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AN AXIOMATIC SETUP FOR ALGORITHMIC HOMOLOGICAL ALGEBRA AND AN ALTERNATIVE APPROACH TO LOCALIZATION

MOHAMED BARAKAT AND MARKUS LANGE-HEGERMANN

*This paper is dedicated to our teacher Professor WILHELM PLESKEN
on the occasion of his 60th birthday.*

ABSTRACT. In this paper we develop an axiomatic setup for algorithmic homological algebra of ABELian categories. This is done by exhibiting all existential quantifiers entering the definition of an ABELian category, which for the sake of computability need to be turned into constructive ones. We do this explicitly for the often-studied example ABELian category of finitely presented modules over a so-called computable ring R , i.e., a ring with an explicit algorithm to solve one-sided (in)homogeneous linear systems over R . For a finitely generated maximal ideal \mathfrak{m} in a commutative ring R we show how solving (in)homogeneous linear systems over $R_{\mathfrak{m}}$ can be reduced to solving associated systems over R . Hence, the computability of R implies that of $R_{\mathfrak{m}}$. As a corollary we obtain the computability of the category of finitely presented $R_{\mathfrak{m}}$ -modules as an ABELian category, without the need of a MORA-like algorithm. The reduction also yields, as a by-product, a complexity estimation for the ideal membership problem over local polynomial rings. Finally, in the case of localized polynomial rings we demonstrate the computational advantage of our homologically motivated alternative approach in comparison to an existing implementation of MORA's algorithm.

1. INTRODUCTION

As finite dimensional constructions in linear algebra over a field k boil down to **solving (in)homogeneous linear systems** over k , the GAUSSIAN algorithm makes the whole theory perfectly computable, provided k itself is. Solving linear systems in GAUSSIAN form, i.e., in reduced echelon form, is a trivial task. And computing the GAUSSIAN form of a linear system is thus a major step towards its solution.

Homological algebra of module categories can be viewed as linear algebra over general rings. Hence, in analogy to linear algebra over a field one would expect that solving linear systems would play an important rule in making the theory computable. [BR08] introduced a data structure for additive functors of module categories useful for an efficient computer implementation. Solving linear systems was used to describe the calculus of such functors in a constructive way. Here we proceed in a more foundational manner. We show that solving linear systems is, as expected, the key to the complete computability of the category of finitely presented modules, *merely* viewed as an ABELian category [HS97, Rot09, Wei94].

We would like to thank the anonymous referee for valuable comments.

In Section 2 we list all the existential quantifiers entering the definition of an ABELIAN category. Turning all of them into algorithms for any given ABELIAN category establishes its computability. This abstract point of view widens the range of applicability of a computer implementation along these lines beyond the context of module categories.

Section 3 addresses the computability of the ABELIAN categories of finitely presented modules over so-called computable rings. A ring R is called **computable** if one can effectively solve (in)homogeneous linear systems over R (cf. Def. 3.2). Proposition 3.1 together with Theorem 3.4 show that, as expected, the computability of the ring together with some simple matrix operations indeed suffice to provide all the algorithms needed to make this category computable *as an ABELIAN category*.

One way to verify the computability of a ring is to find an appropriate substitute for the GAUSSIAN algorithm. Fortunately such substitutes exist for many rings of interest. Beside the well-known HERMITE normal form algorithm for principal ideal rings with computable gcd's, it turns out that appropriate generalizations of the classical GRÖBNER basis algorithm for polynomial rings [Buc06] provide the desired substitute for a wide class of commutative *and* noncommutative rings [Lev05, Rob06].

Although finding a substitute for the GAUSSIAN algorithm, which we will refer to as computing a “**distinguished basis**”¹, is the traditional way to solve linear systems over rings, it is only *one* mean to this end. Indeed, other means do exist:

Let $R_{\mathfrak{m}}$ be the localization of the commutative ring R at a finitely generated maximal ideal $\mathfrak{m} \triangleleft R$. Theorem 4.1 together with Corollary 4.2 show how computations in the ABELIAN category of finitely presented modules over the local ring $R_{\mathfrak{m}}$ can be reduced to computations over the global ring R . In particular, one can avoid computing distinguished bases over the local ring $R_{\mathfrak{m}}$. The idea is very simple. Elements of the local ring $R_{\mathfrak{m}}$ can be viewed as numerator-denominator pairs (n, d) with $n \in R$ and $d \in R \setminus \mathfrak{m}$. Likewise, $(r \times c)$ -matrices over $R_{\mathfrak{m}}$ can be viewed as numerator-denominator pairs (N, d) with $N \in R^{r \times c}$ and $d \in R \setminus \mathfrak{m}$. It is now easy to see that solving (in)homogeneous linear systems over $R_{\mathfrak{m}}$ can simply be done by solving associated systems over R , thus deducing the computability of $R_{\mathfrak{m}}$ from that of R .

In principle, MORA's algorithm, which provides a “distinguished basis”, can replace BUCHBERGER's algorithm for all sorts of computations over localized polynomial rings. This also seems to be the common practice. Nevertheless, a considerable amount of these computations only depend on the category of finitely presented modules over localized polynomial rings *merely* being ABELIAN. From this point of view we show how BUCHBERGER's algorithm suffices to carry out all such constructions and explain in §4.4 why our homologically motivated approach to localization of polynomial rings is computationally superior to an approach based on MORA's algorithm. However, MORA's algorithm remains indispensable when it comes to the computation of HILBERT series of modules over such rings, for example. Still, these modules are normally the outcome of huge homological computations, which often enough only become feasible through our alternative approach. SERRE's intersection formula is a typical example of this situation (cf. Example 6.3).

¹Basis in the sense of a generating set, and *not* in the sense of a free basis.

In Section 5 we will shortly describe our implementation. The examples in Section 6 illustrate the computational advantage of our alternative approach. An existing performant implementation of MORA's algorithm fails for these examples.

The paper suggests a specification which can be used to realize a constructive setting for the homological algebra of further concrete ABELian categories. Realizing this for other ABELian categories is work in progress.

2. BASIC CONSTRUCTIONS IN ABELIAN CATEGORIES

The aim of this section is to list the basic constructions of an ABELian category with enough projectives, which suffice to build all the remaining ones. In case these few basic constructions are computable, it follows that all further constructions become computable as well.

In the list we only want to emphasize the existential quantifiers, that need to be turned into constructive ones. We decided to suppress the universal properties needed to correctly formulate some of the points below, as we assume that they are well-known to the reader.

\mathcal{A} is a **category**:

- (1) For any object M there exists an **identity morphism** 1_M .
- (2) For any two composable morphisms ϕ, ψ there exists a **composition** $\phi\psi$.

\mathcal{A} is a category **with zero**:

- (3) There exists a **zero object** 0 .
- (4) For all objects M, N there exists a **zero morphism** 0_{MN} .

\mathcal{A} is an **additive** category:

- (5) For all objects M, N there exists an **addition**² $(\phi, \psi) \mapsto \phi + \psi$ in the ABELian group $\text{Hom}_{\mathcal{A}}(M, N)$.
- (6) For all objects M, N there exists a **subtraction** $(\phi, \psi) \mapsto \phi - \psi$ in the ABELian group $\text{Hom}_{\mathcal{A}}(M, N)$.
- (7) For all objects A_1, A_2 there exists a **direct sum** $A_1 \oplus A_2$.
- (8) For all pairs of morphisms $\phi_i : A_i \rightarrow M$, $i = 1, 2$ there exists a **coproduct morphism** $\langle \phi_1, \phi_2 \rangle : A_1 \oplus A_2 \rightarrow M$.
- (9) For all pairs of morphisms $\phi_i : M \rightarrow A_i$, $i = 1, 2$ there exists a **product morphism** $\{\phi_1, \phi_2\} : M \rightarrow A_1 \oplus A_2$.

\mathcal{A} is an **ABELian** category:

- (10) For any morphism $\phi : M \rightarrow N$ there exists a **kernel** $\ker \phi \xrightarrow{\kappa} M$, such that
- (11) for any morphism $\tau : L \rightarrow M$ with $\tau\phi = 0$ there exists a *unique lift* $\tau_0 : L \rightarrow \ker \phi$ of τ along κ , i.e., $\tau_0\kappa = \tau$ (cf. A.1.(k).(ii)).
- (12) For any morphism $\phi : M \rightarrow N$ there exists a **cokernel** $N \xrightarrow{\epsilon} \text{coker } \phi$, such that
- (13) for any morphism $\eta : N \rightarrow L$ with $\phi\eta = 0$ there exists a *unique colift* $\eta_0 : \text{coker } \phi \rightarrow L$ of η along ϵ , i.e., $\epsilon\eta_0 = \eta$ (cf. A.1.(c).(ii)).

²In fact, the addition can be recovered from the product, coproduct, composition, and identity morphisms. A direct description of the addition is nevertheless important for computational efficiency.

\mathcal{A} has **enough projectives**:

- (14) P is called projective if for each morphism $\phi : P \rightarrow N$ and each morphism $\epsilon : M \rightarrow N$ with $\text{im } \phi \leq \text{im } \epsilon$ there exists a **projective lift** $\phi_1 : P \rightarrow M$ of a ϕ along ϵ (cf. Def. A.2 and Remark A.3).
- (15) For each object M there exists a **projective hull** $\nu : P \twoheadrightarrow M$.

If for an ABELian category \mathcal{A} we succeed in making the above basic constructions computable, all further constructions which only depend on \mathcal{A} being ABELian will be computable as well.

Definition 2.1. Let \mathcal{A} be an ABELian category.

- (1) We say that \mathcal{A} is **computable as an ABELian category** if the existential quantifiers in (1)-(13) can be turned into constructive ones.
- (2) If additionally the existential quantifiers in (14)-(15) can be turned into constructive ones, then we say that \mathcal{A} is **computable as an ABELian category with enough projectives**.

[Bar09] details a construction of spectral sequences (of filtered complexes) only using the axioms of an ABELian category as detailed above. In particular, all arguments are based on operations on morphisms rather than chasing single elements.

To compute the derived functors of an additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ where \mathcal{A} does not have enough projectives resp. injectives one needs to provide a substitute for projective resp. injective resolutions. The abstract DE RHAM theorem suggests the use of so-called left (resp. right) F -acyclic resolutions (as used in [Har77, Prop. III.6.5], for example).

3. COMPUTABILITY IN ABELIAN CATEGORIES OF FINITELY PRESENTED MODULES

The previous section suggests a short path to ensure computability in an ABELian category. This section follows that path for the often-studied example of module categories.

From now on let $\mathcal{A} := R\text{-fmod}$ be the category of finitely presented *left* R -modules. The category $\text{fmod} - R$ of finitely presented right R -modules is treated analogously. In this section we will show how to make the basic operations of §2 computable. As customary from linear algebra the basic data structure for computations will be finite dimensional matrices over R .

Finitely presented R -modules are in particular finitely generated. Thus, a morphism in $R\text{-fmod}$ can be represented by a finite dimensional matrix, the so-called **representation matrix**, with entries in R .

A free object in $R\text{-fmod}$ is a free module of finite rank r , i.e., a module of the form $R^{1 \times r}$. And since every finitely generated module M is an epimorphic image of some $\nu : F_0 \twoheadrightarrow M$ with $F_0 = R^{1 \times r_0}$, it follows that $R\text{-fmod}$ even has **enough free objects**.

By definition of $R\text{-fmod}$ each object even admits an exact sequence $F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\nu} M$, $F_1 = R^{1 \times r_1}$ being free of finite rank. The morphism ∂_1 is called a **finite free presentation**

of M . If we denote by $\mathbf{M} \in R^{r_1 \times r_0}$ the matrix representing ∂_1 and call it **presentation matrix** of M , then ν induces an isomorphism from

$$\text{coker } \mathbf{M} := R^{1 \times r_0} / R^{1 \times r_1} \mathbf{M} = \text{coker}(R^{1 \times r_1} \xrightarrow{\mathbf{M}} R^{1 \times r_0}) = \text{coker } \partial_1$$

to M . The rows of \mathbf{M} are regarded as **relations** among the r_0 **generators** of M given by the residue classes of the unit row vectors in $R^{1 \times r_0} / R^{1 \times r_1} \mathbf{M}$ (cf. [BR08, §2], [GP08, Def. 2.1.23], [DL06, Def. 1.11]).

Denote by $R - \mathbf{fpres}$ the category of finite left R -presentations with objects being finite dimensional matrices over R , where one identifies two matrices $\mathbf{M} \in R^{r_1 \times r_0}$ and $\mathbf{M}' \in R^{r'_1 \times r_0}$ with the same number of columns to one object, if $R^{1 \times r_1} \mathbf{M} = R^{1 \times r'_1} \mathbf{M}'$, as R -submodules of $R^{1 \times r_0}$. The set $\text{Hom}_{R - \mathbf{fpres}}(\mathbf{M}, \mathbf{L})$ of morphisms between two objects $\mathbf{M} \in R^{r_1 \times r_0}$ and $\mathbf{L} \in R^{r'_1 \times r_0}$ is the set of all $r_0 \times r'_0$ -matrices φ over R with $R^{1 \times r_1} \mathbf{M} \varphi \leq R^{1 \times r'_1} \mathbf{L}$, where one identifies two matrices φ_1 and φ_2 to one morphism, if they induce the same R -module homomorphism from $\text{coker } \mathbf{M}$ to $\text{coker } \mathbf{L}$. Summing up:

Proposition 3.1. $R - \mathbf{fpres} \xrightarrow{\text{coker}} R - \mathbf{fpmo d}$ is an equivalence of categories.

The advantage of $R - \mathbf{fpres}$ is that it can directly be realized on a computer. Hence, describing the basic constructions of §2 in $R - \mathbf{fpres}$ makes the category $R - \mathbf{fpmo d}$ computable.

But we note that for $R - \mathbf{fpmo d}$ or equivalently $R - \mathbf{fpres}$ to be ABELian, cokernels and kernels of morphisms between finitely presented modules need to be finitely presented. This is obvious for cokernels but in general false for kernels:

Assumption (*):

From now on we assume that R is a ring for which the category $R - \mathbf{fpmo d}$ is ABELian.

Left NOETHERian rings are the most prominent rings satisfying this assumption.

3.1. Basic operations for matrices and computable rings. Let \mathbf{A} be an $r_1 \times r_0$ -matrix over R .

3.1.1. (Relative)SyzygiesGenerators. An $\mathbf{X} \in R^{r_2 \times r_1}$ is called a matrix of **generating syzygies (of the rows) of \mathbf{A}** if for all $\mathbf{x} \in R^{1 \times r_1}$ with $\mathbf{x}\mathbf{A} = 0$, there exists a $\mathbf{y} \in R^{1 \times r_2}$ such that $\mathbf{y}\mathbf{X} = \mathbf{x}$. The rows of \mathbf{X} are thus a generating set of the kernel of the map $R^{1 \times r_1} \xrightarrow{\mathbf{A}} R^{1 \times r_0}$. We write

$$\mathbf{X} = \text{SyzygiesGenerators}(\mathbf{A})$$

and say that \mathbf{X} is the most general solution of the homogeneous linear system $\mathbf{X}\mathbf{A} = 0$.

Further let \mathbf{L} be an $r'_1 \times r_0$ -matrix over R . $\mathbf{X} \in R^{r_2 \times r_1}$ is called a matrix of **relative generating syzygies (of the rows) of \mathbf{A} modulo \mathbf{L}** if the rows of \mathbf{X} form a generating set of the kernel of the map $R^{1 \times r_1} \xrightarrow{\mathbf{A}} \text{coker } \mathbf{L}$. We write

$$\mathbf{X} = \text{RelativeSyzygiesGenerators}(\mathbf{A}, \mathbf{L})$$

and say that X is the most general solution of the homogeneous linear system $XA + YL = 0$. This last system is of course equivalent³ to solving the homogeneous linear system $(X \ Y) \begin{pmatrix} A \\ L \end{pmatrix} = 0$ (cf. [BR08, §3.2]).

3.1.2. DecideZero(Effectively) and RightDivide. Further let B be an $r_2 \times r_0$ -matrix over R . Deciding the solvability and solving the inhomogeneous linear system $XA = B$ is equivalent to the construction of matrices N, T such that $N = TA + B$ satisfying the following condition: If the i -th row of B is a linear combination of the rows of A , then the i -th row of N is zero⁴. Hence the inhomogeneous linear system $XA = B$ is solvable (with $X = -T$), if and only if $N = 0$. We write

$$(N, T) = \text{DecideZeroEffectively}(A, B) \quad \text{and} \quad N = \text{DecideZero}(A, B).$$

In case $N = 0$ we write $X = \text{RightDivide}(B, A)$.

Rows of the matrices A and B can be considered as elements of the free module $R^{1 \times r_0}$. Deciding the solvability of the inhomogeneous linear system $XA = B$ for a single row matrix B is thus nothing but the **submodule membership problem** for the submodule generated by the rows of the matrix A . Finding a particular solution X (in case one exists) solves the submodule membership problem **effectively**.

As with relative syzygies we also consider a relative version. In case the inhomogeneous system $XA = B \pmod L$ is solvable, we denote a particular solution by

$$X = \text{RightDivide}(B, A, L).$$

This is equivalent to solving $(X \ Y) \begin{pmatrix} A \\ L \end{pmatrix} = B$. For details cf. [BR08, §3.1.1].

Definition 3.2. A ring R is called **left** (resp. **right**) **computable** if any finite dimensional inhomogeneous linear system $XA = B$ (resp. $AX = B$) over R is effectively solvable in the following sense: There exists *algorithms* computing $\text{SyzygiesGenerators}(A)$ and $\text{DecideZeroEffectively}(A, B)$. R is called **computable** if it is left and right computable.

In other words, a ring R is computable if one can *effectively solve (in)homogeneous linear systems over R* .

Remark 3.3. If the ring R is left computable then the categories $R - \mathbf{fpres}$ and (hence) $R - \mathbf{fpmod}$ are ABELian, and the Assumption (*) is satisfied.

We want to emphasize that all the free modules used in the constructions below are assumed to be given on a *free set of generators*. This is necessary since there is no known algorithm to decide whether a finitely presented module over a computable ring R is free or not. In practice this means that if we need to construct a free left R -module of rank r we simply present it by the empty matrix in $R^{0 \times r}$.

³In practice, however, one can often implement efficient algorithms to compute X without explicitly computing Y .

⁴So we do not require a “normal form”, but only a mechanism to decide if a row is zero modulo some relations.

3.2. Computability of the category of finite presentations.

Theorem 3.4. *Let R be a (left) computable ring and $\mathcal{A} := R\text{-fpres}$ the ABELian category of finite left R -presentations. Then \mathcal{A} is computable as an ABELian category with enough projectives.*

Proof. Using all of the vocabulary introduced so far we show how for the category $R\text{-fpres}$ the 15 operations for ABELian categories listed in §2 can be turned into algorithms.

In the following we denote $\mathbf{M} \in R^{r_1 \times r_0}$ and $\mathbf{N} \in R^{s_1 \times s_0}$ presentation matrices of M and N , respectively. I.e., $M := \text{coker } \mathbf{M}$ and $N := \text{coker } \mathbf{N}$.

$R\text{-fpres}$ is a **category**:

- (1) **IdentityMatrix**: The identity morphism 1_M of $M := \text{coker}(R^{1 \times r_1} \xrightarrow{\mathbf{M}} R^{1 \times r_0})$ is represented by the identity matrix $\mathbf{I}_{r_0} \in R^{r_0 \times r_0}$.
- (2) **Compose**: The composition of two composable morphisms ϕ, ψ represented by the matrices \mathbf{A}, \mathbf{B} is represented by the matrix product \mathbf{AB} .

$R\text{-fpres}$ is a category **with zero**:

- (3) A zero object 0 is presented by an empty matrix in $R^{0 \times 0}$.
- (4) **ZeroMatrix**: The zero morphism 0_{MN} for pairs of objects $M := \text{coker}(R^{1 \times r_1} \xrightarrow{\mathbf{M}} R^{1 \times r_0})$, $N := \text{coker}(R^{1 \times s_1} \xrightarrow{\mathbf{N}} R^{1 \times s_0})$ is represented by the zero matrix $\mathbf{0}_{r_0, s_0} \in R^{r_0 \times s_0}$.

$R\text{-fpres}$ is an **additive** category:

- (5) **AddMat**: The addition of two morphisms $\phi, \psi : M \rightarrow N$ represented by the matrices \mathbf{A}, \mathbf{B} is represented by the matrix sum $\mathbf{A} + \mathbf{B}$.
- (6) **SubMat**: The difference of two morphisms $\phi, \psi : M \rightarrow N$ represented by the matrices \mathbf{A}, \mathbf{B} is represented by the matrix subtraction $\mathbf{A} - \mathbf{B}$.
- (7) **DiagMat**: The direct sum of two objects M and N is presented by the block diagonal matrix $\begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}$.
- (8) **UnionOfRows**: The coproduct morphism $\langle \phi, \psi \rangle$ of two morphisms $\phi : M \rightarrow L$ and $\psi : N \rightarrow L$ represented by the matrices \mathbf{A} and \mathbf{B} is represented by the stacked matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$.
- (9) **UnionOfColumns**: The product morphism $\{\phi, \psi\}$ of two morphisms $\phi : L \rightarrow M$ and $\psi : L \rightarrow N$ represented by the matrices \mathbf{A} and \mathbf{B} is represented by the augmented matrix $(\mathbf{A} \ \mathbf{B})$.

$R\text{-fpres}$ is an **ABELian** category:

- (10) **RelativeSyzygiesGenerators**: To compute the kernel $\ker \phi \xrightarrow{\kappa} M$ of a morphism $\phi : M \rightarrow N$ represented by a matrix \mathbf{A} we do the following: First compute $\mathbf{X} = \text{RelativeSyzygiesGenerators}(\mathbf{A}, \mathbf{N})$, the matrix representing κ . Then $\ker \phi$ is presented by the matrix $\mathbf{K} = \text{RelativeSyzygiesGenerators}(\mathbf{X}, \mathbf{M})$.

- (11) **DecideZeroEffectively**: Let $\tau : L \rightarrow M$ be a morphism represented by a matrix \mathbf{B} and $\kappa : K \hookrightarrow M$ a monomorphism represented by a matrix \mathbf{A} with $\tau\phi = 0$ for $\phi = \text{coker } \kappa$. Then the matrix $\mathbf{X} = \text{RightDivide}(\mathbf{B}, \mathbf{A}, \mathbf{M})$ is a representation matrix for $\tau_0 : L \rightarrow K$, the lift of τ along κ . It is an easy exercise to see that \mathbf{X} indeed represents a *morphism* (cf. [BR08, 3.1.1, case (2)]).
- (12) **UnionOfRows & IdentityMatrix**: The cokernel module $\text{coker } \phi$ of a morphism $\phi : M \rightarrow N = \text{coker } \mathbf{N}$ represented by the matrix \mathbf{A} is presented by the stacked matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{N} \end{pmatrix}$. The natural epimorphism $N \xrightarrow{\epsilon} \text{coker } \phi$ is represented by the identity matrix $\mathbf{I}_{s_0} \in R^{s_0 \times s_0}$.
- (13) Without loss of generality assume that the cokernel module $C = \text{coker } \phi$ is presented according to (12), with \mathbf{I}_{s_0} the representation of the natural epimorphism $\epsilon : N \rightarrow C$. Further let $\eta : N \rightarrow L$ be a morphism represented by \mathbf{B} . Then the colift $\eta_0 : C \rightarrow L$ along ϵ is again given by the matrix \mathbf{B} .

$R - \mathbf{fpres}$ has enough free objects:

- (14) **DecideZeroEffectively**: Let F be a free R -module presented by an empty matrix, i.e., F is given on a set of *free* generators. Further let $\phi : F \rightarrow N$ and $\epsilon : M \rightarrow N$ be morphisms represented by the matrices \mathbf{B} and \mathbf{A} , respectively. The image condition $\text{im } \phi \leq \text{im } \epsilon$ guarantees the existence of the matrix $\mathbf{X} = \text{RightDivide}(\mathbf{B}, \mathbf{A}, \mathbf{N})$, which is a representation matrix of a **free lift** $\phi_1 : F \rightarrow M$ along ϵ (cf. [BR08, 3.1.1, case (1)]).
- (15) **IdentityMatrix**: A free hull $\nu : F \rightarrow M$ of M is given by $F = \text{coker } \mathbf{F}$ with $\mathbf{F} = \mathbf{0} \in R^{0 \times r_0}$ and ν is represented by the identity matrix \mathbf{I}_{r_0} .

□

In the constructive setting we seek, it is necessary to decide if a module is zero and if two morphisms are equal. To decide if $M = \text{coker } \mathbf{M}$ is zero check whether $\text{DecideZero}(\mathbf{I}_{r_0}, \mathbf{M}) = 0$. To decide the equality of two morphisms $\phi, \psi : M \rightarrow N = \text{coker } \mathbf{N}$ represented by \mathbf{A} and \mathbf{B} check whether $\text{DecideZero}(\mathbf{A} - \mathbf{B}, \mathbf{N}) = 0$. This, in turn, enables deciding properties like monic, epic, exactness of two composable morphisms, etc.

As R -modules are sets it makes sense to compute in $M = \text{coker } \mathbf{M}$. This, in turn, requires deciding equality of two elements in $m, m' \in M$, represented by two rows $\mathbf{m}, \mathbf{m}' \in R^{1 \times r_0}$, respectively. This is again achieved by test whether $\text{DecideZero}(\mathbf{m} - \mathbf{m}', \mathbf{M}) = 0$.

3.3. Closed symmetric monoidal ABELIAN categories. The category of R -modules over a *commutative* ring R admits further constructions. The tensor product $M \otimes_R N$ of two R -modules M, N turns the $R - \mathbf{fmod}$ into a symmetric monoidal category. It is even a closed symmetric monoidal category with the homomorphism module $\text{Hom}_R(M, N)$ as an internal Hom object.

For the constructibility of the tensor product and its derived functors Tor_i^R in $R - \mathbf{fpres}$ see [GP08, Example. 7.1.5] or [DL06, Problem 4.7]. For the (internal) Hom module and the higher extension functors Ext_R^c see [GP08, Example 2.1.26], [DL06, Problems 4.5, 4.6],

[KR00, Thm. 3.3.15]. The morphism part of these and other functors is systematically dealt with in [BR08].

3.4. Free and projective modules. Whereas a finitely presented module is free on free generators if and only if its presentation matrix is zero, deciding the freeness of a finitely presented module over a computable ring, let alone computing a free basis, can in general be highly non-trivial. Deciding projectiveness is often easier than deciding freeness: Let $\nu : F_0 \rightarrow M$ be a free presentation of the R -module M . It follows that M is projective if and only if ν admits a section $\sigma : M \hookrightarrow F_0$ (i.e., $\sigma\nu = \text{id}_M$). Finding the section σ for a finitely and freely presented module $M \xleftarrow{\nu} F_0 \xleftarrow{M} F_1$ leads to solving a two-sided inhomogeneous linear system⁵ over R , which can of course be brought to a one-sided inhomogeneous linear system if R is commutative. Hence, testing projectiveness of finitely presented modules over commutative computable rings is constructive [ZL02]. Another simple exercise is to see that an R -module is projective if and only if $\text{Ext}_R^1(M, K_1(M)) = 0$, where $K_1(M)$ is the first syzygy module of M . This can also be turned into an algorithm for commutative computable rings as outlined in §3.3. Without the commutativity assumption SERRE’s Remark [Ser55] states that a module admitting a *finite free* resolution is projective if and only if it is stably free. Shortening the finite free resolution [Lam99, Prop. 5.11] yields a simple proof that can be turned into an algorithm for computing the projective dimension of M (and hence to decide projectiveness) whenever M is a left (resp. right) module over the left (resp. right) computable ring R and a finite free resolution of M is *constructible* [QR07, Algorithm 1]. For a not necessarily commutative ring R with finite global dimension there is yet another approach based on AUSLANDER’s degree of torsion-freeness [AB69], which involves the higher extension modules with values in R of the so-called AUSLANDER dual module of M . This approach is constructive if R is (left *and* right) computable and the finite global dimension of R is explicitly known (cf. [CQR05, Thm. 7]).

The QUILLEN-SUSLIN Theorem states that a polynomial ring $k[x_1, \dots, x_n]$ over a principal ideal domain k is HERMITE, i.e., every stably free $k[x_1, \dots, x_n]$ -module is free. Algorithms were given in [LS92, LW00, GV02] and implementations in [FQ07] and [CQR07]. A constructive version of STAFFORD’s Theorem [Sta78] offers a way to decide freeness and compute a free basis of finitely presented stably free modules over the WEYL algebra $A_n(k)$ and the rational WEYL algebra $B_n(k)$, where k is a computable field of characteristic 0 [QR07]. The JACOBSON normal form [Coh85] offers an alternative algorithm for $B_1(k)$. An implementation can be found in the Maple package *Janet* [BCG+03].

Computability stands for deciding zero and computing syzygies. As mentioned in the Introduction, the axiomatic approach pursued so far underlines the conceptual importance of solving (in)homogeneous linear systems rather than computing a “distinguished basis”, the latter being the traditional way to solve such systems.

⁵ $\mathbf{X} + \mathbf{Y}\mathbf{M} = \mathbf{Id}$, $\mathbf{M}\mathbf{X} = 0$, where \mathbf{X} is a square matrix representing σ and \mathbf{Y} another unknown matrix.

4. COMPUTING OVER LOCAL COMMUTATIVE RINGS

This section provides a simple alternative approach to solve (in)homogeneous linear systems over localizations of commutative computable rings at maximal ideals. The simplicity lies in avoiding the computation of distinguished bases over these local rings. In particular, for localized polynomial rings this approach offers a way to circumvent MORA's basis algorithm. More precisely:

For a commutative computable ring R with a finitely generated maximal ideal \mathfrak{m} we show in Lemma 4.3 and Proposition 4.5 how to reduce solving linear systems over the local ring $R_{\mathfrak{m}}$ to solving linear systems over the "global" ring R . Thus, the computability of R implies that of $R_{\mathfrak{m}}$, and as a corollary, the computability of $R_{\mathfrak{m}} - \mathbf{fpres}$ as an ABELian category.

This reduction leads to a result in complexity analysis of local polynomial rings, formulated in §4.3. Finally, in §4.4 we compare our approach to MORA's algorithm in the case of localized polynomial rings.

Let $1 \in S \subseteq R$ be a multiplicatively closed subset of the commutative ring R . Recall, the **localization** of R at S is defined by $S^{-1}R := \{\frac{r}{s} \mid r \in R, s \in S\} / \sim$, where $\frac{r}{s} \sim \frac{r'}{s'} \Leftrightarrow 0 \in (rs' - sr') \cdot S$. The localization $S^{-1}M$ of an R -module M is defined by $S^{-1}R \otimes_R M$ and the localization $S^{-1}\phi$ of a morphism ϕ maps $\frac{m}{s} \in S^{-1}M$ to $\frac{\phi(m)}{s}$. Note that localization is an **exact functor**. For a prime ideal $\mathfrak{p} \triangleleft R$ the set $S := R \setminus \mathfrak{p}$ is multiplicatively closed and the localization $M_{\mathfrak{p}}$ at \mathfrak{p} is defined as $S^{-1}M$. In this section we will only treat the case when the prime ideal $\mathfrak{p} = \mathfrak{m}$ is **maximal**.

4.1. Computability in the category of finite presentations over localized rings.

Theorem 4.1. *Let R be a commutative computable ring and $\mathfrak{m} = \langle m_1, \dots, m_k \rangle$ a finitely generated maximal ideal in R . Then $R_{\mathfrak{m}}$ is a computable ring.*

The proof of this theorem will be given in §4.2. We first draw the following conclusion:

Corollary 4.2. *Let R and \mathfrak{m} as in Theorem 4.1. Then $R_{\mathfrak{m}} - \mathbf{fpres}$ is ABELian and is computable as an ABELian category with enough projectives.*

Proof. Objects and morphisms in $R_{\mathfrak{m}} - \mathbf{fpres}$ are given by finite matrices with entries in the localized ring $R_{\mathfrak{m}}$, being fractions of elements of R . Each such matrix can be rewritten as a fraction $\frac{\mathbf{A}}{s}$ with a numerator matrix \mathbf{A} over R and a single element $s \in S := R \setminus \mathfrak{m}$ as a common denominator⁶. Using this simplified data structure for matrices over $R_{\mathfrak{m}}$ is mandatory for the proof of Theorem 4.1. The computability of $R_{\mathfrak{m}}$, essential for the constructions (10), (11), and (14), is the statement of Theorem 4.1. We still need to go through the remaining basic constructions listed in the proof of Theorem 3.4 and adapt the required matrix operations to the new data structure:

Points (1)-(4) are covered by taking the identity matrix (with 1 as denominator), composing the matrices $\frac{\mathbf{A}}{s}$ and $\frac{\mathbf{B}}{t}$ to $\frac{\mathbf{AB}}{st}$, taking an empty matrix, and taking the zero matrix (with 1 as denominator), respectively. Points (5)-(9), stemming from the axioms of an

⁶This leads to a simple data structure for matrices over $R_{\mathfrak{m}}$, which will prove advantageous from the standpoint of a computer implementation (cf. §5.1).

additive category, are also easily seen to reduce to the corresponding constructions over the global ring R after writing the involved pairs of matrices with common denominators. The presentation matrix and representation matrix of the corresponding natural epimorphism of the cokernel (12) are again given by stacking matrices after writing them with a common denominator and by taking the identity matrix, respectively. The colift (13) as described above was trivial anyway. The presentation matrix of a free hull (15) of a module M generated by r_0 elements is again given by an empty matrix with r_0 columns, while the identity matrix still represents the natural epimorphism. \square

4.2. Proof of Theorem 4.1. Let R be a commutative computable ring. Further let $\mathfrak{m} = \langle m_1, \dots, m_k \rangle$ a finitely generated maximal ideal in R . For the computability of $R_{\mathfrak{m}}$ we need to compute generating sets of syzygies and to (effectively) solve the submodule membership problem (cf. Def. 3.2). This is the content of Lemma 4.3 and Proposition 4.5.

4.2.1. (Relative)SyzygiesGenerators. A matrix of generating syzygies over R is also a matrix of generating syzygies over $R_{\mathfrak{m}}$:

Lemma 4.3 (Syzygies). *Let $\mathbf{A} \in R_{\mathfrak{m}}^{m \times n}$. Rewrite $\mathbf{A} = \frac{\tilde{\mathbf{A}}}{\tilde{a}}$ with $\tilde{\mathbf{A}} \in R^{m \times n}$ and $\tilde{a} \in R \setminus \mathfrak{m}$. If $\tilde{\mathbf{X}} \in R^{k \times m}$ is a matrix of generating syzygies for $\tilde{\mathbf{A}}$, then the matrix $\mathbf{X} := \frac{\tilde{\mathbf{X}}}{\tilde{a}}$ is a matrix of generating syzygies of \mathbf{A} .*

Proof. Starting from an exact sequence $R^{1 \times k} \xrightarrow{\tilde{\mathbf{X}}} R^{1 \times m} \xrightarrow{\tilde{\mathbf{A}}} R^{1 \times n}$ exactness of the localization functor yields an exact sequence $R_{\mathfrak{m}}^{1 \times k} \xrightarrow{\tilde{\mathbf{X}}/1} R_{\mathfrak{m}}^{1 \times m} \xrightarrow{\tilde{\mathbf{A}}/1} R_{\mathfrak{m}}^{1 \times n}$. Since multiplication with $\frac{1}{\tilde{a}}$ is an isomorphism, the sequence $R_{\mathfrak{m}}^{1 \times k} \xrightarrow{\tilde{\mathbf{X}}/1} R_{\mathfrak{m}}^{1 \times m} \xrightarrow{\mathbf{A}} R_{\mathfrak{m}}^{1 \times n}$ is also exact. \square

This lemma is valid more generally for any multiplicatively closed set S . Of course, the result of the algorithm can have redundant generators and thus might be a non-minimal set of generating syzygies.

Remark 4.4. NAKAYAMA's Lemma implies that a finitely presented $R_{\mathfrak{m}}$ -module $M = \text{coker } \mathbf{M}$ is given on a minimal set of generators, if and only if its presentation matrix \mathbf{M} is unit-free (cf. [CLO05, Chap. 5, Prop. 4.3]). An element $\frac{d}{n} \in R_{\mathfrak{m}}$ is a unit iff the numerator d is not contained in \mathfrak{m} , which is an ideal membership problem in R . Some elementary matrix transformations can now be used to construct unit-free presentation matrices (see for example [BR08, §2]). Another easy consequence of NAKAYAMA's Lemma is the characterization of minimal resolutions over $R_{\mathfrak{m}}$ as the unit-free ones [Eis95, §19.1]. Being able to detect units in the computed syzygies we can use a standard procedure⁷ to compute a unit-free and hence minimal resolutions over $R_{\mathfrak{m}}$.

4.2.2. DecideZero(Effectively). The algorithm for effectively deciding zero is a bit more involved. As seen in the proof of Lemma 4.3, denominators of matrices can be omitted since multiplication with them is an isomorphism. So without loss of generality we assume

⁷Cf. [BR08, §3.2.1], for example.

all denominators 1. To ease the notation let $\mathbf{m} := \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} \in R^{k \times 1}$ denote the column of generators of the maximal ideal \mathbf{m} .

Proposition 4.5 (Submodule membership). *Let $\mathbf{A} = \frac{\tilde{\mathbf{A}}}{1} \in R_{\mathbf{m}}^{m \times n}$ and $\mathbf{b} = \frac{\tilde{\mathbf{b}}}{1} \in R_{\mathbf{m}}^{1 \times n}$ with numerator matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{b}}$ over R . There exists a row matrix $\mathbf{t} \in R_{\mathbf{m}}^{1 \times m}$ with $\mathbf{t}\mathbf{A} + \mathbf{b} = 0$ iff there exists a matrix $\tilde{\mathbf{s}} \in R^{1 \times (m+k)}$ satisfying*

$$(1) \quad \tilde{\mathbf{s}} \begin{pmatrix} \tilde{\mathbf{A}} \\ \mathbf{m}\tilde{\mathbf{b}} \end{pmatrix} + \tilde{\mathbf{b}} = 0.$$

Proof. Write $\tilde{\mathbf{s}} = (\mathbf{y} \quad \mathbf{z})$ with $\mathbf{y} \in R^{1 \times m}$, $\mathbf{z} \in R^{1 \times k}$.

If $\tilde{\mathbf{s}}$ and thus \mathbf{y} and \mathbf{z} exist, we simply set $t := \mathbf{z}\mathbf{m} + 1 \notin \mathbf{m}$ and $\mathbf{t} := \frac{\mathbf{y}}{t}$. For the converse implication write $\mathbf{t} = \frac{\tilde{\mathbf{t}}}{t}$ with $\tilde{\mathbf{t}} \in R^{1 \times m}$ and $t \in R \setminus \mathbf{m}$. Since $t \notin \mathbf{m}$ it has an inverse y modulo \mathbf{m} , i.e., there exist $z_1, \dots, z_k \in R$ with $yt = z_1m_1 + \dots + z_km_k + 1$. We conclude

$$\begin{aligned} \mathbf{t}\mathbf{A} + \mathbf{b} = 0 &\Leftrightarrow \tilde{\mathbf{t}}\tilde{\mathbf{A}} + t\tilde{\mathbf{b}} = 0 \\ &\Leftrightarrow y\tilde{\mathbf{t}}\tilde{\mathbf{A}} + (z_1m_1 + \dots + z_km_k)\tilde{\mathbf{b}} + 1 \cdot \tilde{\mathbf{b}} = 0 \\ &\Leftrightarrow y\tilde{\mathbf{A}} + \mathbf{z}\tilde{\mathbf{b}} + \tilde{\mathbf{b}} = 0 \end{aligned}$$

for $\mathbf{y} := y\tilde{\mathbf{t}}$ and $\mathbf{z} := (z_1 \ \dots \ z_k) \in R^{1 \times k}$. □

This proof is *constructive*. Another short but nonconstructive proof was suggested by our colleague FLORIAN EISELE using NAKAYAMA's Lemma, which is also the most illuminating way to interpret Formula (1). The proof given here is indeed a reduction to the effective submodule membership problem over R . We don't see a way to generalize this proof to non-maximal prime ideals.

Note that we formulate the proposition for a single row matrix \mathbf{b} . For a multi-row matrix \mathbf{B} simply stack the results of the proposition applied to each row \mathbf{b} of \mathbf{B} . Contrary to GRÖBNER basis methods, the proof of the proposition does *not* provide a normal form⁸. Nevertheless the proof yields an *effective* solution of the submodule membership problem (see §3.1.2). Further note that at no step we need to compute any kind distinguished basis over the local ring. The advantages but also the drawbacks of avoiding such a basis will be discussed in §4.4.

Example 4.6. Let $R = k[x]$ for an arbitrary field k and $\mathbf{m} = \langle x \rangle$. We want to compute $\mathbf{t} \in R_{\mathbf{m}}^{1 \times 1}$ with $\mathbf{t}\mathbf{A} + \mathbf{b} = 0$ for $\mathbf{A} = \tilde{\mathbf{A}} := x - x^2$ and $\mathbf{b} = \tilde{\mathbf{b}} := x$ regarded as 1×1 -matrices. In this case the gcd of $\tilde{\mathbf{A}}$ and $\mathbf{m}\tilde{\mathbf{b}}$ coincides with $\tilde{\mathbf{b}}$ and the extended EUCLIDIAN algorithm⁹

⁸This is an instance where the name `DecideZero` makes more sense than `Reduce` or `NormalForm`.

⁹This is a special case of the HERMITE normal form algorithm applied to a one-column matrix.

directly yields the desired coefficients y and z :

$$\begin{aligned} \underbrace{(-1)}_y \cdot \underbrace{(x - x^2)}_{\tilde{A}} + \underbrace{(-1)}_z \cdot \underbrace{(x \cdot x)}_{\tilde{m}} &= \underbrace{-x}_{-\tilde{b}} \Leftrightarrow \underbrace{(-1)}_y \cdot \underbrace{(x - x^2)}_{\tilde{A}} + \underbrace{(-x + 1)}_{t=zm+1} \underbrace{x}_{\tilde{b}} = 0 \\ &\Leftrightarrow \underbrace{\frac{1}{x-1}}_{t=\frac{y}{t}} \underbrace{(x - x^2)}_{\tilde{A}} + \underbrace{x}_{\tilde{b}} = 0 \end{aligned}$$

4.3. Complexity estimation for local polynomial rings. As E. MAYR has shown in [May89], the ideal membership problem over the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ is exponential space complete. Proposition 4.5 implies a result about the complexity of computations over the local polynomial ring $\mathbb{Q}[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle}$: It gives a polynomial time reduction from the ideal membership problem over the localized polynomial ring $\mathbb{Q}[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle}$ to the ideal membership problem over the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$.

Corollary 4.7. *The ideal membership problem over $\mathbb{Q}[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle}$ is solvable in exponential space.*

4.4. A comparison with MORA's algorithm. For a polynomial ring $R := k[x_1, \dots, x_n]$ over a computable field k MORA's algorithm [Mor82] makes the category $R_{\mathfrak{m}}\text{-fpmod}$ for $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ computable. The algorithms suggested in Lemma 4.3 and Proposition 4.5 describe our alternative approach to establish the computability of $R_{\mathfrak{m}}$, whereas MORA's algorithm can be seen as the classical way to this end.

Except for the use of a different reduction method capable of dealing with a local term ordering¹⁰, MORA's algorithm proceeds exactly in the same way as BUCHBERGER's algorithm, where of course leading terms and s -polynomial depend on the chosen term ordering. And since MORA's algorithm computes, as a by product, the leading ideal/module one can read off HILBERT series. MORA's different reduction method minimizes the so-called "ecart" (which measures the distance of a polynomial from being homogeneous) by a kind of elimination procedure, which can be very expensive. This makes MORA's reduction slower than BUCHBERGER's. For a modern treatment of the theory of local standard bases in polynomial rings we refer to [GP08, §1.6, 1.7]. A free implementation can be found in SINGULAR [GPS09].

Hence, we argue that our approach to localization which only requires an implementation of BUCHBERGER's algorithm is computationally superior to a comparable implementation of MORA's algorithm.

MORA's slower reduction is even dwarfed by yet another major issue: Unlike for small input, we experienced that MORA's algorithm does not scale well enough when applied to large matrices. In comparison with the reduction given by Proposition 4.5, we observed that MORA's algorithm creates larger units in $R_{\mathfrak{m}}$, which in subsequent computations have to be interpreted as denominators. And when writing several matrices over $R_{\mathfrak{m}}$ with a common denominator, these large units blow up the entries of the numerator matrices. Clearly, this makes succeeding computations much harder.

¹⁰Local term ordering implies $x_i < 1$ for all i .

Remark 4.8. MORA’s algorithm is still indispensable when it comes to computing HILBERT series (and related invariants), which cannot be computed using our approach. Intermediate homological computations tend to become huge, even if the final result is typically *much* smaller. Our approach is thus suited to get through the intermediate steps, while MORA’s algorithm can then be applied to the smaller result, e.g., to obtain invariants. This is demonstrated in Example 6.3.

5. IMPLEMENTATION AND DATA STRUCTURES

As mentioned in the Introduction, this paper suggests a specification for implementing homological algebra of ABELIAN categories (cf. [CS07]). This specification is realized in the `homalg` project [hom17].

We found the programming language of GAP4 ideally suited to realize this specification. GAP4 provides object oriented and (to some useful extent) functional programming paradigms, classical method selection, multi-dispatching, and last but not least so-called immediate and true-methods, which are extensively used to teach GAP4 how to avoid unnecessary computations by applying mathematical reasoning. All these capabilities build upon a type-system, which is as simple as possible and as sophisticated as needed for the purposes of high level computer algebra [BL98].

The abstract setting of ABELIAN categories is implemented in the `homalg` package [BLH14a] according to §2. Only building upon the basic constructions as abstract operations, the implementation provides routines to compute (co)homology, derived functors, long exact sequences [BR08], CARTAN-EILENBERG resolutions, hyper-derived functors, spectral sequences (of bicomplexes) and the filtration they induce on (co)homology [Bar09], etc.

In order to use these routines for performing computations within a concrete ABELIAN category, the latter only needs to provide its specific implementation of the basic constructions. The specifics of our implementation for the category $R\text{-fmod}$ of finitely presented modules over a computable ring R were detailed in §3, utilizing the natural equivalence $\text{coker} : R\text{-fpres} \xrightarrow{\sim} R\text{-fmod}$. More precisely, the proof of Theorem 3.4 shows how this equivalence of categories is used to translate *all* constructions in the ABELIAN category $R\text{-fmod}$ to operations on matrices. Note that this translation is independent of the computable ring R , a point reflected in our implementation.

This allows the matrices over specific computable rings with all their operations to reside outside GAP4, preferably in a system that has performant implementations of all the matrix operations mentioned in the proof of Theorem 3.4. It turned out that GAP4 does not need to know the content but only few characteristic information about the matrices created during the computations, minimizing the communication between GAP4 and the external system drastically. For further details the interested reader is referred to the documentation of the `homalg` project [hom17].

5.1. `LocalizeRingForHomalg`. The algorithms presented in §4 are implemented in a GAP4-package [GAP17] `LocalizeRingForHomalg` [BLH14b]. The implementation is abstract in the sense that *any* commutative computable ring R supported by a computer algebra

system to which the `homalg` project [hom17] offers an interface can be localized at any of its finitely generated maximal ideals \mathfrak{m} , thus providing a new ring $R_{\mathfrak{m}}$ for the `homalg` project. The package `LocalizeRingForHomalg` additionally includes an interface to the implementation of MORA's algorithm in SINGULAR [GPS09], which can alternatively be used to solve (in)homogeneous linear systems over $R_{\mathfrak{m}}$, making (together with the matrix operations in the proof of Corollary 4.2) the ABELIAN category $R_{\mathfrak{m}} - \mathbf{fpmo}$ d computable.

Our aim to reduce computations over $R_{\mathfrak{m}}$ to ones over R suggests the above used well-adapted data structure for matrices A over $R_{\mathfrak{m}}$: Write A as a fraction $\frac{\tilde{A}}{s}$ with numerator matrix \tilde{A} over R and a single denominator $s \in R \setminus \mathfrak{m}$. Lemma 4.3 and Proposition 4.5, which provide the key algorithms for this reduction, require at least common denominators for each row of the input matrices. But taking common denominators for each row (or column) of a matrix is not suited for matrix multiplication. We saw in the proof of Corollary 4.2 that this data structure uses no more than computations of common denominators to realize the remaining basic operations for matrices over $R_{\mathfrak{m}}$.

The computational aspects of fraction arithmetic are critical for performance issues since writing matrices with a common denominator blows up numerator matrices. It became efficient the moment we started using least common multiples (as far as they exist in R) for common denominators and for canceling fractions representing ring elements.

Solving the submodule membership problem for the same submodule and various different elements occurs very often in homological computations. For Proposition 4.5 this means that the matrix \tilde{A} as part of $\begin{pmatrix} \tilde{A} \\ \mathfrak{m}\tilde{\mathbf{b}} \end{pmatrix}$ enters many computations with different rows $\tilde{\mathbf{b}}$. Thus replacing \tilde{A} by a distinguished basis (over R , if such a basis exists) is a minor optimization. But with \tilde{A} being a distinguished basis one can first check if $\tilde{\mathbf{b}}$ reduces to zero modulo \tilde{A} . If so, then the row vector \mathbf{z} , occurring in the proof of Proposition 4.5, can be assumed zero. Hence, the row vector \mathbf{t} , occurring in the statement of the proposition, has 1 as denominator. This heuristic succeeds remarkably often and prevents the creation of unnecessary denominators, which would propagate through the remaining computations.

6. EXAMPLES

The examples below are computed using several packages from the `homalg` project [hom17], all written in GAP4 [GAP17]. Here we use SINGULAR [GPS09] as one of the most performant GRÖBNER basis engines with an existing interface in the `homalg` project.

Example 6.1. Let $R := \mathbb{Q}[a, b, c, d, e]$ and M the R -module given by 4 generators satisfying the 9 relations given below, i.e., presented by a 9×4 -matrix A :

```
gap> LoadPackage( "RingsForHomalg" );;
gap> R := HomalgFieldOfRationalsInSingular( ) * "a,b,c,d,e";
<An external ring residing in the CAS Singular>
gap> A := HomalgMatrix( "[\
> 2*a+c+d+e-2,2*a+c+d+e-2,2*a+c+d+e-2,0,\
> 2*c*d+d^2+2*c*e+2*d*e-3*e^2-2*c-d-6*e+1,\
> 2*c*d+d^2+2*c*e+2*d*e-3*e^2-2*c-d-6*e+1,\
> 2*c*d+d^2+2*c*e+2*d*e-3*e^2-2*c-d-6*e+1,0,\
```

```

> -4*a+2*b-c-d-e+2, -4*a+2*b-c-d-e+2, \
> -c+d+e+2, 4*a*d-2*b*d+2*d^2+2*d*e, \
> c^2-d-1, c^2-d-1, c^2-d-1, 0, \
> 4*d*e^2-d^2+2*c*e+4*d*e-3*e^2-2*c+3*d-6*e+5, \
> 4*d*e^2-d^2+2*c*e+4*d*e-3*e^2-2*c+3*d-6*e+5, \
> 4*d*e^2-d^2+2*c*e+4*d*e-3*e^2-2*c+3*d-6*e+5, 0, \
> 0, b^2+a+c+d+e, 0, b^2*e+a*e+c*e+d*e+e^2, \
> 0, b^2*d+a*d+c*d+d^2+d*e, 0, 0, \
> 0, a*b^2+a^2+a*c+a*d+a*e, 0, 0, \
> 4*b^3*d-4*d^3-12*d^2*e-32*c*e^2+12*e^3+21*d^2\
> -42*c*e+40*d*e+27*e^2+8*c-9*d+16*e-17, \
> -4*a*b*d-4*b*c*d-4*b*d^2-4*d^3-4*b*d*e-12*d^2*e-32*c*e^2+\
> 12*e^3+21*d^2-42*c*e+40*d*e+27*e^2+8*c-9*d+16*e-17, \
> -12*d^2*e-32*c*e^2+12*e^3+21*d^2-42*c*e+44*d*e+\
> 27*e^2+8*c-9*d+16*e-17, -4*b^3+4*d^2+4*e\
> ]", 9, 4, R );
<A 9 x 4 matrix over an external ring>
gap> LoadPackage( "Modules" );;
gap> M := LeftPresentation( A );
<A left module presented by 9 relations for 4 generators>

```

Let $R_0 := R_{\mathfrak{m}}$ denote the localized ring at the maximal ideal $\mathfrak{m} = \langle a, b, c, d, e \rangle$ in R corresponding to the “origin” in $\mathbb{A}^5(\mathbb{Q})$. We now want to compute (following [Bar09]) the purity filtration (=equidimensional filtration) of the localized $R_{\mathfrak{m}}$ -module

$$M_0 := M_{\mathfrak{m}} = R_{\mathfrak{m}} \otimes_R M.$$

One possible approach would be to compute the purity filtration for the global R -module M and to localize the resulting filtration afterwards¹¹. But since the module M is also supported¹² at components not including the origin it is clear that computing the global purity filtration will automatically accumulate any structural complexity of M at these components as well. The algorithm suggested in [Bar09] for computing the purity filtration starts by resolving M . And indeed, an early syzygy computation during the resolution of M failed to terminate within a reasonable time.

The computation for the localized module M_0 over the local ring R_0 did not terminate using MORA’s algorithm either, where it gets stuck in a basis computation at an early stage.

However, the purity filtration can be computed for the localized module M_0 using the approach suggested in §4 within seconds:

```

gap> LoadPackage( "LocalizeRingForHomalg" );;
gap> R0 := LocalizeAtZero( R );
<A local ring>
gap> M0 := R0 * M;
<A left module presented by 9 relations for 4 generators>
gap> ByASmallerPresentation( M0 );
<A left module presented by 10 relations for 3 generators>
gap> filt0 := PurityFiltration( M0 );

```

¹¹Justifying this statement is left to the reader.

¹²Recall, $\text{supp } M := \{\mathfrak{p} \in \text{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

```

<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
  0:  <A zero left module>
  -1: <A cyclic reflexively pure grade 1 left module presented by
1 relation for a cyclic generator>
  -2: <A reflexively pure grade 2 left module presented by 7 relations for
2 generators>
  -3: <A cyclic reflexively pure grade 3 left module presented by
3 relations for a cyclic generator>
of
<A non-pure grade 1 left module presented by 10 relations for 3 generators>>

```

The following command computes an explicit isomorphism between a new module (equipped with a triangular presentation compatible with the purity filtration) and our original module M_0 .

```

gap> m := IsomorphismOfFiltration( filt0 );
<An isomorphism of left modules>
gap> FilteredModule := Source( m );
<A left module presented by 7 relations for 4 generators>
gap> Display( FilteredModule );
_ [1,1],0, 0, _ [1,4],
0, _ [2,2],0, 0,
0, 0, _ [3,3],0,
0, _ [4,2],_ [4,3],-b-1/2,
0, 0, 0, _ [5,4],
0, 0, 0, _ [6,4],
0, 0, 0, _ [7,4]
/(4*a*b*d^3-2*b^2*d^3+2*b*d^3*e+2*a*d^3-b*d^3+d^3*e-4*a*b*d+2*b^2*d+2*b*d^2-2*\
b*d*e-2*a*d+b*d+d^2-d*e-2*b-1)

Cokernel of the map

R^(1x7) --> R^(1x4), ( for R := Q[a,b,c,d,e]_< a, b, c, d, e > )

currently represented by the above matrix

```

SINGULAR suppressed the relatively big entries of the triangular presentation matrix. We use the following command to make them visible. We see that in fact all fractions cancel except for one entry which retains a denominator:

```

gap> EntriesOfHomalgMatrix( MatrixOfRelations( FilteredModule ) );
gap> ListToListList( last, 7, 4 );
[ [ (b^2+a+c+d+e)/1, 0/1, 0/1, -1/1 ],
  [ 0/1, (2*a-b+d+e)/1, 0/1, 0/1 ],
  [ 0/1, 0/1, (b^3-d^2-e)/1, 0/1 ],
  [ 0/1, (2*b^2*d*e+b*d*e+2*c*d*e+d^2*e+d*e^2)/1,
    (2*a*b*e+b^2*e+2*b*c*e+2*b*d*e+2*d^2*e+2*b*e^2+a*e+c*e+d*e+3*e^2)/1,
    (-1/2)/(2*a*d^3-b*d^3+d^3*e-2*a*d+b*d+d^2-d*e-1) ],
  [ 0/1, 0/1, 0/1, d/1 ],
  [ 0/1, 0/1, 0/1, a/1 ],
  [ 0/1, 0/1, 0/1, (b^4-b^3*e-b*e+e^2)/1 ] ]

```

Remark 6.2. One would wonder why local computations (as in Example 6.1 above and Example 6.3 below) using the approach suggested in §4 could be faster than the global ones, although the local syzygies computation in Lemma 4.3 is nothing but a global syzygies computation and the effective solution of the local submodule membership problem in Proposition 4.5 is again nothing but the effective solution of an adapted global one. This has to do with the fact that the local ring $R_{\mathfrak{m}}$ has many more units than R , leading to the structural simplifications mentioned in Remark 4.4. Furthermore, $\text{DecideZero}_{R_{\mathfrak{m}}}(\mathbf{B}, \mathbf{A})$ often equals zero even if $\text{DecideZero}_R(\tilde{\mathbf{B}}, \tilde{\mathbf{A}})$ (for the corresponding numerator matrices $\tilde{\mathbf{B}}, \tilde{\mathbf{A}}$) does not (cf. §3.1.2). For the geometric interpretation of the last statement let $\langle \tilde{\mathbf{C}} \rangle$ denote the submodule of the free R -module generated by the rows of the matrix $\tilde{\mathbf{C}}$. Then the maximal ideal \mathfrak{m} often enough does not lie in the support of the nontrivial R -subfactor module $\langle \tilde{\mathbf{B}} \rangle / \langle \tilde{\mathbf{A}} \rangle$, i.e., the $R_{\mathfrak{m}}$ -subfactor $\langle \mathbf{B} \rangle / \langle \mathbf{A} \rangle$ is trivial although its global counterpart $\langle \tilde{\mathbf{B}} \rangle / \langle \tilde{\mathbf{A}} \rangle$ is not. So one can roughly say that zero modules occur more frequently in local homological computations than in global ones, making the former faster, in general.

Example 6.3. SERRE's intersection multiplicity formula of two ideals $I, J \triangleleft R$ at a prime ideal $\mathfrak{p} \triangleleft R$ (cf. [Har77, Thm. A.1.1])

$$i(I, J; \mathfrak{p}) = \sum_i (-1)^i \text{length} \left(\text{Tor}_i^{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I_{\mathfrak{p}}, R_{\mathfrak{p}}/J_{\mathfrak{p}}) \right)$$

offers a nice demonstration of Remark 4.8.

Let $R := \mathbb{F}_5[x, y, z, v, w]$ with maximal ideal $\mathfrak{p} = \mathfrak{m} = \langle x, y, z, v, w \rangle$. We use the package `LocalizeRingForHomalg` to define the two localized rings $R_0 = S_0 := R_{\mathfrak{m}}$. The ring R_0 utilizes Lemma 4.3 and Proposition 4.5, whereas S_0 uses MORA's algorithm to solve (in)homogeneous linear systems.

```
gap> LoadPackage( "RingsForHomalg" );;
gap> R := HomalgRingOfIntegersInSingular( 5 ) * "x,y,z,v,w";;
gap> LoadPackage( "LocalizeRingForHomalg" );;
gap> R0 := LocalizeAtZero( R );;
gap> S0 := LocalizePolynomialRingAtZeroWithMora( R );;
```

The ideals I and J are the intersection of ideals similar to those in [Har77, Example. A.1.1.1] with ideals not supported at zero:

```
gap> i1 := HomalgMatrix( "[ \
> x-z, \
> y-w \
> ]", 2, 1, R );;
gap> i2 := HomalgMatrix( "[ \
> y^6*v^2*w-y^3*v*w^20+1, \
> x*y^4*z^4*w-z^5*w^5+x^3*y*z^2-1 \
> ]", 2, 1, R );;
gap> LoadPackage( "Modules" );;
gap> I := Intersect( LeftSubmodule( i1 ), LeftSubmodule( i2 ) );;
gap> IO := R0 * I;
<A torsion-free left ideal given by 4 generators>
gap> OIO := FactorObject( IO );
<A cyclic left module presented by yet unknown relations for a cyclic generator>
```

```

gap> j1 := HomalgMatrix( "[ \
> x*z, \
> x*w, \
> y*z, \
> y*w, \
> v^2 \
> ]", 5, 1, R );;
gap> j2 := HomalgMatrix( "[ \
> y^6*v^2*w-y^3*v*w^2+1, \
> x*y^4*z^4*w-z^5*w^5+x^3*y*z^2-1, \
> x^7 \
> ]", 3, 1, R );;
gap> J := Intersect( LeftSubmodule( j1 ), LeftSubmodule( j2 ) );;
gap> J0 := R0 * J;;
gap> OJ0 := FactorObject( J0 );
<A cyclic left module presented by yet unknown relations for a cyclic generator>

```

Computing the Tor-modules over S_0 (using MORA's algorithm) or globally¹³ over R both did not terminate within a week. But over R_0 (using the approach suggested in §4) the computation terminates in few seconds:

```

gap> T0 := Tor( OI0, OJ0 );
<A graded homology object consisting of 3 left modules at degrees [ 0 .. 2 ]>

```

As mentioned in Remark §4.8, our approach cannot produce a “distinguished basis” for the presentation matrices of the resulting Tor-modules, from which their HILBERT series can be read off. The sum of the coefficients of the HILBERT series of the module $\text{Tor}_i^{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/I_{\mathfrak{m}}, R_{\mathfrak{m}}/J_{\mathfrak{m}})$ is nothing but its dimension as a $R/\mathfrak{m} \cong \mathbb{F}_5$ vector space, which in this case coincides with the length. But now we can apply MORA's algorithm to the already computed presentation matrices of the modules $\text{Tor}_i^{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/I_{\mathfrak{m}}, R_{\mathfrak{m}}/J_{\mathfrak{m}})$:

```

gap> TOMora := S0 * T0;
<A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> List( ObjectsOfComplex( TOMora ), AffineDegree );
[ 6, 2, 0 ]

```

Thus the intersection multiplicity at \mathfrak{m} is $6 - 2 + 0 = 4$.

Of course, getting rid of the irrelevant¹⁴ primary components of I and/or J would simplify the computations. But we were not able to compute a primary decomposition for I with the computer algebra system SINGULAR, and computing one for J took more than seven minutes, which is still longer than the few seconds needed by our approach.

7. CONCLUSION

The above axiomatic setup for algorithmic homological algebra in the categories of finitely presented modules, merely viewed as ABELian categories, only requires solving

¹³ $\text{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N) = S^{-1}\text{Tor}_i^R(M, N)$ for multiplicatively closed subsets $S \subset R$, cf. [Rot09, Prop. 7.17 or Cor. 10.72].

¹⁴I.e., those primary ideals \mathfrak{q} with $\mathfrak{q} \not\subset \mathfrak{m}$.

(in)homogeneous linear systems and does not enforce the introduction of any notion of distinguished basis. This abstraction motivated a constructive approach to the homological algebra of such module categories over commutative rings localized at maximal ideals, an approach in which global computations replace local ones.

For local polynomial rings this shows that BUCHBERGER’s algorithm provides an alternative to MORA’s algorithm, an alternative which for some examples may even lead to a remarkable gain in computational efficiency. MORA’s algorithm remains indispensable when it comes to computing HILBERT series. It is hence a mixture of both algorithms that proves more useful in practice.

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We would like to thank Sebastian Posur for spotting a typo in the proof of Lemma 4.3.

APPENDIX A. EXISTENTIAL QUANTIFIERS IN ABELIAN CATEGORIES

In this appendix we will briefly recall the *existential quantifiers* for lift, colift, and projective lift appearing in §2.

Let \mathcal{A} be a category. We use the following convention¹⁵ for composing morphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M, L) \times \mathrm{Hom}_{\mathcal{A}}(L, N) & \rightarrow & \mathrm{Hom}_{\mathcal{A}}(M, N), \\ (\phi, \psi) & \mapsto & \phi\psi \end{array}$$

Definition A.1 ([HS97, §II.6, p. 61]). Let \mathcal{A} be a category with 0 and $\phi : M \rightarrow N$ a morphism.

- (k) A morphism $\kappa : K \rightarrow M$ is called “the” **kernel** of $\phi : M \rightarrow N$ if
- (i) $\kappa\phi = 0$, and
 - (ii) for all objects L and all morphisms $\tau : L \rightarrow M$ with $\tau\phi = 0$ there exists a *unique* morphism $\tau_0 : L \rightarrow K$, such that $\tau = \tau_0\kappa$. τ_0 is called the **lift** of τ along κ .

It follows from the uniqueness of the lift τ_0 that κ is a *monomorphism*.

$$\begin{array}{ccccc} & L & & & \\ & \searrow & & & \\ & & 0 & & \\ \tau_0 \downarrow & & \searrow & & \\ & K & \xrightarrow{\kappa} & M & \xrightarrow{\phi} & N \\ & & & & & \end{array}$$

K is called “the” **kernel object** of ϕ . Depending on the context $\ker \phi$ sometimes stands for the morphism κ and sometimes for the object K .

- (c) A morphism $\epsilon : M \rightarrow C$ is called “the” **cokernel** of $\phi : M \rightarrow N$ if
- (i) $\phi\epsilon = 0$, and
 - (ii) for all objects L and all morphisms $\eta : N \rightarrow L$ with $\phi\eta = 0$ there exists a *unique* morphism $\eta_0 : C \rightarrow L$, such that $\eta = \epsilon\eta_0$. η_0 is called the **colift** of η along ϵ .

¹⁵This differs from the convention followed in [HS97, §II.1, p. 41].

It follows from the uniqueness of the colift η_0 that ϵ is an *epimorphism*.

$$\begin{array}{ccccc}
 & L & & & \\
 & \uparrow & \swarrow 0 & & \\
 & \eta_0 \vdots & \eta & & \\
 & C & \xleftarrow{\epsilon} & N & \xleftarrow{\phi} & M
 \end{array}$$

C is called “the” **cokernel object** of ϕ . Depending on the context $\text{coker } \phi$ sometimes stands for the morphism ϵ and sometimes for the object C .

Definition A.2. An object P in a category \mathcal{A} is called **projective**, if for each epimorphism $\epsilon : M \rightarrow N$ and each morphism $\phi : P \rightarrow N$ there exists a morphism $\phi_1 : P \rightarrow M$ with $\phi_1 \epsilon = \phi$.

$$\begin{array}{ccc}
 P & & \\
 \phi_1 \downarrow & \searrow \phi & \\
 M & \xrightarrow{\epsilon} & N
 \end{array}$$

We call ϕ_1 a **projective lift** of ϕ along ϵ .

Remark A.3. A supposedly more general form of the projective lift is often used in *ABELian* categories. The assumption of ϵ being epic can be relaxed in the following way: Let \mathcal{A} be an *ABELian* category and $\epsilon : M \rightarrow N$ a morphism with $\epsilon : M \xrightarrow{\pi} I \xrightarrow{\iota} N$. According to the homomorphism theorem (cf. [HS97, Prop. II.9.6],[ML95, Thm. IX.2.1]) there exists an essentially unique decomposition of ϵ into an epic π and a monic ι . Further let $\beta : N \rightarrow L$ be a morphism with kernel ι (in other words, $M \xrightarrow{\epsilon} N \xrightarrow{\beta} L$ is an **exact sequence**), $P \in \mathcal{A}$ projective object, and $\phi : P \rightarrow N$ a morphism with $\phi\beta = 0$. This last condition expresses that the **image subobject**¹⁶ of ϕ is “contained” in the image subobject of ϵ . It easily follows that there exists a **projective lift** ϕ_1 along ϵ making the following diagram commutative:

$$\begin{array}{ccccc}
 P & & & & \\
 \phi_1 \downarrow & \searrow \phi & \searrow 0 & & \\
 M & \xrightarrow{\epsilon} & N & \xrightarrow{\beta} & L
 \end{array}$$

The projective lift ϕ_1 is constructed in two steps:

$$\begin{array}{ccccc}
 & P & & & \\
 \phi_1 \swarrow & \downarrow \phi_0 & \searrow \phi & \searrow 0 & \\
 M & \xrightarrow{\pi} & I & \xrightarrow{\iota} & N & \xrightarrow{\beta} & L
 \end{array}$$

First construct ϕ_0 as the lift of ϕ along the monomorphism ι . ϕ_1 is then the projective lift of ϕ_0 along the epimorphism π .

¹⁶For details see [Rot09, Def. of subobject, p. 306].

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