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AN AUTONOMOUS SYSTEM WHOSE SOLUTIONS ARE UNIFORMLY ULTIMATELY BOUNDED BUT NOT UNIFORMLY BOUNDED

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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Consider the ordinary differential equation

$$(1) \qquad \dot{x} = f(t, x)$$

for a continuous function $f(t, x)$ on $I \times R^n$, $I = [0, \infty)$. Then, the following properties are well established for (1) (cf. [2], [4]), where $x(t)$ denotes an arbitrary solution of (1):

(i) If there exists a continuous function $L(t, s, r)$ on I^3 such that

$$(2) \qquad \|x(t)\| \leqq L(t, s, \|x(s)\|)\,, \qquad s \leqq t < T\,,$$

as long as $x(t)$ exists, then every solution of (1) is continuable up to $t = T$.

(ii) Conversely, if every solution is continuable up to $t = T$, then there is an $L(t, s, r)$ for which (2) holds.

(iii) Especially, the solution operator $T(t, s)$ defined by $x(t) = T(t, s)x(s)$ (under the uniqueness) is completely continuous, that is, for any bounded set $B \subset R^n$, $T(t, s)B$ is bounded if $T(t, s)$ is defined on B.

(iv) If the solutions of (1) are uniformly ultimately bounded, then they are uniformly bounded when the system (1) is autonomous.

It is quite natural to expect the same properties for functional differential equations

$$(3) \qquad \dot{x}(t) = f(t, x_t)$$

replacing $x(s)$, $\|x(s)\|$ etc. in (i) $\sim$ (iv) by x_s, $\|x_s\| = \sup\{\|x(s+u)\|: -h \leqq u \leqq 0\}$ etc., respectively, where $f(t, \phi)$ is assumed to be continuous on $I \times C([-h, 0], R^n)$. However, in general, it turns out that none of them are true for (3) without additional conditions on $f(t, \phi)$. A counterexample to the assertion (i) for (3) is given by Yorke [3], while it turns out to hold if $f(t, \phi)$ is completely continuous [1]. On the other hand, Henry (cf. [1]) obtained a counterexample to (iii). The aim of this paper is to present an autonomous scalar equation

$$(4) \qquad \dot{x}(t) = f(x_t)\,,$$

which is proved to be a counterexample to (iv) even though $f(\phi)$ is completely continuous on $X = C([-1, 0], R^1)$ and satisfies a local Lipschitz condition there. Here, we note that (iii) implies (ii), while (ii) implies (iv).

First of all, consider a series of functions:

$$\tau\colon X \to X\,, \quad [\tau(\phi)](s) = \min\{1, \max\{-1, \phi(s)\}\}\,,$$
$$\mu\colon X \to R\,, \quad \mu(\phi) = \min\{\phi(s)\colon s \in [-1, 0]\}\,,$$
$$\delta\colon X \times (0, \infty) \to (0, 1]\,,$$
$$\delta(\phi, \varepsilon) = \sup\{\delta \leqq 1\colon |t - s| \leqq \delta \Rightarrow |\phi(t) - \phi(s)| \leqq \varepsilon\}\,,$$
$$G, \rho\colon R \to R,\ G(x) = \max\{0, x - 1\},\ \rho(x) = (\max\{0, x\})^2\,,$$

and set

$$F(\phi; \Delta_s) = \sum_{k=1}^{\sharp\Delta_s} G(\phi(t_k) - \phi(t_{k-1})),\ g(\phi; s) = \sup_{\Delta_s} F(\phi; \Delta_s)\,,$$
$$f(\phi) = \min\{g(\tau(\phi); 0), 2\phi(0)^2\} - \rho(\mu(\phi)) + \rho(\mu(-\phi))\,,$$

where Δ_s denotes a partition of the interval $[s - 1, 0]$

$$(5) \qquad \Delta_s\colon s - 1 \leqq t_0 \leqq t_1 \leqq \cdots \leqq t_N \leqq 0$$

for $s \in [0, 1]$ and $\sharp\Delta_s = N$ for the partition Δ_s given by (5).

We shall state several lemmas. We omit the proofs of some of them since they are trivial.

LEMMA 1. *If* $\|\phi - \psi\| < \varepsilon/3$ *for an* $\varepsilon > 0$, *then* $\delta(\psi, \varepsilon) \geqq \delta(\phi, \varepsilon/3)$.

LEMMA 2. *If* $G(\phi(t_k) - \phi(t_{k-1})) = 0$ *and* $G(\phi(t_{k+1}) - \phi(t_k)) = 0$, *then* $F(\phi; \Delta_s)$ *does not decrease by removing* t_k *from the partition* Δ_s.

The partition Δ_s is said to be *principal to* ϕ if any two of $\{t_k\}_{k=0}^N$ are not identical and if $G(\phi(t_k) - \phi(t_{k-1})) \neq 0$ or $G(\phi(t_{k+1}) - \phi(t_k)) \neq 0$ for any $k = 0, 1, \cdots, N$, where we understand $t_{-1} = t_0$ and $t_{N+1} = t_N$.

LEMMA 3. *Let* Δ_s *be principal to* ϕ. *Then,* $\sharp\Delta_s < 2/\delta(\phi, 1) + 1$.

PROOF. Since $G(\phi(t_k) - \phi(t_{k-1})) \neq 0$ or $G(\phi(t_{k+1}) - \phi(t_k)) \neq 0$, we have $\phi(t_k) - \phi(t_{k-1}) > 1$ or $\phi(t_{k+1}) - \phi(t_k) > 1$, and hence $t_k - t_{k-1} > \delta(\phi, 1)$ or $t_{k+1} - t_k > \delta(\phi, 1)$. Therefore, $t_{k+1} - t_{k-1} > \delta(\phi, 1)$ for all k, which yields the conclusion. q.e.d.

LEMMA 4. *For any* $\phi \in X$ *there exists a partition* $\Delta_s(\phi)$, *which is principal to* ϕ, *such that* $g(\phi; s) = F(\phi; \Delta_s(\phi))$.

PROOF. By Lemmas 2 and 3, we have

$$g(\phi;\, s) = \sup\{F(\phi;\, \Delta_s) \colon \Delta_s \text{ is principal to } \phi,\ \sharp\,\Delta_s \leqq N\}\,,$$

where N is the largest integer less than $2/\delta(\phi, 1) + 1$. Let $D = \{(t_0, t_1, \cdots, t_N) \in R^{N+1};\ s - 1 \leqq t_0 \leqq t_1 \leqq \cdots \leqq t_N \leqq 0\}$. Clearly D is compact. For a fixed $\phi \in X$ and any $\xi = (t_0, t_1, \cdots, t_N) \in D$ set $H(\xi) = F(\phi;\, \Delta_s)$, where Δ_s is the partition given by (5). Since $H(\xi)$ is continuous on D, $g(\phi;\, s) = \sup\{H(\xi) \colon \xi \in D\} = H(\xi^*)$ for a $\xi^* = (t_0^*, t_1^*, \cdots, t_N^*) \in D$. Then, $\Delta_s^* \colon s - 1 \leqq t_0^* \leqq t_1^* \leqq \cdots \leqq t_N^* \leqq 0$ is a partition of $[s - 1, 0]$, for which $g(\phi;\, s) = F(\phi;\, \Delta_s^*)$. By Lemma 2 there exists a partition $\Delta_s(\phi)$, principal to ϕ, such that $F(\phi;\, \Delta_s^*) \leqq F(\phi;\, \Delta_s(\phi)) \leqq \sup_{\Delta_s} F(\phi;\, \Delta_s)$, that is, $F(\phi;\, \Delta_s(\phi)) = g(\phi;\, s)$.
q.e.d.

LEMMA 5. *$g(\phi;\, s)$ is continuous in $(\phi, s) \in X \times [0, 1]$, non-increasing in s and satisfies a local Lipschitz condition in ϕ.*

PROOF. By Lemma 4 we have $g(\phi;\, s) = F(\phi;\, \Delta_s(\phi))$ for a partition $\Delta_s(\phi)$ principal to ϕ, and set $N(\phi) = \sharp\,\Delta_s(\phi)$. Easily, we have $g(\phi;\, s) - g(\psi;\, s) \leqq F(\phi;\, \Delta_s(\phi)) - F(\psi;\, \Delta_s(\phi)) \leqq \sum_{k=1}^{N(\phi)} \{|\phi(t_k) - \psi(t_k)| + |\phi(t_{k-1}) - \psi(t_{k-1})|\} \leqq 2N(\phi)\|\phi - \psi\|$. Similarly, $g(\phi;\, s) - g(\psi;\, s) \geqq -2N(\psi)\|\phi - \psi\|$. On the other hand, by Lemmas 1 and 3 we have $N(\phi) \leqq 2/\delta(\phi, 1) + 1 \leqq 2/\delta(\xi, 1/3) + 1$ if $\phi \in U(\xi) = \{\phi \colon \|\phi - \xi\| < 1/3\}$. Therefore, $|g(\phi;\, s) - g(\psi;\, s)| \leqq L(\xi)\|\phi - \psi\|$ if $\phi, \psi \in U(\xi)$, where $L(\xi) = 4/\delta(\xi, 1/3) + 2$, that is, $g(\phi;\, s)$ is locally Lipschitz continuous in ϕ.

Since $\Delta_s(\phi)$ is also a partition of $[t - 1, 0]$ for $t \leqq s$, we have $g(\phi;\, t) \geqq F(\phi;\, \Delta_s(\phi)) = g(\phi;\, s)$, that is, $g(\phi;\, s)$ is non-increasing in s. Finally, let the partition $\Delta_s(\phi)$ be given by (5), and set $t_0^* = \max\{t_0, t - 1\}$ for a given $t \geqq s$. Since $\Delta_s(\phi)$ is principal to ϕ, we must have $\phi(t_1) - \phi(t_0) > 1$ which yields $t_1 - t_0 \geqq \delta(\phi, 1)$. Hence, if $t - s \leqq \delta(\phi, 1)$, then $t_0^* \leqq t_1$ and $\Delta_t \colon t_0^* \leqq t_1 \leqq \cdots \leqq t_N$ becomes a partition of $[t - 1, 0]$. Therefore, $g(\phi;\, s) \geqq g(\phi;\, t) \geqq F(\phi;\, \Delta_t) = F(\phi;\, \Delta_s(\phi)) + G(\phi(t_1) - \phi(t_0^*)) - G(\phi(t_1) - \phi(t_0)) \geqq g(\phi;\, s) - |\phi(t_0^*) - \phi(t_0)|$. From this it follows that $|g(\phi;\, t) - g(\phi;\, s)| < \varepsilon$ if $|t - s| < \min\{\delta(\phi, 1), \delta(\phi, \varepsilon)\}$, since $|t_0^* - t_0| \leqq |t - s|$. Thus, $g(\phi;\, s)$ is continuous in s and, hence, in (ϕ, s).
q.e.d.

LEMMA 6. *$f(\phi)$ is completely continuous and satisfies a local Lipschitz condition. Moreover, $f(\phi) = -\rho(\mu(\phi))$ when $\mu(\phi) \geqq 0$, $f(\phi) = \rho(\mu(-\phi))$ when $\mu(-\phi) \geqq 0$, and $f(\phi) \geqq 0$ if $\mu(\phi) \leqq 0$.*

PROOF. The first part immediately follows from Lemma 5, since $\|\tau(\phi) - \tau(\psi)\| \leqq \|\phi - \psi\|$ and $|\mu(\phi) - \mu(\psi)| \leqq \|\phi - \psi\|$. If $\mu(\phi) \geqq 0$, then $1 \geqq [\tau(\phi)](s) \geqq 0$ for all $s \in [-1, 0]$. Hence, $g(\tau(\phi);\, 0) = 0$ since $[\tau(\phi)](t) - [\tau(\phi)](s) \leqq 1$ for all $t, s \in [-1, 0]$. Similarly, $g(\tau(\phi);\, 0) = 0$ if $\mu(-\phi) \geqq 0$. Thus, the rest of the proof is also immediate.
q.e.d.

LEMMA 7. *Let* $x(t)$ *be continuous on* $[-1, a]$, $0 < a \leqq 1$, *continuously differentiable on* $[0, a]$, *and suppose that* $\dot{x}(t) \geqq -M$ *on* $[0, a]$. *Then,* $g(\tau(x_0); t) \leqq g(\tau(x_t); 0) \leqq g(\tau(x_0); t) + M/2 + 2$ *for* $t \in [0, a]$.

PROOF. Let Δ_0: $-1 \leqq t_0 \leqq t_1 \leqq \cdots \leqq t_N \leqq 0$ be a partition of $[-1, 0]$, and choose n so that $t_{n-2} \leqq -t \leqq t_{n-1}$ for a given $t \in [0, a] \subset [0, 1]$. Then, Δ_t: $t_0 + t \leqq t_1 + t \leqq \cdots \leqq t_{n-2} + t$ is a partition of $[t-1, 0]$, and Δ: $t_{n-1} \leqq \cdots \leqq t_N$ is a partition of $[-1, 0]$. Clearly, if $\Delta_t = \Delta_t(\tau(x_0))$ as given in Lemma 4, then $g(\tau(x_0); t) = F(\tau(x_0); \Delta_t) \leqq F(\tau(x_t); \Delta_0) \leqq g(\tau(x_t); 0)$. On the other hand, if $\Delta_0 = \Delta_0(\phi)$ for $\phi = \tau(x_t)$ mentioned in Lemma 4, then $g(\phi; 0) = F(\phi; \Delta_0) \leqq F(\tau(x_0); \Delta_t) + F(\phi; \Delta) + G(\phi(t_{n-1}) - \phi(t_{n-2})) \leqq g(\tau(x_0); t) + F(\phi; \Delta) + 1$. Here, since $-1 \leqq \phi(s) \leqq 1$ for $s \in [-t, 0]$ and Δ_0 is principal to ϕ, $\phi(t_k) - \phi(t_{k-1})$ takes different sign alternately for $k = n, \cdots, N$ and $\phi(t_N) - \phi(t_{N-1}) > 1$. Therefore, by setting m to be the largest integer less than or equal to $(N-n)/2$, we have

$$\begin{aligned} F(\phi; \Delta) &= \sum_{k=0}^{m} G(\phi(t_{N-2k}) - \phi(t_{N-2k-1})) \leqq \left[\sum_{k=0}^{m} \{\phi(t_{N-2k}) - \phi(t_{N-2k-1})\}\right] \Big/ 2 \\ &\leqq \left[\phi(t_N) - \sum_{k=1}^{m} \{\phi(t_{N-2k+1}) - \phi(t_{N-2k})\} - \phi(t_{N-2m-1})\right] \Big/ 2 \\ &\leqq \left[2 + M \sum_{k=1}^{m} (t_{N-2k+1} - t_{N-2k})\right] \Big/ 2 \leqq (2+M)/2 . \end{aligned}$$

The conclusion follows immediately. q.e.d.

Now, we shall go back to the equation (4). Then, we have the following theorems.

THEOREM 1. *The solutions of* (4) *are unique for the initial value problem and continuable up to* $t = \infty$.

PROOF. By Lemma 6, the uniqueness is trivial, and any solution $x(t)$ is continuable unless $\|x_t\| \to \infty$. Because (4) is autonomous, it is sufficient to prove that the solution $x(t)$ satisfying $x_0 = \phi$ is continuable up to $t = 1$ for any $\phi \in X$. Since $\mu(x_t) \leqq \|\phi\|$ for $t \leqq 1$, we have $f(x_t) \geqq -\rho(\|\phi\|)$ $(= -M)$ as long as $x(t)$ exists and $t \leqq 1$. Hence, Lemma 7 shows that $g(\tau(x_t); 0) \leqq g(\tau(\phi); 0) + M/2 + 2$, and, therefore, $|f(x_t)| \leqq g(\tau(x_t); 0) + \rho(\|\phi\|) \leqq g(\tau(\phi); 0) + M/2 + 2 + M$ $(= M^*)$ as long as $x(t)$ exists and $t \leqq 1$. Thus, $\phi(0) - M^*t \leqq x(t) \leqq \phi(0) + M^*t$, which prevents $\|x_t\| \to \infty$ in $[0, 1]$, and hence $x(t)$ is continuable up to $t = \infty$. q.e.d.

THEOREM 2. *The solutions of* (4) *are not uniformly bounded on a finite interval.*

PROOF. Put

$$\phi^k(t) = \begin{cases} \cos[1 - 1/(2t+1)^2], & |2t+1| \geqq 1/k \\ \cos[1 - k^2], & |2t+1| \leqq 1/k, \end{cases}$$

for $k \geqq 3$ and $t \in [-1, 0]$. Clearly, $\phi^k \in X$, $\|\phi^k\| = 1$, $\tau(\phi^k) = \phi^k$ and $g(\phi^k; 1/2) \geqq m_k$, where $m_k = (k^2 - 1)/(2\pi) - 1$. Let $x^k(t)$ be the solution of (4) satisfying $x_0 = \phi^k$. Then, by Lemma 7 $g(\tau(x_t^k); 0) \geqq g(\tau(\phi^k); 1/2) \geqq m_k$ if $0 \leqq t \leqq 1/2$, and $\rho(\mu(x_t^k)) = \rho(\mu(-x_t^k)) = 0$ if $0 \leqq t \leqq 1/2$. Hence, $x^k(t)$ is a solution of $\dot{x} = 2x^2$, $x(0) = \phi^k(0) = 1$, that is, $x^k(t) = 1/(1 - 2t)$ as long as $t < 1/2$ and $|x^k(t)| \leqq \sqrt{m_k/2}$. This shows that there is a $t_k \in [0, 1/2)$ such that $x^k(t_k) = \sqrt{m_k/2}$, which diverges as $k \to \infty$. Thus, the solutions of (4) are not uniformly bounded. q.e.d.

LEMMA 8. *Let $x(t)$ be a solution of* (4) *starting at $t = 0$. Then, either $\mu(x_t) \geqq 0$ for all $t \geqq 3$, or $\mu(-x_t) > 0$ for all $t \geqq 3$.*

PROOF. Let $\sigma = \inf\{t \geqq 1: \mu(-x_t) \leqq 0\}$. Note that if there is no such σ, we have $\mu(-x_t) > 0$ for all $t \geqq 1$ and we are done. Also note that $\dot{x}(t) \geqq 0$ on $[1, \sigma]$ by Lemma 6, since $\mu(-x_t) \geqq 0$ implies $\mu(x_t) \leqq 0$. Suppose that $\sigma > 2$. Then, $\mu(-x_\sigma) = -x(\sigma) = 0$, and $x(t) < 0$ on $[1, \sigma)$. Moreover, we can find an $s \in [\sigma - 1, \sigma)$ so that $0 < \mu(-x_t) \leqq \mu(-x_s) = -x(s) < 1$ for all $t \in [s, \sigma)$. Hence, $\dot{x}(t) = \rho(\mu(-x_t)) \leqq \rho(\mu(-x_s)) = x(s)^2$ on $[s, \sigma)$ by Lemma 6. Therefore, $x(\sigma) \leqq x(s) + x(s)^2(\sigma - s) \leqq x(s) + x(s)^2 < 0$, which yields a contradiction. Thus, we have $\sigma \leqq 2$, and hence $x(s^*) \geqq 0$ for an $s^* \in [0, \sigma] \subset [0, 2]$. Suppose that $\mu(x_t) < 0$ for all $t \in [s^*, s^* + 1]$. Then, especially there is an $s \in [s^*, s^* + 1]$ such that $x(s) < 0$. Since $x(s^*) \geqq 0$ and $x(s) < 0$, we can find a $t \in [s^*, s]$ so that $x(t) < 0$ and $\dot{x}(t) < 0$, which contradicts Lemma 6. Therefore, there exists an $s \in [s^*, s^* + 1] \subset [0, 3]$ for which $\mu(x_s) \geqq 0$. Suppose that $\mu(x_t) < 0$ for a $t > s$, that is, $x(t^*) < 0$ for some $t^* > s$. Then, we can find a $t \in (s, t^*]$ so that $x(t) < 0$ and $\dot{x}(t) < 0$, since $x(s) \geqq 0$, which again contradicts Lemma 6. Thus, we have $\mu(x_t) \geqq 0$ for all $t \geqq s$. q.e.d.

THEOREM 3. *The solutions of* (4) *are uniformly ultimately bounded with an arbitrarily small bound.*

PROOF. Lemma 8 shows that any solution $x(t)$ of (4) starting at $t = 0$ satisfies $\mu(x_t) \geqq 0$ $(t \geqq 3)$ or $\mu(-x_t) > 0$ $(t \geqq 3)$. First of all, suppose that $\mu(x_t) \geqq 0$ for all $t \geqq 3$. Then, by Lemma 6 we have $\dot{x}(t) = -\rho(\mu(x_t)) \leqq 0$ for $t \geqq 3$, and hence $\mu(x_t) = x(t) \geqq 0$ for all $t \geqq 4$. Therefore, $x(t)$ satisfies $\dot{x}(t) = -x(t)^2$, that is, $x(t) = x(4)/\{1 + x(4)(t - 4)\}$ for all $t \geqq 4$. Now, let it be the case that $\mu(-x_t) > 0$ for $t \geqq 3$. From Lemma 6 it follows that $\dot{x}(t) = \rho(\mu(-x_t)) \geqq 0$ there. Hence, $\mu(-x_t) = -x(t) > 0$ for $t \geqq 4$, that is, $\dot{x}(t) = x(t)^2$. Therefore, we have $x(t) =$

$x(4)/\{1 - x(4)(t - 4)\}$ for $t \geqq 4$. In both cases, we have $|x(t)| < \varepsilon$ if $t \geqq 4 + 1/\varepsilon$, which completes the proof. q.e.d.

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