# An axiomatization of minimal curb sets*

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**Abstract.** Norde et al. [Games Econ. Behav. 12 (1996) 219] proved that none of the equilibrium concepts in the literature on equilibrium selection in finite strategic games satisfying existence is consistent. A transition to set-valued solution concepts overcomes the inconsistency problem: there is a multiplicity of consistent set-valued solution concepts that satisfy nonemptiness and recommend utility maximization in one-player games. The minimal curb sets of Basu and Weibull [Econ. Letters 36 (1991) 141] constitute one such solution concept; this solution concept is axiomatized in this article.

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## 1. Introduction

The notion of consistency for solutions of noncooperative games was introduced by Peleg and Tijs (1996) and Peleg et al. (1996). Consistency essentially requires that if a nonempty set of players commits to playing according to a certain solution, the remaining players in the reduced game should not have an incentive to deviate from it either. This appears to be a minimal requirement on a solution concept (see also Aumann, 1987, pp. 478-479): given that others play the game according to a certain solution, the solution concept should recommend you to do the same.

Yet, the axiom has a dramatic impact: Norde et al. (1996) proved that the unique point-

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valued[2] solution concept for the set of strategic games satisfying consistency, in combination with standard utility maximizing behavior in one-player games and nonemptiness, is the Nash equilibrium concept. In particular, none of the concepts from the extensive equilibrium refinement literature satisfying nonemptiness is consistent. As Aumann states in an interview (van Damme, 1998, p. 204), this is something to "chalk up against selection theory". Also Barry O'Neill (2004, p. 215) calls this a "surprising result" which "seems to challenge the whole project" of equilibrium refinement: "It seems hard for refinement advocates to dismiss consistency, since it is so close to the basic rationale for the Nash equilibrium".

Dufwenberg et al. (2001) show by means of examples that a transition to set-valued solution concepts overcomes the inconsistency problem: there is a multiplicity of consistent set-valued solution concepts that satisfy nonemptiness and recommend utility maximization in one-player games. The minimal curb sets of Basu and Weibull (1991) constitute one such a solution concept. Minimal curb sets are of central importance in the literature on strategic adjustment, since many intuitively appealing adjustment processes eventually settle down in a minimal curb set; cf. Hurkens (1995), Young (1998), and Fudenberg and Levine (1998).

Building on the papers cited earlier, which strive for characterizations of existing solution concepts in terms of consistency and other properties or axioms, we provide a similar axiomatization of minimal curb sets. Section 2 contains definitions and notation. Section 3 describes properties of set-valued solution concepts. It is shown that the set-valued solution concept that assigns to each game its collection of minimal curb sets satisfies these properties (Prop. 3.1); indeed, it is the only one (Thm. 4.1). Moreover, the properties are logically independent (Prop. 4.2). Section 5 contains variants and extensions of the main result.

## 2. Notation and definitions

Weak set inclusion is denoted by $\subseteq$, proper set inclusion by $\subset$. A ***game*** is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where $N$ is a nonempty, finite set of players, each player $i \in N$ has a nonempty, finite set of pure strategies (or actions) $A_i$ and a von Neumann-Morgenstern utility function $u_i : A \to \mathbb{R}$, where $A = \times_{j \in N} A_j$. The set of all games is denoted by $\Gamma$. The ***subgame*** obtained from $G$ by restricting the action set of each player $i \in N$ to a subset $X_i \subseteq A_i$ is

[2] A point-valued solution concept assigns to each game a collection of strategy profiles, i.e., a set of points in the strategy space of the game. A set-valued solution concept assigns to each game a collection of product sets of strategies, i.e., a set of product sets in the strategy space of the game. Set-valued solution concepts include: the set of rationalizable strategies (Bernheim, 1984), persistent retracts (Kalai and Samet, 1984), minimal curb sets (Basu and Weibull, 1991), and minimal prep sets (Voorneveld, 2004, 2005). Also in cooperative game theory such set-valued solutions arise. A familiar example is the solution concept that assigns to each transferable utility game its possibly empty collection of von Neumann and Morgenstern stable sets; see Osborne and Rubinstein (1994, Section 14.2) for a textbook treatment.

denoted — with a minor abuse of notation from restricting the domain of the payoff functions $u_i$ to $\times_{i\in N} X_i$ — by $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$. The set of mixed strategies of player $i \in N$ with support in $X_i \subseteq A_i$ is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. As usual, $(a_i, \alpha_{-i})$ is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$. For $i \in N$ and $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(A_j)$,

$$BR_i(\alpha_{-i}) = \arg\max_{a_i\in A_i} u_i(a_i, \alpha_{-i})$$

is the set of pure best responses of player $i$ against $\alpha_{-i}$.

A ***set-valued solution concept*** $\varphi$ on $\Gamma$ is a correspondence that assigns to each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ a collection $\varphi(G)$ of product sets in $A$, i.e., each element of $\varphi(G)$ (if there is any) is a set $X = \times_{i\in N} X_i$ with $X_i \subseteq A_i$ for each $i \in N$. We call elements $X \in \varphi(G)$ ***solutions*** of $G$.

A ***curb set*** (Basu and Weibull, 1991; 'curb' is mnemonic for 'closed under rational behavior') of a game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ is a nonempty product set $X = \times_{i\in N} X_i \subseteq A$ such that for each $i \in N$ and each belief $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(X_j)$ of player $i$, the set $X_i$ contains *all* best responses of player $i$ against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(X_j): \quad BR_i(\alpha_{-i}) \subseteq X_i.$$

A curb set $X$ is ***minimal*** if no curb set is a proper subset of $X$. The set-valued solution concept that assigns to each game its collection of minimal curb sets is denoted by min-curb. Hence, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$:

$$\text{min-curb}(G) = \{X \subseteq A : X \text{ is a minimal curb set of } G\}.$$

Similarly,

$$\text{curb}(G) = \{X \subseteq A : X \text{ is a curb set of } G\}.$$

We occasionally refer to minimal prep sets (Voorneveld, 2004; 'prep' is short for 'preparation'). A ***prep set*** of $G$ is a nonempty product set $X = \times_{i\in N} X_i \subseteq A$ such that for each $i \in N$ and each belief $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(X_j)$ of player $i$, the set $X_i$ contains *at least one* best response of player $i$ against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(X_j): \quad BR_i(\alpha_{-i}) \cap X_i \neq \emptyset.$$

A prep set $X$ is ***minimal*** if no prep set is a proper subset of $X$. The set-valued solution concept that assigns to each game its collection of minimal prep sets is denoted by min-prep. Hence, for each game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$:

$$\text{min-prep}(G) = \{X \subseteq A : X \text{ is a minimal prep set of } G\}.$$

Similarly,

$$\text{prep}(G) = \{X \subseteq A : X \text{ is a prep set of } G\}.$$

**Example 2.1** In the two-player game $G$ in Figure 1, min-curb($G$) $= \{\{T, B\} \times \{L, R\}\}$, min-prep($G$) $= \{\{T\} \times \{L\}\}$, and the set of Nash equilibria consists of all mixed strategy profiles $(\alpha T + (1-\alpha)B, L)$ with $\alpha \in [1/2, 1]$. The pure Nash equilibrium $(T, L)$ can be obtained by iterated elimination of weakly dominated actions; in this example, this is exactly the outcome predicted by the game's unique minimal prep set $\{T\} \times \{L\}$. ◁

| | $L$ | $R$ |
|---|---|---|
| $T$ | $1,1$ | $1,0$ |
| $B$ | $1,0$ | $0,1$ |

Figure 1: Differences between min-curb, min-prep, and Nash equilibria.

## 3. Properties of set-valued solution concepts

We provide properties of set-valued solution concepts and show that min-curb satisfies these properties. Variants are discussed in Section 5. Throughout this section, $\varphi$ is an arbitrary set-valued solution concept on $\Gamma$. The first three properties are well-known from Peleg and Tijs (1996), Peleg et al. (1996), and Norde et al. (1996) for point-valued solutions like the Nash equilibrium concept and are simply restated for set-valued solution concepts. *Nonemptiness* requires that the solution concept assigns to each game a nonempty collection of solutions. *One-person rationality* requires that in one-player games, the solution simply consists of the set of utility maximizers.

**Nonemptiness:** $\varphi(G) \neq \emptyset$ for each $G \in \Gamma$.

**One-person rationality:** for each one-player game $G = \langle \{i\}, A_i, u_i \rangle \in \Gamma$ it holds that $\varphi(G) = \{\arg\max_{a_i \in A_i} u_i(a_i)\}$.

The idea behind *consistency* is that if some players commit to playing according to a certain solution, the remaining players should have an incentive to do so too. This requires appropriate ways to model: (a) the reduced game that arises if some players commit to a certain behavior, (b) the absence of incentives to deviate, i.e., the statement that the solution of the original game gives rise to a solution of the reduced game.

Different models of these issues yield different forms of *consistency*. In this article we use the notion of reduced games as defined by Peleg and Tijs (1996), Peleg et al. (1996), and Norde et al. (1996): Given a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with at least two players and a mixed strategy profile $\alpha \in \times_{i \in N} \Delta(A_i)$, fix a coalition $S \subset N, S \neq \emptyset$, and suppose that the players in $N \setminus S$ commit to playing their part of $\alpha$. The ***reduced game*** w.r.t. $S$ and $\alpha$ is the game $G^{S,\alpha} = \langle S, (A_i)_{i \in S}, (v_i)_{i \in S} \rangle \in \Gamma$ where only players $i \in S$ choose from their set of pure strategies $A_i$, while their payoff functions reduce to $v_i : \times_{j \in S} A_j \to \mathbb{R}$ defined as $v_i(\cdot) = u_i(\cdot, \alpha_{N \setminus S})$, i.e., the payoff in the original game, given that members of $N \setminus S$ play $\alpha_{N \setminus S} = (\alpha_j)_{j \in N \setminus S}$ in accordance with $\alpha$.

The next step models the statement that a solution of the original game gives rise to a solution of the reduced game. Consider a solution $X \in \varphi(G)$ of $G \in \Gamma$. Playing according to $X$ implies restricting attention to mixed strategy profiles $\alpha \in \times_{i \in N} \Delta(X_i)$. Fix some coalition $S \subset N, S \neq \emptyset$, of players and suppose that the members of $N \setminus S$ commit to such a strategy profile $\alpha$, thus yielding the reduced game $G^{S,\alpha}$. *Consistency* now requires that the initial solution $X \in \varphi(G)$ yields a solution of the reduced game in the following sense: the reduced game $G^{S,\alpha}$ has a solution in $\times_{j \in S} X_j$, the relevant part of $X \in \varphi(G)$.

> **Consistency:** for each $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, each $X = \times_{i \in N} X_i \in \varphi(G)$, each $\alpha \in \times_{i \in N} \Delta(X_i)$, each $S \subset N, S \neq \emptyset$, there is a solution $Y \in \varphi(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} X_j$.

The other properties are specific for set-valued solution concepts, but remain standard.

> **Nonnestedness:** for each $G \in \Gamma$, there are no $X, Y \in \varphi(G)$ with $X \subset Y$.

Many common set-valued solution concepts satisfy *nonnestedness*, including those defined by product sets of actions which: (a) survive some iterated elimination process, for instance of strictly/weakly dominated actions, or, in the case of rationalizability, of never-best replies, or (b) are minimal or maximal sets with some desirable property, including persistent retracts (so-called minimal absorbing retracts, see Kalai and Samet, 1984, pp. 134-135), minimal curb/prep sets, the product set of all minimax/maximin actions in two-person zero-sum games, the product set of all rationalizable actions (the so-called maximal tight curb set, see Basu and Weibull, 1991, p. 145), or the largest consistent set of Chwe (1994, pp. 313-318; his use of the word 'consistent' is unrelated to our notion of *consistency*).

The next property, *satisfaction*, is a simple revealed-preference property. A product set of strategies is called satisfactory, given the solution concept $\varphi$, if players can credibly commit to playing actions from that set if they believe that others do so: it always contains a solution of the associated reduced game. Given such credible commitment, *satisfaction*[3] states that a way of finding solutions of the original game is to solve the subgame restricted to a satisfactory set.

[3]The adjective 'satisfactory' describes a property of product sets, the noun '*satisfaction*' describes a property of a solution concept.

Formally, consider a game $G \in \Gamma$ with at least two players and a product set $X = \times_{i \in N} X_i \subseteq A$. Such a set is called ***satisfactory under*** $\varphi$ if for each $\alpha \in \times_{i \in N} \Delta(X_i)$ and each $S \subset N, S \neq \emptyset$, there exists a $Y \in \varphi(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} X_j$.

> **Satisfaction:** for each $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with $|N| \geq 2$ and each $X \subseteq A$ which is satisfactory under $\varphi$, one has $\varphi(\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle) \subseteq \varphi(G)$.

This property is reminiscent of the converse consistency axiom of Peleg and Tijs (1996) and Peleg et al. (1996), which roughly states that if a solution candidate always yields a solution in the associated reduced games, it is indeed a solution of the original game. Note that *satisfaction* is much weaker: satisfactory sets need not be contained in the solution of the game.

**Proposition 3.1** *The set-valued solution concept* min-curb *satisfies nonemptiness, one-person rationality, consistency, nonnestedness, and satisfaction.*

**Proof.** *Nonemptiness*: Let $G \in \Gamma$. As the entire strategy space $A$ is a curb set, the collection of curb sets is nonempty, finite and partially ordered by set inclusion. Consequently, a minimal curb set of $G$ exists.
*One-person rationality*: Let $G = \langle \{i\}, A_i, u_i \rangle \in \Gamma$ be a one-player game. In a one-player game, the set of best responses is simply the set of maximizers of the utility function. Hence, $X_i \subseteq A_i$ is a curb set of $G$ if and only if $\arg\max_{a_i \in A_i} u_i(a_i) \subseteq X_i$; it is a minimal curb set of $G$ if and only if $X_i = \arg\max_{a_i \in A_i} u_i(a_i)$. So min-curb$(G) = \{\arg\max_{a_i \in A_i} u_i(a_i)\}$.
*Consistency*: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, $X = \times_{i \in N} X_i \in$ min-curb$(G)$, $\alpha \in \times_{i \in N} \Delta(X_i)$, and $S \subset N, S \neq \emptyset$. To show: there is a $Y \in$ min-curb$(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} X_j$. Since $X \in$ min-curb$(G)$, it follows that $\times_{j \in S} X_j \in$ curb$(G^{S,\alpha})$. Since $\times_{j \in S} X_j \in$ curb$(G^{S,\alpha})$ and there are only finitely many curb sets in $G^{S,\alpha}$, it contains a minimal one: there is a $Y \in$ min-curb$(G^{S,\alpha})$ with $Y \subseteq \times_{j \in S} X_j$.
*Nonnestedness*: Holds by minimality.
*Satisfaction*: Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ with $|N| \geq 2$. Let $X \subseteq A$ be a satisfactory set under min-curb. To show:

$$\text{min-curb}(\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle) \subseteq \text{min-curb}(G). \tag{1}$$

We first show that $X \in$ curb$(G)$. Let $i \in N$ and $\alpha \in \times_{j \in N} \Delta(X_j)$. Since $X$ is a satisfactory set under min-curb, there is a $Y \in$ min-curb$(G^{\{i\},\alpha})$ with $Y \subseteq X_i$. But $G^{\{i\},\alpha}$ is the one-player game $\langle \{i\}, A_i, v_i \rangle \in \Gamma$ with $v_i(a_i) = u_i(a_i, \alpha_{-i})$ for each $a_i \in A_i$. Hence, min-curb$(G^{\{i\},\alpha}) = \{\arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i})\}$, so $Y = \arg\max_{a_i \in A_i} u_i(a_i, \alpha_{-i}) \subseteq X_i$, i.e., $X_i$ contains all best replies to the belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j)$. Since this holds for arbitrary $i \in N$ and $\alpha \in \times_{j \in N} \Delta(X_j)$, it holds by definition that $X \in$ curb$(G)$.

We now prove (1) by contradiction: let $Y \in$ min-curb$(\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle)$. Since $X \in$ curb$(G)$, we also have $Y \in$ curb$(G)$. If $Y \notin$ min-curb$(G)$, there is a $Z \in$ min-curb$(G)$ with $Z \subset Y$. But since $Z \in$ min-curb$(G)$, it is also a curb set of the subgame $G' = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, contradicting that $Y \in$ min-curb$(G')$. Conclude that (1) holds. □

## 4. Axiomatization

In this section, we show that min-curb is the unique solution concept satisfying the properties in Section 3 and that these properties are logically independent.

**Theorem 4.1** *The unique set-valued solution concept on* $\Gamma$ *satisfying nonemptiness, one-person rationality, consistency, nonnestedness, and satisfaction is* min-curb*.*

**Proof.** Proposition 3.1 shows that min-curb satisfies the properties. Let $\varphi$ be a set-valued solution concept on $\Gamma$ that also satisfies them. To show: $\varphi(G) =$ min-curb$(G)$ for each $G \in \Gamma$. We do so by induction on the number of players. In a one-player game $G = \langle \{i\}, A_i, u_i\rangle \in \Gamma$, it follows from *one-person rationality* of $\varphi$ and min-curb that

$$\varphi(G) = \text{min-curb}(G) = \{\arg\max_{a_i \in A_i} u_i(a_i)\}.$$

Next, let $n \in \mathbb{N}$ and assume that $\varphi$ and min-curb coincide on all games in $\Gamma$ with at most $n$ players. Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \Gamma$ have $n+1$ players.

**Step 1:** $\varphi(G) \subseteq$ curb$(G)$.

Let $X \in \varphi(G), i \in N$, and $\alpha_{-i} \in \times_{j\in N\setminus\{i\}} \Delta(X_j)$. To show: $BR_i(\alpha_{-i}) \subseteq X_i$. Let $\beta \in \times_{j\in N}\Delta(X_j)$ be a mixed strategy profile with $\beta_{-i} = \alpha_{-i}$. By *consistency* of $\varphi$, there is a solution $Y \in \varphi(G^{\{i\},\beta})$ with $Y \subseteq X_i$. The game $G^{\{i\},\beta}$ is the one-player game $\langle \{i\}, A_i, v_i\rangle \in \Gamma$ with $v_i(a_i) = u_i(a_i, \beta_{-i}) = u_i(a_i, \alpha_{-i})$ for each $a_i \in A_i$. By *one-person rationality* of $\varphi$, it follows that

$$\varphi(G^{\{i\},\beta}) = \{\arg\max_{a_i\in A_i} v_i(a_i)\} = \{\arg\max_{a_i\in A_i} u_i(a_i, \alpha_{-i})\},$$

i.e., the unique solution of the reduced game $G^{\{i\},\beta}$ is the set of best replies of $i$ in the game $G$ against the belief $\alpha_{-i}$:

$$Y = \arg\max_{a_i\in A_i} u_i(a_i, \alpha_{-i}) \subseteq X_i,$$

as we had to show.

**Step 2:** If $X \in$ min-curb$(G)$, then $X$ is a satisfactory set under $\varphi$.

Let $X \in$ min-curb$(G), \alpha \in \times_{i\in N}\Delta(X_i)$, and $S \subset N, S \neq \emptyset$. By induction, $\varphi(G^{S,\alpha}) =$ min-curb$(G^{S,\alpha})$. By *consistency* of min-curb, there is a $Y \in$ min-curb$(G^{S,\alpha})$ with $Y \subseteq \times_{i\in S} X_i$.

Combining these two results, we find that there is a $Y \in \varphi(G^{S,\alpha})$ with $Y \subseteq \times_{i \in S} X_i$. Hence, $X$ is a satisfactory set under $\varphi$.

**Step 3:** If $X \in$ min-curb$(G)$, then there is a $Y \in \varphi(G)$ with $Y \subseteq X$.

Let $X \in$ min-curb$(G)$. By Step 2, $X$ is a satisfactory set under $\varphi$. Since $\varphi$ satisfies *non-emptiness* and *satisfaction*, it follows that

$$\emptyset \neq \varphi(\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle) \subseteq \varphi(G). \tag{2}$$

So let $Y \in \varphi(\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle)$. Then $Y \subseteq X$, and by (2): $Y \in \varphi(G)$.

**Step 4:** $\varphi(G) \subseteq$ min-curb$(G)$.

Let $X \in \varphi(G)$. By Step 1, $X \in$ curb$(G)$. Suppose $X \notin$ min-curb$(G)$: there is a $Y \in$ min-curb$(G)$ with $Y \subset X$. By Step 3, there is a $Z \in \varphi(G)$ with $Z \subseteq Y$. But since $Z \subseteq Y \subset X$ and $X, Z \in \varphi(G)$, we have a contradiction with the assumption that $\varphi$ is *nonnested*. Conclude that $X \in$ min-curb$(G)$.

**Step 5:** min-curb$(G) \subseteq \varphi(G)$.

Let $X \in$ min-curb$(G)$. By Step 3, there is a $Y \subseteq X$ with $Y \in \varphi(G)$. By Step 1, $Y \in$ curb$(G)$. Since $X \in$ min-curb$(G)$ and $Y \subseteq X$ is a curb set, it follows that $Y = X$, i.e., $X = Y \in \varphi(G)$.

Combining Steps 4 and 5, conclude that $\varphi(G) =$ min-curb$(G)$ also for the $(n+1)$-player game $G$. By induction: $\varphi(G) =$ min-curb$(G)$ for each $G \in \Gamma$. □

**Proposition 4.2** *The axioms in Theorem 4.1 are logically independent.*

We show this by means of five set-valued solution concepts, each violating exactly one of the five axioms in Theorem 4.1. Since the verification that these concepts satisfy the given properties proceeds along the same lines as the proof of Proposition 3.1, we only show explicitly which axiom is violated. Solution concepts $\varphi_1$ to $\varphi_5$ are defined, for each game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, as follows:

$$\begin{aligned}
\varphi_1(G) &= \begin{cases} \text{min-curb}(G) & \text{if } G \text{ is a one-player game,} \\ \emptyset & \text{otherwise.} \end{cases} \\
\varphi_2(G) &= \text{min-prep}(G). \\
\varphi_3(G) &= \begin{cases} \text{min-curb}(G) & \text{if } G \text{ is a one-player game,} \\ \{\times_{i \in N}\{a_i\} \mid \forall i \in N : a_i \in A_i\} & \text{otherwise.} \end{cases} \\
\varphi_4(G) &= \begin{cases} \text{min-curb}(G) & \text{if } G \text{ is a one-player game,} \\ \text{curb}(G) & \text{otherwise.} \end{cases} \\
\varphi_5(G) &= \begin{cases} \text{min-curb}(G) & \text{if } G \text{ is a one-player game,} \\ \{A\} & \text{otherwise.} \end{cases}
\end{aligned}$$

The solution concept $\varphi_1$ satisfies all properties in Theorem 4.1, except *nonemptiness*: $\varphi_1(G) = \emptyset$ for each game $G \in \Gamma$ with two or more players.

The solution concept $\varphi_2$ satisfies all properties in Theorem 4.1, except *one-person rationality*: in the one-player game $G = \langle \{1\}, \{a, b\}, u_1 \rangle$ with $u_1(a) = u_1(b)$, we have

$$\varphi_2(G) = \text{min-prep}(G) = \{\{a\}, \{b\}\} \neq \{\{a, b\}\} = \{\arg\max_{c \in \{a,b\}} u_1(c)\}.$$

The solution concept $\varphi_3$ satisfies all properties in Theorem 4.1, except *consistency*: in the game $G$ in Figure 2, we have $X = \{T\} \times \{R\} \in \varphi_3(G)$. Consider the belief $(T, R)$ in which player 1 chooses $T$ with probability one and player 2 chooses $R$ with probability one. In the reduced game $G^{\{1\},(T,R)} = \langle \{1\}, \{T, B\}, v_1 \rangle$ with $v_1(T) = v_1(B) = 0$, we have

$$\varphi_3(G^{\{1\},(T,R)}) = \text{min-curb}(G^{\{1\},(T,R)}) = \{\{T, B\}\},$$

so $X_1 = \{T\}$ does not contain a solution of the reduced game $G^{\{1\},(T,R)}$.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $1, 1$ | $0, 0$ |
| $B$ | $0, 0$ | $0, 0$ |

Figure 2: A simple two-player game $G$.

The solution concept $\varphi_4$ satisfies all properties in Theorem 4.1, except *nonnestedness*: in the game $G$ in Figure 2, we have $\varphi_4(G) = \text{curb}(G) = \{\{T\} \times \{L\}, \{T, B\} \times \{L, R\}\}$ with $\{T\} \times \{L\} \subset \{T, B\} \times \{L, R\}$.

The solution concept $\varphi_5$ satisfies all properties in Theorem 4.1, except *satisfaction*: in the two-player game $G$ in Figure 2, $\{T\} \times \{L\}$ is a satisfactory set under $\varphi_5$, but in the subgame $G'$ restricted to $\{T\} \times \{L\}$, we have $\varphi_5(G') = \{\{T\} \times \{L\}\} \not\subseteq \{\{T, B\} \times \{L, R\}\} = \varphi_5(G)$.

## 5. Variants and extensions

**(a)** In Theorem 4.1, *nonnestedness* can be replaced by the following property:

**Decisiveness:** for each $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ and $X \in \varphi(G)$:

$$\varphi(\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle) = \{X\}.$$

The intuition behind *decisiveness* is that the solution concept takes some argument to its logical conclusion: given a solution $X$ of a game, the solution of the subgame restricted to $X$ is not refined further. Note that min-curb satisfies *decisiveness*. *Nonnestedness* is used only in Step

4 of Theorem 4.1, the proof of which now becomes as follows: Let $X \in \varphi(G)$. By Step 1, $X \in \text{curb}(G)$. Let $Y \in \text{min-curb}(G)$ with $Y \subseteq X$. Then also $Y \in \text{min-curb}(\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle)$. By Step 3 applied to the subgame $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, there is a $Z \in \varphi(\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle)$ with $Z \subseteq Y$. *Decisiveness* of $\varphi$ implies that $\varphi(\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle) = \{X\}$. Conclude that $X = Z \subseteq Y \subseteq X$, i.e., $X = Y \in \text{min-curb}(G)$, proving Step 4.

The set-valued solution concepts $\varphi_1$ to $\varphi_5$ can be used to show that the new axiom system, with *decisiveness* instead of *nonnestedness*, uses logically independent properties; see Figure 5 for a summary.

**(b)** Since most of the literature on minimal curb sets involves mixed extensions of finite strategic games, we took this to be our domain $\Gamma$. This finiteness assumption is not necessary: we essentially need $\Gamma$ to be closed w.r.t. certain subgames and reduced games, and that each game in $\Gamma$ has a nonempty collection of minimal curb sets. In particular, defining curb sets and the properties in Section 3 in terms of product sets $X = \times_{i\in N} X_i$ where each component $X_i$ is a nonempty *compact* set of pure strategies, our analysis carries through also on the domain of games where each strategy space is assumed to be compact in some Euclidean space and utility functions are continuous, the domain on which Basu and Weibull (1991) establish existence of minimal curb sets.

**(c)** Rationality requires decision makers in one-player games to choose utility maximizing actions. That is the motivation behind the standard *one-person rationality* axiom in the consistency literature. For set-valued solution concepts, it plays a role whether one pools the utility maximizers within a single set or considers them separately. For instance, in the one-player game $G = \langle\{1\}, \{a, b\}, u_1\rangle$ with $u_1(a) = u_1(b)$, we have $\text{min-curb}(G) = \{\{a, b\}\}$, whereas $\text{min-prep}(G) = \{\{a\}, \{b\}\}$: curb sets require all 'best replies' to be present, prep sets require the presence of at least one. An intuitive modification of the *one-person rationality* axiom in Section 3 would therefore be:

$$\text{For each one-player game } G = \langle\{i\}, A_i, u_i\rangle \in \Gamma : \varphi(G) = \{\{b_i\} : b_i \in \arg\max_{a_i \in A_i} u_i(a_i)\}. \quad (3)$$

Rewriting our earlier results yields an axiomatization of min-prep:

**Theorem 5.1** *The unique set-valued solution concept on $\Gamma$ satisfying nonemptiness, one-person rationality as in (3), consistency, nonnestedness, and satisfaction is* min-prep.

**Proposition 5.2** *The axioms in Theorem 5.1 are logically independent.*

The proofs are virtually identical to those of Propositions 3.1, 4.2, and Theorem 4.1 by interchanging, firstly, curb and prep and, secondly, min-curb and min-prep. They are therefore omitted.

In analogy with the remark under **(a)**, *nonnestedness* can be replaced with *decisiveness*; the axioms remain logically independent.

**(d)** Basu and Weibull (1991) briefly consider socalled minimal curb* sets, a 'cautious' variant of minimal curb sets in which players are assumed to abstain from choosing weakly dominated actions.

Formally, let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, let $i \in N$, and let $a_i \in A_i$. Recall that $a_i$ is ***weakly dominated*** if there is a mixed strategy $\alpha_i \in \Delta(A_i)$ such that $u_i(a_i, a_{-i}) \leq u_i(\alpha_i, a_{-i})$ for each $a_{-i} \in \times_{j \in N \setminus \{i\}} A_j$, with strict inequality for some $a_{-i}$. The set of actions of player $i$ that are not weakly dominated (sometimes referred to as ***admissible***) is denoted by $A_i^*$.

A ***curb**** ***set*** of $G$ is a nonempty product set $X = \times_{i \in N} X_i \subseteq A$ such that for each $i \in N$ and each belief $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j)$ of player $i$, the set $X_i$ contains *all* admissible best responses of player $i$ against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j) : \quad BR_i(\alpha_{-i}) \cap A_i^* \subseteq X_i.$$

A curb* set $X$ is ***minimal*** if no curb* set is a proper subset of $X$. The set-valued solution concept that assigns to each game its collection of minimal curb* sets is denoted by min-curb*. Hence, for each game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$:

$$\text{min-curb}^*(G) = \{X \subseteq A : X \text{ is a minimal curb}^* \text{ set of } G\}.$$

It is easily verified that min-curb* satisfies *nonemptiness*, *one-person rationality*, and *nonnestedness*. All other axioms, however, are violated. The main reason for this is that the weak dominance relation may change if one goes from the original game to reduced games or subgames; for instance, an action that is admissible in the original game may be weakly dominated in a reduced game. This indicates intuitively that *consistency* may be violated; we show this formally below and also indicate violations of *satisfaction* and *decisiveness*.

The solution concept min-curb* does not satisfy *consistency*: in the game $G$ in Figure 3 we have $X = \{T, B\} \times \{L\} \in \text{min-curb}^*(G)$. Consider the belief $(B, L)$ in which player 1 chooses $B$ with probability one and player 2 chooses $L$ with probability one. In the reduced game $G^{\{2\},(B,L)} = \langle \{2\}, \{L, C, R\}, v_2 \rangle$ with $v_2(L) = v_2(C) = 0, v_2(R) = -1$, action $C$ is no longer weakly dominated and $\text{min-curb}^*(G^{\{2\},(B,L)}) = \{\{L, C\}\}$. So $X_2 = \{L\}$ does not contain a solution of the reduced game $G^{\{2\},(B,L)}$.

The solution concept min-curb* does not satisfy *satisfaction*: in the game $G$ in Figure 3, $\{T, B\} \times \{L, C\}$ is a satisfactory set under min-curb*, but in the subgame $G'$ restricted to $\{T, B\} \times \{L, C\}$ we have $\text{min-curb}^*(G') = \{\{T\} \times \{L\}\} \not\subseteq \{\{T, B\} \times \{L\}\} = \text{min-curb}^*(G)$.

The solution concept min-curb* does not satisfy *decisiveness*: in the game $G$ in Figure 4 we have $\text{min-curb}^*(G) = \{\{T, M\} \times \{L, C\}\}$. But in the subgame $G'$ restricted to $\{T, M\} \times \{L, C\}$, $M$ is weakly dominated by $T$ and $\text{min-curb}^*(G') = \{\{T\} \times \{L\}\} \neq \{\{T, M\} \times \{L, C\}\}$.

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $T$ | $1,1$ | $1,0$ | $0,-1$ |
| $B$ | $1,0$ | $0,0$ | $1,-1$ |

Figure 3: min-curb* satisfies neither *consistency*, nor *satisfaction*.

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $T$ | $1,1$ | $1,0$ | $0,-1$ |
| $M$ | $1,0$ | $0,1$ | $1,-1$ |
| $B$ | $-1,0$ | $-1,1$ | $-1,-1$ |

Figure 4: min-curb* does not satisfy *decisiveness*.

**(e)** Figure 5 summarizes which axioms are satisfied by the key solution concepts in this paper.

| | min-curb | $\varphi_1$ | min-prep ($\varphi_2$) | $\varphi_3$ | $\varphi_4$ | $\varphi_5$ | min-curb* |
|---|---|---|---|---|---|---|---|
| *nonemptiness* | + | − | + | + | + | + | + |
| *one-person rationality* | + | + | − | + | + | + | + |
| *consistency* | + | + | + | − | + | + | − |
| *nonnestedness* | + | + | + | + | − | + | + |
| *satisfaction* | + | + | + | + | + | − | − |
| *decisiveness* | + | + | + | + | − | + | − |
| *one-person rationality* as in (3) | − | − | + | − | − | − | − |

Figure 5: Solution concepts and the axioms they do (+) or do not (−) satisfy.

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