V. EXAMPLE

We now illustrate the proposed method to make the eigenvalues of matrix A nonresonant through state feedback.

Example:

Step 1) Let $n = 3$.

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad K = [k_3,\, k_2,\, k_1]. \tag{31}$$

$A = F - GK$ and the characteristic equation of A is given by $\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 = 0$.

Step 2) Choose $r_{01} = 1$, $r_{11} = 6$, $r_{21} = 10$ and $r_{31} = 6$. The unique polynomial correspondingly is given by $\mu^3 + 6\mu^2 + 11\mu + 6 = 0$ having roots -1, -2 and -3 which are resonant of orders 2 and 3 [note: $(2)(-1) + (0)(-2) + (0)(-3) = -2$, $(1)(-1) + (1)(-2) = -3$ and $(3)(-1) + (0)(-2) + (0)(-3) = -3$].

Step 3) In order to satisfy (25), as per Theorem 4, we choose $r = 3 - 2\sqrt{2}$ and $\delta = \sqrt{2}$, and transform $\mu \mapsto \lambda$ according to

$$\mu = \frac{2(\lambda + 1)}{\lambda + 2} \tag{32}$$

resulting in the characteristic equation

$$\lambda^3 + \frac{133}{30}\lambda^2 + \frac{98}{15}\lambda + \frac{16}{5} = 0$$

which has nonresonant roots of $(-3/2)$, $-(4/3)$ *and* $(-8/5)$.

Step 4) The state feedback is then given by$K = [\frac{16}{5}\ \frac{98}{15}\ \frac{133}{30}]$.

VI. CONCLUSION

The problem of transforming a dynamic system (11) into normal form is an old one. In the reduction to the normal form, the role played by the resonance of the eigenvalues of the matrix A in (11), is well known. In the context of control system of (17), however, the control input u can be used to provide state feedback so as to ensure the nonresonance of the A matrix. This, in turn, can lead to the satisfaction of the linearization conditions. Moreover, the eigenvalues of the linear system can be arbitrarily placed inside a scalable circular region in the left half of the complex plane. This region itself can be chosen to meet other design criteria, such as, relative stability. The actual computation of the transformation will, however, involve solving a series of generalized homological equations.

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An Averaging Approach to Chattering

Leonid M. Fridman

***Abstract*—The singularly perturbed relay control systems (SPRCS) as mathematical models of chattering in the small neighborhood of the switching surface in sliding mode systems are examined. Sufficient conditions for existence and stability of fast periodic solutions to the SPRCS are found. It is shown that the slow motions in such SPRCS are approximately described by equations derived from equations for the slow variables of SPRCS by averaging along fast periodic motions. It is shown that in the general case, when the equations of a plant contain relay control nonlinearly, the averaged equations do not coincide with the equivalent control equations or with the Filippov's definition for the sliding motions in the reduced system; however, in the linear case, they coincide.**

***Index Terms*—Averaging, periodic solutions, singularly perturbed systems, sliding mode control, variable structure systems.**

I. INTRODUCTION

The chattering phenomenon is one of the actual problems in modern sliding mode control theory. The presence of actuators is one of the basic reasons for chattering in sliding mode control systems [8], [1]. The behavior of sliding mode systems with fast actuators and inertial sensors is described by singularly perturbed relay control systems (SPRCS). In such a case, the following two qualitatively different types of motions can occur in the original SPRCS and in the reduced system:

- in the original SPRCS, there are no stable first-order sliding modes;
- the sufficient conditions for existence of a stable first-order sliding mode hold for the reduced system describing the behavior of a plant with ideal actuators and sensors [3], [4].

If the original SPRCS contains a sliding mode of the third order and greater, then the sliding modes are unstable [4]. In such systems, fast periodic oscillations can occur [3].

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The general model of sliding mode control systems with fast actuators has the following form (see [1] and [4]):

$$\mu\, dz/dt = g(z, s, x, u(s))$$
$$ds/dt = h_1(z, s, x, u(s)) \quad dx/dt = h_2(z, s, x, u(s)) \tag{I.1}$$

where $z \in \mathbf{R}^m$, $s \in \mathbf{R}$, $x \in \mathbf{R}^n$, $u(s) = \text{sign}\,(s)$ and g, h_1, h_2 are smooth functions of their arguments. Under such assumptions, system (I.1) can describe, for instance, the behavior of control systems such that variables x, s describe plant dynamics, the vector z describes dynamics of the fast actuator, where μ is the actuator time constant.

Letting $\mu = 0$ and expressing $\overline{z} = \varphi(s, x, u(s))$ from the equation $g(\overline{z}, s, x, u(s)) = 0$ according to the formula $\overline{z} = \varphi(s, x, u(s))$, we obtain the reduced system

$$ds/dt = h_1(\varphi(s, x, u(s)), s, x, u(s)) = H_1(s, x, u(s))$$
$$dx/dt = h_2(\varphi(s, x, u(s)), s, x, u(s)) = H_2(s, x, u(s)) \tag{I.2}$$

describing the behavior of the plant when actuator is ideal.

Suppose that

1) Original system (I.1) has no stable first-order sliding mode. This means that (almost everywhere on $s = 0$)
   a) either there is no sliding mode and
   $$h_1(z, 0, x, 1)h_1(z, 0, x, -1) > 0 \tag{I.3}$$
   b) or the sliding mode is unstable and consequently $h_1(z, 0, x, 1) > 0$ and $h_1(z, 0, x, -1) < 0$.
2) The measure of the domain $\mathbf{S} = \{x\colon H_1(0, x, 1) < 0, H_1(0, x, -1) > 0\} \subset \mathbf{R}^n$ is not zero, which means that the reduced system (I.2) has a stable first-order sliding domain.

This note presented deals with the chattering phenomenon in (I.2) from the viewpoint of the averaging approach and specific features of fast periodic solutions of system (I.1). This note consists of three sections. Mathematical tools for the analysis of periodic solutions of SPRCS are developed in Section II. In Section III these tools are used to analyze the behavior of sliding mode control systems with fast actuators.

II. Mathematical Tools

A. Problem Formulation

In this section, we consider the existence and stability of the fast periodic solutions to the SPRCS of the form

$$\mu\, dz/dt = g(z, \xi, x, u(\xi))$$
$$\mu\, d\xi/dt = h_1(z, \xi, x, u(\xi)), \quad dx/dt = h_2(z, \xi, x, u(\xi)) \tag{1}$$

where $z \in \mathbf{R}^m$, $\xi \in \mathbf{R}$, $x \in \mathbf{R}^n$, $u(\xi) = \text{sign}\,(\xi)$, and g, h_1, h_2 are smooth functions of their arguments, ξ here is the auxiliary variable, and the relation between ξ and s will be given in Section III. Introducing the "fast time" $\tau = t/\mu$ into (1), we obtain

$$dz/d\tau = g(z, \xi, x, u(\xi))$$
$$d\xi/d\tau = h_1(z, \xi, x, u(\xi)), \quad dx/d\tau = \mu h_2(z, \xi, x, u(\xi)). \tag{2}$$

For smooth singularly perturbed system, the existence and stability of the first approximation of the fast periodic solution was investigated in [7] using linearization method. For the relay control systems, this method is not useful.

In this section, we develop the mathematical tools to explore the fast periodic oscillations of (1), (2). The Poincare maps (see, for example, [5]) are to be employed. It is possible to consider the results of this section as some generalization of [7] for systems (1) and (2).

B. Some Properties of the Poincare Map, Generated by SPRCS

Let us denote the variation domain as Z, X with variables (z, ξ, x) and x.

Suppose that the following conditions are true:

1^0 $h_1, h_2, g \in \mathbf{C}^2[\overline{Z} \times [-1, 1]]$;

2^0 all the trajectories of
$$dz/d\tau = g(z, \xi, x, u(\xi)), \quad d\xi/d\tau = h_1(z, \xi, x, u(\xi)) \tag{3}$$
which start outside the surface $\xi = 0$ for all $x \in \overline{X}$ cross $\xi = 0$ at the point $(z, 0, x)$, where sliding mode does not exist and condition (I.3) holds;

3^0 system (3) for all $x \in \overline{X}$ has an isolated orbitally asymptotically stable solution $(z_0(\tau, x), \xi_0(\tau, x))$ with period $T(x)$.

Let us denote as $z_0^{\pm}(\tau, z, x)$ and $\xi_0^{\pm}(\tau, z, x)$, the solutions of system (3) with the initial conditions $z_0^{\pm}(0, z, x) = z$, $\xi_0^{\pm}(0, z, x) = 0$ for $\xi > 0$ and $\xi < 0$. Then, it is possible to define the Poincare map of the set $V = \{(z, x)\colon h_1(z, 0, x, 1) > 0\}$ on the surface $\xi = 0$ into itself, which is generated by (3) in the form (see Fig. 1) $R(z, x) = z_0^{-}(\Theta_0, z_0^{+}(\theta_0, z, x), x)$, where $\theta_0(z, x)$, $\Theta_0(z, x)$ are the smallest positive roots of equations
$$\xi_0^{+}(\theta_0, z, x) = 0, \quad \xi_0^{-}(\Theta_0, z_0^{+}(\theta, z, x), x) = 0.$$
The roots of this equations exist for all $x \in X$ according to condition 2^0.

Suppose that

4^0 the Poincare map $R(z, x)$ has a fixed point $z^*(x)$ corresponding to $(z_0(\tau, x), \xi_0(\tau, x))$;

5^0 provided that for all $x \in \overline{X}$ $\lambda_i(x)$ $(i = 1, \ldots, m)$, the eigenvalues of the matrix $(\partial R/\partial z)(z^*(x), x)$, satisfy $|\lambda_i(x)| \neq 1$;

6^0 the averaged system $dx/dt = p(x)$, where
$$dx/dt = p(x) \tag{4}$$
$$p(x) = \frac{1}{T(x)} \int_0^{T(x)} \times h_2(z_0(\tau, x), \xi_0(\tau, x), x, u(\xi_0(\tau, x))\, d\tau$$
has an isolated equilibrium point x_0 such that $p(x_0) = 0$, $\det \frac{dp}{dx}(x_0) \neq 0$.

Let us denote as $z^{\pm}(\tau, z, x, \mu)$ and $\xi^{\pm}(\tau, z, x, \mu)$, the solutions of (2) with the initial conditions $z^{\pm}(0, z, x, \mu) = z$, $\xi^{\pm}(0, z, x, \mu) = 0$ for $\xi > 0$ and $\xi < 0$.

The Poincare map of the domain V of the surface $\xi = 0$, generated by (2), has the following form:

$$\begin{aligned}\Phi(z, x, \mu) &= (\Phi_1(z, x, \mu), \Phi_2(z, x, \mu)) \\ &= (z^{-}(\Theta, z^{+}(\theta, z, x, \mu), x^{+}(\theta, z, x, \mu), \mu) \\ &\qquad x^{-}(\Theta, z^{+}(\theta, z, x, \mu), x^{+}(\theta, z, x, \mu), \mu))\end{aligned}$$

where the functions $\theta(z, x, \mu)$ and $\Theta(z, x, \mu)$ are determined by equations

$$\xi^{+}(\theta, z, x, \mu) = 0$$
$$\xi^{-}(\Theta, z^{+}(\theta, z, x, \mu) \quad x^{+}(\theta, z, x, \mu), \mu) = 0.$$

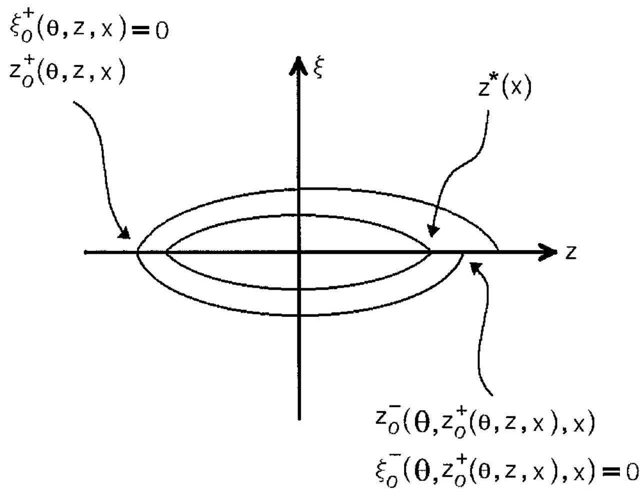

Fig. 1. Poincare map $R(z, x)$.

This means that

$$\Phi_1(z, x, 0) = R(z, x), \quad \Phi_2(z, x, 0) = x. \tag{5}$$

The surface $\xi = 0$ is the surface without stable sliding for (3). This means that there exists a neighborhood of the point $(z^*(x_0), x_0)$ on the surface $\xi = 0$, where $\max\{|d\xi^+/d\theta|, |d\xi^-/d\Theta|\} > 0$. From condition 1^0 and the implicit function theorem, it follows that for some small μ_0, the functions Φ, θ, Θ have continuous derivatives in some set $U \times [0, \mu_0]$ on the surface $\xi = 0$. This means that we can consider the function Φ as the Poincare map of the set $U \times [0, \mu_0]$ on the surface $\xi = 0$ into itself. Moreover, according to (5) and condition 6^0 we can rewrite $\Phi(z, x, \mu)$ in the form

$$\Phi(z, x, \mu) = (\Phi_1(z, x, \mu), x + \mu\overline{Q}(z, x, \mu))$$

where $\Phi_1(z, x, \mu)$, $\overline{Q}(z, x, \mu)$ are sufficiently smooth functions and

$$\begin{aligned} \Phi_1(z^*(x), x, 0) &= z^*(x), \quad \overline{Q}(z^*(x), x, 0) = T(x)p(x) \\ \overline{Q}(z^*(x_0), x_0, 0) &= T(x_0)p(x_0) = 0. \end{aligned}$$

Let us introduce new variables for the function Φ using the formula $\eta = z - z^*(x)$. Then the Poincare map Φ takes the form

$$\begin{aligned} \Psi(\eta, x, \mu) &= (\Psi_1(\eta, x, \mu), \quad \Psi_2(\eta, x, \mu)) \\ &= (\Phi_1(\eta + z^*(x), x, \mu) - z^*(x) \\ &\qquad x + \mu\overline{Q}(\eta + z^*(x), x, \mu)) \end{aligned} \tag{6}$$

and consequently $\Psi(0, x, 0) = (0, x)$.

C. Existence of Fast Periodic Solution

Theorem 1: Under conditions 1^0–6^0, (1) has an isolated periodic solution with the period $\mu(T(x_0) + O(\mu))$ near the circle $(z_0(t/\mu, x_0), \xi_0(t/\mu, x_0), x_0)$.

Proof: We will prove a periodic solution to exist as being the fixed point $(\eta^*(\mu), x^*(\mu))$ of the map Ψ. Let us rewrite the existence conditions of this fixed point in the form

$$\begin{aligned} G(\eta^*, x^*, \mu) &= \begin{pmatrix} G_1(\eta^*, x^*, \mu) \\ G_2(\eta^*, x^*, \mu) \end{pmatrix} \\ &= \begin{pmatrix} \eta^* - \Psi_1(\eta^*, x^*, \mu) \\ \dfrac{1}{\mu}[x^* - \Psi_2(\eta^*, x^*, \mu)] \end{pmatrix} = 0. \end{aligned} \tag{7}$$

It is necessary to take into account that for $\mu = 0$ $\eta^*(0) = 0$, $x^*(0) = x_0$ and $G_2(0, x_0, 0) = -T(x_0)p(x_0) = 0$ and consequently for $\mu = 0$ conditions (7) are fulfilled. Moreover, taking into account that $G_1(0, x, 0) = 0$ for all $x \in \overline{X}$ we can conclude that $\frac{\partial G_1}{\partial x}(0, x_0, 0) = 0$. Let us compute the Jacobian of function G with respect to variables η, x at $\mu = 0$.

$$\begin{aligned} &\frac{\partial G}{\partial(\eta, x)}(0, x_0, 0) \\ &= \begin{vmatrix} I_m - \dfrac{\partial R}{\partial z}(z^*(x_0), x_0) & 0 \\ \dfrac{\partial G_2}{\partial \eta}(0, x_0, 0) & -T(x_0)\dfrac{\partial p}{\partial x}(x_0) \end{vmatrix} \neq 0. \end{aligned}$$

This means that there exists an isolated fixed point $(z^*(\mu), x^*(\mu))$ of the map G, which corresponds to the periodic solution of systems (1) and (2), and in this case $z^*(\mu) = z^*(x_0) + O(\mu)$, $x^*(\mu) = x_0 + O(\mu)$. ∎

D. Stability in the First Approximation

Assume that

7^0 the eigenvalues $\lambda_i(x_0)$ of the matrix $(\partial R/\partial z)(z(x_0), x_0)$ satisfy the inequalities $|\lambda_i(x_0)| < 1$ $(i = 1, \ldots, m)$;

8^0 the eigenvalues $\nu_j(x_0)$, $j = 1, \ldots, n$ of the matrix $\frac{dp}{dx}(x_0)$ satisfy the inequalities

$$\mathbf{Re}\,\nu_j(x_0) < 0.$$

Theorem 2: Under conditions 1^0–8^0 the periodic solution of (1), (2) is orbitally asymptotically stable.

Proof: Let us find the derivatives Ψ by variables η, x, as shown in the equation at the bottom of the page. Consequently the matrix $\Gamma(\eta, x, \mu)$ has at the small vicinity of $(0, x_0, 0)$ two groups of eigenvalues

$$\begin{aligned} &\lambda_i(x_0) + O(\mu), \qquad i = 1, \ldots, m \\ &\qquad 1 + \mu T(x_0)\nu_j(x_0) + \mu o(\mu), \qquad j = 1, \ldots, n. \end{aligned}$$

This means that under conditions of Theorem 2 there exists some neighborhood of $(0, x_0, 0)$, where Ψ is contraction map and the corresponding fast periodic solution of systems (1), (2) is orbitally asymptotically stable. ∎

$$\frac{\partial \Psi}{\partial(\eta, x, \mu)} = \Gamma(\eta, x, \mu) = \begin{bmatrix} I_m - \dfrac{\partial R}{\partial z}(x_0) + O(\mu) & O(\mu) \\ \dfrac{\partial \Psi_2}{\partial \eta}(0, x_0, 0) + O(\mu) & I_m + \mu T(x_0)\dfrac{\partial p}{\partial x}(x_0) + O(\mu) \end{bmatrix}.$$

III. Analysis of Averaged Equations in Sliding Mode with Fast Actuators

A. The Averaged Equations of Systems that Linear in Relay Control

In this section, we consider the SPRCS, which linearly depend on relay control. We will show that if the fast periodic solution of SPRCS exists, then the averaged equations which describe the slow motions in such SPRCS and the equations which describe the sliding motion in the reduced systems coincide.

Let us consider the system

$$\begin{aligned}\mu\, dz/dt &= A(s, x)z + f_1(s, x) + K_1(s, x)u(s)\\ ds/dt &= B(s, x)z + f_2(s, x) + K_2(s, x)u(s)\\ dx/dt &= D(s, x)z + f_3(s, x) + K_3(s, x)u(s) \qquad (8)\end{aligned}$$

where $z \in \mathbf{R}^m$, $s \in \mathbf{R}$, $x \in \mathbf{R}^n$, $u(s) = \text{sign}\,(s)$ and f_i, K_i $(i = 1, 2, 3)$ are smooth functions of their arguments. Suppose that there is no stable first order sliding mode in (8). This means that $K_2(0, x) \geq 0$. Setting $\mu = 0$ and expressing z_0 from the first equation of (8), according to the formula $z_0 = -A^{-1}(s, x)[f_1(s, x) + K_1(s, x)u(s)]$, we obtain the reduced system

$$\begin{aligned}ds/dt =& -B(s, x)A^{-1}(s, x)f_1(s, x) + f_2(s, x)\\ &- [B(s, x)A^{-1}(s, x)K_1(s, x) - K_2(s, x)]u(s)\\ dx/dt =& D(s, x)A^{-1}(s, x)f_1(s, x) + f_3(s, x)\\ &- [D(s, x)A^{-1}(s, x)K_1(s, x) - K_3(s, x)]u(s). \qquad (9)\end{aligned}$$

Suppose that sufficient conditions for the existence of sliding mode are found for this system, then

$$B(0, x)A^{-1}(0, x)K_1(0, x) - K_2(0, x) > 0. \qquad (10)$$

Equations which describe the sliding motion in the reduced system have the form

$$\begin{aligned}dx/dt =& -D(0, x)A^{-1}(0, x)f_1(0, x) + f_3(0, x)\\ &- [D(0, x)A^{-1}(0, x)K_1(0, x) - K_3(0, x)]\\ &\times (u(s) - u_{eq}(x))\\ u_{eq}(x) =& [B(0, x)A^{-1}(0, x)K_1(0, x) - K_2(0, x)]^{-1}\\ &\times [-B(0, x)A^{-1}(0, x)f_1(0, x) + f_2(0, x)]. \qquad (11)\end{aligned}$$

Let us show that the averaged equations for the original system (8) coincide with (11).

Suppose that the following conditions are true:

(∗) $\mathbf{Re}\quad \mathbf{Spec}\, A(0, x) < 0$ for all $x \in \overline{X}$;

(∗∗) the measure of the sliding mode domain $\mathbf{S} \subset X$ for (9) in which $|u_{eq}(x)| < 1$ is nonzero.

Let us denote by $\mathbf{V}(\mathbf{S})$ the attractive domain of $\mathbf{S}$ for (9). This means that all solutions of (9) starting from $\mathbf{V}(\mathbf{S})$ reach the sliding domain $\mathbf{S}$ in a finite time and with a finite number of switchings.

Consider a solution of (8) starting from the point (z^*, s^*, x^*), $((s^*, x^*) \in \mathbf{V}(\mathbf{S}))$. Condition (∗) ensures that at least before the switching moment the fast variables of (8) will be stable. Then, according to the boundary layer method (see for example [9]), one can conclude that the solution of the original system (8) reaches the $O(\mu)$ neighborhood of the $\mathbf{S}$. This means that it is reasonable to consider only solutions of (8) with initial conditions

$$z(0, \mu) = z^0, \quad s(0, \mu) = \mu s^0, \quad x(0, \mu) = x^0$$

which are located in the $O(\mu)$ vicinity of the sliding domain $\mathbf{S}$. Following [3], let us increase $1/\mu$ times the neighborhood of the discontinuity surface $s = 0$ in (8) and let the variable $\xi = s/\mu$. Then, (8) is rewritten in the form

$$\begin{aligned}\mu\, dz/dt &= A(\mu\xi, x)z + f_1(\mu\xi, x) + K_1(\mu\xi, x)u(\xi)\\ \mu\, d\xi/dt &= B(\mu\xi, x)z + f_2(\mu\xi, x) + K_2(\mu\xi, x)u(\xi)\\ dx/dt &= D(\mu\xi, x)z + f_3(\mu\xi, x) + K_3(\mu\xi, x)u(\xi). \qquad (12)\end{aligned}$$

In this case, the system which describes the fast motions in (12) has, analogous to (3), the form

$$\begin{aligned}dz/d\tau &= A(0, x)z + f_1(0, x) + K_1(0, x)u(\xi)\\ d\xi/d\tau &= B(0, x)z + f_2(0, x) + K_2(0, x)u(\xi)\\ &(x\text{-parameter}). \qquad (13)\end{aligned}$$

Suppose that (13) has the isolated periodic solution $(z_0(\tau, x), \xi_0(\tau, x))$.

Introducing the new variables

$$\eta = z + A^{-1}(0, x)[f_1(0, x) + K_1(0, x)u_{eq}(x)]$$

for (13), we have

$$\begin{aligned}d\eta/d\tau &= A(0, x)\eta + K_1(0, x)\overline{u}(\xi, x)\\ d\xi/d\tau &= B(0, x)\eta + K_2(0, x)\overline{u}(\xi, x)\\ \overline{u}(\xi, x) &= u(\xi) - u_{eq}(x). \qquad (14)\end{aligned}$$

Lemma 1 [6]: If there exists a solution of (14) of the period $T(x)$ then

$$\int_0^{T(x)} \eta_0(\tau, x)\, d\tau = 0$$
$$\frac{1}{T(x)}\int_0^{T(x)} u(\eta_0(\tau, x))\, d\tau = \frac{\theta^*(x) - \Theta^*(x)}{T(x)} = u_{eq}(x).$$

Let us consider (8). If the conditions of Theorem 2 are true for (12), then there exists an isolated periodic solution $(z(\tau, \mu), \xi(\tau, \mu), x(\tau, \mu))$, which corresponds to the periodic solution $(\eta_0(\tau, x), \xi_0(\tau, x))$ of (14). Moreover,

$$\int_0^{T(x)} z_0(\tau, x)\, d\tau = A^{-1}(0, x)(f_1(0, x) + K_1(0, x)u_{eq}(x)).$$

This means that the averaged equations which approximately describe the behavior of the slow motions in (8) coincide with (11) for the sliding motions in the reduced system.

B. Example

Consider a mathematical model of a control system with an actuator and the overall relative degree 3

$$\dot{x} = -x - u, \quad \dot{s} = z_1 \qquad (15)$$

$$\mu\dot{z}_1 = z_2, \quad \mu\dot{z}_2 = -2z_1 - 3z_2 - u. \qquad (16)$$

Here $z_1, z_2, s, x \in \mathbf{R}$, $u(s) = \text{sign}\,(s)$, μ is the actuator time constant. The fast motions taking place in (15), (16) are described by the system

$$\frac{d\xi}{d\tau} = z_1, \quad \frac{dz_1}{d\tau} = z_2, \quad \frac{dz_2}{d\tau} = -2z_1 - 3z_2 - u$$
$$u = \text{sign}\,(\xi). \tag{17}$$

Then, the solution of (17) for $\xi > 0$ with initial condition $\xi(0) = 0$, $z_1(0) = z_{10}$, $z_2(0) = z_{20}$ is as follows:

$$\begin{aligned}\xi(\tau) =& \tfrac{3}{2} z_{10} - 2z_{10}e^{-\tau} + \tfrac{1}{2} z_{10}e^{-2\tau} + \tfrac{1}{2} z_{20} - z_{20}e^{-\tau}\\ &+ \tfrac{1}{2} z_{20}e^{-2\tau} - \tfrac{1}{2}\tau + \tfrac{3}{4} - e^{-\tau} + \tfrac{1}{4} e^{-2\tau};\\ z_1(\tau) =& 2z_{10}e^{-\tau} - z_{10}e^{-2\tau} + z_{20}e^{\tau} - z_{20}e^{-2\tau}\\ &- \tfrac{1}{2} + e^{-\tau} - \tfrac{1}{2} e^{-2\tau}\\ z_2(\tau) =& 2z_{10}e^{-2\tau} - 2z_{10}e^{-\tau} - z_{20}e^{-\tau}\\ &+ 2z_{20}e^{-2\tau} - e^{-\tau} + e^{-2\tau}.\end{aligned}$$

Consider the point mapping $\Xi(z_1, z_2)$ of the domain $z_1 > 0$, $z_2 > 0$ on the switching surface $\xi = 0$ into the domain $z_1 < 0$, $z_2 < 0$ with sign $(\xi) > 0$, generated by (17). Then

$$\begin{aligned}\Xi(z_1, z_2) =& (\Xi_1(z_1, z_2), \Xi_2(z_1, z_2));\\ \Xi_1(z_1, z_2) =& 2z_1e^{-T} - z_1e^{-2T} + z_2e^{-T} - z_2e^{-2T}\\ &- \tfrac{1}{2} + e^{-T} - \tfrac{1}{2} e^{-2T};\\ \Xi_2(z_1, z_2) =& 2z_1e^{-2T} - 2z_1e^{-T} - z_2e^{-T}\\ &+ 2z_2e^{-2T} - e^{-T} + e^{-2T}\end{aligned}$$

where $T(z_1, z_2)$is the smallest root of equation

$$\begin{aligned}\xi(T(z_1, z_2)) =& \tfrac{3}{2} z_1 - 2z_1e^{-T} + \tfrac{1}{2} z_1e^{-2T} + \tfrac{1}{2} z_2 - z_2e^{-T}\\ &+ \tfrac{1}{2} z_2e^{-2T} - \tfrac{1}{2} T + \tfrac{3}{4} - e^{-T} + \tfrac{1}{4} e^{-2T} = 0.\end{aligned}$$

The system (17) is symmetric with respect to the point $\xi = z_1 = z_2 = 0$. Thus, the initial condition $(0, z_1^*, z_2^*)$ and the semi-period $T^* = T(z_1^*, z_2^*)$ for the periodic solution of (17) are determined by the equations

$$\Xi(z_1^*, z_2^*) = -(z_1^*, z_2^*), \quad \xi(T(z_1^*, z_2^*)) = 0. \tag{18}$$

Equations (18) and (18) have positive solution

$$T^* \approx 2.2755, \quad z_1^* \approx 0.3241, \quad z_2^* \approx 0.1654$$

corresponding to the existence of a $2T^*$-periodic solution in (17). Calculating the value of Frechet derivative $\partial\Xi/\partial z$ at (z_1^*, z_2^*), using the found value of T^*, we achieve

$$\frac{\partial\Xi}{\partial z}(z_1^*, z_2^*) = J = \begin{bmatrix} -0.4686 & -0.1133 \\ 0.3954 & 0.0979 \end{bmatrix}.$$

The eigenvalues of matrix J are -0.3736 and 0.0029. That implies existence and asymptotic stability of the periodic solution of (17). The averaged equation for (15), (16) is $\dot{x} = -x$, and it has the asymptotically stable equilibrium point $x = 0$. Hence, (15), (16) has an orbitally asymptotically stable periodic solution, which lies in the $O(\mu)$-neighborhood of the switching surface.

C. The Systems Containing the Relay Control Nonlinearly

Suppose that the behavior of a control system with a fast actuator is described by equations

$$dx/dt = ((2z_1)^4 - (2z_1)^2 + \beta)x, \quad ds/dt = z_1$$
$$\mu\, dz_1/dt = z_2, \quad \mu dz_2/dt = -2z_1 - 3z_2 - u \tag{19}$$

where $x, s \in \mathbf{R}$ describing the behavior of the plant, $z_1, z_2 \in \mathbf{R}$ are the variables of the actuator, $u(s)=$ sign (s), $0< \beta < 1$, and μ is the actuator time constant.If we take $\mu = 0$ in (19) we will have

$$ds/dt = -u/2, \quad dx/dt = (u^4 - u^2 + \beta)x. \tag{20}$$

A stable first order sliding mode exists in (20). Both the classical extension of the definition of solutions by Filippov [2] and the equivalent control method [8] coincide. These motions are described by the equation $dx/dt = \beta x$. The zero solution of this equation is unstable for $\beta > 0$.

At the same time, in (17), a fast periodic solution occurs. Let us denote $z_1(\tau)$ as the first coordinate of $2T^*$ periodic solution (17). If $\gamma > \beta$ are selected so that

$$-\gamma = \int_0^{2T^*} [(2z_1)^4(\tau) - (2z_1)^2(\tau)]\, d\tau < -\beta < 0,$$

then the averaged equation has the form $dx/dt = -(\gamma - \beta)x$. The zero solution of this equation is asymptotically stable. This means that (19) has an asymptotically orbitally stable periodic solution in the $O(\mu)$ neighborhood of the point $s = x = 0$. The averaged equation does not coincide with the equations of the equivalent control method and Filippov's definition of the solution.

IV. Conclusion

The singularly perturbed relay control systems as mathematical models of chattering in the small neighborhood of the switching surface in sliding mode systems are examined.

Sufficient conditions for existence and stability of fast periodic solutions to the SPRCS are found. It is shown that the slow motions in such SPRCS are approximately described by equations derived from the equations for the slow variables of SPRCS by averaging along fast periodic motions.

The analysis of oscillations in the small neighborhood of the sliding surface in the sliding mode control systems has shown that

- in the general case, when the original SPCSC contains the relay control nonlinearly, the averaged equations do not coincide with the equivalent control equations or the Filippov's definition, which describes the motions in the sliding mode in the reduced system;
- in the case, when the original SPCSC contain the relay control linearly, the averaging equations and equations which describe the reduced system motions in the sliding mode coincide.

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Fundamental Limits in Robustness and Performance for Unstable, Underactuated Systems

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***Abstract*—We derive bounds on the $\mathcal{H}_\infty$ norm of a weighted output sensitivity function and a weighted output complementary sensitivity function for an underactuated, multiple-input–multiple-output (MIMO), linear, time-invariant (LTI), unstable plant in feedback with an internally stabilizing, LTI controller. These bounds indicate a limit to the achievable robustness and performance for such systems and thus, are valuable tools for performing system design tradeoffs.**

***Index Terms*—Fundamental system limits, robustness, underactuated systems.**

I. Introduction

In control system design, common measures of the effect of system uncertainties on closed-loop robustness and performance of a system are the $\mathcal{H}_\infty$ norm of the sensitivity function, which we denote $S(s)$, and the complementary sensitivity function, denoted $T(s)$. Unfortunately, in every system, there are limits and tradeoffs which restrict the minimization of these sensitivities. The investigation of limits on achievable sensitivity reduction has a long history beginning with Bode's integral theorem [1], [2] and Zames [3]. Since that time there has been an explosion of results providing sensitivity reduction limits for a larger class of plants including multiple-input–multiple-output (MIMO) systems as well as plants with instabilities and/or nonminimum phase zeros [4]–[9]. Until recently, the additional limitations present in nonsquare systems have not received much attention. However, there have been a few exceptions worth noting, for example [10], [11].

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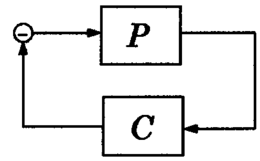

Fig. 1. Closed-loop system configuration.

In this note, we consider limits in achievable robustness and performance for underactuated, MIMO, linear, time-invariant (LTI), unstable systems which are internally stabilized by an LTI controller in feedback. We develop lower bounds on the $\mathcal{H}_\infty$ norm of both the weighted output sensitivity and weighted output complementary sensitivity functions of the closed-loop system. We then discuss ways to calculate the bounds and use the bounds to derive intuitive insight into the nature of the system limitations.

II. System Preliminaries

For the development which follows, we will refer to the feedback configuration of Fig. 1 where we are given an LTI plant P of dimension $N_o \times N_i$ with $N_o > N_i$ having unstable poles $p_0, \ldots, p_{N_p}$, and an internally stabilizing, LTI controller C in feedback around P.

Let $[A, B; C, D]$ be a state-space representation of our system $P(s)$. Then for each unstable pole p_i of our plant $P(s)$ we define an output direction vector $x_{p_i} \in \mathbb{C}^{N_o}$, $\|x_{p_i}\|_2 = 1$ as $x_{p_i} := C\zeta$ where ζ satisfies $A\zeta = p_i\zeta$ [12]. In addition, since $P(s)$ is underactuated $(N_o > N_i)$, for every $s_0 \in \mathbb{C}^+$, where $\mathbb{C}^+ := \{s \in \mathbb{C}: \mathcal{R}e(s) > 0\}$, $P(s_0)$ has a nontrivial left nullspace (an $N_o - N_i$ dimensional subspace of $\mathbb{C}^{N_o}$). Thus, for every $s_0 \in \mathbb{C}^+ \setminus \{p_0, \ldots, p_{N_p}\}$ there exists at least one output null direction vector $u_{s_0} \in \mathbb{C}^{N_o}$ defined as a vector satisfying $\|u_{s_0}\|_2 = 1$ and $u_{s_0}^* P(s_0) = 0$. Both analytical and numerical methods of calculating output direction vectors such as x_{p_i} and u_{s_0} have been outlined in [12].

We now define two new transfer matrices as follows:

$$P_{N_p}(s) := B_{N_p}^{-1}(s)P(s), \qquad P_{s_0}(s) := B_{s_0}(s)P(s) \tag{1}$$

where $B_{N_p}(s)$ and $B_{s_0}(s)$ are the all-pass filters

$$\begin{aligned} B_{N_p}(s) &:= \prod_{i=0}^{N_p} B_{p_i}(s) \\ &:= \prod_{i=0}^{N_p} \left(I + \frac{2\,\mathcal{R}e(p_i)}{s - p_i}\, \hat{x}_{p_i}\hat{x}_{p_i}^* \right) \end{aligned} \tag{2}$$

$$B_{s_0}(s) := I + \frac{2\,\mathcal{R}e(s_0)}{s - s_0}\, u_{s_0}u_{s_0}^* \tag{3}$$

and where u_{s_0} is any output null direction vector corresponding to $P(s_0)$ and $\hat{x}_{p_i}$ is the output direction vector corresponding to the pole p_i for the transfer function

$$\begin{cases} P & \text{for } i = 0 \\ P_{p_{(i-1)}} & \text{for } i = 1, \ldots, N_p \end{cases} \tag{4}$$

where $P_{p_{(i-1)}} := (\prod_{j=0}^{i-1} B_{p_j})^{-1} P$. In general, $\hat{x}_{p_i}$ will be different from x_{p_i} except for $i = 0$ [12]. The transfer matrices $B_{N_p}(s)$ and $B_{s_0}(s)$ will be used to express the subsequent bounds.

Now, if P is in feedback with the LTI, internally stabilizing controller C as shown in Fig. 1, we can define the output loop transfer function $L = PC$ and the output sensitivity and output complementary sensitivity functions for our system, respectively, as follows: $S =$