BULL. AUSTRAL. MATH. SOC. MOS 3402, 3440
VOL. 1 (1969), 391-395.

An autonomous system of differential equations in the plane

R. F. Matlak

In the present note the equation $y'' = x^{1-m} y^m$ is reduced, under appropriate conditions, to a quadratic autonomous system of differential equations in the plane. In pursuance of this new approach, the main geometric features of this autonomous system are determined and a method of solving it is outlined.

1. Introduction

The equation

$$y'' = x^{1-m} y^m \qquad (' = \frac{d}{dx}) , \tag{1}$$

where m is real, and both x and y are positive, has been investigated extensively by both mathematicians and physicists. (See Bellman [3], Chandrasekhar [4], Davis [6], Hille [7]-[9].) The treatments of various aspects of (1) have, however, met with considerable difficulties, and may, by and large, be considered only a partial success.

In the present note, we suggest a new way to approach the equation based on the geometric theory of differential equations. (See Lefschetz [11], Poincaré [12].) This includes a geometric characterisation of an autonomous system in the plane, obtained from (1) by a suitable transformation, and a method of solution of the system. From the results thus obtained we hope to be able to examine the behaviour of the original equation in a more systematic fashion. This next step, by no means easy,

Received 24 July 1969. The author gratefully acknowledges the encouragement and advice of Professor Einar Hille and Professor George Szekeres. This work forms part of his Ph.D. thesis at the University of New South Wales.

must, however, be postponed for some time.

2. The autonomous system and its geometric properties

With a view to further discussion, let $m \neq -1, 1, 2, 3$. Assuming that $y' \neq 0$ and applying the transformation

$$(2) \qquad \xi = \frac{xy'}{y}\ , \quad \eta = \frac{x^{2-m}\, y^{m}}{y'}\ , \quad t = \ln|x|$$

appropriate to equations of the Emden-Fowler type (see Coppel [5]) to the equation (1), we obtain

$$(3) \qquad \begin{aligned} \frac{d\xi}{dt} &= \xi[1 - \xi + \eta] &&= P(\xi,\eta)\ , \\ \frac{d\eta}{dt} &= \eta[(2-m) + m\xi - \eta] &&= Q(\xi,\eta)\ , \text{ say,} \end{aligned}$$

which is a quadratic autonomous system in the (ξ,η)-plane with t as the independent variable.

It follows from the general theory of systems of differential equations that through every point of the (ξ,η)-plane passes a unique trajectory (path) of the system. Moreover, by virtue of a result established by Bautin [2] (see also Coppel [5]) in a more general case, no trajectory of (3) can be a closed curve (cycle).

Using the methods of the geometric theory of differential equations we next study the critical points of the system (3).

There are four such points $0, U, V, W$, say, specified by

$$(4) \qquad 0 \equiv (0,\ 0)\ , \quad U \equiv (1,\ 0)\ , \quad V \equiv (0,\ 2-m)\ , \quad W \equiv \left(\frac{3-m}{1-m},\ \frac{2}{1-m}\right)\ ,$$

representing stationary solutions of (3) in the finite part of the (ξ,η)-plane. Moreover, since ξ, η are, respectively, factors of the ξ-, η-components of the vector field of (3), the coordinate axes consist of paths of the system (the "point"-paths $0, U, V$ included). Now if $A(C)$ is the coefficient matrix of the linearised system associated with (3) at a given point C of (4) then

https://doi.org/10.1017/S0004972700042301 Published online by Cambridge University Press

$$A(O) = \begin{bmatrix} 1 & 0 \\ 0 & 2-m \end{bmatrix}, \qquad A(U) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

(5)

$$A(V) = \begin{bmatrix} 3-m & 0 \\ m(2-m) & m-2 \end{bmatrix}, \quad A(W) = \frac{1}{1-m}\begin{bmatrix} m-3 & 3-m \\ 2m & -2 \end{bmatrix}.$$

With regard to (3), the identities (5) imply that:-

(i) The origin O is a node or a saddle point according as $m < 2$ or $m > 2$;

(ii) the unit point, U, of the ξ-axis is *always* a saddle point;

(iii) the point V is a saddle point or an unstable node according as $(m-2)(m-3)$ is positive or negative, and becomes a one-tangent node when $m = \frac{5}{2}$;

(iv) the point W is a saddle point if $1 < m < 3$; it becomes a node when $m_1 < m < 1$ or $3 < m < m_2$, where

$$m_1 \doteq \tfrac{1}{7}(11-8\sqrt{2}) = -0.0448 \text{ , } m_2 \doteq \tfrac{1}{7}(11+8\sqrt{2}) = 3.1858 .$$

If $m < m_1$ or $m > m_2$, W is a focus which is unstable if $m_2 < m < 5$, and stable otherwise. When $m = m_1$ or $m = m_2$, W becomes a one-tangent node.

Moreover, the system (3) possesses three critical points at infinity specified as the points X_∞, F_∞ and Y_∞ , say, on the ξ-axis, the line $\eta = \frac{1}{2}(m+1)\xi$ and the η-axis, respectively. Similarly as before we find that, with regard to (3),

(v) X_∞ is a node or a saddle point according as $m > -1$ or $m < -1$;

(vi) F_∞ is a node or a saddle point according as $|m| < 1$ or $|m| > 1$;

(vii) Y_∞ is *always* a node.

The above results thus yield seven non-singular cases of the system (3) which, on putting *s*, *n* and *f* for the saddle point, node and focus singularities, respectively, can be set out as below.

https://doi.org/10.1017/S0004972700042301 Published online by Cambridge University Press

Case	Inequality	0	U	V	W	X_∞	F_∞	Y_∞
1.	$m < -1$	n	s	s	f	s	n	n
2.	$-1 < m < m_1$	n	s	s	f	n	s	n
3.	$m_1 < m < 1$	n	s	s	n	n	s	n
4.	$1 < m < 2$	n	s	s	s	n	n	n
5.	$2 < m < 3$	s	s	n	s	n	n	n
6.	$3 < m < m_2$	s	s	s	n	n	n	n
7.	$m_2 < m$	s	s	s	f	n	n	n

$$[m_1 = \tfrac{1}{7}(11-8\sqrt{2}) \doteq -0.0448 \ , \quad m_2 = \tfrac{1}{7}(11+8\sqrt{2}) \doteq 3.1858] \ .$$

Since cases 1 and 2 are topologically equivalent, as are cases 4, 5 and 6, the system (3) yields essentially four distinct types of Poincaré maps (phase-portraits) in the extended (ξ,η)-plane.

3. A method of solution of the system

One can solve the system (3) by applying one of the standard transformations related to Abel's differential equations of the second kind (see Kamke [*10*]). Indeed, the transformation

$$\zeta = \xi[1 - \xi + \eta] \tag{6}$$

reduces $\dfrac{d\eta}{d\xi} = \dfrac{Q(\xi,\eta)}{P(\xi,\eta)}$ to

$$\zeta\frac{d\zeta}{d\xi} - [(4-m) + (m-3)\xi]\zeta + \xi(1-\xi)[(3-m) + (m-1)\xi] = 0 \tag{7}$$

to which, owing to its simple form, the standard processes of power series expansion can be readily applied.

If $m = 5$ (and then only) equation (7) admits of an integrating factor of the form $|\alpha(\xi) + \beta(\xi)\zeta|^a$, with $a = 1$ or $a = -\frac{1}{2}$,

$$\alpha(\xi) = \frac{1}{a}(a+1)c\int[(4-m) + (m-3)\xi]d\xi = \frac{1}{a}(a+1)c\xi(\xi-1) \ , \text{ and } \ \beta(\xi) = c \ ,$$

a constant of integration (see Abel [*1*]).

References

[1] Niels Henrik Abel, *Sur l'équation differentielle* $(y+s)dy + (p+qy+ry^2)dx = 0$ (Oeuvres, Vol. 2; Christiania, 1881. Reprinted Johnson, New York, 1965).

[2] N.N. Bautin, "On periodic solutions of a system of differential equations" (Russian), *Akad. Nauk. SSSR. Prikl. Mat. Meh.* 18 (1954), 128.

[3] R. Bellman, *Stability theory of differential equations* (McGraw-Hill, New York, 1953).

[4] S. Chandrasekhar, *An introduction to the study of stellar structure* (Chicago University Press, 1939. Reprinted Dover, New York, 1957).

[5] W.A. Coppel, "A survey of quadratic systems", *J. Differential Equations* 2 (1966), 293-304.

[6] Harold T. Davis, *Introduction to nonlinear differential and integral equations* (Dover, New York, 1962).

[7] E. Hille, "On the Thomas-Fermi equation", *Proc. Nat. Acad. Sci. USA* 62 (1969), 7-10.

[8] E. Hille, "Some aspects of the Thomas-Fermi equation", *J. Analyse Math.* (to appear).

[9] E. Hille, "Aspects of Emden's equation", (to appear).

[10] E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen, I (Gewöhnliche Differentialgleichungen)*, 7th ed. (Akad. Verlagsgesellschaft, Leipzig, 1961).

[11] Solomon Lefschetz, *Differential equations: Geometric theory*, 2nd ed. (Interscience, New York, 1963).

[12] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle* (Oeuvres, Vol. 1; Gauthier-Villars, Paris, 1928).

Macquarie University,
North Ryde, New South Wales.