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An Automated Method of Gravity Interpretation

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Summary

The interpretation of an observed gravity anomaly in terms of an anomalous mass with irregular outline and with uniform density contrast requires the solution of a non-linear problem. It is possible to iterate this non-linear problem by means of a linear approximation, provided some assumption is made about one of the surfaces of the anomalous mass. This paper gives such a method.

If there are m observations of a gravity anomaly and if the anomalous mass is assumed to be subdivided into n two-dimensional rectangular blocks ($n \leqslant m$) then a set of linear equations can be solved—directly if $m = n$, and by least squares if $m > n$—to give a system of blocks of variable density contrast which satisfy, or nearly satisfy in the case of the least squares solution, the observed gravity anomaly. These blocks are then transformed to give blocks of uniform density contrast. Because the gravity effect is non-linear the transformed blocks will not usually satisfy the observed anomaly. It is, therefore, necessary to adjust the model using the same general method.

Two computer programs applying respectively to structures with inward dipping contacts and to structures with outward dipping contacts have been developed. The formulae used in the programs apply to two-dimensional structures, but three-dimensional structures are approximated by end corrections.

1. Introduction

Consider, in two dimensions, a Cartesian system in which the gravity anomaly lies along the horizontal x-axis and has the value $\Delta g(x)$ at the point $(x, 0)$, the z-axis points vertically downwards, and the distribution of mass causing the anomaly is represented by a closed body whose surface is cut either twice or not at all by any vertical line. Then the usual problem of gravity interpretation is the solution of the integral equation:

$$\Delta g(x) = \int_{-\infty}^{\infty} K[(x-\xi), \zeta_1(\xi), \zeta_2(\xi)]\rho(\xi)\, d\xi, \tag{1}$$

where K is the kernel function giving the gravity effect, $\Delta g(x)$, per unit of density contrast of a two-dimensional sheet with upper surface $\zeta_1(\xi)$ and lower surface $\zeta_2(\xi)$ and density contrast $\rho(\xi)$. This is an inverse problem which is non-linear if either $\zeta_1(\xi)$ or $\zeta_2(\xi)$ is the unknown function and is linear if $\rho(\xi)$ is the unknown function (Bott 1967). In this paper a method of using the linear solution to iterate the non-linear problem is presented. This method is an extension of that of Bott (1960) and

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uses the solution of linear equations, by matrix methods, to obtain the successive approximations to the non-linear system. Two computer programs called 'Sedimentary Basins' and 'Granite Bodies' respectively have been developed. The first approximates structures with flat upper surfaces and inward dipping contacts and the second approximates structures with outward sloping contacts and flat lower surfaces. In both programs it is assumed that the density contrast of the anomalous mass is uniform. The formulae used apply to two-dimensional structures, but three dimensional structures are approximated by means of end corrections (Nettleton 1940).

2. The linear solution using matrices

In practice, the gravity anomaly is known only at m discrete points over a limited portion of the Earth's surface. If the anomalous mass is sub-divided into n $(n \leqslant m)$ two-dimensional rectangular blocks then the integral equation (1) can be approximated by the finite summation:

$$\Delta g_i = K_{ij}\rho_j, \quad (i=1 \ldots m;\ j=1 \ldots n), \tag{2}$$

where the repeated subscript j indicates that, for each value of i, j must be summed over all its possible values. The kernel function can be calculated using the formula giving the gravity effect of two-dimensional rectangular blocks (Heiland 1940, p. 152). The formula is

$$\Delta g = 2G\rho\left[-(x-\xi_1)\ln\frac{r_2}{r_1} + (x-\xi_2)\ln\frac{r_4}{r_3} + \zeta_2(\phi_1-\phi_3) - \zeta_1(\phi_2-\phi_4)\right], \tag{3}$$

where the notation has the meaning given in Fig. 1. If the depth to the centre of the block is twice the thickness of the block or more, a good approximation to equation (3) can be made by assuming the mass to be concentrated in a horizontal plane through the centre of block. Equation (3) then reduces to

$$\Delta g = 2G\rho(\zeta_2-\zeta_1)\,\theta, \tag{4}$$

where $\theta = \phi_1 - \phi_3 = \phi_1 - \phi_4$. This has the advantage of being much more amenable to speedy computation.

Equation (2) defines a system of m equations in n unknowns. If $m=n$ the solution is obtained directly. In matrix notation it is

$$\boldsymbol{\rho} = \mathbf{K}^{-1}\,\mathbf{g}, \tag{5}$$

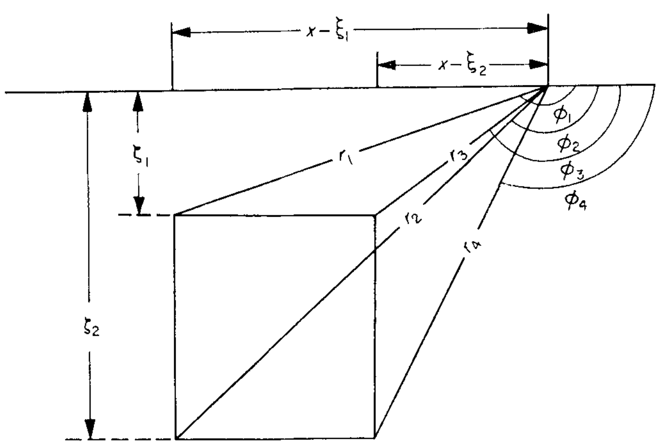

Fig. 1. The geometry and gravity effect of a two-dimensional rectangular block.

where $\mathbf{K}^{-1}$ is the inverse of the kernel array and $\mathbf{g}$ is the array of observed anomalies. If $m>n$ then the method of least squares must be used. For this $Q=\sum_{i=1}^{m}(\Delta g_i-K_{ij}\rho_j)^2$ is minimized. This requires that

$$\frac{\partial Q}{\partial \rho_r}=-2\sum_{i=1}^{m}K_{ir}(\Delta g_i-K_{ij}\rho_j)=0, \quad (r=1 \ldots n).$$

This can be rearranged and the summation notation dropped to give

$$K_{ir}\,\Delta g_i=K_{ir}K_{ij}\,\rho_j.$$

In matrix notation this can be written:

$$\mathbf{K}^T\mathbf{g}=\mathbf{K}^T\mathbf{K}\,\boldsymbol{\rho},$$

where $\mathbf{K}^T$ is the transpose of the matrix $\mathbf{K}$. This is a system of n equations in n unknowns which has the solution

$$\boldsymbol{\rho}=[\mathbf{K}^T\mathbf{K}]^{-1}\mathbf{K}^T\mathbf{g}. \tag{6}$$

The solution of (5) or (6) specifies a system of rectangular blocks, of variable density ρ_j, or a system of horizontal two dimensional sheets of variable mass/unit area σ_j, which satisfies the given gravity anomaly. The corresponding distribution of blocks of uniform density contrast ρ can be approximately estimated by assuming either the upper surface or the lower surface of the blocks to be fixed and transforming on the basis that the mass of the blocks is kept constant. Since the gravity effect is non-linear with depth, this system of transformed blocks cannot satisfy the observed anomaly, even approximately, unless the differences in thickness between the transformed and the untransformed blocks are all relatively small. Consequently it is usually necessary to adjust the set of transformed blocks. The details of the adjustment vary with the type of structure approximated, but it can be done using the linear approximation, provided precautions are taken to avoid instability.

The next sections are devoted to the application of this linear theory to produce structures whose outlines approximate sedimentary basins and granite bodies as defined earlier. Given the local anomaly at m discrete points over the Earth's surface, the depth to the upper surface and the density contrast ρ, the problem can be reduced to (a) first estimating a distribution of blocks of uniform density contrast, and (b) adjusting the model to satisfy the given anomaly. Except for minor variations the method used to provide the first estimate is the same for both programs. Both programs permit the interpreter to provide an estimate of the anomalous mass distribution and thus will execute step (b) only.

Examples from both programs are presented in Section (5).

3. The first estimate

The first step involves the estimation of the thickness of the blocks of uniform density contrast and the arrangement of their distribution so that the outline approximates that of a sedimentary basin or a granite body.

Initially the mass giving rise to the anomaly is assumed to be concentrated in a thin horizontal sheet along the upper surface of the structure. The solution of either (5) or (6) then provides an equivalent layer as shown in Fig. 2 (c) in which the mass/unit area given by the matrix method is compared with that obtained by the $(\sin x)/x$ method (Tomada & Aki 1955). The mass/unit area shown in Fig. 2 (b) was calculated from the observed anomaly using blocks 2000 metres wide. The two methods agree to within one or two per cent, except at the ends of the profile. This disagreement at the ends is due to the assumption in the $(\sin x)/x$ method that the observed anomaly is zero beyond the ends of the profile.

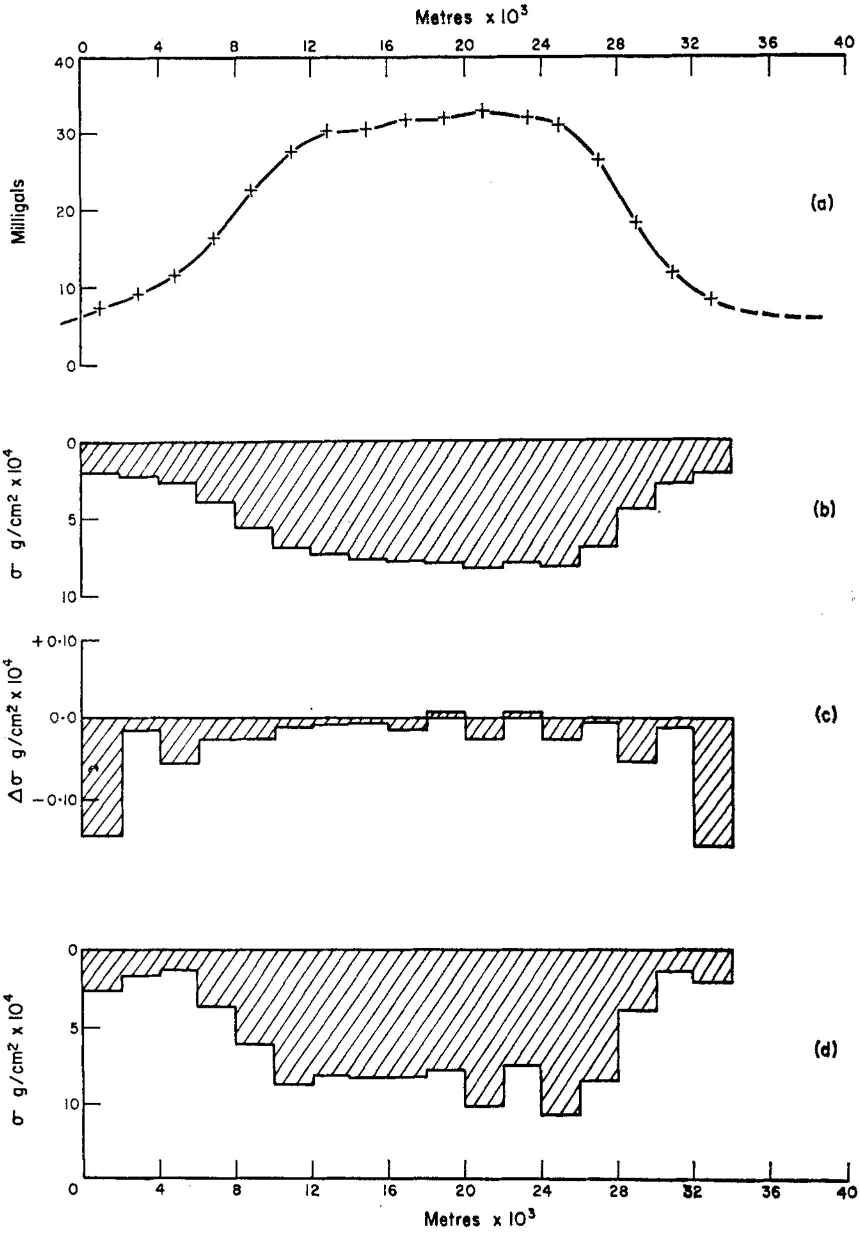

FIG. 2. The equivalent layer and the matrix method. (a) shows the observed anomaly and (b) the mass/unit area at a depth of 500 metres deduced by the matrix method using a block width of 2000 metres. (c) gives the differences in mass/unit area between the matrix method and the method $(\sin x)/x$. The mass at a depth of 2000 metres (block spacing 2000 metres) is shown in (d). Note the onset of instability when the depth to the upper surface equals the block width.

Fig. 2(d) gives the equivalent layer when the depth to the upper surface equals the block width. Incipient instability is shown. If the upper surface is placed at a depth greater than the block width, instability becomes more pronounced. Because of this, the practice of choosing block widths greater than the depth of the upper surface has been followed.

The last part of this first estimate requires that blocks of uniform density contrast ρ be obtained. In the sedimentary basin program the upper surface is known and the lower surface is unknown. The latter is estimated using the relation

$$\zeta'_{2j} = \zeta_{1j} + \sigma_j/\rho, \quad (j = 1 \ldots n). \tag{7}$$

In the granite bodies program the upper surface is given for a specified block (the kth). The upper surface of all other blocks and the flat lower surface are unknown. The lower surface of all blocks is estimated using the relation

$$\zeta'_{2j} = \zeta_{1k} + \sigma_k/\rho, \quad (j = 1 \ldots n). \tag{8a}$$

The upper surfaces are then obtained by:

$$\zeta'_{1j} = \zeta'_{2j} - \sigma_j/\rho, \quad (j \neq k). \tag{8b}$$

4. The adjustment of the model

In the adjustment of the first estimate a method which converges quickly and remains stable is desired. Stability has proved more of a problem with the sedimentary basin program. Consequently two methods of adjustment have been incorporated into this program. Only one method has been used in the granite bodies program.

The first method of adjusting the sedimentary basin model is as follows:

(a) calculate the gravity effect of the system of blocks resulting from the previous iteration and obtain the residual anomaly at each point of observation,

(b) place a thin sheet at the base of each block and, using the residual anomalies, calculate the σ_j by the matrix method, and

(c) transform the σ_j to give the adjustment in terms of a small block of density contrast ρ—to be added or subtracted from that given by the first estimate according to the sign of σ_j.

This process can be continued until the residuals are within the limits desired. This method (method 1) converges quickly and usually only one or two adjustments are necessary. However, it can be unstable occasionally. Because the mass necessary to adjust the blocks is assumed to be concentrated in a thin sheet at the base of each block and because the gravity effect is non-linear with depth, the amount of mass to be removed from any block is always overestimated. When large 'negative' adjustments are required the system can become unstable, since in satisfying the residual anomalies it must compensate by adding in mass elsewhere. A second type of instability arises when the observed anomaly cannot be fitted to any model with the specified density contrast and surface. This occurs if the upper surface is too deep or the density contrast is too low.

A more stable, but more slowly converging method (method 2) uses the entire transformed block given by the first estimate to adjust the model. This averages the adjustment throughout the whole block and consequently instability is unlikely. After each adjustment a new set of transformed blocks can be obtained using the relation:

$$\zeta'_{2j} = \zeta_{1j} + (\zeta_{2j} - \zeta_{1j})\rho_j/\rho, \quad (j = 1 \ldots n), \tag{9}$$

where the primed co-ordinate again indicates the estimated value. This method does not require a knowledge of the residuals and a solution is given when ρ_j is everywhere equal to or nearly equal to the assumed density contrast ρ. However, the residuals do indicate directly the quality of the solution and are always computed in practice.

In the granite bodies program only the upper surface of the kth block is fixed and it is usually necessary to adjust the upper surface of all other blocks and the depth to the base of the structure. Method 1 is adapted to granite bodies as follows:

(a) determine the residual anomalies,

(b) place a thin sheet along the upper surface of each block (the kth excluded) and across the base of the structure, and

(c) calculate the σ_j and transform using the relation

$$\zeta'_{1j} = \zeta_{1j} - \sigma_j/\rho, \quad (j \neq k) \tag{10a}$$

for the upper surfaces and

$$\zeta'_{2j} = \zeta_{2j} + \sigma_k/\rho \tag{10b}$$

for the lower surface.

Thus far, method 1, as applied to granite bodies, has not shown instability. The reason for this seems to be that the adjustment is made to the upper surface of the model (except for the wide kth sheet) and is usually relatively small. The kth sheet at the base of the structure is usually wider than the depth to it and will not reflect the presence of local components in the observed anomaly.

Both programs will continue to adjust the model until either: (a) the residuals are everywhere less than that specified by the interpreter or (b) the number of adjustments desired by the interpreter has been effected. After each iteration the dimensions of the model and the residuals are printed out.

5. Examples

Since the sedimentary basin program gives a result that is nearly identical to that produced by Bott's (1960) method, the application given here is aimed at demonstrating a procedure that can be used for buried structures. In such a situation there is usually no direct information available regarding the overall width of the structure. Although it is possible to make a guess at the width of the structure and then proceed with the program, it may be more prudent to proceed as follows:

(a) assume all blocks have the same width, preferably one that is greater than the depth to the upper surface,

(b) centre one of the blocks under the extreme value of the observed anomaly, and

(c) assume the blocks extend from this central point to either edge of the anomaly.

This approach is demonstrated in Fig. 3 for a hypothetical structure at the base of the crust. A block 60 km wide and 4 km thick was placed at a depth of 35 km. The anomaly it produced, assuming the density contrast to be 0·5 g/cm^3, is shown in Fig. 3(a). The procedure outlined above, for a block width of 50 km, was used. The structure derived (Fig. 3(d)) satisfied the given anomaly to within 0·3 mgal everywhere and provides a reasonable interpretation.

The granite bodies program has been applied to the anomaly over the Weardael granite in northeast England. Although not exposed, the presence of a granite has been established by drilling at Rookhope in County Durham. The local anomaly shown in Fig. 4(a) was taken from Bott (1956) with the background level of the regional field assumed to be 10 mgal. The structure was approximated by 10 blocks, each of which was 3200 metres wide. The upper surface of block 6 (Table 1) was fixed at 400 metres. All blocks were assumed to have a strike length of 10 000 metres in either direction from the plane of the profile. The suggested outline of the granitic mass (Fig. 4(c)) is roughly that of a trapezium with the north face dipping at about 45° and the south face dipping steeply. The depth to the base of the structure is about one

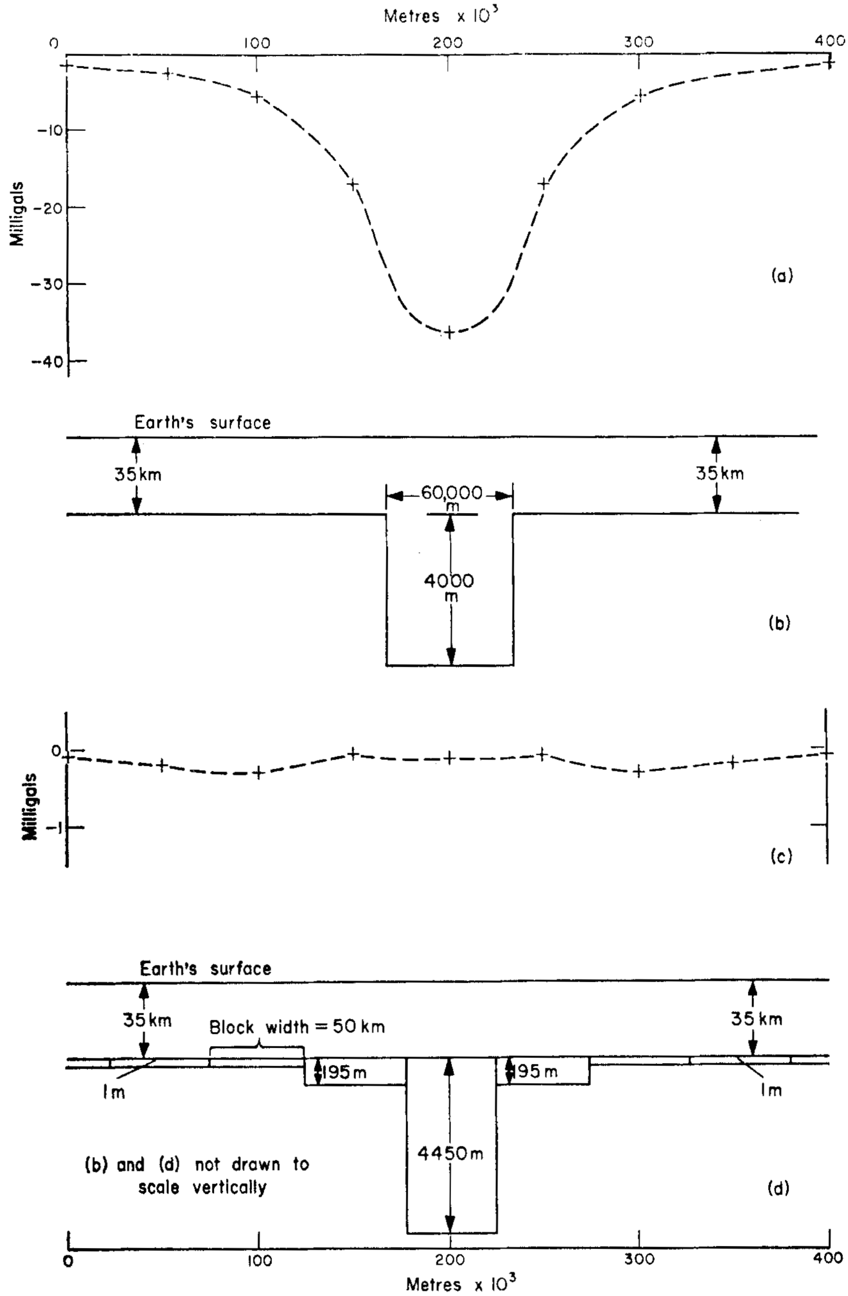

FIG. 3. The application of the matrix method to buried structures using a hypothetical structure at the base of the Earth's crust as an example. The structure assumed is shown in (b) and the anomaly it produces in (a). The model produced by the sedimentary basin program is shown in (d) and the residuals in (c).

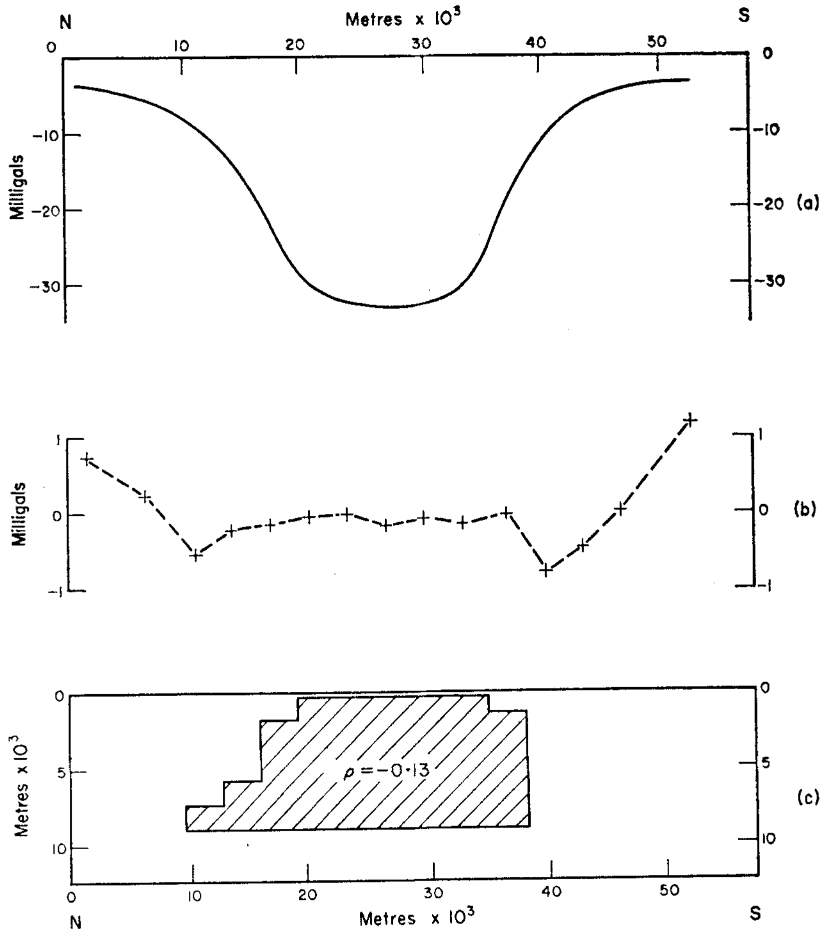

FIG. 4. The interpretation of the Weardale granite in northeastern England using the granite bodies program. The model is assumed to have a strike length of 10 000 metres in either direction from the plane of the profile. The local anomaly is shown in (a), the residuals in (b) and the model in (c).

Table 1

Model of Weardale granite

Block*	Left side (metres)	Right side (metres)	Upper surface (metres)	Lower surface (metres)
1	9600	12800	7400	8970
2	12800	16000	5730	8970
3	16000	19200	1770	8970
4	19200	22400	380	8970
5	22400	25600	490	8970
6	25600	28800	400†	8970
7	28800	32000	380	8970
8	32000	35200	420	8970
9	35200	38400	1470	8970
10	38400	41600	8969	8970

* All blocks assumed to have a strike length of 10 km in both directions normal to plane of profile.

† This surface not adjusted.

kilometre greater than that obtained by Stacey (1965) who assumed a two-dimensional structure.

The residual anomalies (Fig. 4(b)) are systematically positive at the ends of the profile. This suggests that either the assumed background level is in error by about a milligal or the assumed density contrast is too low. It is probable that the background level is in error for reasons given by Bott & Masson-Smith (1957).

6. Discussion

The method presented has two main disadvantages: (1) the approximation of the anomalous mass by rectangular blocks; and (2) the desirability to consider block widths which are equal to or greater than the depth to the upper surface of the structure. It may be possible to alter the program to produce structures which are polygonal in outline. Disadvantage (2) may exclude the possibility of investigating a given anomaly in detail.

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