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An axiomatic definition of holonomy

A. Caetano[†] and R. F. Picken[‡]

[†] Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, R. Ernesto de Vasconcelos, Bloco C1, 1700 Lisboa, Portugal
and
Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal

[‡] Departamento de Matemática, Instituto Superior Técnico, Avenida Rovisco Pais, 1096 Lisboa Codex, Portugal
and
Centro de Física da Matéria Condensada, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal

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Abstract

A group of loops $\mathcal{GL}^\infty(M)$ is associated to every smooth pointed manifold M using a strong homotopy relation. It is shown that the holonomy of a connection on a principal G-bundle may be presented as a group morphism $\mathcal{H} : \mathcal{GL}^\infty(M) \to G$ and that every such morphism satisfying a natural smoothness condition is the holonomy of some unique connection up to isomorphism.

1 Introduction

Let M be a smooth, connected and paracompact manifold and $* \in M$, a fixed point. The choice of a space of paths or loops on M depends on the kind of properties of M one is interested in. For instance, if we are interested in topological properties, we may choose the space $\Omega^0(M)$ of continuous loops defined on $[0,1]$ and based at $* \in M$ and consider the quotient of $\Omega^0(M)$ by the equivalence relation given by continuous homotopies based at $* \in M$. This gives rise to a set, $\pi_1(M)$, which acquires a group structure through the natural composition of loops, and this *fundamental group* is a topological invariant of M. If, however, we are interested in the geometrical properties of a *principal G-bundle* $\pi : P \to M$ equipped with a *connection* ∇ we could follow Barrett in [Ba] and choose the loop space $\Omega(M)$ of continuous piecewise smooth loops based at $* \in M$ quotiented by *thin homotopy* (see the next section). Our purpose here is to point out some disadvantages of this approach and to show how a different choice of loop space, together with an alternative homotopy equivalence, enables one to repeat Barrett's construction eliminating those disadvantages.

Our main concept is that of *holonomy*. In differential geometry one associates to every connection defined on a principal G-bundle, together with a choice of base point in the bundle, a holonomy, i.e. a map from a space of piecewise smooth loops to the group G. The Ambrose-Singer theorem [AmS] shows that the holonomy contains useful information about the bundle and the connection, in particular about the curvature. Barrett's work shows that the holonomy actually contains, in a suitable sense, all the information about the bundle and the connection.

Holonomy is the mathematical structure underlying a number of physical phenomena both in classical mechanics (the Hannay angle [H]) and in quantum mechanics (the Bohm-Aharonov effect [BoA] and Berry's phase [Be]). In Yang-Mills theory there has been considerable interest in using the gauge-invariant traced holonomy or *Wilson loop* as a complementary variable to the connection. Wilson loops played a key part in Witten's approach to knot invariants using quantum Yang-Mills theory [W]. Since the introduction of the Ashtekar connection [As1], Wilson loops have also been intensively used as non-local variables in the description of classical and quantum gravity [RS].

In what follows we first discuss Barrett's and our own approach to exhibiting holonomy as a morphism from a group of loops to G (sections 2 to 6). The main definition of a holonomy as such a morphism subject to a smoothness condition is given in section 7. After stating an important property of holonomy due to Barrett [Ba], in section 8, and examining the use of a different basepoint (section 9), we describe the construction of a principal G-bundle from a holonomy map (sections 10 and 11). A connection on this bundle is then constructed through the intermediate device of a lifting function (sections 12 to 14). Finally, in section 15, the main conclusion is stated, namely that holonomies are in one-to-one correspondence with triples consisting of a principal G-bundle, a connection on this bundle and a point in the fibre over $*$, up to isomorphism. Some final comments are presented in section 16.

2 Thin homotopy

A *loop* l is a continuous piecewise smooth map $l : [0,1] \to M$ such that $l(0) = l(1) = *$. The space of all such loops will be denoted $\Omega(M)$. Following Barrett [Ba] two loops $l, k \in \Omega(M)$ are said to be *thinly homotopic* if there exists a map $H : [0,1] \times [0,1] \longrightarrow M$ such that:

(1) $H(s,.) \in \Omega(M), \forall s \in [0,1]$
(2) $H(0,t) = l(t) \wedge H(1,t) = k(t), \forall t \in [0,1]$
(3) $H([0,1] \times [0,1]) \subseteq l([0,1]) \cup k([0,1])$
(4) H *is continuous and piecewise smooth for some paving of* $[0,1] \times [0,1]$ *consisting of polygons*

The thin homotopy relation is not an equivalence relation because transitivity fails. Nevertheless we can build a minimal extension of this relation to form an equivalence relation: we think of the thin homotopy as a subset of $\Omega(M) \times \Omega(M)$ and consider the intersection of all subsets of $\Omega(M) \times \Omega(M)$ which are equivalence relations and contain the thin homotopy. This intersection is the required minimal extension. It may be redefined in a useful way: two loops $l, k \in \Omega(M)$ will be said to be *thinly equivalent*, written $l \overset{\sigma}{\sim} k$ if there exists a finite sequence $l_1, l_2, \ldots, l_n \in \Omega(M)$ such that $l_1 = l$, $l_n = k$ and l_i is thinly homotopic to l_{i+1} for $i = 1, 2, \ldots, n-1$. The nice feature of thin equivalence is that the usual composition and inversion of paths go over to the quotient $\Omega(M)/\overset{\sigma}{\sim}$ to give rise to a group structure. The proof of this statement consists merely of copying the construction of the fundamental group $\pi_1(M)$.

Given a Lie group G and a principal G-bundle $\pi : P \to M$ equipped with a connection ∇ one can perform *parallel transport* along each loop $l \in \Omega(M)$. Together with the right G-action on P and the fixing of a point $p_0 \in \pi^{-1}(*)$ this operation defines the *holonomy* $\mathcal{H} : \Omega(M) \longrightarrow G$ in the following way: let $l^{\uparrow} : [0,1] \to P$ be the unique *horizontal lift* of l such that $l^{\uparrow}(0) = p_0$; then $\mathcal{H}(l)$ is the unique element of G satisfying the equation $l^{\uparrow}(0) = l^{\uparrow}(1)\mathcal{H}(l)$.

For Barrett's approach to work it is essential to ensure that the holonomy goes over to the quotient by thin equivalence to define a group morphism $\mathcal{H}_\sigma : \Omega(M)/\overset{\sigma}{\sim} \longrightarrow G$. The delicate step consists in proving that if $l, k \in \Omega(M)$ are loops related by a thin homotopy $H : [0,1] \times [0,1] \longrightarrow M$ then parallel transport along l gives the same result as parallel transport along k. For this purpose it is necessary to consider the pull-back connection $H^*\nabla$ defined on H^*P and this only makes sense if H is smooth throughout $[0,1] \times [0,1]$, and hence, if l and k themselves are smooth over $[0,1]$. Since generalizing the notion of pull-back connections of smooth maps to continuous piecewise smooth maps is not straightforward, it is natural to look for a more suitable space of loops to replace $\Omega(M)$.

3 Smooth paths

There is a family of smooth *reparametrizations* that will often be used to show that the spaces of paths we shall introduce are suited to our purposes. These reparametrizations derive from the celebrated smooth function f defined by $f(t) = e^{-1/t}$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$. The function g defined by $g(t) = f(t)/(f(t) + f(1-t))$ is smoothly increasing on $\mathbf{R}$, equals 0 on $]-\infty, 0]$ and equals 1 on $[1, +\infty[$. Notice that the function g may be regarded as a smooth path on the manifold $[0,1]$ which spends much of its 'time' sitting at 0 and 1. Using this function g and similar functions derived from it one can reparametrize any continuous piecewise smooth path so that it becomes smooth. Of course, such reparametrizations are non-regular. The image of such a smoothed path may still be a broken curve. It is also possible to smooth the 'corners' of a continuous piecewise smooth function by changing it slightly in a finite number of intervals, which may be as narrow as one likes.

Given a smooth path $p : [0,1] \to M$, an instant $t_0 \in [0,1]$ will be called a *sitting instant* if there exists an $\epsilon > 0$ such that the map p is constant on $[0,1] \cap]t_0 - \epsilon, t_0 + \epsilon[$. Notice that one can create new sitting instants on the domain of the path p using the reparametrizations previously mentioned.

We shall be performing the usual operation of *composition* of smooth paths only in the following way: Given two smooth paths $p : [0,1] \to M$ and $q : [0,1] \to M$ such that $p(1) = q(0)$ and 1 and 0 are sitting instants for p and q respectively, then $pq : [0,1] \to M$ is the path given by $pq(t) = p(2t)$ for $0 \leq t \leq 1/2$ and $pq(t) = q(2t-1)$ for $1/2 < t \leq 1$. Notice that pq is a smooth path. The *inversion* of paths shall be given by the usual formula $p^{-1}(t) = p(1-t)$ for $t \in [0,1]$.

We denote by $P^\infty(M)$ the space of all smooth paths in M having 0 and 1 as sitting instants. $P^\infty(M,*)$ shall be the subspace of $P^\infty(M)$ consisting of paths which start at $* \in M$. Because every piecewise smooth path may be reparametrized to become smooth the paths in $P^\infty(M,*)$ reach everywhere on M, since M is connected. Let $\Omega^\infty(M)$ be the subspace of $P^\infty(M,*)$ consisting of paths which are *loops*, i.e. paths whose ends are $* \in M$. The main advantage of the slightly unusual definition of these spaces is that the operations of composition and inversion of loops are well defined in $\Omega^\infty(M)$.

4 Intimacy

Given two paths $l, k \in P^\infty(M)$ we say that they are *intimate*, and write $l \overset{\ell}{\sim} k$, if there exists a map $H : [0,1] \times [0,1] \rightarrow M$ such that:

(a) *H is smooth throughout* $[0,1] \times [0,1]$

(b) $\mathrm{rk}(DH_{(s,t)}) \leq 1$, $\forall (s,t) \in [0,1] \times [0,1]$

(c) *there exists* $0 < \epsilon < 1/2$ *such that*

$$\begin{aligned} 0 \leq s \leq \epsilon &\Rightarrow H(s,t) = l(t) \\ 1-\epsilon \leq s \leq 1 &\Rightarrow H(s,t) = k(t) \\ 0 \leq t \leq \epsilon &\Rightarrow H(s,t) = l(0) \\ 1-\epsilon \leq t \leq 1 &\Rightarrow H(s,t) = l(1) \end{aligned}$$

Such a map H will be called a *rank-one-homotopy.*

It follows from the definition that whenever $l \overset{\ell}{\sim} k$ then $l(0) = k(0) \wedge l(1) = k(1)$, i.e. intimate paths have the same endpoints. Reparametrizing and glueing rank-one-homotopies one can easily check that intimacy is an equivalence relation on $P^\infty(M)$.

Using measure theory methods it is possible to prove that the image of a smooth path cannot fill a 2-dimensional submanifold [F]. Then, according to condition (3) in the definition of a thin homotopy H (see section 2) and taking into account the inverse function theorem we may conclude that H satisfies condition (b) for rank-one-homotopies. Thus the intimacy relation is a weakening of the thin homotopy relation.

5 The group of loops

By reparametrizing and glueing rank-one-homotopies we may show that there is a perfect matching between intimacy and the usual operations performed with paths, i.e. given paths l, p, k, q such that $l \overset{\ell}{\sim} k$ and $p \overset{\ell}{\sim} q$ then $lp \overset{\ell}{\sim} kq$ whenever such products are defined, and also $l^{-1} \overset{\ell}{\sim} k^{-1}$.

Now let $p \in P^\infty(M)$. Consider the function $\alpha : [0,1] \rightarrow [0,1]$ defined by $\alpha(t) = 2t$ for $t \in [0,1/2]$ and by $\alpha(t) = 2 - 2t$ for $t \in]1/2, 1]$. Clearly $pp^{-1}(t) = p(\alpha(t))$. As mentioned in section 3 the function α may be slightly adjusted in a neighbourhood of $t = 1/2$ to become a smooth function α_∞. Because $t = 1/2$ is a sitting instant, taking this neighbourhood sufficiently small we still have $pp^{-1}(t) = p(\alpha_\infty(t))$. Again according to section 3 it is possible to construct a smoothly increasing function $\beta : [0,1] \to [0,1]$ such that $\beta(t) = 0$ for $t \in [0,1/3]$ and $\beta(t) = 1$ for $t \in [2/3,1]$. Using this function we construct $H : [0,1] \times [0,1] \to M$ given by $H(s,t) = p(\beta(s)\alpha_\infty(t))$. H is a rank-one-homotopy making pp^{-1} intimate with the trivial loop based at $p(0)$.

Next let $l \in P^\infty(M)$ be a path and $\theta : [0,1] \to [0,1]$ a smooth function such that $\theta(0) = 0$ and $\theta(1) = 1$. The reparametrized path $l \circ \theta$ still belongs to $P^\infty(M)$. Let us take again the function $\beta : [0,1] \to [0,1]$ introduced above and consider $H : [0,1] \times [0,1] \to M$ defined by $H(s,t) = l(t + \beta(s)(\theta(t) - t))$. H is a rank-one-homotopy making l and $l \circ \theta$ intimate.

Suppose $p, q, r \in P^\infty(M)$ are such that $(pq)r$ is defined. Let us consider the function $\theta : [0,1] \to [0,1]$ given by $\theta(t) = t/2$ for $0 \leq t \leq 1/2$, $\theta(t) = t - 1/4$ for $1/2 < t \leq 3/4$ and $\theta(t) = 2t - 1$ for $3/4 < t \leq 1$. θ has been chosen so that $(pq)r(\theta(t)) = p(qr)(t)$ over $[0,1]$. θ may be adjusted in small neighbourhoods of $1/2$ and $3/4$ to become a smooth function θ_∞ in such a way that the equality $(pq)r(\theta_\infty(t)) = p(qr)(t)$ still holds. The previous paragraph shows that $(pq)r$ and $p(qr)$ are intimate paths.

Suppose p is a path and T is the trivial path based at $p(1)$. It is possible to find a smooth reparametrization of p which makes this path intimate with pT. A similar conclusion is true for the trivial path based at $p(0)$.

To summarize we may say that the usual operations of composition, inversion and reparametrization of paths go over to the quotient by intimacy to give rise to an 'algebra' of classes of paths where associativity holds, trivial paths act as identities and 'tails' of the form pp^{-1} may be included or thrown away. Restricting the intimacy relation and the operations with paths to $\Omega^\infty(M)$ these results may be expressed in the following way:

Theorem 1 *The operations of composition and inversion defined on the loop space $\Omega^\infty(M)$ go over to the quotient by intimacy $\Omega^\infty(M)/\overset{\mathcal{L}}{\sim}$ to produce a group.*

This 'group of loops' will be denoted by $\mathcal{GL}^\infty(M)$. If we omit condition (b) in the definition of rank-one-homotopy the whole group construction that follows still works and the group emerging will be $\pi_1(M)$. To check this we just have to go through some technical results about approximations of continuous paths and homotopies by means of smooth ones. Such results may be found in a very detailed fashion in [G]. Since rank-one-homotopy is stronger than normal homotopy, a group epimorphism $\mathcal{C} : \mathcal{GL}^\infty(M) \to \pi_1(M)$ arises in a canonical way. When M has dimension 1 all homotopies become rank-one-homotopies so that this epimorphism becomes an isomorphism. Thus, in particular, $\mathcal{GL}^\infty(\mathbf{R}) = \{1\}$ and $\mathcal{GL}^\infty(\mathbf{S}^1) = \mathbf{Z}$.

6 Holonomy as a group morphism

Let G be a Lie group, $\pi : P \to M$ a principal G-bundle and ∇ a connection on this bundle. Let $H : [0,1] \times [0,1] \to M$ be a rank-one-homotopy between the loops $l, k \in \Omega^\infty(M)$. Consider the pull-back bundle $\pi' : H^*P \to [0,1] \times [0,1]$ equipped with the pull-back connection $H^*\nabla$ and the associated G-bundle morphism $\hat{H} : H^*P \to P$ as well as the commutative diagram built by these maps:

$$\begin{array}{ccc} H^*P & \xrightarrow{\hat{H}} & P \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ [0,1] \times [0,1] & \xrightarrow[H]{} & M \end{array}$$

Differentiating this diagram at a point $p \in H^*P$ we obtain a commutative diagram consisting of linear maps, which can be enlarged using the horizontal projection $h : T_{\hat{H}(p)}P \to H_{\hat{H}(p)}$ defined by ∇, and further enlarged using the restriction of the differential $D\pi(\hat{H}(p))$ to $H_{\hat{H}(p)}$, which is an isomorphism. The result is the following diagram, which is still commutative:

$$\begin{array}{ccccc} T_pH^*P & \xrightarrow{D\hat{H}(p)} & T_{\hat{H}(p)}P & \xrightarrow{h} & H_{\hat{H}(p)} \\ {\scriptstyle D\pi'(p)}\downarrow & & \downarrow{\scriptstyle D\pi(\hat{H}(p))} & \swarrow{\scriptstyle \cong} & \\ \mathbf{R}^2 & \xrightarrow[DH(\pi'(p))]{} & T_{\pi(\hat{H}(p))}M & & \end{array}$$

Let ω be the Lie-algebra-valued 1-form associated to ∇, defined on the tangent bundle TP, and let Ω be its curvature. Given $u, v \in T_pH^*P$ the curvature of $H^*\nabla$ is expressed by $(\hat{H}^*\Omega)_p(u,v) = \Omega_{\hat{H}(p)}(D\hat{H}(p)(u), D\hat{H}(p)(v)) = d\omega_{\hat{H}(p)}(h(D\hat{H}(p)(u)),$

$h(D\hat{H}(p)(v))) = 0$ where the last equality comes from the skew symmetry of $d\omega$ and the fact that the vectors $h(D\hat{H}(p)(u))$ and $h(D\hat{H}(p)(v))$ belong to the same subspace of $H_{\hat{H}(p)}$ with dimension ≤ 1. This last assertion arises from the commutativity of the diagram and from $\operatorname{rk} DH(\hat{\pi}(p)) \leq 1$. Since $H^*\nabla$ is flat and $[0,1] \times [0,1]$ is simply connected we may conclude that H^*P is a trivial bundle and $H^*\nabla$ is a trivial connection. Because of this, parallel transport along the boundary $\partial([0,1] \times [0,1])$ is trivial. This boundary is mapped by H onto $lTk^{-1}T$, where T is the trivial loop. Since $\hat{H}$ maps horizontal lifts with respect to $H^*\nabla$ into horizontal lifts with respect to ∇, we arrive at the conclusion that the parallel transport along $lTk^{-1}T \overset{\varrho}{\sim} lk^{-1}$ is trivial, that is, the holonomy of l equals the holonomy of k. Thus we have:

Theorem 2 *The holonomy of a connection ∇ may be expressed by means of a group morphism $\mathcal{H}_\nabla : \mathcal{GL}^\infty(M) \to G$.*

In [As2] a notion of equivalence between loops is proposed, known as Ashtekar equivalence, which in our framework may be expressed as follows: two loops $l, k \in \Omega^\infty(M)$ are equivalent iff $\mathcal{H}_\nabla(l) = \mathcal{H}_\nabla(k)$ for every connection ∇ on every principal G-bundle over M for every G. Theorem 2 shows that the intimacy relation is stronger than the Ashtekar relation.

7 An axiomatic definition of holonomy

We define a *smooth family of loops* to be a map $\psi : U \subseteq \mathbf{R}^n \to \Omega^\infty(M)$ defined on a open subset $U \subseteq \mathbf{R}^n$ such that the function ψ' defined on $U \times [0,1]$ by $\psi'(x,t) = \psi(x)(t)$ is smooth.

Definition 1 *A holonomy is a group morphism $\mathcal{H} : \mathcal{GL}^\infty(M) \to G$ such that for every smooth family of loops $\psi : U \subseteq \mathbf{R}^n \to \Omega^\infty(M)$ the composition*

$$U \subset \mathbf{R}^n \xrightarrow{\psi} \Omega^\infty(M) \xrightarrow{\text{proj}} \mathcal{GL}^\infty(M) \xrightarrow{\mathcal{H}} G$$

where proj *is the natural projection, is smooth throughout $U \subseteq R^n$.*

In [KN, vol.1, page 74] a proof can be found that every holonomy $\mathcal{H}_\nabla$ associated to a connection ∇ verifies the smoothness condition demanded in the previous definition.

We remark that a vast class of holonomies is given by group morphisms $h : \pi_1(M) \to G$ since the composition $h \circ \mathcal{C} : \mathcal{GL}^\infty(M) \to G$, where $\mathcal{C}$ is the canonical morphism $\mathcal{C} : \mathcal{GL}^\infty(M) \to \pi_1(M)$, defines a holonomy (the smoothness condition is easily checked). It is well known that such holonomies arise from *flat connections*.

Now the obvious question is: which holonomies come from general connections? Our answer will be Barrett's answer: all of them. This fact ensures that our definition of holonomy is indeed a proper definition of holonomy.

8 A property of holonomy

There is a result about holonomies which will be useful later and whose proof, due to Barrett, is remarkable:

Theorem 3 *The trivial loop extremizes every holonomy $\mathcal{H} : \mathcal{GL}^\infty(M) \to G$, that is, given a smooth family of loops $\psi : [0,1] \to \Omega^\infty(M)$ such that $\psi(0)$ is the trivial loop, then $\frac{d(\mathcal{H}\circ\psi)}{ds}(0) = 0$, where we also denote by ψ the composition of ψ with the natural projection* $\mathrm{proj} : \Omega^\infty(M) \to \mathcal{GL}^\infty(M)$.

Proof: In a neighbourhood of $s = 0$ all the loops $\psi(t)$ are contained in the domain of a chart for the manifold M. Thus we may restrict ourselves to the case $M = \mathbf{R}^n$ and $* = \vec{0} \in \mathbf{R}^n$. We may express ψ as

$$\psi(s)(t) = \psi'(s,t) = (\psi_1(s,t), \psi_2(s,t), \ldots, \psi_n(s,t))$$

and build from it the function $\varphi' : [0,1]^n \times [0,1] \to \mathbf{R}^n$ given by

$$\varphi'(s^1, s^2, \ldots, s^n, t) = (\psi_1(s^1,t), \psi_2(s^2,t), \ldots, \psi_n(s^n,t))$$

so that $\varphi(s^1,\ldots,s^n)(t) = \varphi'(s^1,\ldots,s^n,t)$ defines a smooth family of loops $\varphi : [0,1]^n \to \Omega^\infty(\mathbf{R}^n)$. Using the diagonal map $D : [0,1] \to [0,1]^n$ given by $D(s) = (s,s,\ldots,s)$ we can relate φ to ψ through $\psi = \varphi \circ D$. 'Forgetting' once more $\mathrm{proj} : \Omega^\infty(M) \to \mathcal{GL}^\infty(M)$ we may calculate:

$$\frac{d(\mathcal{H}\circ\psi)}{ds}(0) = \frac{d(\mathcal{H}\circ\varphi\circ D)}{ds}(0) =$$

$$= \sum_{i=1}^{n} \frac{\partial(\mathcal{H}\circ\varphi)}{\partial s^i}(\vec{0}) = 0$$

The final equality is due to the fact that all the partial derivatives in the sum are zero. Let us see why for the first partial derivative: to calculate $\frac{\partial(\mathcal{H}\circ\varphi)}{\partial s^1}(\vec{0})$ all that matters is the behaviour of the function $\mathcal{H}\circ\varphi$ on the first coordinate axis: $\mathcal{H}(\varphi(s^1,0,0,\ldots,0))$. But the loops $\varphi(s^1,0,\ldots,0)$ are given by $\varphi'(s^1,0,\ldots,0,t) = (\psi_1(s^1,t),0,\ldots,0)$ so they are intimate with the trivial loop. To check this consider the previously-used, smoothly increasing function $\beta : [0,1] \to [0,1]$ which equals 0 on $[0,1/3]$ and equals 1 on $[2/3,1]$ and consider the rank-one-homotopy given by $H(s,t) = (\beta(s)\psi_1(s^1,t),0,\ldots,0)$. Thus $\mathcal{H}\circ\varphi$ is constant on the first coordinate axis which implies that its first partial derivative at $\vec{0}$ is zero. □

9 Recentering a holonomy

Let $\mathcal{H} : \mathcal{GL}^\infty(M) \to G$ be a holonomy and let $m \in M$ be another fixed point. Denote by $\mathcal{GL}^\infty(M,m)$ the analogue of $\mathcal{GL}^\infty(M)$ but using $m \in M$ as a fixed point instead of $*$. With the help of a path $p \in P^\infty(M,*)$ such that $p(1) = m$, $\mathcal{H}$ may be *recentered* at $m \in M$ using the following procedure: we define $\tilde{\mathcal{H}} : \mathcal{GL}^\infty(M,m) \to G$ by $\tilde{\mathcal{H}}(l) = \mathcal{H}(plp^{-1})$. It is clear that $\tilde{\mathcal{H}}$ satisfies the smoothness requirement for being a holonomy. Thus we check the group morphism property: $\tilde{\mathcal{H}}(l)\tilde{\mathcal{H}}(k) = \mathcal{H}(plp^{-1})\mathcal{H}(pkp^{-1}) = \mathcal{H}(plp^{-1}(pkp^{-1}))$. According to section 5 $plp^{-1}(pkp^{-1})$ is intimate with $p(lk)p^{-1}$. But $\mathcal{H}(p(lk)p^{-1}) = \tilde{\mathcal{H}}(lk)$ so the morphism requirement is satisfied. Of course, if $\mathcal{H} = \mathcal{H}_\nabla$ is the holonomy of a connection ∇ then $\tilde{\mathcal{H}}$ is the holonomy of ∇ using $m \in M$ as the fixed base point and a fixed point in the fibre $\pi^{-1}(m)$ related to $p_0 \in \pi^{-1}(*)$ by belonging to the same horizontal lift of p (see section 2).

10 Reconstruction of the bundle

The construction we describe in this section has been used by Anandan [An] and Barrett [Ba]. Given a holonomy $\mathcal{H} : \mathcal{GL}^\infty(M) \to G$ we consider the following equivalence relation on $P^\infty(M,*) \times G$:

$$(p,g) \sim (q,h) \quad \text{iff} \quad p(1) = q(1) \ \wedge \ h = \mathcal{H}(qp^{-1})g$$

In order to check that this is indeed an equivalence relation we must remember that $\mathcal{H}$ is a group morphism, and use some rank-one-homotopies of the type mentioned in section 5.

We shall denote the equivalence class of $(p,g) \in P^\infty(M,*) \times G$ by $[(p,g)]$. We have got a quotient set $B = P^\infty(M,*) \times G/\sim$, a natural projection $\pi : B \to M$ given by $\pi([(p,g)]) = p(1)$ and a natural free right action of G on B, given by $[(p,g)]h = [(p,gh)]$, whose orbits are the fibers of π. Using the relation $\sim$ it is easy to check that both the projection and the right G-action are well defined.

Of course we are trying to build a principal G-bundle $\pi : B \to M$. What is missing is a family of *local trivializations* compatible with the given right G-action.

11 Local trivializations for the bundle

One can choose an atlas for M consisting of charts $\phi_\alpha : U_\alpha \to \mathbf{R}^n$ whose images are the unit open ball $B(\vec{0}, 1)$, where $n = \dim M$. Consider again the function $\beta : [0,1] \to [0,1]$ which equals 0 on $[0, 1/3]$, 1 on $[2/3, 1]$ and increases smoothly over $[0,1]$. The map $[0,1] \times B(\vec{0},1) \to B(\vec{0},1)$ which takes (s,x) to $\beta(s)x$ is a smooth inverse retraction of the ball $B(\vec{0},1)$ along straight lines. This inverse retraction may be transported to each U_α by means of the corresponding ϕ_α. This permits us to associate to each $m \in U_\alpha$ a path γ_α^m which starts at $\phi_\alpha^{-1}(\vec{0})$ and ends at $m \in U_\alpha$. The properties of the function β ensure that the path γ_α^m 'sits' at its two ends, whilst its parameter is in $[0, 1/3]$ and $[2/3, 1]$. Now we fix a path $p \in P^\infty(M, *)$ such that $p(1) = \phi_\alpha^{-1}(\vec{0})$. Notice that the elements of $\pi^{-1}(U_\alpha)$ are all of the form $[(p\gamma_\alpha^m, g)]$ so that the equality

$$T_\alpha([(p\gamma_\alpha^m, g)]) = (m, g)$$

establishes a bijection $T_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ that is compatible with the right-G action on B. Of course this local trivialization depends on the choice of the chart ϕ_α and the path p. Thus, in order to check the compatibility of local trivializations we must suppose we have U_β such that $U_\alpha \cap U_\beta \neq \emptyset$ and another trivialization $T_\beta : \pi^{-1}(U_\beta) \to U_\beta \times G$ built from $\phi_\beta : U_\beta \to B(\vec{0},1)$ and $q \in P^\infty(M, *)$ with $\phi_\beta(q(1)) = \vec{0}$:

$$T_\beta([(p\gamma_\alpha^m, g)]) = T_\beta([(q\gamma_\beta^m, \mathcal{H}(q\gamma_\beta^m(\gamma_\alpha^m)^{-1}p^{-1})g)]) =$$

$$= (m, \mathcal{H}(q\gamma_\beta^m(\gamma_\alpha^m)^{-1}p^{-1})g) \quad \text{for} \quad m \in U_\alpha \cap U_\beta$$

Hence for $m \in U_\alpha \cap U_\beta$ we have:

$$T_\beta \circ T_\alpha^{-1}(m, g) = (m, \mathcal{H}(q\gamma_\beta^m(\gamma_\alpha^m)^{-1}p^{-1})g)$$

Because of the smoothness properties of $\mathcal{H}$ we may conclude that $T_\beta \circ T_\alpha^{-1}$ is smooth. Thus the family of all T_α constitutes the required family of local trivializations.

Although we have initially followed the approach suggested by [Ba] for the construction of the bundle $\pi : B \to M$ the local trivializations we have used are closer to the ideas of *radial gauge* put forward by Lewandowski who also works out a reconstruction of connections from holonomies [L]. Lewandowski's approach is not, however, directly comparable to ours, since it is based on the weaker Ashtekar relation between loops (see the remark after theorem 2).

12 Lifting paths

For the reconstruction of the connection we will use the fact that the bundle $\pi : B \to M$ has a natural 'inbuilt' *lifting* of paths on $P^\infty(M,*)$, as Barrett points out in [Ba]. Our task is, however, made easier by our different choice of local trivializations.

We pick once again the by-now-familiar function $\beta : [0,1] \to [0,1]$. For each path $p \in P^\infty(M,*)$ and each $k \in [0,1]$ we define the path $p_k : [0,1] \to M$ by $p_k(t) = p(k\beta(t))$ for all $t \in [0,1]$. Clearly p_k belongs to $P^\infty(M,*)$ and its endpoint is $p_k(1) = p(k)$. p_0 is the trivial loop.

For each $p \in P^\infty(M,*)$ and each $g \in G$ we define the *lifting of* p to be the path $\hat{p}_g : [0,1] \to B$ given by

$$\hat{p}_g(k) = [(p_k, g)]$$

Clearly $\pi \circ \hat{p}_g = p$ so $\hat{p}_g$ is indeed a lift. To check the smoothness, given $a \in [0,1]$ and $m = p(a)$ let us consider the local trivialization T_α built from p_a and from an appropriate chart defined on an open neighbourhood U_α of $m = p(a)$:

$$T_\alpha(\hat{p}_g(k)) = T_\alpha([(p_k, g)]) = T_\alpha([(p_a\gamma_\alpha^{p(k)}, \mathcal{H}(p_a\gamma_\alpha^{p(k)}p_k^{-1})g)]) =$$

$$= (p(k), \mathcal{H}(p_a\gamma_\alpha^{p(k)}p_k^{-1})g)$$

The function of k given by this last expression is clearly smooth.

13 Lifting tangent vectors

Now suppose we have another path $q \in P^\infty(M,*)$ and $b \in [0,1]$ such that

$$q(b) = p(a) \quad \text{and} \quad \frac{dq}{dt}(b) = \frac{dp}{dt}(a)$$

We will show that the unique lift $\hat{q}_h$ such that $\hat{q}_h(b) = \hat{p}_g(a)$ satisfies

$$\frac{d\hat{q}_h}{dt}(b) = \frac{d\hat{p}_g}{dt}(a)$$

The condition $\hat{q}_h(b) = \hat{p}_g(a)$ implies $h = \mathcal{H}(q_b{p_a}^{-1})g$, as is easily checked. Using the same local trivialization as before we have:

$$\begin{aligned}
T_\alpha(\hat{p}_g(t)) &= (p(t), \mathcal{H}(p_a\gamma_\alpha^{p(t)}p_t^{-1})g) \\
T_\alpha(\hat{q}_h(t)) &= (q(t), \mathcal{H}(p_a\gamma_\alpha^{q(t)}q_t^{-1})h) \\
&= (q(t), \mathcal{H}(p_a\gamma_\alpha^{q(t)}q_t^{-1})\mathcal{H}(q_bp_a^{-1})g) \\
&= (q(t), \mathcal{H}(p_a\gamma_\alpha^{q(t)}q_t^{-1}q_bp_a^{-1})g)
\end{aligned}$$

Now we check the equality of the derivatives

$$\frac{d}{dt}_{|t=b} T_\alpha(\hat{q}_h(t)) \quad \text{and} \quad \frac{d}{dt}_{|t=a} T_\alpha(\hat{p}_g(t))$$

The assumptions about p and q ensure that the first components, i.e. the $T_m M$ components, of these derivatives are indeed equal. Next we look at the second components:

$$\frac{d}{dt}_{|t=b} \mathcal{H}(p_a \gamma_\alpha^{q(t)} q_t^{-1} q_b p_a^{-1}) \quad \text{and} \quad \frac{d}{dt}_{|t=a} \mathcal{H}(p_a \gamma_\alpha^{p(t)} p_t^{-1})$$

Notice that in both expressions we omitted the right-multiplication by $g \in G$ which is valid if we just want to check their equality.

Consider the holonomy $\tilde{\mathcal{H}}$ which is obtained from $\mathcal{H}$ by recentering at $p(a)$ with the help of the path p_a as explained in section 9:

$$\mathcal{H}(p_a \gamma_\alpha^{q(t)} q_t^{-1} q_b p_a^{-1}) = \tilde{\mathcal{H}}(\gamma_\alpha^{q(t)} q_t^{-1} q_b)$$

Let us start by supposing that $t > b$, i.e. we will only be concerned with the right derivative. We know from section 3 that the restriction of q_t to $[b, t]$ may be reparametrized to become a member of $P^\infty(M)$. Section 5 presents several rank-one-homotopies which may be used to check that $\gamma_\alpha^{q(t)} q_t^{-1} q_b$ is intimate with $\gamma_\alpha^{q(t)} q_{|[b,t]}^{-1}$, where the aforementioned reparametrization is assumed in the latter path. Thus we end up with $\tilde{\mathcal{H}}(\gamma_\alpha^{q(t)} q_{|[b,t]}^{-1})$. When $t \to b^+$, however, the loops $\gamma_\alpha^{q(t)} q_{|[b,t]}^{-1}$ shrink to a trivial loop so that we may apply theorem 3 to $\tilde{\mathcal{H}}$ which gives

$$\frac{d}{dt}_{|t=b^+} \tilde{\mathcal{H}}(\gamma_\alpha^{q(t)} q_{|[b,t]}^{-1}) = 0$$

This reasoning may be slightly modified in order to reach the same conclusion for the left derivative. Furthermore this whole reasoning which was used to study the second component of $T_\alpha(\hat{q}_h(t))$ may be repeated to reach a similar conclusion about the second component of $T_\alpha(\hat{p}_g(t))$:

$$\frac{d}{dt}_{|t=b} \mathcal{H}(p^a \gamma_\alpha^{q(t)} q_t^{-1} q_b p_a^{-1}) = \frac{d}{dt}_{|t=a} \mathcal{H}(p_a \gamma_\alpha^{p(t)} p_t^{-1}) = 0$$

Thus we are finally in a position to state that:

$$\frac{d}{dt}_{|t=b} T_\alpha(\hat{q}_h(t)) = \frac{d}{dt}_{|t=a} T_\alpha(\hat{p}_g(t))$$

Since T_α is a local trivialization it follows that:

$$\frac{d\hat{q}_h}{dt}(b) = \frac{d\hat{p}_g}{dt}(a)$$

14 Reconstructing the connection

The previous section shows that $\frac{d\hat{p}_g}{dt}(a)$ does not depend on the choice of p but just on the choice of $\frac{dp}{dt}(a)$. Thus the lifting construction gives rise to well defined maps

$$\Gamma_b : T_{\pi(b)}M \to T_bB$$

for all $b \in B$. In order to construct the connection we have the following propositions:

Proposition 4 *The family of linear spaces* $\{\mathrm{Im}\,\Gamma_b\}_{b\in B}$ *is a horizontal distribution on the bundle* $\pi : B \to M$.

Proof: In the previous section it was shown that $\frac{d}{dt}_{|t=a}T_\alpha(\hat{p}_g(t)) = (\frac{dp}{dt}(a), 0)$, which guarantees that Γ_b is linear and injective for all $b \in B$. Furthermore, since $\pi \circ \hat{p}_g = p$ we have $D\pi_{\hat{p}_g(a)}(\frac{d\hat{p}_g}{dt}(a)) = \frac{dp}{dt}(a)$ so $\mathrm{Ker}D\pi_b \bigcap \mathrm{Im}\Gamma_b = \{0\}$ for all $b \in B$. □

Proposition 5 *The family of linear spaces* $\{\mathrm{Im}\,\Gamma_b\}_{b\in B}$ *defines a G-connection on the bundle* $\pi : B \to M$.

Proof: It only remains to check the *G-invariance* of the spaces $\Gamma_{\hat{p}_g(a)}$ and the *smoothness* of their distribution. For the G-invariance it is enough to remark that for each $h \in G$ the liftings are related by $\hat{p}_{gh}(t) = \hat{p}_g(t)h$ so that when we differentiate both sides we obtain:

$$\frac{d\hat{p}_{gh}}{dt}(a) = DR_h(\hat{p}_g(a))(\frac{d\hat{p}_g}{dt}(a))$$

where $R_h : B \to B$ is right multiplication by $h \in G$.

To check the smoothness we take a local trivialization $T_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ built from a path $p \in P^\infty(M, *)$ and a local chart $\phi_\alpha : U_\alpha \to B(\vec{0}, 1)$ such that $\phi_\alpha(p(1)) = \vec{0}$. Suppose $\sigma(s) \in P^\infty(M, *)$ is a smooth family of paths, where $s = (s^1, s^2, \ldots, s^{n-1}) \in [0,1]^{n-1}$ is the family parameter and $n = \dim M$. Thus $\sigma' : [0,1]^n \to M$ defined by $\sigma'(s,t) = \sigma(s)(t)$ for all $(s,t) \in [0,1]^n$ is smooth. It is possible to build $\sigma(s)$ in such a way that σ' restricted to $[0,1]^{n-1} \times [1/3, 2/3]$ is an embedding into U_α, so that the velocity vector field $\frac{d}{dt}\sigma(s)(t)$ is a smooth vector field over a certain region of U_α. Let us consider the lifts of the paths $\sigma(s)$. These are given by $\Sigma(s,h)(t) = [(\sigma(s)_t, h)]$ for each $h \in G$. In order to see that $\Sigma(s,h)(t)$ defines a smooth map $\Sigma : [0,1]^{n-1} \times G \times [0,1] \to B$ which when restricted to $[0,1]^{n-1} \times G \times [1/3, 2/3]$ becomes an embedding we look at the local trivialization and calculate

$$T_\alpha(\Sigma(s,h)(t)) = T_\alpha([(\sigma(s)_t, h)]) =$$

$$= T_\alpha([(p\gamma_\alpha^{\sigma(s)(t)}, \mathcal{H}(p\gamma_\alpha^{\sigma(s)(t)}\sigma(s)_t{}^{-1})h)]) = (\sigma(s)(t), \mathcal{H}(p\gamma_\alpha^{\sigma(s)(t)}\sigma(s)_t{}^{-1})h)$$

where the last expression shows that $T_\alpha(\Sigma(s,h)(t))$ is an embedding for $t \in [1/3, 2/3]$. So $\frac{d}{dt}\Sigma(s,h)(t)$ is a *horizontal smooth vector field* over a region of $\pi^{-1}(U_\alpha)$. It is possible to build another $n-1$ such vector fields to obtain a *smooth field of frames* for the horizontal distribution. □

15 The correspondence between holonomies and connections

Combining the trivial loop $T \in \Omega^\infty(M)$ and the identity $e \in G$ gives rise to a particular point in the total space of our bundle: $[(T,e)] \in B$. Using this distinguished point to determine the holonomy $\mathcal{H}_\Gamma$ of Γ we find $\mathcal{H}_\Gamma = \mathcal{H}$, where $\mathcal{H}$ is the starting holonomy of section 14, which was our purpose.

Suppose now that the starting holonomy $\mathcal{H}$ is the holonomy $\mathcal{H}_\nabla$ of a connection ∇ defined on a principal G-bundle $\pi' : P \to M$, with the help of a fixed point $p_0 \in \pi'^{-1}(*)$. Consider the map

$$\phi : P^\infty(M,*) \times G \to P$$
$$(p,g) \mapsto \hat{p}(1)g$$

where $\hat{p}$ is the horizontal lift of p with respect to ∇ starting at p_0. ϕ descends to the quotient by the relation "$\sim$" defined in section 10, to define a map $\tilde{\phi} : B \to P$ which takes $[(T,e)]$ to p_0. Using the local trivializations for the reconstructed bundle presented in section 11 it is possible to check that $\tilde{\phi}$ is a diffeomorphism. Furthermore $\tilde{\phi}$ maps fibres into fibres preserving the basepoints and G-right actions so it is an M-isomorphism of principal G-bundles. By construction $\tilde{\phi}$ maps horizontal lifts into horizontal lifts so it must transform Γ into ∇. Thus we are finally in a position to say that the holonomy contains all the information about a bundle and a connection, as stated in more rigorous terms in the following theorem:

Theorem 6 *The reconstruction procedure of sections 10 to 14 establishes a one-to-one correspondence between holonomies and triples consisting of a principal G-bundle, a connection on this bundle and a point in the fiber over $*$, up to isomorphism.*

16 Final remarks

Results similar to the last theorem have, of course, been known for a long time (see for instance [K]). Our main contribution has been to facilitate the 'surgery' with paths by using smooth paths which 'sit' at their ends, and the intimacy relation between them. Furthermore we were able to bypass some of the subtle but nonetheless important difficulties in the Barrett [Ba] approach, and simplify the reconstruction of the connection, whilst taking over the main lines of his construction. Although the above-mentioned class of paths and loops may appear restrictive, it actually captures the essence of the problem, as far as holonomy is concerned, since any piecewise smooth path can be reparametrized, by means of a non-regular reparametrization, to become an element of this class, without changing the holonomy.

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