Int J Game Theory (2005) 33: 355–366
DOI: 10.1007/s00182-005-0204-x

International Journal of Game Theory


# An axiomatization of the consistent non-transferable utility value

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**Abstract.** The Maschler–Owen consistent value for non-transferable utility games is axiomatized, by means of a marginality axiom.

**JEL Classification:** C7, D7

## 1. Introduction

A general *non-transferable utility game in coalitional form* — an *NTU-game* for short — is given by its set of players and the sets of outcomes that are feasible for each subset ("coalition") of players; see Section 2.

A central solution concept for coalitional games is that of *value*, originally introduced by Shapley [1953] for games with *transferable utility* (or *TU-games* for short). The value has been extended in different ways to NTU-games;[1] the most notable *NTU-values* are due to Harsanyi [1963], Shapley [1969],[2] and Maschler and Owen [1992].[3]

The original approach to value (starting with Shapley [1953]) was *axiomatic*: characterizing the value uniquely by an appropriate set of "axioms." In what concerns the NTU-values, axiomatizations have been provided by

Previous versions: October 2003 (Center for the Study of Rationality DP-337), December 2004. Research partially supported by a grant of the Israel Academy of Sciences and Humanities. The author thanks Andreu Mas-Colell, Bezalel Peleg, Peter Sudhölter, the referees, and the editor for their comments and suggestions.

[1]For recent comparisons of NTU-values, see de Clippel, Peters and Zank [2004] and Hart [2004].

[2]The Shapley NTU-value is sometimes referred to as the "$\lambda$-transfer value."

[3]The Maschler–Owen NTU-value is called the "consistent NTU-value." For a noncooperative approach to this value, see Hart and Mas-Colell [1996]; for a first analysis in large market games, see Leviatan [2003].

Aumann [1985] for the Shapley NTU-value, by Hart [1985] for the Harsanyi NTU-value, and by de Clippel, Peters and Zank [2004] for the Maschler–Owen consistent NTU-value.

We will provide here (in Section 4) another axiomatization for the Maschler–Owen consistent NTU-value, which is based on an axiom of *marginality*, originally introduced by Young [1985] for the Shapley TU-value and used by Hart [1994] for the subclass of hyperplane games. The axiom of MARGINALITY requires the value of a player to depend only on his "marginal contributions." While the definition of "marginal contributions" in the TU-case is straightforward, the generalization to the NTU-case is not so; see Section 3. The surprising fact about the marginalistic axiomatizations is that no axioms of linearity or additivity are needed.

The reader is referred to the chapters on value in the *Handbook of Game Theory*, in particular Winter [2002] and McLean [2002], for further discussion and references.

Some notations: $\mathbb{R}$ is the real line; for a finite set $S$, the number of elements of $S$ is denoted $|S|$; the $|S|$-dimensional Euclidean space with coordinates indexed by $S$ (or, equivalently, the set of real functions on $S$) is $\mathbb{R}^S$; and $\Delta(S) = \{x = (x^i)_{i\in S} \in \mathbb{R}^S : \sum_{i\in S} x^i = 1 \text{ and } x^i \geq 0 \text{ for all } i \in S\}$ is the $(|S| - 1)$-dimensional unit simplex on $S$. Finally, $A \subset B$ denotes *weak* inclusion (i.e., possibly $A = B$).

## 2. Preliminaries

### *2.1. Games*

A *non-transferable utility game in coalitional form* is a pair $(N, V)$, where $N$ — the set of *players* — is a finite set, and $V$ — the *coalitional function* — is a mapping that associates to each *coalition* $S \subset N$ a set $V(S) \subset \mathbb{R}^S$ of *feasible payoff vectors* for $S$. An element $x = (x^i)_{i\in S}$ of $V(S)$ is interpreted as follows: there exists an outcome that is feasible for the coalition $S$ whose utility to player $i$ is $x^i$ (for each $i$ in $S$). Thus $V(S)$ is the set of utility combinations that are feasible for the coalition $S$.

We make the following standard assumptions: for each nonempty coalition $S$, the set $V(S)$ is

1. A nonempty strict subset of $\mathbb{R}^S$;
2. Closed, convex, and comprehensive (i.e., $y \leq x$ and $x \in V(S)$ imply $y \in V(S)$).
3. "Uniformly positively smooth": at each $x$ on $\partial V(S)$, the (Pareto efficient) boundary of $V(S)$, there exists a unique supporting hyperplane to $V(S)$ (i.e., there exists a unique[4] $\lambda \equiv \lambda(x) \in \Delta(S)$ such that $V(S) \subset \{y \in \mathbb{R}^S : \lambda \cdot y \leq \lambda \cdot x\}$); moreover, there exists a $\delta > 0$ (which may depend on the set $V(S)$ but not on $x$) such that $\lambda^i \geq \delta$ for all $i \in S$. We will refer to $\lambda$ as "the normal to $V(S)$ at $x$," and we will denote by $\widehat{V}_x(S)$ the resulting half-space that contains $V(S)$, i.e., $\widehat{V}_x(S) := \{y \in \mathbb{R}^S : \lambda \cdot y \leq \lambda \cdot x\}$.

[4] $\lambda$ must be nonnegative since $V(S)$ is comprehensive; we normalize it so that $\sum_{i\in S} \lambda^i = 1$.

Two special classes of NTU-games are the TU-games and the hyperplane games:

- An NTU-game $(N, V)$ is a *transferable utility game* (or *TU-game* for short) if for each coalition $S$ there exists a number $v(S)$ such that $V(S) = \{y \in \mathbb{R}^S : \sum_{i \in S} y^i \leq v(S)\}$; we will denote this game as $(N, V)$ or $(N, v)$ interchangeably.
- An NTU-game $(N, V)$ is a *hyperplane game*[5] (or *H-game* for short) if for each coalition $S$ there exists a number $v(S)$ and a strictly positive vector $\mu_S \in \mathbb{R}^S$ such that $V(S) = \{y \in \mathbb{R}^S : \sum_{i \in S} \mu_S^i y^i \leq v(S)\}$.

### 2.2. *The Shapley TU-value*

The Shapley [1953] value for TU-games, which we denote $\varphi_{\mathrm{TU}}$, is defined as follows. Let $(N, v)$ be a TU-game. For each player $i$ in $N$ and each coalition $S$ containing $i$, let[6]

$$D^i_{\mathrm{TU}}(S, v) := v(S) - v(S \backslash i) \tag{1}$$

be the *marginal contribution of player i to coalition S in the TU-game* $(N, v)$. Then the *Shapley TU-value* of player $i$ in the game $(N, v)$ is

$$\varphi^i_{\mathrm{TU}}(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} D^i_{\mathrm{TU}}\left(P^i_\pi \cup i, v\right), \tag{2}$$

where: $n := |N|$ is the number of players; $\Pi(N)$ is the set of all $n!$ orders on the set $N$; and $P^i_\pi := \{j \in N : \pi(j) < \pi(i)\}$ is the set of predecessors of $i$ in the order $\pi$. Equivalently,[7]

$$\varphi^i_{\mathrm{TU}}(N, v) = \mathsf{E}\left[D^i_{\mathrm{TU}}(\mathbf{Z}, v)\right], \tag{3}$$

where $\mathbf{Z} \equiv \mathbf{Z}_{N,i}$ is the random coalition $P^i_\pi \cup i$ obtained from a random order $\pi$ chosen uniformly in $\Pi(N)$; thus

$$\mathsf{P}[\mathbf{Z} = Z] = \frac{(z-1)!(n-z)!}{n!} \tag{4}$$

for each coalition $Z$ of size $z := |Z|$ that contains player $i$.

### 2.3. *The consistent H-value*

Maschler and Owen [1989] have defined a "consistent value" $\varphi_{\mathrm{H}}$ on the class of hyperplane games. In Hart [1994, Proposition 4.1] it is shown that it may be given also by a formula parallel to (3),[8] once the marginal contributions are defined in an appropriate manner.

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[5] For an interpretation of hyperplane games as "prize games," see Hart [1994].

[6] For simplicity we write $S \backslash i$ and $S \cup i$ instead of the more cumbersome $S \backslash \{i\}$ and $S \cup \{i\}$, respectively.

[7] $\mathsf{E}$ and $\mathsf{P}$ denote expectation and probability, respectively.

[8] The original definition parallels (2); see Maschler and Owen [1989] or Hart [1994] for further details.

Let $(N, V)$ be a hyperplane game, $S \subset N$ a coalition, and $i$ a player in $S$. To determine the contribution of $i$ to $S$ one needs to compare the two *sets* $V(S)$ and $V(S\backslash i)$ (whereas in the TU-case these are just two real numbers — hence formula (1)). The idea is to "summarize" the set $V(S\backslash i)$ by one payoff vector; the natural candidate is the value $\varphi_{\mathrm{H}}(S\backslash i, V)$ of the subgame[9] $(S\backslash i, V)$. We thus define the *marginal contribution of player i to coalition S in the H-game* $(N, V)$ by

$$D^i_{\mathrm{H}}(S, V) := \max\{\xi \in \mathbb{R} : (\varphi_{\mathrm{H}}(S\backslash i, V), \xi) \in V(S)\} \tag{5}$$

(the "max" is well-defined since $V(S)$ is a closed half-space with strictly positive slope).[10] Thus $D^i_{\mathrm{H}}(S, V)$ is the maximal payoff $i$ can get in $V(S)$ once the other players in $S\backslash i$ receive their payoffs according to the value of the $(S\backslash i)$-subgame.

**Proposition 1.** *The Maschler–Owen consistent H-value* $\varphi_{\mathrm{H}}$ *is given by*

$$\varphi^i_{\mathrm{H}}(N, V) = \mathsf{E}\left[D^i_{\mathrm{H}}(\mathbf{Z}, V)\right], \tag{6}$$

*for each hyperplane game* $(N, V)$, *where* $D^i_{\mathrm{H}}$ *is defined in* (5) *and* $\mathbf{Z} \equiv \mathbf{Z}_{N,i}$ *is a random coalition as in* (4).

*Proof.* Hart [1994], formula (4.9) in Proposition 4.1. ■

The H-value $\varphi_{\mathrm{H}}$ is thus determined inductively: formula (6) together with (5) gives the value of $(N, V)$ once the values of the subgames $(N\backslash i, V)$ for all $i \in N$ are known.

### *2.4. The consistent NTU-value*

To extend the definition of value to the general class of NTU-games, one uses the concept of a *payoff configuration* (see Hart [1985]), which is a collection $\mathbf{x} = (x_S)_{S\subset N} \in \prod_{S\subset N} \mathbb{R}^S$ consisting of a payoff vector $x_S \in \mathbb{R}^S$ for each coalition $S \subset N$. Given a game $(N, V)$, a payoff configuration $\mathbf{x} = (x_S)_{S\subset N}$ is *efficient* if $x_S \in \partial V(S)$ for all coalitions $S \subset N$. An efficient payoff configuration $\mathbf{x}$ generates from $(N, V)$ a hyperplane game $(N, \widehat{V}_{\mathbf{x}})$ which is obtained by enlarging each $V(S)$ up to the unique supporting hyperplane to $V(S)$ at $x_S$, i.e., $\widehat{V}_{\mathbf{x}}(S) := \widehat{V}_{x_S}(S)$; we will call $\left(N, \widehat{V}_{\mathbf{x}}\right)$ the "H-cover of $(N, V)$ at $\mathbf{x}$."

A payoff configuration $\mathbf{x} = (x_S)_{S\subset N}$ is a *Maschler–Owen consistent NTU-value* of the game $(N, V)$ if it is efficient and it satisfies

$$x_S = \varphi_{\mathrm{H}}\left(S, \widehat{V}_{\mathbf{x}}\right)$$

for each coalition $S \subset N$. In short:[11] $\mathbf{x}$ is a consistent NTU-value of $(N, V)$ if and only if $\mathbf{x}$ is the H-value of the H-cover of $(N, V)$ at $\mathbf{x}$. We will write

[9] When $S \subset N$, the *subgame* $(S, V)$ of $(N, V)$ is obtained by restricting the coalitional function $V$ to the subsets of $S$.

[10] $D^i_{\mathrm{H}}(S, V)$ depends on the solution $x_{S\backslash i} := \varphi_{\mathrm{H}}(S\backslash i, V)$ of the subgame, and so a more precise notation would be $D^i_{\mathrm{H}}(S, V; \varphi_{\mathrm{H}})$ or $D^i_{\mathrm{H}}(S, V; x_{S\backslash i})$. However, this is not needed here since the H-value $\varphi_{\mathrm{H}}$ is unique.

[11] A statement on payoff configurations should be understood to hold coordinatewise (i.e., for each $S$).

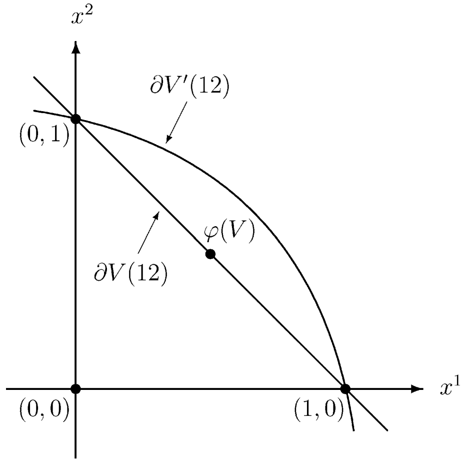


**Fig. 1.** Two games with the same $D_H$-marginal contributions

$\Phi(N, V)$ for the *set* of consistent NTU-values of the game $(N, V)$ (it is a set of payoff configurations).[12] It is immediate that in the special case that $(N, V)$ is a TU-game or a hyperplane game (and so $\widehat{V}_{\mathbf{x}} = V$), there exists a unique consistent value $\mathbf{x}$, whose $S$-coordinate is the value of the subgame $(S, V)$; that is, $\Phi(N, V) = \{(\varphi(S, V))_{S \subset N}\}$, where $\varphi$ stands for $\varphi_{TU}$ or $\varphi_H$, respectively.

## 3. Marginal contributions for NTU-games

We come now to the question of how to extend the definition of marginal contribution to NTU-games, so that the NTU-value equals the expected marginal contribution, as in (3) and (6). A first attempt is to use the same formula (5) as in the H-case. However, consider the TU-game $(N, V) \equiv (N, v)$ with $N = \{1, 2\}$, $v(1) = v(2) = 0$ and $v(12) = 1$, and the NTU-game $(N, V')$ which is identical to $(N, V)$ except for the grand coalition, where $V'(12)$ is a *strictly* convex set whose Pareto efficient boundary contains the points $(1, 0)$ and $(0, 1)$ (see Figure 1). The marginal contributions according to formula (5) (which reduces to (1) in the TU-case) are identical in the two games: $D^i_H(\{i\}, V) = 0 = D^i_H(\{i\}, V')$ and $D^i_H(N, V) = 1 = D^i_H(N, V')$ for $i = 1, 2$. However, the Shapley value of $(N, V)$, which equals $\mathsf{E}[D^i_H(\mathbf{Z}, V)] = \mathsf{E}[D^i_H(\mathbf{Z}, V')]$, *cannot* be an NTU-value of $(N, V')$, since it is not efficient in $(N, V')$.

Therefore we cannot use definition (5) for NTU-games. Moreover, varying the Pareto efficient boundary of $V'(N)$ (while keeping the two points $(1, 0)$ and $(0, 1)$ there) suggests that, in order for the value to equal the expected marginal contribution, these marginal contributions should be evaluated *at the solution point* (rather than far away from it, as in $D_H(N, V')$). This leads us to the following definition.

Let $(N, V)$ be an NTU-game and let $\mathbf{x} = (x_S)_{S \subset N}$ be an efficient payoff configuration. For each coalition $S$ containing player $i$ and each $\varepsilon > 0$, let

[12] Under our assumptions, consistent values exist and need not be unique; see Sections 5.2 and 5.3.

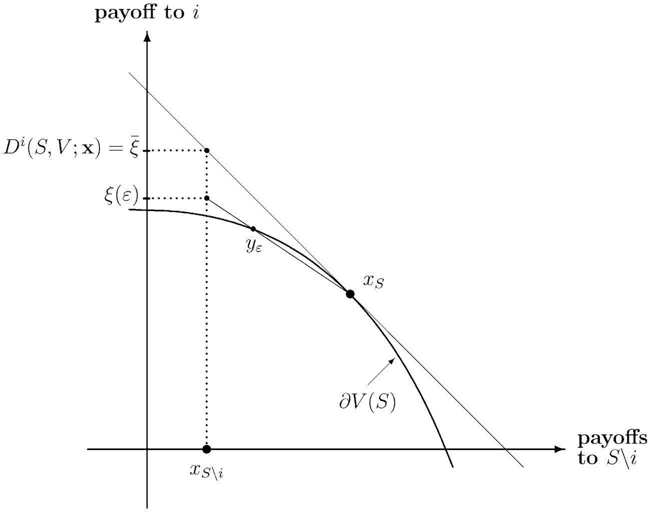


**Fig. 2.** Defining marginal contribution

$$\xi(\varepsilon) := \sup\{\zeta \in \mathbb{R} : (1-\varepsilon)x_S + \varepsilon(x_{S\backslash i}, \zeta) \in V(S)\}, \text{ and}$$
$$\bar{\xi} := \lim_{\varepsilon \searrow 0} \xi(\varepsilon) \equiv \sup_{\varepsilon>0} \xi(\varepsilon). \tag{7}$$

We define the *marginal contribution of player i to coalition S at* **x** as

$$D^i(S, V; \mathbf{x}) := \bar{\xi}. \tag{8}$$

See Figure 2 and the discussion below. The marginal contribution $D^i$ may be viewed as a *directional derivative* of the boundary of $V(S)$ at $x_S$ in the direction $x_{S\backslash i}$ (and $D^i_{\mathrm{H}}$, as a directional "discrete" derivative). Geometrically, $D^i$ is obtained by means of the supporting hyperplane to $V(S)$ at $x_S$.

**Lemma 2:** *$\bar{\xi}$ is well-defined, and it satisfies $\lambda \cdot (x_{S\backslash i}, \bar{\xi}) = \lambda \cdot x_S$, where $\lambda$ is the normal to $V(S)$ at $x_S$.*

*Proof.* $\xi(\varepsilon)$ is well-defined since all the supporting hyperplanes to $V(S)$ have uniformly positive slopes and $V(S)$ is closed; the limit $\bar{\xi}$ is well-defined — and it equals the supremum of the $\xi(\varepsilon)$ — since $V(S)$ is convex. Let $y_\varepsilon := (1-\varepsilon)x_S + \varepsilon(x_{S\backslash i}, \xi(\varepsilon)) \in \partial V(S)$, and let $\lambda$ and $\lambda_\varepsilon$ be the normals to $V(S)$ at $x_S$ and $y_\varepsilon$, respectively. Then $\lambda \cdot y_\varepsilon \leq \lambda \cdot x_S$ and $\lambda_\varepsilon \cdot x_S \leq \lambda_\varepsilon \cdot y_\varepsilon$, from which it follows in the limit as $\varepsilon \searrow 0$ that $\lambda \cdot (x_{S\backslash i}, \bar{\xi}) \leq \lambda \cdot x_S$ and $\lambda \cdot x_S \leq \lambda \cdot (x_{S\backslash i}, \bar{\xi})$ (for the latter we use $\lambda_\varepsilon \to \lambda$, since $\lambda$ is the *unique* normal at $x_S$). ■

Lemma 2 implies that if $x_{S\backslash i} = \varphi_{\mathrm{H}}\left(S\backslash i, \widehat{V}_{\mathbf{x}}\right)$ then

$$D^i(S, V; \mathbf{x}) = D^i_{\mathrm{H}}\left(S, \widehat{V}_{\mathbf{x}}\right). \tag{9}$$

Therefore (8) reduces to (5) in the hyperplane case, and to (1) in the TU-case. Note that in the general NTU-case the marginal contribution of $i$ to $S$ depends on the payoff vectors of the two coalitions $S$ and $S\backslash i$, i.e., $x_S$ and $x_{S\backslash i}$;

these are not needed in the TU-case, and only $x_{S\backslash i}$ is needed in the H-case. One may refer to $D^i$ as a "marginal marginal contribution": one "marginal" refers as usual to the added player $i$, and the other "marginal" to the fact that only infinitesimal changes in the payoff vector $x_S$ are considered in (7).

To interpret (7)–(8), assume that the probability of player $i$ leaving the coalition $S$ is $\varepsilon > 0$. The expected payoff (according to $\mathbf{x}$) of every other player $j \neq i$ in $S$ is then $(1-\varepsilon)x_S^j + \varepsilon x_{S\backslash i}^j = y_\varepsilon^j$, so player $i$ can get at most $y_\varepsilon^i$, where $y_\varepsilon \in \partial V(S)$. Now $y_\varepsilon^i = (1-\varepsilon)x_S^i + \varepsilon\xi(\varepsilon)$, which translates into $i$ getting $x_S^i$ when he stays in $S$ (with probability $1-\varepsilon$) and $\xi(\varepsilon)$ when he leaves $S$ (with probability $\varepsilon$). Therefore $\xi(\varepsilon)$ may be viewed as the contribution of $i$ to $S$ when the probability of leaving is $\varepsilon$ (note that when $i$ leaves for sure, i.e., $\varepsilon = 1$, it is precisely $D_{\mathrm{H}}^i$), and we take the limit $\bar{\xi}$ of $\xi(\varepsilon)$ as the probability $\varepsilon$ converges to zero. This ties in nicely with the noncooperative bargaining model of Hart and Mas-Colell [1996] (in particular Proposition 1 and Theorem 5), where, following a rejected proposal, the current proposer has a small probability $\varepsilon = 1-\rho$ of leaving the "active" coalition; in the limit, as $\varepsilon \to 0$, one obtains the consistent NTU-value. An argument of the same kind appears in the justification of the Shapley NTU-value by Myerson [1991] (see the discussion on pages 475–476), where there is a small probability that a player will reject the value.

**Theorem 3.** *Let $(N,V)$ be an NTU-game and let $\mathbf{x} = (x_S)_{S\subset N}$ be an efficient payoff configuration for $(N,V)$. Then $\mathbf{x}$ is a Maschler–Owen consistent NTU-value of $(N,V)$ if and only if*

$$x_S^i = \mathsf{E}\left[D^i(\mathbf{Z}_{S,i}, V; \mathbf{x})\right] \tag{10}$$

*for all coalitions $S \subset N$ and all players $i \in S$, where $\mathbf{Z}_{S,i}$ is a random subcoalition of $S$ containing player $i$ (i.e., $\mathsf{P}\left[\mathbf{Z}_{S,i} = Z\right] = (z-1)!(s-z)!/s!$, where $s := |S|$ and $z := |Z|$).*

Note that the characterization of Theorem 3 dispenses with the need to deal first with the consistent value for hyperplane games (see also Hart and Mas-Colell [1996, Proposition 4]).

*Proof.* Let $\mathbf{x}$ be an efficient payoff configuration for $(N,V)$ that satisfies (10). For each $i \in S \subset N$, if $x_{S\backslash i} = \varphi_{\mathrm{H}}\left(S\backslash i, \widehat{V}_{\mathbf{x}}\right)$ then $D^i(S,V;\mathbf{x}) = D_H^i\left(S, \widehat{V}_{\mathbf{x}}\right)$ by (9), and so, by (10) and (6),

$$x_S^i = \mathsf{E}\left[D^i(\mathbf{Z}_{S,i}, V; \mathbf{x})\right] = \mathsf{E}\left[D_{\mathrm{H}}^i\left(\mathbf{Z}_{S,i}, \widehat{V}_{\mathbf{x}}\right)\right] = \varphi_{\mathrm{H}}^i\left(S, \widehat{V}_{\mathbf{x}}\right).$$

Induction on $S$ (starting with the singleton coalitions) yields $\mathbf{x} \in \Phi(N,V)$.

Conversely, if $\mathbf{x} \in \Phi(N,V)$ then

$$x_S^i = \varphi_{\mathrm{H}}^i\left(S, \widehat{V}_{\mathbf{x}}\right) = \mathsf{E}\left[D_{\mathrm{H}}^i\left(\mathbf{Z}_{S,i}, \widehat{V}_{\mathbf{x}}\right)\right] = \mathsf{E}\left[D^i(\mathbf{Z}_{S,i}, V; \mathbf{x})\right]$$

by (6) and (9). ■

## 4. Axiomatization

In this section we extend the "marginalistic axiomatization" of value initiated by Young [1985] in the TU-case, and provided in the hyperplane case by Hart [1994].

A *solution function* is a mapping $\Psi$ that associates to each NTU-game $(N, V)$ a set of payoff configurations $\Psi(N, V)$. The axioms we will use are:

- EFFICIENCY: For each game $(N, V)$, every $\mathbf{x} \in \Psi(N, V)$ is efficient, i.e., $x_S \in \partial V(S)$ for all coalitions $S \subset N$.
- MARGINALITY: Let $(N, V)$ and $(N, W)$ be two games with the same set of players, let $i \in N$ be a player, and let $\mathbf{x} \in \Psi(N, V)$ and $\mathbf{y} \in \Psi(N, W)$. If $D^i(S, V; \mathbf{x}) = D^i(S, W; \mathbf{y})$ for all coalitions $S \subset N$ containing $i$, then $x_S^i = y_S^i$ for all coalitions $S \subset N$ containing $i$.
- TU-NONEMPTINESS: For each TU-game $(N, v)$, the set $\Psi(N, v)$ is not empty.
- TU-EQUAL-TREATMENT: If $(N, v)$ is a TU-game with $v(S \cup i) = v(S \cup j)$ for all coalitions $S \subset N \backslash \{i, j\}$ (i.e., if players $i$ and $j$ are *substitutes* in $(N, v)$) and $\mathbf{x} \in \Psi(N, v)$, then $x_S^i = x_S^j$ for all coalitions $S \subset N$ containing both $i$ and $j$.

Our result is:

**Theorem 4.** *The maximal solution function $\Psi$ that satisfies* EFFICIENCY, MARGINALITY, TU-NONEMPTINESS, *and* TU-EQUAL-TREATMENT *is the Maschler–Owen consistent NTU-solution $\Phi$.*

Thus:

1. $\Phi$ satisfies the four axioms; and
2. If $\Psi$ satisfies the four axioms, then $\Psi(N, V) \subset \Phi(N, V)$ for all games $(N, V)$.

*Proof*. The first claim is immediate (MARGINALITY follows from (10)). To prove the second claim, let $\Psi$ satisfy the four axioms. On the subclass of TU-games these axioms characterize the Shapley TU-value by the result of Young [1985].[13] More precisely, adapting the Proof of Theorem 2 of Young [1985] to our setup of payoff configurations yields[14] $\Psi(N, v) = \{(\varphi_{\mathrm{TU}}(S, v))_{S \subset N}\}$. Let now $(N, V)$ be a general game, $i \in N$ a player, and $\mathbf{x} \in \Psi(N, V)$. Define a TU-game $(N, w)$ as follows: $w(S) := D^i(S, V; \mathbf{x})$ if the coalition $S$ contains player $i$, and $w(S) := 0$ otherwise. Then $D^i(S, V; \mathbf{x}) = D^i_{\mathrm{TU}}(S, w)$ for all $S$ containing $i$, and so by MARGINALITY and (3)

$$x_S^i = \varphi^i_{\mathrm{TU}}(S, w) = \mathsf{E}\left[D^i_{\mathrm{TU}}\left(\mathbf{Z}_{S,i}, w\right)\right] = \mathsf{E}\left[D^i\left(\mathbf{Z}_{S,i}, V; \mathbf{x}\right)\right]$$

for all $S$ containing $i$. This applies to each player $i$, and therefore $\mathbf{x}$ is a consistent NTU-value of $(N, V)$ (recall Theorem 3); hence indeed $\Psi(N, V) \subset \Phi(N, V)$. ∎

[13] The result of Young [1985, Theorem 2] is stated with the axiom of STRONG MONOTONICITY (which is: $D^i(S, v) \geq D^i(S, w)$ for all $S$ implies $\psi^i(N, v) \geq \psi^i(N, w)$) instead of MARGINALITY. However, as noted on page 71 there, the proof uses only MARGINALITY.

[14] For example, let $(N, u_0)$ be the "zero game," i.e., $u_0(S) = 0$ for all $S$. All players are substitutes, therefore every $\mathbf{x} \in \Psi(N, u_0)$ satisfies $x_S^i = x_S^j$ for all $S$ and all $i, j \in S$ by TU-EQUAL-TREATMENT. Together with EFFICIENCY we get $x_S^i = 0$ for all $S$ and all $i \in S$, or $\mathbf{x} = \mathbf{0}$ (the zero payoff configuration). Hence $\Psi(N, u_0) = \{\mathbf{0}\}$ by TU-NONEMPTINESS. The other steps in Young's proof are adapted in a similar way.

## 5. Additional results

We conclude with a number of remarks and additional results.

### 5.1. Independence of the axioms

To prove that the four axioms are independent we will show that dropping any one of the axioms allows solution functions $\Psi$ that do not satisfy $\Psi \subset \Phi$.

Indeed:

- Without EFFICIENCY: Let $\Psi(N, V) = \{\mathbf{0}\}$ for all games, where $\mathbf{0}$ denotes the zero payoff configuration (i.e., $\mathbf{0}^i_S = 0$ for all $i \in S \subset N$).[15]
- Without MARGINALITY: For each coalition $S$ let $\sigma(S, V)$ be the "equal split" efficient payoff vector in $V(S)$, i.e., $\sigma(S, V) \in \partial V(S)$ and $\sigma^i(S, V) = \sigma^j(S, V)$ for all $i, j \in S$ (it is unique by comprehensiveness). Let $\Psi(N, V) = \{(\sigma(S, V))_{S \subset N}\}$.
- Without TU-NONEMPTINESS: Let $(N, v_0)$ be a TU-game where no two players are substitutes, and let $\mathbf{x}$ be an efficient payoff configuration that is different from the Shapley value configuration. Let $\Psi(N, v_0) = \{\mathbf{x}\}$ and $\Psi(N, V)$ the empty set for all other games $(N, V)$.
- Without TU-EQUAL-TREATMENT: Use a nonsymmetric distribution for $\mathbf{Z}$ throughout — for instance, let $\pi_0$ be a fixed order on $N$ and take $\mathbf{Z}_{N,i} \equiv P^i_{\pi_0} \cup i$ and $\mathbf{Z}_{S,i} \equiv \mathbf{Z}_{N,i} \cap S$ in (10) (e.g., for $\pi_0$ the natural order on $N = \{1, 2, \ldots, n\}$, the payoff of player $i$ in a TU-game $(N, v)$ is $v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$).

Finally, maximality is needed, since any $\Psi$ with $\Psi(N, V) \subset \Phi(N, V)$ for all games $(N, V)$ and $\Psi(N, v) = \Phi(N, v)$ for all TU-games $(N, v)$ satisfies the four axioms; in particular, to make $\Psi$ satisfy also NONEMPTINESS (see below), let $\Psi(N, V) = \Phi(N, V)$ for all games $(N, V)$ except an NTU-game $(N, V_0)$ that possesses more than one consistent value,[16] for which $\Psi(N, V_0)$ is taken to be a nonempty strict subset of $\Phi(N, V_0)$.

### 5.2. Nonemptiness

Every game satisfying our assumptions of Subsection 2.1 has a consistent value. Indeed, Maschler and Owen [1992, Theorem 3.3] prove such a claim under their assumptions (A1)–(A6). Now our games satisfy (A1)–(A5) but not (A6), which requires that[17] $0^S \in V(S)$ for all $S$. However, adding a fixed amount $C$ to the payoffs of each player (i.e., replacing each $V(S)$ with $V(S) + \{Ce_S\}$, where $e^i_S = 1$ for each $i \in S$) clearly does not affect the existence of consistent values (it just adds $C$ to the value payoffs as well) — so (A6) is not needed. Thus one could use the stronger NONEMPTINESS axiom instead of TU-NONEMPTINESS.

[15] Another possibility — which has the advantage that it always yields feasible outcomes — is to subtract a positive amount, say, 1, from all payoffs in all the consistent NTU-values; i.e., let $\Psi(N, V) = \{\mathbf{x} - \mathbf{1} : \mathbf{x} \in \Phi(N, V)\}$, where $(\mathbf{x} - \mathbf{1})^i_S = x^i_S - 1$ for all $i \in S \subset N$.

[16] E.g., the games mentioned at the beginning of Section 5.3 below.

[17] $0^S$ denotes the zero vector in $\mathbb{R}^S$.

### 5.3. Equal treatment and symmetry

The consistent NTU-value does *not* satisfy EQUAL TREATMENT for general NTU-games: Example 3.2 of Owen [1994] is a three-player symmetric game (i.e., all players are substitutes), which nevertheless has consistent NTU-values that are not symmetric (i.e., the payoffs of the three players are not equal).[18] Of course, $\Phi$ satisfies the weaker SYMMETRY axiom, which says that if the game does not change when interchanging two players $i$ and $j$, then the solution does not change either.

Formally, given two players $i, j \in N$, let $\tau : N \to N$ be the transposition $\tau(i) = j, \tau(j) = i$, and $\tau(k) = k$ for all $k \neq i, j$; for a coalition $S \subset N$, let $\tau S := \{\tau(k) : k \in S\}$; for a payoff vector $x \in \mathbb{R}^S$, define $\tau x \in \mathbb{R}^{\tau S}$ by $(\tau x)^{\tau(k)} = x^k$ for each $k \in S$; and finally, for a payoff configuration $\mathbf{x} = (x_S)_{S \subset N}$, let $\tau \mathbf{x}$ be the payoff configuration whose $\tau S$-coordinate is $\tau x_S$.

The players $i$ and $j$ are *substitutes* in the game $(N, V)$ if: (i) for each coalition $S$ containing both $i$ and $j$, the set $V(S)$ is symmetric, i.e., $x \in V(S)$ if and only if $\tau x \in V(S)$; and (ii) for each coalition $S$ that contains neither $i$ nor $j$, the sets $V(S \cup i)$ and $V(S \cup j)$ are superposable, i.e., $x \in V(S \cup i)$ if and only if $\tau x \in V(S \cup j)$. (This is clearly an extension of the definition of substitutes in the TU-case, where only (ii) matters.)

- SYMMETRY: If players $i$ and $j$ are substitutes in a game $(N, V)$, then $\mathbf{x} \in \Psi(N, V)$ if and only if $\tau \mathbf{x} \in \Psi(N, V)$.

One gets:

**Proposition 5:** *The maximal solution function* $\Psi$ *that satisfies* EFFICIENCY, MARGINALITY, TU-NONEMPTINESS, *and* SYMMETRY *is the Maschler–Owen consistent NTU-solution* $\Phi$.

*Proof.* We will show that SYMMETRY and MARGINALITY imply TU-EQUAL-TREATMENT[19] — whence the result follows from Theorem 4. Indeed, let $i, j$ be substitutes in a TU-game $(N, v)$, and let $\mathbf{x} \in \Psi(N, v)$. Symmetry implies that $\tau \mathbf{x} \in \Psi(N, v)$. Now $D^i(S, v; \mathbf{x}) = v(S) - v(S \backslash i) = D^i(S, v; \tau \mathbf{x})$ for all $S$ containing $i$, therefore $x_S^i = (\tau \mathbf{x})_S^i$ for all such $S$ by MARGINALITY. If $S$ also contains $j$, then $\tau S = S$ and $(\tau \mathbf{x})_S^i = x_S^j$, so $x_S^i = x_S^j$. ∎

### 5.4. Maximality

To obtain a set of axioms that characterize the consistent value $\Phi$ without the maximality requirement, one can use the following axiom:

- INDEPENDENCE OF IRRELEVANT ALTERNATIVES: Let $(N, V)$ and $(N, W)$ be two games with the same set of players, and let $\mathbf{x} \in \Psi(N, W)$. If $V(S) \subset W(S)$ and $x_S \in V(S)$ for all $S \subset N$ then $\mathbf{x} \in \Psi(N, V)$.

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[18] A simpler example — though the game is not monotonic — is the two-person game of Figure 1(B) in Hart and Mas-Colell [1996, page 367] when $r^1 = r^2$ and $V(12)$ is a symmetric set.

[19] This argument is due to Peter Sudhölter (personal communication).

Thus, if a solution $\mathbf{x}$ of the larger game $W$ is in fact feasible also in the smaller game $V$, then it is also a solution of the smaller game $V$.

**Proposition 6:** *A solution function* $\Psi$ *satisfies* EFFICIENCY, MARGINALITY, NON-EMPTINESS,[20] TU-EQUAL-TREATMENT, *and* INDEPENDENCE OF IRRELEVANT ALTERNATIVES *if and only if it is the Maschler–Owen consistent NTU-solution* $\Phi$.

*Proof.* We have $\Psi \subset \Phi$ by Theorem 4. If $(N, V)$ is a hyperplane game then $\Phi(N, V)$ is a singleton, so $\Psi(N, V) = \Phi(N, V)$ by NONEMPTINESS. For a general game $(N, V)$, if $\mathbf{x} \in \Phi(N, V)$ then $\mathbf{x} \in \Phi\left(N, \widehat{V}_{\mathbf{x}}\right) = \Psi\left(N, \widehat{V}_{\mathbf{x}}\right)$ (since $\left(N, \widehat{V}_{\mathbf{x}}\right)$ is a hyperplane game), and $x_S \in V(S) \subset \widehat{V}_{\mathbf{x}}(S)$ for all $S$, so $\mathbf{x} \in \Psi(N, V)$ by INDEPENDENCE OF IRRELEVANT ALTERNATIVES. ■

## 5.5. Monotonic games

Our axiomatic characterization also holds when restricted to the class of monotonic games — as is the case for Young [1985] for TU-games[21] and Hart [1994] for hyperplane games. An NTU-game $(N, V)$ is *monotonic* if $V(T) \times \{0^{T \setminus S}\} \subset V(S)$ for all $T \subset S \subset N$; for TU-games, this is $v(T) \leq v(S)$. In a monotonic game, the marginal contributions are always nonnegative; i.e., for every efficient payoff configuration $\mathbf{x}$ we have $D^i(S, V; \mathbf{x}) \geq 0$ for all $i \in S \subset N$ (indeed: $\left(x_{S \setminus i}, 0\right) \in V(S)$ by monotonicity and thus $\xi(\varepsilon) \geq 0$ for all $\varepsilon > 0$; see Figure 2). The counterpart of Theorem 4 is

**Proposition 7.** *The maximal solution function* $\Psi$ *on the class of monotonic games that satisfies* EFFICIENCY, MARGINALITY, TU-NONEMPTINESS, *and* TU-EQUAL-TREATMENT *is the Maschler–Owen consistent NTU-solution* $\Phi$.

*Proof.* The same as the Proof of Theorem 4, except that we now use the result of Young [1985] for monotonic TU-games (see Footnote 21), and we construct the TU-game $w$ so that it will be monotonic: for each coalition $S$ that contains player $i$ let $w(S) := \sum_{T \subset S, T \ni i} D^i(S, V; \mathbf{x})$ and $w(S \setminus i) := w(S) - D^i(S, V; \mathbf{x})$. ■

## 5.6. Fixed set of players

The domain of games we consider allows the set of players to be *any* finite set $N$. It is easy to see that our axiomatizations apply also for each *fixed* $N$ separately.

[20] We need NONEMPTINESS rather than TU-NONEMPTINESS; in fact, NONEMPTINESS for hyperplane games suffices.

[21] The modification of the Proof of Theorem 2 of Young [1985] that is described on page 71 there for the class of superadditive games applies also to the class of monotonic games.

## References

[1] Aumann RJ (1985) An Axiomatization of the Non-Transferable Utility Value. Econometrica 53:599–612
[2] de Clippel G, Peters H, Zank H (2004) Axiomatizing the Harsanyi Solution, the Symmetric Egalitarian Solution and the Consistent Shapley Solution for NTU-Games. International Journal of Game Theory 33:145–158
[3] Harsanyi JC (1963) A Simplified Bargaining Model for the *n*-Person Cooperative Game. International Economic Review 4:194–220
[4] Hart S (1985) An Axiomatization of Harsanyi's Non-Transferable Utility Solution. Econometrica 53:1295–1313
[5] Hart S (1994) On Prize Games. In: Megiddo N (ed.), Essays in Game Theory. Springer-Verlag, 111–121
[6] Hart S (2004) A Comparison of Non-Transferable Utility Values. Theory and Decision 56:35–46
[7] Hart S, Mas-Colell A (1996) Bargaining and Value. Econometrica 64:357–380
[8] Leviatan S (2003) Consistent Values and the Core in Continuum Market Games with Two Types of Players. International Journal of Game Theory 31:383–410
[9] Maschler M, Owen G (1989) The Consistent Shapley Value for Hyperplane Games. International Journal of Game Theory 18:389–407
[10] Maschler M, Owen G (1992) The Consistent Shapley Value for Games without Side Payments. In: Selten R (ed.), Rational Interaction. Springer-Verlag, 5–12
[11] McLean R (2002) Values of Non-Transferable Utility Games. In: Aumann RJ, Hart S (eds.), Handbook of Game Theory, with Economic Applications, Vol. 3. Elsevier, 2077–2120
[12] Myerson RB (1991) Game Theory. Harvard University Press
[13] Owen G (1994) The Non-Consistency and Non-Uniqueness of the Consistent Value. In: Megiddo N (ed.), Essays in Game Theory. Springer-Verlag, 155–162
[14] Shapley LS (1953) A Value for *n*-Person Games. In: Kuhn HW, Tucker AW (eds.), Contributions to the Theory of Games II (Annals of Mathematics Studies 28). Princeton University Press, 307–317
[15] Shapley LS (1969) Utility Comparison and the Theory of Games, in La Décision. Editions du CNRS, Paris, 251–263
[16] Winter E (2002) The Shapley Value. In: Aumann RJ, Hart S (eds.), Handbook of Game Theory, with Economic Applications, Vol. 3. Elsevier, 2025–2054
[17] Young HP (1985) Monotonic Solutions of Cooperative Games. International Journal of Game Theory 14:65–72