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Abstract

In this paper we provide an axiomatization of the Shapley value for TU-games using a *fairness* property. This property states that if to a game we add another game in which two players are *symmetric* then their payoffs change by the same amount. We show that the Shapley value is characterized by this fairness property, efficiency and the null player property. These three axioms also characterize the Shapley value on important subclasses of games, such as the class of simple games or the class of apex games.

Keywords: TU-game, Shapley value, fairness, simple games.
JEL classification number: C71

1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility* –or simply a TU-game– being a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v: 2^N \to \mathbb{R}$ is a *characteristic function* such that $v(\emptyset) = 0$. Since we take the set of players N to be fixed, we represent a TU-game by its characteristic function v. The collection of all characteristic functions on N is denoted by $\mathcal{G}^N$.

A (single valued) solution for TU-games is a function $f: \mathcal{G}^N \to \mathbb{R}^N$ which assigns an $|N|$-dimensional real vector to every TU-game. This vector can be seen as a distribution of the payoffs that can be obtained by cooperation over the individual players in the game. A famous solution is the *Shapley value* (Shapley (1953a)). Various axiomatizations of the Shapley value have been given. In this paper we provide an axiomatization of the Shapley value using efficiency, the null player property and a *fairness* property. This last property states that if to a game $v \in \mathcal{G}^N$ we add a game $w \in \mathcal{G}^N$ in which players i and j are *symmetric* then the payoffs of players i and j change by the same amount, i.e., if $w(S \cup \{i\}) = w(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$ then $f_i(v + w) - f_i(v) = f_j(v + w) - f_j(v)$.

This concept of fairness is related to fairness as introduced by Myerson (1977) for games in which the possibilities of coalition formation in a TU-game are limited because of the fact that players are part of a limited *communication structure*. In that model fairness means that deleting a communication relation between two players has the same effect on both their payoffs. A similar fairness axiom is used in van den Brink (1997) for games in which the cooperation possibilities in a TU-game are limited because the players are part of a hierarchical *permission structure* in which there are players who need permission from certain other players before they are allowed to cooperate. In that model fairness means that deleting a permission relation between two players has the same effect on both their payoffs. In van den Brink (1995a) a fairness axiom for *relational power measures* for *directed graphs*[1] is introduced. In that context fairness means that deleting a relation between two nodes in a digraph changes their relational power by the same amount.

As already noted by Dubey (1975), axiomatizations of the Shapley value on $\mathcal{G}^N$ not necessarily characterize the Shapley value on important subclasses of games such as the class of *simple games*. A TU-game v is simple if $v(S) \in \{0,1\}$ for all $S \subset N$. It turns out that efficiency, the null player property, and fairness also characterize the Shapley value on the class of simple games. Van den Brink (1995a) shows that these three axioms characterize the Shapley value on the even smaller class of *apex games*.

Besides the literature on fairness started in Myerson (1977), this paper also is related to the axiomatization of the Shapley value by efficiency, symmetry and strong monotonicity given in Young (1985). A solution satisfies *strong monotonicity* if for every pair of games $v, w \in \mathcal{G}^N$ and $i \in N$, the payoff of i in v is at least equal to its payoff in w if the *marginal contribution* of player i to any coalition in v is at least equal to its corresponding marginal contribution in w, i.e., $f_i(v) \geq f_i(w)$ if $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subset N \setminus \{i\}$. As argued by Chun (1991), it is sufficient to require that $f_i(v) = f_i(w)$ if $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subset N \setminus \{i\}$. So, strong monotonicity essentially compares the payoff of a player if we add a game in which this player is a *null player*, while fairness compares the change in payoff of two players if we add a game in which these players are *symmetric*.

[1]A directed graph is a pair (N, D) where N is a finite set of nodes and $D \subset N \times N$ is a binary relation on N. A relational power measure for directed graphs is a function that assigns real values to all nodes in a directed graph. For a general discussion about relational power measures for directed graphs we refer to van den Brink (1994).

The paper is organized as follows. In Section 2 we define fairness and show that the Shapley value is the unique solution on $\mathcal{G}^N$ that satisfies efficiency, the null player property and fairness. We also show that these three axioms characterize the Shapley value on the class of simple games. We end Section 2 by comparing fairness with strong monotonicity and *balanced contributions* as considered in, e.g., Myerson (1980) and Hart and Mass-Colell (1989). In Section 3 we generalize the characterization of the Shapley value to *weighted Shapley values* as considered in, e.g., Shapley (1953b) and Kalai and Samet (1987). Finally, there is an appendix that discusses *components* in TU-games which are used in the proof of the main theorem.

2 An axiomatization of the Shapley value

In this section we provide an axiomatization of the Shapley value using efficiency, the null player property and fairness. The *Shapley value* (Shapley (1953a)) is the function $Sh\colon \mathcal{G}^N \to \mathbb{R}^N$ given by

$$Sh_i(v) = \sum_{S \ni i} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N,$$

with *dividends* $\Delta_v(S) = \sum_{T \subset S} (-1)^{|S|-|T|} v(T)$ for all $S \subset N$ (see Harsanyi (1959)). We first state the well-known efficiency and null player axioms for solutions $f\colon \mathcal{G}^N \to \mathbb{R}^N$. Player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $v(S) = v(S \setminus \{i\})$ for all $S \subset N$.

Axiom 2.1 (Efficiency) *For every $v \in \mathcal{G}^N$ it holds that $\sum_{i \in N} f_i(v) = v(N)$.*

Axiom 2.2 (Null player property) *If $i \in N$ is a null player in game $v \in \mathcal{G}^N$ then $f_i(v) = 0$.*

Players $i, j \in N$ are *symmetric* in $v \in \mathcal{G}^N$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$. Fairness states that if to a game $v \in \mathcal{G}^N$ we add a game $w \in \mathcal{G}^N$ in which players i and j are symmetric, then the payoffs of players i and j change by the same amount.

Axiom 2.3 (Fairness) *If $i, j \in N$ are symmetric players in $w \in \mathcal{G}^N$, then*

$$f_i(v + w) - f_i(v) = f_j(v + w) - f_j(v) \text{ for all } v \in \mathcal{G}^N.$$

It is easy to verify that every solution that satisfies symmetry and additivity also satisfies fairness. A solution $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies *symmetry* if $i, j \in N$ being symmetric players in $v \in \mathcal{G}^N$ implies that $f_i(v) = f_j(v)$. Solution $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies *additivity* if for every pair of games $v, w \in \mathcal{G}^N$ it holds that $f(v + w) = f(v) + f(w)$, where $(v + w) \in \mathcal{G}^N$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$.

Proposition 2.4 *If $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies symmetry and additivity, then f also satisfies fairness.*

PROOF
Let $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfy symmetry and additivity. Further, suppose that $i, j \in N$ are symmetric in $w \in \mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it then holds that

$$f_i(v + w) - f_i(v) = f_i(v) + f_i(w) - f_i(v) = f_i(w) =$$

$$= f_j(w) = f_j(w) + f_j(v) - f_j(v) = f_j(v + w) - f_j(v),$$

where the first and sixth equality follow from additivity, and the third equality follows from symmetry of f. Thus, f satisfies fairness.

$\square$

It is known that the Shapley value is characterized by efficiency, the null player property, symmetry and additivity. By Proposition 2.4 it thus also satisfies fairness. A solution that satisfies fairness need not satisfy symmetry nor additivity. This can be seen from the solution $f: \mathcal{G}^N \to \mathbb{R}^N$ given by $f_1(v) = Sh_1(v) + 1$ and $f_i(v) = Sh_i(v) - \frac{1}{|N|-1}$ for $i \in N \setminus \{1\}$. This solution satisfies fairness but does not satisfy symmetry nor additivity.

Now we state the main result of the paper.

Theorem 2.5 *A solution $f: \mathcal{G}^N \to \mathbb{R}^N$ is equal to the Shapley value if and only if it satisfies efficiency, the null player property, and fairness.*

For transparancy we split the proof that there can be at most one solution satisfying efficiency, the null player property, and fairness in two parts[2]. For every $v \in \mathcal{G}^N$ we define

$$D(v) = \{T \subset N \mid \Delta_v(T) \neq 0\}, \text{ and } d(v) = |D(v)|. \tag{1}$$

So, every $v \in \mathcal{G}^N$ can be expressed as $v = \sum_{T \in D(v)} \Delta_v(T) u_T$, where u_T is the *unanimity game* of coalition $T \subset N$, i.e., $u_T(S) = 1$ if $S \supset T$, and $u_T(S) = 0$ otherwise. We first consider games for which there are at most two coalitions with a non-zero dividend.

Lemma 2.6 *Let $\mathcal{G}_2^N := \{v \in \mathcal{G}^N \mid d(v) \leq 2\}$. There can be at most one solution $f\colon \mathcal{G}_2^N \to \mathbb{R}^N$ that satisfies efficiency, the null player property, and fairness.*

PROOF

Suppose that $f\colon \mathcal{G}_2^N \to \mathbb{R}^N$ satisfies efficiency, the null player property, and fairness. Let $v \in \mathcal{G}_2^N$.

If $d(v) = 0$ then v is the *null game*, i.e., $v(S) = 0$ for all $S \subset N$. The null player property then implies that $f_i(v) = 0$ for all $i \in N$.

If $d(v) = 1$ then v is a multiple of the unanimity game of some coalition $T \subset N$, i.e., $v = c_T u_T$ for some $T \subset N$ and $c_T \in \mathbb{R}$, $c_T \neq 0$. The null player property implies that $f_i(c_T u_T) = 0$ for all $i \in N \setminus T$. Fairness implies that there exists a constant $c^* \in \mathbb{R}$ such that $f_i(c_T u_T) = c^*$ for all $i \in T$. With efficiency it then follows that $c^* = \frac{c_T}{|T|}$. So, in this case $f_i(c_T u_T) = \begin{cases} \frac{c_T}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$

If $d(v) = 2$ then v is the sum of two unanimity games, i.e., $v = c_T u_T + c_H u_H$, with $c_T \neq 0$, $c_H \neq 0$. We distinguish the following three cases:

[2]In van den Brink (1995b) the Shapley value is characterized by fairness and *component efficiency*. A coalition $B \subset N$ is a *component* in $v \in \mathcal{G}^N$ if $v(S) = v(S \cap B) + v(S \setminus B)$ for all $S \subset N$. A solution $f\colon \mathcal{G}^N \to \mathbb{R}^N$ satisfies component efficiency if for every $v \in \mathcal{G}^N$ and component B in v it holds that $\sum_{i \in B} f_i(v) = v(B)$. Clearly, this property implies efficiency and the null player property. However, efficiency and the null player property do not imply component efficiency. A solution that satisfies efficiency and the null player property, but does not satisfy component efficiency is the *normalized Banzhaf value* as characterized in van den Brink and van der Laan (1998).

1. Suppose that $T \cap H \neq \emptyset$.

   Clearly, the null player property implies that $f_i(v) = 0$ for all $i \in N \setminus (T \cup H)$.

   Take a $j \in T \cap H$ and let $f_j(v) = c^*$.

   Fairness implies that $f_i(v) - f_i(c_H u_H) = f_j(v) - f_j(c_H u_H)$ for all $i \in T$. Similarly, it follows that $f_i(v) - f_i(c_T u_T) = f_j(v) - f_j(c_T u_T)$ for all $i \in H$. Since $d(c_H u_H) = d(c_T u_T) = 1$, we already determined that

   (i) $f_i(c_H u_H) = 0$ and $f_i(c_T u_T) = \frac{c_T}{|T|}$ for all $i \in T$;

   (ii) $f_i(c_T u_T) = 0$ and $f_i(c_H u_H) = \frac{c_H}{|H|}$ for all $i \in H$.

   But then

$$f_i(v) = \begin{cases} f_j(v) - f_j(c_H u_H) + f_i(c_H u_H) = c^* - \frac{c_H}{|H|} & \text{if } i \in T \setminus H \\ f_j(v) - f_j(c_T u_T) + f_i(c_T u_T) = c^* - \frac{c_T}{|T|} & \text{if } i \in H \setminus T \\ f_j(v) - f_j(c_H u_H) + f_i(c_H u_H) = c^* & \text{if } i \in T \cap H \\ 0 & \text{otherwise.} \end{cases}$$

   With efficiency it follows that $\sum_{i \in N} f_i(v) = |T \cup H| c^* - \frac{|T \setminus H|}{|H|} c_H - \frac{|H \setminus T|}{|T|} c_T$ must be equal to $c_T + c_H$. Thus, $c^* = \frac{|T| + |H \setminus T|}{|T||T \cup H|} c_T + \frac{|H| + |T \setminus H|}{|H||T \cup H|} c_H = \frac{c_T}{|T|} + \frac{c_H}{|H|}$, is uniquely determined, and so are all $f_i(v)$, $i \in N$,

$$f_i(v) = \begin{cases} \frac{c_T}{|T|} & \text{if } i \in T \setminus H \\ \frac{c_H}{|H|} & \text{if } i \in H \setminus T \\ \frac{c_T}{|T|} + \frac{c_H}{|H|} & \text{if } i \in T \cap H \\ 0 & \text{otherwise.} \end{cases}$$

2. Suppose that $T \cap H = \emptyset$ and $T \cup H \neq N$.

   The null player property implies that $f_i(v) = 0$ for all $i \in N \setminus (T \cup H)$.

   Take a $j \in N \setminus (T \cup H)$. (Note that, by assumption, there is at least one null player.)

For every $i \in T$, fairness and the fact that $f_j(v) = f_j(c_T u_T) = 0$, imply that $f_i(v) - f_i(c_T u_T) = f_j(v) - f_j(c_T u_T) = 0$. Since $d(c_T u_T) = 1$, we already determined that $f_i(c_T u_T) = \frac{c_T}{|T|}$ for $i \in T$. Thus, $f_i(v) = f_i(c_T u_T) = \frac{c_T}{|T|}$ for all $i \in T$.

Similarly, it follows that $f_i(v) = \frac{c_H}{|H|}$ for all $i \in H$. So, $f(v)$ is also uniquely determined in this case.

3. Suppose that $T \cap H = \emptyset$ and $T \cup H = N$.

   Note that $d(v) = 2$ implies that $|N| \geq 2$. We distinguish the following two cases with respect to $|N|$:

   A. We first consider the case that $|N| \geq 3$. Suppose without loss of generality that $|T| \geq 2$. Take a $j \in T$ and $h \in H$. Further, define the game $w \in \mathcal{G}^N$ by

   $$w = v + c_T u_{(T \setminus \{j\}) \cup \{h\}} = c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}} + c_H u_H.$$

   Let $f_h(w) = c^*$. Fairness implies that $f_j(w) - f_j(c_H u_H) = f_h(w) - f_h(c_H u_H)$. Since $d(c_H u_H) = 1$, we already determined that $f_j(c_H u_H) = 0$ and $f_h(c_H u_H) = \frac{c_H}{|H|}$. So, $f_j(w) = f_h(w) - f_h(c_H u_H) + f_j(c_H u_H) = c^* - \frac{c_H}{|H|}$.
   For every $i \in H \setminus \{h\}$, fairness implies that $f_i(w) - f_i(c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}}) = f_h(w) - f_h(c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}})$. Since $c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}}$ is as considered under 1 (i.e., $T \cap ((T \setminus \{j\}) \cup \{h\}) \neq \emptyset$,) we have that $f_i(w) = f_h(w) - f_h(c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}}) + f_i(c_T u_T + c_T u_{(T \setminus \{j\}) \cup \{h\}}) = c^* - \frac{c_T}{|T|}$ for every $i \in H \setminus \{h\}$.

   For every $i \in T \setminus \{j\}$, fairness implies that $f_i(w) - f_i(c_T u_{(T \setminus \{j\}) \cup \{h\}}) = f_j(w) - f_j(c_T u_{(T \setminus \{j\}) \cup \{h\}})$. Since $d(c_T u_{(T \setminus \{j\}) \cup \{h\}}) = 1$, we already determined that $f_i(c_T u_{(T \setminus \{j\}) \cup \{h\}}) = \frac{c_T}{|T|}$ for $i \in T \setminus \{j\}$, and $f_j(c_T u_{(T \setminus \{j\}) \cup \{h\}}) = 0$. Thus, $f_i(w) = f_j(w) - f_j(c_T u_{(T \setminus \{j\}) \cup \{h\}}) + f_i(c_T u_{(T \setminus \{j\}) \cup \{h\}}) = c^* - \frac{c_H}{|H|} + \frac{c_T}{|T|}$.

So, we determined that

$$f_i(w) = \begin{cases} c^* & \text{if } i = h \\ c^* - \frac{c_H}{|H|} & \text{if } i = j \\ c^* - \frac{c_T}{|T|} & \text{if } i \in H \setminus \{h\} \\ c^* - \frac{c_H}{|H|} + \frac{c_T}{|T|} & \text{if } i \in T \setminus \{j\}. \end{cases}$$

With efficiency it follows that $\sum_{i\in N} f_i(w) = |N|c^* - \frac{|T|}{|H|}c_H + \frac{(|T|-|H|)}{|T|}c_T$ must be equal to $2c_T + c_H$. Thus, $c^* = \frac{2|T|-|T|+|H|}{|T||N|}c_T + \frac{|H|+|T|}{|H||N|}c_H = \frac{c_T}{|T|} + \frac{c_H}{|H|}$ is uniquely determined, and so are all $f_i(w)$, $i \in N$,

$$f_i(w) = \begin{cases} \frac{c_T}{|T|} + \frac{c_H}{|H|} & \text{if } i = h \\ \frac{c_T}{|T|} & \text{if } i = j \\ \frac{c_H}{|H|} & \text{if } i \in H \setminus \{h\} \\ \frac{2c_T}{|T|} & \text{if } i \in T \setminus \{j\}. \end{cases}$$

Next we determine the values $f_i(v)$, $i \in N$. Let $f_h(v) = c^{**}$. Fairness implies that $f_i(v) = c^{**}$ for all $i \in H$.

For every $i \in T \setminus \{j\}$, fairness implies that $f_i(v) - f_i(w) = f_h(v) - f_h(w)$, and thus $f_i(v) = c^{**} - f_h(w) + f_i(w) = c^{**} - \frac{c_H}{|H|} + \frac{c_T}{|T|}$ for $i \in T \setminus \{j\}$.

Since fairness also implies that $f_j(v) = f_i(v)$ for all $i \in T\setminus\{j\}$, and $T\setminus\{j\} \neq \emptyset$ it follows that $f_j(v) = c^{**} - \frac{c_H}{|H|} + \frac{c_T}{|T|}$. So,

$$f_i(v) = \begin{cases} c^{**} & \text{if } i \in H \\ c^{**} - \frac{c_H}{|H|} + \frac{c_T}{|T|} & \text{if } i \in T \end{cases}$$

Efficiency determines that $\sum_{i\in N} f_i(v) = |N|c^{**} - \frac{|T|}{|H|}c_H + c_T$ must be equal to $c_T + c_H$, and thus $c^{**} = \frac{|H|+|T|}{|H||N|}c_H = \frac{c_H}{|H|}$. So, all $f_i(v)$, $i \in N$, are uniquely determined,

$$f_i(v) = \begin{cases} \frac{c_H}{|H|} & \text{if } i \in H \\ \frac{c_T}{|T|} & \text{if } i \in T. \end{cases}$$

B. Suppose that $|N| = 2$, i.e., $N = \{i,j\}$ and $v = c_i u_{\{i\}} + c_j u_{\{j\}}$. Suppose without loss of generality that $c_i \geq c_j$ and let $f_j(v) = c^*$. The null player property implies that $f_j((c_i - c_j)u_{\{i\}}) = 0$. With efficiency it then follows that $f_i((c_i - c_j)u_{\{i\}}) = (c_i - c_j)$.

Fairness implies that $f_i(v) - f_i((c_i - c_j)u_{\{i\}}) = f_j(v) - f_j((c_i - c_j)u_{\{i\}})$, and thus $f_i(v) = c^* + c_i - c_j$. Efficiency then implies that $f_i(v) + f_j(v) = 2c^* + c_i - c_j$ must be equal to $c_i + c_j$, and thus $c^* = c_j$. So, $f_i(v) = c^* + c_i - c_j = c_i$ and $f_j(v) = c^* = c_j$ are uniquely determined.

□

To prove that there can be at most one solution satisfying efficiency, the null player property, and fairness on $\mathcal{G}^N$, we define connectedness of players in a TU-game.

Definition 2.7 *Players $i, j \in N$ are* **connected** *in $v \in \mathcal{G}^N$ if there exists a sequence of coalitions $(T^1, \ldots, T^m)$ such that*

(i) $i \in T^1,\ j \in T^m$;

(ii) $T^k \cap T^{k+1} \neq \emptyset$ *for all* $k \in \{1, \ldots, m-1\}$;

(iii) $\Delta_v(T^k) \neq 0$ *for all* $k \in \{1, \ldots, m\}$.

A coalition $B \subset N$ such that all $i, j \in N$ are connected to each other in game $v \in \mathcal{G}^N$ is called a **connected coalition** *in v. A connected coalition B in $v \in \mathcal{G}^N$ is a* **maximal connected coalition** *if every $T \supset B$, $T \neq B$, is not a connected coalition in v.*

Thus, two players $i, j \in N$ are connected in game v if there exists a sequence of 'active' coalitions from player i to player j such that every coalition in this sequence has a non-empty intersection with its neighbouring coalitions. A coalition $B \subset N$ is a maximal connected coalition in $v \in \mathcal{G}^N$ if and only if the following two conditions are satisfied:

(i) for every $i, j \in B$ it holds that i and j are connected in v;

(ii) for every $i \in B$ and $j \in N \setminus B$ it holds that i and j are not connected in v.

For a discussion of connected coalitions we refer to the appendix of this paper.

Proof of Theorem 2.5

It is well-known that the Shapley value satisfies efficiency and the null player property. Since the Shapley value satisfies symmetry and additivity, it follows from Propositon 2.4 that it also satisfies fairness.

Now, suppose that $f\colon \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, the null player property, and fairness. Let $v \in \mathcal{G}^N$.

We show that $f(v)$ is uniquely determined by induction on the number $d(v)$ (defined in equation (1)). By Lemma 2.6, $f(v)$ is uniquely determined for all games v with $d(v) \leq 2$.

Proceeding by induction, assume that $f(v')$ is uniquely determined for all $v' \in \mathcal{G}^N$ with $d(v') \leq k$ $(k \geq 2)$, and let $d(v) = k+1$. We distinguish the following three cases with respect to $|\mathcal{B}(v)|$, where $\mathcal{B}(v)$ denotes the partition of N into maximal connected coalitions in v:

1. Suppose that $|\mathcal{B}(v)| = 1$ (meaning that N is a connected coalition in v). (Note that $d(v) \geq 3$ implies that $|N| \geq 2$).

   Take a $j \in N$. We show that $f_j(v)$ is uniquely determined in the following three steps.

   (a)

   We define the sets $T^k,\ k \in \{0\} \cup \mathbb{N}$, as follows:

   - $T^0 = \{j\}$;
   - for every $k \in \mathbb{N}$

$$T^k = \left\{ i \in N \setminus \bigcup_{l=0}^{k-1} T^l \;\middle|\; \begin{array}{c} \text{there exists a } T \subset N \text{ such that} \\ T \cap \left(\bigcup_{l=0}^{k-1} T^l\right) \neq \emptyset, \\ T \ni i, \text{ and } \Delta_v(T) \neq 0 \end{array} \right\}.$$

(If $i \in T^k$ then we can say that i is connected to j through $k-1$ other players.)

(b)

For every $k \in \mathbb{N}$ with $N \setminus \bigcup_{l=0}^{k-1} T^l \neq \emptyset$, we show that $T^k \neq \emptyset$.

On the contrary, suppose that $T^k = \emptyset$. Let $i \in N \setminus \bigcup_{l=0}^{k-1} T^l$.

Since by assumption $T^k = \emptyset$, it holds by definition of the sets T^k that there exists no $T \subset N$ such that $T \cap \left(\bigcup_{l=0}^{k-1} T^l\right) \neq \emptyset$, $T \ni i$, and $\Delta_v(T) \neq 0$.

Thus, for every $T \subset N$ such that $T \cap \left(\bigcup_{l=0}^{k-1} T^l\right) \neq \emptyset$ and $T \ni i$ it holds that $\Delta_v(T) = 0$. But then i and j are not connected in v. This is in contradiction with $|\mathcal{B}(v)| = 1$. Thus, $T^k \neq \emptyset$.

From this it follows that there exists an $m \in \mathbb{N}$ such that

(a) $T^k \neq \emptyset$ for all $k \in \{0, \ldots, m\}$,

(b) $T^k \cap T^l = \emptyset$ for all $k, l \in \{0, \ldots, m\}$, $k \neq l$, and

(c) $\bigcup_{k=0}^{m} T^k = N$.

Thus $T^0, \ldots, T^m$ is a partition of N consisting of non-empty sets only.

(c)

Suppose that $f_j(v) = c^*$ for some value $c^* \in \mathbb{R}$. Next we determine for every $i \in N \setminus \{j\}$ the value $f_i(v)$ as a function of c^* by the following procedure:

Step 1 Let $k = 1$ and $c_j = 0$ (and thus $f_j(v) = c^* + c_j$). Goto Step 2.

Step 2 By definition of the set T^k, for every $i \in T^k$ there exists an $h \in \bigcup_{l=0}^{k-1} T^l$ and a $T \subset N$ such that $T \supset \{i, h\}$ and $\Delta_v(T) \neq 0$.

Fairness implies that $f_i(v) - f_i(v - \Delta_v(T)u_T) = f_h(v) - f_h(v - \Delta_v(T)u_T)$.

With the induction hypothesis and the fact that we already determined the value $c_h \in \mathbb{R}$ for which $f_h(v) = c^* + c_h$ this yields that

$$f_i(v) = c^* + c_h - f_h(v - \Delta_v(T)u_T) + f_i(v - \Delta_v(T)u_T) = c^* + c_i, \tag{2}$$

where $c_i := c_h - f_h(v - \Delta_v(T)u_T) + f_i(v - \Delta_v(T)u_T)$ is known.

GOTO STEP 3.

STEP 3 If $k = m$ then STOP.

Else let $k = k + 1$. GOTO STEP 2.

Since $T^0, \ldots, T^m$ is a partition of N consisting of non-empty sets only, this procedure determines all values c_i, $i \in N$. Efficiency then implies that $\sum_{i \in N} f_i(v) = |N| \cdot c^* + \sum_{i \in N} c_i = v(N)$. From this it follows that the value c^* is uniquely determined, and thus the values $f_i(v)$, $i \in N$, are uniquely determined by equation (2).

2. If $|\mathcal{B}(v)| \geq 3$, then take a $j \in N$ and suppose that $f_j(v) = c^*$. For every $i \in N \setminus \{j\}$ there is a $T \in D(v)$ with $T \cap \{i, j\} = \emptyset$. Fairness then implies that $f_i(v) = c^* - f_j(v - \Delta_v(T)u_T) + f_i(v - \Delta_v(T)u_T)$. By the induction hypothesis we determined all $f_i(v - \Delta_v(T)u_T)$, $i \in N$. Efficiency then uniquely determines c^*, and thus all $f_i(v)$, $i \in N$.

3. Finally, suppose that $|\mathcal{B}(v)| = 2$, i.e., $\mathcal{B}(v) = \{B^1, B^2\}$. Suppose without loss of generality that $|B^2| \geq 2$. Take a $j \in B^1$, and suppose that $f_j(v) = c^*$. Fairness then implies that for every $i \in B^1$ there is some $T^2 \in D(v)$, $T^2 \subset B^2$, with $f_i(v) = c^* - f_j(v - \Delta_v(T^2)u_{T^2}) + f_i(v - \Delta_v(T^2)u_{T^2})$.

   Take a $T \subset B^2$, $T \in D(v)$ and $h \in B^2 \setminus T$. (Such an h exists by assumption.) Fairness implies that $f_h(v) = c^* - f_j(v - \Delta_v(T)u_T) + f_h(v - \Delta_v(T)u_T)$.

   Finally, for every $i \in B^2 \setminus \{h\}$ there is a $T^1 \in D(v)$ with $T^1 \subset B_1$, and thus fairness implies that $f_i(v) = f_h(v) - f_h(v - \Delta_v(T^1)u_{T^1}) + f_j(v - \Delta_v(T^1)u_{T^1})$ for $i \in B^2 \setminus \{h\}$. Efficiency and the induction hypothesis again uniquely determine c^*, and thus all values $f_i(v)$, $i \in N$.

Thus, there can be at most one solution $f\colon \mathcal{G}^N \to \mathbb{R}^N$ that satisfies efficiency, the null player property, and fairness. Since the Shapley value satisfies these axioms, f must be equal to the Shapley value.

$\square$

The independence of the three axioms of Theorem 2.5 can be illustrated by the following three well-known solutions:

1. The *Banzhaf value* $\beta: \mathcal{G}^N \to \mathbb{R}^N$ given by

$$\beta_i(v) = \frac{1}{2^{|N|-1}} \sum_{S \ni i} (v(S) - v(S \setminus \{i\})) \text{ for all } i \in N,$$

   satisfies the null player property and fairness. It does not satisfy efficiency. The Banzhaf value is introduced in Banzhaf (1965) for simple games. Characterizations of the Banzhaf value for TU-games can be found in, e.g., Lehrer (1988) and Haller (1994).

2. The *egalitarian rule* $\gamma: \mathcal{G}^N \to \mathbb{R}^N$ given by

$$\gamma_i(v) = \frac{v(N)}{|N|} \text{ for all } i \in N,$$

   satisfies efficiency and fairness. It does not satisfy the null player property.

3. The *normalized Banzhaf value* $\overline{\beta}: \mathcal{G}^N \to \mathbb{R}^N$ given by

$$\overline{\beta}_i(v) = \frac{\beta_i(v)}{\sum_{j \in N} \beta_j(v)} v(N) \text{ for all } i \in N,$$

   satisfies efficiency and the null player property. It does not satisfy fairness. A characterization of the normalized Banzhaf value can be found in van den Brink and van der Laan (1998).

Fairness and Young's strong monotonicity do not imply one another, as can be seen from the following examples:

1. The egalitarian rule $\gamma: \mathcal{G}^N \to \mathbb{R}^N$ satisfies fairness but does not satisfy strong monotonicity.

2. Let $f: \mathcal{G}^N \to \mathbb{R}^N$ be given by $f_i(v) = \begin{cases} Sh_i(v) & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$

   This solution satisfies strong monotonicity but does not satisfy fairness.

As noted by Dubey (1975), axioms that caharacterize the Shapley value on $\mathcal{G}^N$ need not characterize the Shapley value on the class of simple games. A TU-game $v \in \mathcal{G}^N$ is a simple game if $v(S) \in \{0,1\}$ for all $S \subset N$. However, efficiency, the null player property, and fairness do characterize the Shapley value on the class $\mathcal{G}_S^N$ which consists of all simple games on N. If we restrict ourselves to $\mathcal{G}_S^N$ then efficiency and the null player property are required only for simple games. Fairness is required for pairs of simple games $v, w \in \mathcal{G}_S^N$ for which the sum game $(v+w)$ also is a simple game.

Theorem 2.8 *A solution $f: \mathcal{G}_S^N \to \mathbb{R}^N$ is equal to the Shapley value if and only if it satisfies efficiency, the null player property, and fairness on $\mathcal{G}_S^N$.*

Proof

The Shapley value satisfies efficiency, the null player property and fairness on $\mathcal{G}_S^N$ since it satisfies these properties on $\mathcal{G}^N \supset \mathcal{G}_S^N$.

Now, suppose that $f: \mathcal{G}_S^N \to \mathbb{R}^N$ satisfies efficiency, the null player property and fairness on $\mathcal{G}_S^N$, and let $v \in \mathcal{G}_S^N$. We define

$$D^s(v) = \{T \subset N \mid v(T) = 1\}, \text{ and } d^s(v) = |D^s(v)|.$$

We show that $f(v) = Sh(v)$ by induction on the number $d^s(v)$.
If $d^s(v) = 0$ then v is the null game, and the null player property implies that $f_i(v) = 0$ for all $i \in N$. Thus, $f(v) = Sh(v)$ in this case.
If $d^s(v) = 1$ then v is the *standard game*[3] of some coalition $T \subset N$, i.e., $v = b_T$ for some $T \subset N$. Now we first consider the corresponding unanimity game u_T. (Note that every unanimity game is a simple game.) Similarly as in the proof of Lemma 2.6 it follows that $f_i(u_T) = \frac{1}{|T|}$ if $i \in T$, and $f_i(u_T) = 0$ otherwise.
Next we determine $f(b_T)$ by the following procedure:

[3]The standard game b_T of coalition $T \subset N$ is given by $b_T(S) = 1$ if $S = T$, and $b_T(S) = 0$, otherwise.

STEP 1 Let $H^0 = N \setminus T$, $k = 0$, $v^0 = u_T$, and $c^0 = \frac{1}{|T|}$ (and thus $f_i(v^0) = c^0$ for all $i \in T$ and $f_i(v^0) = 0$ for all $i \in H^0$). GOTO STEP 2.

STEP 2 IF $H^k = \emptyset$ then STOP.

ELSE take a $j \in H^k$ and define $v^{k+1} = v^k - \sum_{\substack{S \subset N \\ S \supset T \cup \{j\}}} b_S$.

The null player property implies that $f_i(v) = 0$ for all $i \in H^k \setminus \{j\}$.

Since all $f_i(u_T)$ are equal for $i \in T$, fairness implies that there exists a $c^{k+1} \in \mathbb{R}$ such that $f_i(v^{k+1}) = c^{k+1}$ for all $i \in T$.

Applying fairness to v^k and v^{k+1} yields for every $i \in T$ that $f_j(v^{k+1}) = f_i(v^{k+1}) - f_i(v^k) + f_j(v^k) = c^{k+1} - c^k + f_j(v^k)$.

Applying fairness to u_T and v^{k+1} then also yields that $f_i(v^{k+1}) = f_j(v^{k+1})$ for all $i \in N \setminus (T \cup H^k)$.

So,

$$f_i(v^{k+1}) = \begin{cases} c^{k+1} & \text{, if } i \in T \\ c^{k+1} - c^k + f_j(v^k) & \text{, if } i \in (N \setminus (T \cup H^k)) \cup \{j\}) \\ 0 & \text{, if } i \in H^k \setminus \{j\}. \end{cases}$$

Since c^{k+1} is the only unkown, efficiency uniquely determines c^{k+1}, and thus $f(v^{k+1})$. GOTO STEP 3.

STEP 3 Let $k = k + 1$ and $H^k = H^{k-1} \setminus \{j\}$. GOTO STEP 2.

By this procedure we have determined $f(b_T)$.

If $d^s(v) = 2$ then $v = b_T + b_H$ for some $T, H \subset N$, $T \neq H$, and $f(v)$ is determined in a way similar as the case $d(v) = 2$ in the proof of Lemma 2.6, but with the role of unanimity games replaced by standard games. Besides replacing unanimity games by standard games, we also should avoid the null player property since this property cannot be used in this case. We do this as follows. In the case $T \cap H \neq \emptyset$, assume without loss of generality that $T \setminus H \neq \emptyset$. Take an $h \in T \setminus H$. For every $i \in N \setminus (T \cup H)$, fairness implies that $f_i(v) - f_i(b_T) = f_h(v) - f_h(b_T)$. Since $f_i(b_T)$ and $f_h(b_T)$ are known, and $f_h(v)$ is expressed as c^* plus a known constant, we also have expressed $f_i(v)$ as c^* plus a known constant. Efficiency, again determines c^*, and thus $f(v)$ is determined.

In a similar way the null player property can be avoided in case $T \cap H = \emptyset$, $T \cup H \neq N$. (In the third and last case, $T \cap H = \emptyset$, $T \cup H = N$, the null player property is not used.)

Proceeding by induction we assume that $f(v') = Sh(v')$ for all $v' \in \mathcal{G}_S^N$ with $d^s(v') \leq k$ ($k \geq 2$), and let $d^s(v) = k+1$. Again, it can be shown that $f(v)$ is uniquely determined by replacing unanimity games in the proof of Theorem 2.5 by standard games, and replacing connected coalitions in v by standard connected coalitions in v. Here, we define two players $i, j \in N$ to be **standard connected** in v if there exists a sequence of coalitions $(T^1, \ldots, T^m)$ such that

(i) $i \in T^1,\ j \in T^m$;

(ii) $T^k \cap T^{k+1} \neq \emptyset$ for all $k \in \{1, \ldots, m-1\}$;

(iii) $v(T^k) \neq 0$ for all $k \in \{1, \ldots, m\}$.

Then we denote by $\mathcal{B}^s(v)$ the partition of N into maximal standard connected coalitions, where $B \subset N$ is a **maximal standard connected coalition** in v if and only if the following two conditions are satisfied:

(i) for every $i, j \in B$ it holds that i and j are standard connected in v;

(ii) for every $i \in B$ and $j \in N \setminus B$ it holds that i and j are not standard connected in v.

□

For the smaller class of *apex games*, van den Brink (1995a) shows that efficiency, the null player property and fairness characterize the Shapley value on this class of games. The apex game $a_{j,J}$, $j \in N$, $J \subset N \setminus \{j\}$, assigns the value one to every coalition that either contains J or contains the apex player j and at least one player from J. All other coalitions are assigned the value zero. This fairness property for apex games states that making a non-apex player a null player changes the payoffs of this non-apex player and the apex player by the same amount if $|J| \geq 2$, i.e., $f_i(a_{j,J}) - f_i(a_{j,J\setminus\{i\}}) = f_j(a_{j,J}) - f_j(a_{j,J\setminus\{i\}})$ for all $i \in J$, $|J| \geq 2$.

Although the purpose of this paper is to characterize the Shapley value on classes of games with fixed player set N, we conclude this section by comparing fairness with the concept of balanced contributions as considered in, e.g., Myerson (1980) and Hart and Mas-Colell (1989). This property is stated for games with variable sets of players. In order to state this property, we therefore denote in this paragraph a TU-game as a pair (N, v), and by $\mathcal{G}$ we denote the collection of all TU-games. For a characteristic function v on N and coalition $T \subset N$ we denote by v_T the *restricted* characteristic function on T given by $v_T(S) = v(S)$ for all $S \subset T$. A solution on $\mathcal{G}$ is a function f that assigns to every game $(N, v) \in \mathcal{G}$ an $|N|$-dimensional real vector representing a payoff distribution over the players in N. (Thus, to games with player sets of different size such a solution assigns vectors of different dimension.) A solution on $\mathcal{G}$ has *balanced contributions* if for every $(N, v) \in \mathcal{G}$ and $i, j \in N$ it holds that[4] $f_i(N, v) - f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) = f_j(N, v) - f_j(N \setminus \{i\}, v_{N\setminus\{i\}})$.

It is easy to verify that the egalitarian rule satisfies fairness[5] but does not have balanced contributions. Under the assumptions that a solution f on $\mathcal{G}$ satisfies *single player efficiency*[6] and *permutation neutrality*[7] it holds that f satisfies fairness if it has balanced contributions.

Proposition 2.9 *If f is a solution for TU-games that satisfies single player efficiency, permutation neutrality and has balanced contributions, then f satisfies fairness.*

[4] For convenience we write $f(N, v)$ instead of $f((N, v))$.

[5] Here fairness is defined on the class $\mathcal{G}$ in a straightforward manner.

[6] A solution f on $\mathcal{G}$ satisfies single player efficiency if it is efficient for all 1-player games, i.e., if $f_i(\{i\}, v) = v(\{i\})$ for all $(N, v) \in \mathcal{G}$ with $|N| = 1$.

[7] A solution f on $\mathcal{G}$ satisfies permutation neutrality if for every $(N, v) \in \mathcal{G}$, $\overline{N} \supset N$, permutation $\pi: \overline{N} \to \overline{N}$, and player $i \in N$ with $\pi(i) = i$ it holds that $f_i(N, v) = f_i(\pi N, \pi v)$, where $\pi N = \bigcup_{j \in N} \pi(j)$, and the characteristic function πv on πN is given by $\pi v(S) = v(\bigcup_{j \in S} \pi(j))$ for all $S \subset \pi N$.

Proof

Let f be a solution on $\mathcal{G}$ that satisfies single player efficiency, permutation neutrality, and has balanced contributions. Further, let $(N,v),(N,w) \in \mathcal{G}$, and $i,j \in N$ be such that i and j are symmetric in w. We show that f satisfies fairness by induction on the number of players in N.

If $|N| = 1$ then single player efficiency implies that $f_i(N,v) = v(\{i\})$ for $i \in N$.

If $|N| = 2$, i.e., $N = \{i,j\}$, then f having balanced contributions implies that

$$\begin{aligned}
f_i(N,v+w) - f_j(N,v+w) &= f_i(N \setminus \{j\}, (v+w)_{N\setminus\{j\}}) - f_j(N \setminus \{i\}, (v+w)_{N\setminus\{i\}}) \\
&= (v+w)(\{i\}) - (v+w)(\{j\}) = v(\{i\}) - v(\{j\}) \\
&= f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) - f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) \\
&= f_i(N,v) - f_j(N,v).
\end{aligned}$$

Proceeding by induction we assume that $f_i(N',v'+w') - f_i(N',v') = f_j(N',v'+w') - f_j(N',v')$ for all $(N',v') \in \mathcal{G}$ with $|N'| \leq k$, and i,j symmetric in w'.

Let $|N| = k+1$, and $h \in N \setminus \{i,j\}$. (Such an h exists since $|N| \geq 3$.) Since f has balanced contributions it holds that

$$f_i(N,v+w) - f_h(N,v+w) = f_i(N \setminus \{h\}, (v+w)_{N\setminus\{h\}}) - f_h(N \setminus \{i\}, (v+w)_{N\setminus\{i\}}),$$

and

$$f_j(N,v+w) - f_h(N,v+w) = f_j(N \setminus \{h\}, (v+w)_{N\setminus\{h\}}) - f_h(N \setminus \{j\}, (v+w)_{N\setminus\{j\}}).$$

Using the induction hypothesis and the facts that f has balanced contributions and satisfies permutation neutrality it then follows that

$$\begin{aligned}
f_i(N,v+w) - f_j(N,v+w) &= f_i(N \setminus \{h\}, (v+w)_{N\setminus\{h\}}) - f_j(N \setminus \{h\}, (v+w)_{N\setminus\{h\}}) \\
&= f_i(N \setminus \{h\}, v_{N\setminus\{h\}}) - f_j(N \setminus \{h\}, v_{N\setminus\{h\}})
\end{aligned}$$

$$
\begin{aligned}
&= f_i(N,v) - f_h(N,v) + f_h(N \setminus \{i\}, v_{N\setminus\{i\}}) \\
&\quad -f_j(N,v) + f_h(N,v) - f_h(N \setminus \{j\}, v_{N\setminus\{j\}}) \\
&= f_i(N,v) - f_j(N,v).
\end{aligned}
$$

Thus, f satisfies fairness.

$\square$

3 Weighted Shapley values

In the literature various kinds of *weighted Shapley values* have been studied. An example of such a weighted Shapley value is the one considered in Shapley (1953b). This weighted Shapley value is the function $Sh^w \colon \mathcal{G}^N \times \mathrm{I\!R}^N_{++} \to \mathrm{I\!R}^N$ given by

$$Sh^w_i(v,\lambda) = \sum_{S \ni i} \left(\frac{\lambda_i}{\sum_{j\in S} \lambda_j} \right) \Delta_v(S) \quad \text{for all } i \in N,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{I\!R}^N_{++}$ is a vector that assigns positive *weights* to the players in N. If $\lambda_i = 1$ for all $i \in N$ then $Sh^w(v,\lambda) = Sh(v)$ for all $v \in \mathcal{G}^N$. Thus, Sh^w is a generalization of the Shapley value.

For functions $f \colon \mathcal{G}^N \times \mathrm{I\!R}^N_{++} \to \mathrm{I\!R}^N$, efficieny and the null player property can be generalized in a straigthforward way.

Axiom 3.1 (λ-efficiency) *For every $v \in \mathcal{G}^N$, and $\lambda \in \mathrm{I\!R}^N_{++}$ it holds that $\sum_{i\in N} f_i(v,\lambda) = v(N)$.*

Axiom 3.2 (λ-null player property) *For every $v \in \mathcal{G}^N$ and $\lambda \in \mathrm{I\!R}_{++}$ it holds that $f_i(v,\lambda) = 0$ if i is a null player in v.*

Thus, the sum of the payoffs that are assigned to the players in N is equal to the worth $v(N)$ irrespective of the weights that are assigned to the players. Similarly, a null player always gets a zero payoff, irrespective of the weights. We generalize fairness in the following way.

Axiom 3.3 (**λ-fairness**) *Let $\lambda \in \mathbb{R}^N_{++}$, and let $i, j \in N$ be symmetric in $w \in \mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it holds that $\lambda_j(f_i(v+w) - f_i(v)) = \lambda_i(f_j(v+w) - f_j(v))$.*

Note that for $\lambda \in \mathbb{R}^N_{++}$ with $\lambda_i = 1$ for all $i \in N$, these axioms boil down to the corresponding axioms stated in Section 2.

Theorem 3.4 *A function $f: \mathcal{G}^N \times \mathbb{R}^N_{++} \to \mathbb{R}^N$ is equal to the weighted Shapley value Sh^w if and only if it satisfies λ-efficiency, the λ-null player property, and λ-fairness.*

The proof is obtained by adapting the proof of Theorem 2.5 in a straightforward way (in particular, the use of fairness), and is therefore omitted.

Another type of weighted Shapley value has been considered in Kalai and Samet (1987). A *weight system* is a pair $\omega = (\lambda, \Sigma)$, where $\lambda \in \mathbb{R}^N_{++}$ is a vector of weights and $\Sigma = (S^1, \ldots, S^m)$ is an *ordered* partition of N. Let $\mathcal{S}^N$ denote the collection of all ordered partitions of N. For every $\Sigma = (S^1, \ldots, S^m) \in \mathcal{S}^N$ and $S \subset N$ we denote $k_\Sigma(S) = \max\{k \in \{1, \ldots, m\} \mid S \cap S^k \neq \emptyset\}$, and $K_\Sigma(S) = S \cap S^{k_\Sigma(S)}$. The *KS-weighted Shapley value* is the function $Sh^{ks}: \mathcal{G}^N \times \mathbb{R}^N_{++} \times \mathcal{S}^N \to \mathbb{R}^N$ given by

$$Sh^{ks}_i(v, \lambda, \Sigma) = \sum_{\substack{S \subset N \\ K_\Sigma(S) \ni i}} \left(\frac{\lambda_i}{\sum_{j \in K_\Sigma(S)} \lambda_j} \right) \Delta_v(S) \text{ for all } i \in N.$$

If $\Sigma = (N)$ then $Sh^{ks}(v, \lambda, \Sigma) = Sh^w(v, \lambda)$ for all $v \in \mathcal{G}^N$ and $\lambda \in \mathbb{R}^N_{++}$. Thus, Sh^{ks} is a generalization of Sh^w (and thus also a generalization of Sh). We generalize λ-efficiency, the λ-null player property, and λ-fairness in the following way.

Axiom 3.5 (**λ, Σ-efficiency**) *For every $v \in \mathcal{G}^N$, $\lambda \in \mathbb{R}^N_{++}$, and $\Sigma \in \mathcal{S}^N$ it holds that $\sum_{i \in N} f_i(v, \lambda, \Sigma) = v(N)$.*

Axiom 3.6 (**λ, Σ-null player property**) *For every $v \in \mathcal{G}^N$, $\lambda \in \mathbb{R}_{++}$ and $\Sigma \in \mathcal{S}^N$ it holds that $f_i(v, \lambda, \Sigma) = 0$ if i is a null player in v.*

Thus, the sum of the payoffs that are assigned to the players in N again is equal to the worth $v(N)$ irrespective of the weights that are assigned to the players and the way the players are ordered in the partition Σ. Similarly, null players earn nothing irrespective of the weights and the ordering of the players in the partition Σ.

Axiom 3.7 **(λ, Σ-fairness)** *Let $\lambda \in \mathbb{R}^N_{++}$, $\Sigma = (S^1, \ldots, S^m) \in \mathcal{S}^N$, and let $i \in S^k$ and $j \in S^l$, $k, l \in \{1, \ldots, m\}$, be symmetric in $w \in \mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it holds that*

(i) $\lambda_j(f_i(v+w) - f_i(v)) = \lambda_i(f_j(v+w) - f_j(v))$ *if* $k = l$;

(ii) $f_i(v+w) - f_i(v) = 0$ *if* $k < l$.

Again, by adapting the proof of Theorem 2.5 it can be shown that the KS-weighted Shapley value is the unique function $f: \mathcal{G}^N \times \mathbb{R}^N_{++} \times \mathcal{S}^N \to \mathbb{R}^N$ that satisfies λ, Σ-efficiency, the λ, Σ-null player property, and λ, Σ-fairness.

Appendix: connected coalitions and components in TU-games

Maximal connected coalitions as used in the proof of Theorem 2.5 coincide with minimal components in TU-games. Components in TU-games are already considered in, e.g., Aumann and Drèze (1974) and Chang and Kan (1994). A coalition $B \subset N$ is a *component* in game $v \in \mathcal{G}^N$ if it acts 'independently' of the players in $N \setminus B$, in the sense that the worth of any coalition $S \subset N$ is equal to the worth that can be obtained by those players in S who also belong to B plus the worth that can be obtained by the coalition of other players in S. A component is called a *minimal* component in a game if all its strict subsets are not components in that game.

Definition 3.8 *Let $v \in \mathcal{G}^N$. Coalition $B \subset N$ is a* **component** *in v if*

$$v(S) = v(S \cap B) + v(S \setminus B) \quad \textit{for all } S \subset N.$$

Coalition $B \subset N$ is a **minimal** *component in v if B is a component in v and every $T \subset B$, $T \neq B$, is not a component in v.*

It is easy to show that the empty set and the 'grand coalition' N are components in every $v \in \mathcal{G}^N$. Moreover, for every pair of components in v it holds that their union and their intersection both are components in v. (This is shown in van den Brink (1995b). For completeness we give the proof below.)

Theorem A.1 For every $v \in \mathcal{G}^N$ it holds that (i) $\emptyset$ and N are components in v, and (ii) if $B^1, B^2 \subset N$ are components in v then $B^1 \cup B^2$ and $B^1 \cap B^2$ are components in v.

Proof
Let $v \in \mathcal{G}^N$. Then

(i) for every $S \subset N$ it holds that $v(S \cap \emptyset) + v(S \setminus \emptyset) = v(\emptyset) + v(S) = v(S)$, and $v(S \cap N) + v(S \setminus N) = v(S) + v(\emptyset) = v(S)$.

(ii) for every pair of components $B^1, B^2 \subset N$ in v, repeatedly applying the definition of a component yields that for every $S \subset N$ it holds that

$$v(S \cap (B^1 \cup B^2)) + v(S \setminus (B^1 \cup B^2)) =$$

$$= v((S \cap (B^1 \cup B^2)) \cap B^1) + v((S \cap (B^1 \cup B^2)) \setminus B^1) + v((S \setminus B^1) \setminus B^2)$$

$$= v(S \cap B^1) + v((S \cap B^2) \setminus B^1) + v(S \setminus B^1) - v((S \setminus B^1) \cap B^2)$$

$$= v(S \cap B^1) + v(S \setminus B^1) = v(S),$$

and

$$v(S \cap (B^1 \cap B^2)) + v(S \setminus (B^1 \cap B^2)) =$$

$$v((S \cap B^1) \cap B^2) + v((S \setminus (B^1 \cap B^2)) \cap B^1) + v((S \setminus (B^1 \cap B^2)) \setminus B^1) =$$

$$= v(S \cap B^1) - v((S \cap B^1) \setminus B^2) + v((S \setminus B^2) \cap B^1) + v(S \setminus B^1) =$$

$$= v(S \cap B^1) + v(S \setminus B^1) = v(S).$$

$\square$

From this theorem it follows that for every game v there is a unique partition $\mathcal{B} = (B^1, \ldots, B^m)$ of N such that every $B^k,\ k \in \{1, \ldots, m\}$, in this partition is a minimal component in v.

Components in game v can be characterized by connectedness of players. It turns out that a coalition $B \subset N$ is a component in game v if and only if for every pair of players $i \in B$ and $j \in N \setminus B$ it holds that i and j are *not* connected in v.

Lemma A.2 Let $v \in \mathcal{G}^N$. Coalition $B \subset N$ is a component in v if and only if for every $i \in B$ and $j \in N \setminus B$ it holds that i and j are not connected in v.

Proof

Let $v \in \mathcal{G}^N$.

Only if

Let B be a component in v, and let $i \in B$ and $j \in N \setminus B$. Suppose that i and j are connected in v. Then there exists an $S \subset N$ such that

(i) $S \cap B \neq \emptyset,\ S \not\subset B,\ \Delta_v(S) \neq 0$;

(ii) $\Delta_v(T) = 0$ for every $T \subset S$ with $T \cap B \neq \emptyset$ and $T \not\subset B$.

(Note that it is not necessary that $S \ni i$ nor $T \ni j$.) Since $v = \sum_{T \subset N} \Delta_v(T) u_T$ it holds that

$$\begin{aligned}
v(S \cap B) + v(S \setminus B) &= \sum_{T \subset S \cap B} \Delta_v(T) + \sum_{T \subset S \setminus B} \Delta_v(T) \\
&\neq \sum_{T \subset S \cap B} \Delta_v(T) + \sum_{T \subset S \setminus B} \Delta_v(T) + \Delta_v(S) \\
&= \sum_{T \subset S \cap B} \Delta_v(T) + \sum_{T \subset S \setminus B} \Delta_v(T) + \sum_{\substack{T \subset S \\ T \cap B \neq \emptyset, T \not\subset B}} \Delta_v(T) \\
&= \sum_{T \subset S} \Delta_v(T) = v(S).
\end{aligned}$$

This is in contradiction with B being a component in v. Thus, i and j are not connected in v.

If

Let $B \subset N$. Suppose that for every $i \in B$ and $j \in N \setminus B$ it holds that i and j are not connected in v. Then $\Delta_v(T) = 0$ for all $T \subset N$ with $T \cap B \neq \emptyset$ and $T \not\subset B$. But then, for every $S \subset N$, it holds that

$$\begin{aligned} v(S \cap B) + v(S \setminus B) &= \sum_{T \subset S \cap B} \Delta_v(T) + \sum_{T \subset S \setminus B} \Delta_v(T) = \\ &= \sum_{T \subset S \cap B} \Delta_v(T) + \sum_{T \subset S \setminus B} \Delta_v(T) + \sum_{\substack{T \subset S \\ T \cap B \neq \emptyset, T \not\subset B}} \Delta_v(T) \\ &= \sum_{T \subset S} \Delta_v(T) = v(S). \end{aligned}$$

Thus, B is a component in v.

□

Moreover, component B is a minimal component in v if and only if for every pair of players $i, j \in B$ it holds that i and j are connected in v.

Lemma A.3 Let $B \subset N$ be a component in $v \in \mathcal{G}^N$. Then B is a minimal component in v if and only if for every $i, j \in B$ it holds that i and j are connected in v.

Proof

Let $v \in \mathcal{G}^N$ and let $B \subset N$ be a component in v.

Only if

Suppose that there exist $i, j \in B$ such that i and j are not connected in v.
Let $H := \{h \in B \mid i \text{ and } h \text{ are connected in } v\}$. Then $B \setminus H \ni j$.
By definition of connectedness it holds that $\Delta_v(T) = 0$ for every $T \subset N$ with $T \cap H \neq \emptyset$ and $T \not\subset H$. By B being a component in v and $H \subset B$ it follows that for every $S \subset N$

it holds that

$$\begin{aligned} v(S\cap H)+v(S\setminus H) &= v(S\cap H)+v((S\setminus H)\cap B)+v((S\setminus H)\setminus B) \\ &= v(S\cap H)+v(S\cap(B\setminus H))+v(S\setminus B) \\ &= \sum_{T\subset S\cap H}\Delta_v(T)+\sum_{T\subset S\cap(B\setminus H)}\Delta_v(T)+v(S\setminus B) \\ &= \sum_{T\subset S\cap H}\Delta_v(T)+\sum_{T\subset S\cap(B\setminus H)}\Delta_v(T)+\sum_{\substack{T\subset S\cap B\\ T\cap H\neq\emptyset, T\not\subset H}}\Delta_v(T)+v(S\setminus B) \\ &= \sum_{T\subset S\cap B}\Delta_v(T)+v(S\setminus B)=v(S\cap B)+v(S\setminus B)=v(S). \end{aligned}$$

Thus, H is a component in v. Since $H\subset B$, $H\neq B$, this implies that B is not a minimal component in v.

If

Suppose that B is not a minimal component in v, i.e., there exists a $T\subset B$, $T\neq B$, such that T is a component in v. From Lemma A.2 it then follows that for every $i\in T$ and $j\in B\setminus T$ it holds that i and j are not connected in v.

□

Combining Lemma's A.2 and A.3 yields the following theorem.

Theorem A.4 Let $v\in\mathcal{G}^N$. Then B is a minimal component in v if and only if B is a maximal connected coalition in v.

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