m+2}}\right) = U_{m+1,n} + V_{m+1,n} + n\, o\left(\frac{1}{n}\right). \tag{28}$$

where

$$\begin{aligned}
U_{m+1,n} &= \frac{(-1)^m\,\pi^{2m}}{(2m+1)!} \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\left(1 - e^{\frac{-k^2}{n\tau^2}}\right)}{\frac{k^2}{n}}, \\
V_{m+1,n} &= (-1)^{m+1}\,\pi^{2m+2} \sum_{j=2}^{m+1} \frac{(-1)^j}{\pi^{2j}\,(2m-2j+3)!} \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\left(1 - e^{\frac{-k^2}{n\tau^2}}\right)}{\frac{k^{2j}}{n}}.
\end{aligned}$$

We endeavour to prove that the limit when n goes to infinity of (28) is equal to zero.
Indeed, by (10) and [1, Proposition 7], is

$$\lim_{n\to\infty} U_{m+1,n} = \lim_{n\to\infty} \frac{(-1)^m\,\pi^{2m}}{(2m+1)!}\,\frac{1}{\tau^2}\,L\left(\frac{1}{n\tau^2}\right) = \frac{(-1)^m\,\pi^{2m}}{2\tau^2\,(2m+1)!}. \tag{29}$$

On the other hand, by the Weierstrass's M–criterion is

$$\lim_{n\to\infty} V_{m+1,n} =$$

$$= (-1)^{m+1}\pi^{2m+2}\sum_{j=2}^{m+1}\frac{(-1)^j}{\pi^{2j}(2m-2j+3)!}\sum_{k\in\mathbb{N}}\lim_{n\to\infty}(-1)^{k+1}\frac{\left(1-e^{\frac{-k^2}{n\tau^2}}\right)}{\frac{k^{2j}}{n}}$$

$$= (-1)^{m+1}\pi^{2m+2}\sum_{j=2}^{m+1}\frac{(-1)^j}{\pi^{2j}(2m-2j+3)!}\sum_{k\in\mathbb{N}}\frac{(-1)^{k+1}}{\tau^2 k^{2j-2}}$$

$$= \frac{(-1)^m\pi^{2m}}{\tau^2}\sum_{j=1}^{m}\frac{(-1)^j}{\pi^{2j}(2m-2j+1)!}\sum_{k\in\mathbb{N}}\frac{(-1)^{k+1}}{k^{2j}}$$

and applying Lemma 2 for $r = m$ is

$$\lim_{n\to\infty} V_{m+1,n} = \frac{(-1)^{m+1}\,\pi^{2m}}{2\tau^2\,(2m+1)!}. \tag{30}$$

So, taking limits when n goes to infnity in (28) and using (29) and (30) is

$$\lim_{n\to\infty} n\left(\frac{d^{\tau}_{m+1,n}}{2} - \frac{C^{\tau}_{m+1,n}}{\tau^{2m+2}}\right) = 0.$$

Finally, by the Induction Principle, we conclude that the result works for every m . ∎

Proposition 9. *For every $n \in \mathbb{N}$ is held*

$$D^{\tau}_{m,n} = \begin{cases} \dfrac{1}{2} & \text{if } m = 0, \\[2ex] \dfrac{(-1)^m}{2\,n^m\,m!} + o\left(\dfrac{1}{n}\right) & \text{if } m \geq 1. \end{cases} \tag{31}$$

Proof. The value $\frac{1}{2}$ for $m = 0$ is directly obtained from the definitions given by (24), (25) and (26) because

$$D^{\tau}_{0,n} = B^{\tau}_0\,C^{\tau}_{0,n} = \frac{1}{2}.$$

Let $m = 1$. In the same way that in the previous case is

$$D^{\tau}_{1,n} = C^{\tau}_{1,n} + \frac{B^{\tau}_1}{2} = C^{\tau}_{1,n} - \frac{(\pi\tau)^2}{12}$$

and by Proposition 8 and (27) we conclude that

$$\begin{aligned} D^\tau_{1,n} &= \frac{\tau^2\, d^\tau_{1,n}}{2} - \frac{(\pi\tau)^2}{12} + o\left(\frac{1}{n}\right) = \frac{\tau^2}{2}\left(-\frac{1}{n\tau^2} + \frac{\pi^2}{6}\right) - \frac{(\pi\tau)^2}{12} + o\left(\frac{1}{n}\right) \\ &= -\frac{1}{2n} + o\left(\frac{1}{n}\right). \end{aligned}$$

Finally, let $m \geq 2$. Using the same method as in previous cases is

$$\begin{aligned} D^\tau_{m,n} &= C^\tau_{m,n} + \sum_{p=1}^{m-1} B^\tau_p\, C^\tau_{m-p,n} + \frac{B^\tau_m}{2} \\ &= \frac{\tau^{2m}\, d^\tau_{m,n}}{2} + \sum_{p=1}^{m-1} B^\tau_p \left(\frac{\tau^{2m-2p}\, d^\tau_{m-p,n}}{2} + o\left(\frac{1}{n}\right)\right) + \frac{B^\tau_m}{2} + o\left(\frac{1}{n}\right) \\ &= \frac{\tau^{2m}}{2}\left(\frac{(-1)^m}{n^m\,\tau^{2m}\, m!} - \sum_{j=0}^{m-1} d^\tau_{j,n}\,\frac{B^\tau_{m-j}}{\tau^{2m-2j}}\right) \\ &\quad + \sum_{p=1}^{m-1} B^\tau_p\, \frac{\tau^{2m-2p}\, d^\tau_{m-p,n}}{2} + \sum_{p=1}^{m-1} B^\tau_p\, o\left(\frac{1}{n}\right) + \frac{B^\tau_m}{2} + o\left(\frac{1}{n}\right). \end{aligned}$$

Now, we separate the term of the sum corresponding to $j = 0$ and we make a change in the sum index in p so,

$$\begin{aligned} D^\tau_{m,n} &= \frac{(-1)^m}{2\, n^m\, m!} - \frac{1}{2}\sum_{j=1}^{m-1} d^\tau_{j,n}\,\tau^{2j}\, B^\tau_{m-j} \\ &\quad + \sum_{p=1}^{m-1} B^\tau_p\, \frac{\tau^{2m-2p}\, d^\tau_{m-p,n}}{2} + o\left(\frac{1}{n}\right) \\ &= \frac{(-1)^m}{2\, n^m\, m!} + o\left(\frac{1}{n}\right), \end{aligned}$$

ending the proof. ∎

Note 10. *We underline that the new expression of $D^\tau_{m,n}$ provides by Proposition 9 not depend on τ.*

The next result will be useful in the proof of Theorem 1.

Lemma 11. *Let $m, i \in \mathbb{N}$ be such that $i \leq m$, and let $\Delta_{m,i}$ be the set given by*

$$\Delta_{m,i} = \left\{\alpha = (\alpha_1, \ldots, \alpha_m);\ \alpha_r \in \mathbb{N} \cup \{0\}\ ,\ \sum_{r=1}^{m} \alpha_r = i\ ,\ \sum_{r=1}^{m} r\alpha_r = m\right\}.$$

Then

(i) $\Delta_{m,1} = \{(0, 0, \ldots, 1)\}$ *and* $\Delta_{m,m} = \{(m, 0, \ldots, 0)\}$.

(ii) *Let* $i \in \mathbb{N}$ *be such that* $2 \le i \le m$*, let* $\alpha = (\alpha_r)_{r=1}^m \in \Delta_{m,i}$*. If* $\alpha_r \neq 0$ *is held*

$$\min\{r\alpha_r - 1, \alpha_r\} = \begin{cases} r\alpha_r - 1 & \text{if } r = 1, \\ \alpha_r & \text{if } r \ge 2. \end{cases}$$

Proof. Part (i) follows from the solution of the systems which define the sets $\Delta_{m,1}$ and $\Delta_{m,m}$.
For part (ii), let $\alpha = (\alpha_r)_{r=1}^m \in \Delta_{m,i}$ with $\alpha_r \neq 0$. If $r = 1$ the result is trivial. Assume $r \ge 2$. Since $1 \le (r-1)\alpha_r$ is $\alpha_r \le r\alpha_r - 1$. ∎

At this point we are ready for proving our main result.

Proof of Theorem 1. The statement that we have to prove is: for every $p > 0$ and $\tilde{\tau} > 0$

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{-p\frac{k^2}{n\tilde{\tau}^2}}\operatorname{sinc}(s\tilde{\tau} - k)\right)^n = e^{-ps^2} \tag{32}$$

for every $s \in \mathbb{R}$. Moreover this convergence is uniformly on compact sets. Indeed, let $\tau = \dfrac{\tilde{\tau}}{\sqrt{p}}$ and $t = s\sqrt{p}$, then (32) can be written in the form

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} e^{\frac{-k^2}{n\tau^2}}\operatorname{sinc}(\tau t - k)\right)^n = e^{-t^2}.$$

Thus, using the notation introduced by (9), we have to prove

$$\lim_{n\to\infty}(g(t,n))^n = e^{-t^2} \tag{33}$$

uniformly.
By Theorem 6 we know that $\lim\limits_{n\to\infty}(g(t,n))^n$ is an analytic function. Therefore to prove (33) is enough to show that the power series representation of both maps coincide. Moreover, we note that is enough to prove that the convergence is held for all $t \in \left(0, \dfrac{1}{\tau}\right)$.
Using (24), (25), (26) and the power series representation of $\sin(\pi\tau t)$ the map $g(t,n)$ can be written in the form (see [1, proof of Theorem 2])

$$g(t,n) = \operatorname{sinc}(\tau t) + \frac{2\tau t\sin\pi\tau t}{\pi}\sum_{k\in\mathbb{N}}\frac{(-1)^k}{\tau^2t^2 - k^2}\,e^{\frac{-k^2}{n\tau^2}} = 2\sum_{m=0}^{\infty} D^{\tau}_{m,n}\,t^{2m}.$$

Therefore

$$(g(t,n))^n = 2^n\left(\sum_{m=0}^{\infty} D^{\tau}_{m,n}\,t^{2m}\right)^n = 2^n\sum_{m=0}^{\infty} E^{\tau}_{m,n}\,t^{2m} \tag{34}$$

where, considering the sets $\Delta_{m,i}$ described by Lemma 11, is
(35)
$$E^{\tau}_{m,n}=\begin{cases}(D^{\tau}_{0,n})^n & \text{if } m=0,\\ \displaystyle\sum_{i=1}^{m}\sum_{\alpha\in\Delta_{m,i}}(D^{\tau}_{0,n})^{n-i}\frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\prod_{j=1}^{m}(D^{\tau}_{j,n})^{\alpha_j} & \text{if } m\geq 1.\end{cases}$$

Therefore our objective is reduced to prove that for every $m\geq 0$ is held

$$\lim_{n\to\infty}2^nE^{\tau}_{m,n}=\frac{(-1)^m}{m!}. \tag{36}$$

For $m=0$, the result follows directly by (35) and by the definitions (24), (25) and (26) because
(37)
$$\lim_{n\to\infty}2^nE^{\tau}_{0,n}=\lim_{n\to\infty}2^n(D^{\tau}_{0,n})^n=\lim_{n\to\infty}2^n(B^{\tau}_0\,C^{\tau}_{0,n})^n=\lim_{n\to\infty}2^n\frac{1}{2^n}=1.$$

For $m=1$ and $m=2$ by Lemma 11 part (i) is

$$\begin{aligned}E^{\tau}_{1,n}&=n(D^{\tau}_{0,n})^{n-1}\,D^{\tau}_{1,n},\\ E^{\tau}_{2,n}&=n(D^{\tau}_{0,n})^{n-1}\,D^{\tau}_{2,n}+\frac{n(n-1)}{2}(D^{\tau}_{0,n})^{n-2}\,(D^{\tau}_{1,n})^2.\end{aligned}$$

Now, with the two previous expressions and equality (31) is

$$\begin{aligned}E^{\tau}_{1,n}&=\frac{n}{2^{n-1}}\left(\frac{-1}{2n}+o\left(\frac{1}{n}\right)\right),\\ E^{\tau}_{2,n}&=\frac{n}{2^{n-1}}\left(\frac{1}{4n^2}+o\left(\frac{1}{n}\right)\right)+\frac{n(n-1)}{2^{n-1}}\left(\frac{-1}{2n}+o\left(\frac{1}{n}\right)\right)^2.\end{aligned}$$

Therefore
(38)
$$\begin{aligned}\lim_{n\to\infty}2^nE^{\tau}_{1,n}&=\lim_{n\to\infty}\left(-1+2n\,o\left(\frac{1}{n}\right)\right)=-1\\ \lim_{n\to\infty}2^nE^{\tau}_{2,n}&=\lim_{n\to\infty}\left[\frac{1}{2n}+2n\,o\left(\frac{1}{n}\right)+\frac{n-1}{2n}+2n(n-1)\,o\left(\frac{1}{n^2}\right)\right]\\ &=\frac{1}{2}.\end{aligned}$$

Note that the results (37) and (38) coincide with ([1, Theorem 2]).
Let $m\geq 3$. First of all we change $D^{\tau}_{j,n}$ in (35) for the expression (31) given by Proposition 9 obtaining

$$E^{\tau}_{m,n}=\sum_{i=1}^{m}\sum_{\alpha\in\Delta_{m,i}}\frac{1}{2^{n-i}}\;\frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\prod_{j=1}^{m}\left[\frac{(-1)^j}{2n^jj!}+o\left(\frac{1}{n}\right)\right]^{\alpha_j}.$$

Since

$$\left[\frac{(-1)^j}{2\,n^j\,j!}+o\left(\frac{1}{n}\right)\right]^{\alpha_j}=\frac{(-1)^{j\alpha_j}}{n^{j\alpha_j}\,(j!)^{\alpha_j}\,2^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right),$$

using this equality in the previous expression we can writte $E^{\tau}_{m,n}$ in the form

$$\begin{aligned} E^{\tau}_{m,n} &= \frac{(-1)^m}{2^n}\sum_{i=1}^{m}\sum_{\alpha\in\Delta_{m,i}}\frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!} \\ &\qquad \prod_{j=1}^{m}\left[\frac{1}{n^{j\alpha_j}\,(j!)^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right)\right]. \end{aligned}$$

Spliting the sum into three parts and applying Lemma 11 part (i) for the terms $i=1$ and $i=m$, is

$$2^n E^{\tau}_{m,n}=(-1)^m\left(F_{m,n}+\sum_{i=2}^{m-1}G_{i,m,n}+H_{m,n}\right), \tag{39}$$

where

$$\begin{aligned} F_{m,n} &= \sum_{\alpha\in\Delta_{m,1}}\frac{n}{\alpha_1!\dots\alpha_m!}\prod_{j=1}^{m}\left(\frac{1}{n^{j\alpha_j}\,(j!)^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right)\right) \\ &= n\left(\frac{1}{n^m\,m!}+o\left(\frac{1}{n}\right)\right), \\ G_{i,m,n} &= \sum_{\alpha\in\Delta_{m,i}}\frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\prod_{j=1}^{m}\left(\frac{1}{n^{j\alpha_j}\,(j!)^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right)\right), \\ H_{m,n} &= \sum_{\alpha\in\Delta_{m,m}}\frac{n(n-1)\dots(n-m+1)}{\alpha_1!\dots\alpha_m!}\prod_{j=1}^{m}\left(\frac{1}{n^{j\alpha_j}\,(j!)^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right)\right) \\ &= \frac{n(n-1)\dots(n-m+1)}{m!}\left(\frac{1}{n^m}+o\left(\frac{1}{n^m}\right)\right). \end{aligned}$$

Clearly from the previous expressions is

$$\lim_{n\to\infty}F_{m,n}=0$$

and

$$\lim_{n\to\infty}H_{m,n}=\frac{1}{m!}.$$

For computing the limit of $G_{i,m,n}$ we split the term $j=1$ from the others. Note that for every $j\geq 2$ such that $\alpha_j=0$, the corresponding factor is 1, and if $\alpha_j\neq 0$, by Lemma 11 (ii), we have

$$\begin{aligned} \frac{1}{n^{j\alpha_j}\,(j!)^{\alpha_j}}+o\left(\frac{1}{n^{\alpha_j}}\right) &= o\left(\frac{1}{n^{j\alpha_j-1}}\right)+o\left(\frac{1}{n^{\alpha_j}}\right) \\ &= o\left(\frac{1}{n^{\min\{j\alpha_j-1,\alpha_j\}}}\right)=o\left(\frac{1}{n^{\alpha_j}}\right). \end{aligned}$$

Thus, $G_{i,m,n}$ can be written in the form

$$\begin{aligned}
G_{i,m,n} &= \sum_{\alpha\in\Delta_{m,i}} \frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\left[\left(\frac{1}{n^{\alpha_1}}+o\left(\frac{1}{n^{\alpha_1}}\right)\right)\prod_{j=2}^{m} o\left(\frac{1}{n^{\alpha_j}}\right)\right] \\
&= \sum_{\alpha\in\Delta_{m,i}} \frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!} \\
&\quad \left[\left(\frac{1}{n^{\alpha_1}}+o\left(\frac{1}{n^{\alpha_1}}\right)\right) o\left(\frac{1}{n^{\sum_{j=2}^{m}\alpha_j}}\right)\right] \\
&= \sum_{\alpha\in\Delta_{m,i}} \frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\left[\left(\frac{1}{n^{\alpha_1}}+o\left(\frac{1}{n^{\alpha_1}}\right)\right) o\left(\frac{1}{n^{i-\alpha_1}}\right)\right] \\
&= \sum_{\alpha\in\Delta_{m,i}} \frac{n(n-1)\dots(n-i+1)}{\alpha_1!\dots\alpha_m!}\, o\left(\frac{1}{n^{i}}\right),
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} G_{i,m,n} = 0.$$

So, taking limits when n goes to infinity in the expression (39), if $m \geq 3$ then

$$\lim_{n\to\infty} 2^n E^{\tau}_{m,n} = \frac{(-1)^m}{m!}. \tag{40}$$

Therefore, by the results from (37), (38) and (40), we have obtained that for every $m \geq 0$

$$\lim_{n\to\infty} 2^n E^{\tau}_{m,n} = \frac{(-1)^m}{m!}.$$

So, taking limits in the expression (34) is

$$\lim_{n\to\infty} (g(t,n))^n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} t^{2m}.$$

Note that we have proved the pointwise convergence to the Gaussian map. The uniform convergence on compact sets is guaranteed by Theorem 3. ∎

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