An CUSUM Test with Observation-Adjusted Control Limits in Change Detection

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ABSTRACT

In this paper, we not only propose an new optimal sequential test of sum of logarithmic likelihood ratio (SLR) but also present the CUSUM sequential test (control chart, stopping time) with the observation-adjusted control limits (CUSUM-OAL) for monitoring quickly and adaptively the change in distribution of a sequential observations. Two limiting relationships between the optimal test and a series of the CUSUM-OAL tests are established. Moreover, we give the estimation of the in-control and the out-of-control average run lengths (ARLs) of the CUSUM-OAL test. The theoretical results are illustrated by numerical simulations in detecting mean shifts of the observations sequence.

Keywords: Optimal sequential test, CUSUM-OAL test, change detection.

1 INTRODUCTION

In order to quickly detect a change in distribution of observations sequence without exceeding a certain false alarm rate, a great variety of sequential tests have been proposed, developed and applied to various fields since Shewhart (1931) proposed a control chart method, see, for example, Siegmund (1985), Basseville and Nikiforov (1993), Lai (1995, 2001), Stoumbos *et al.* (2000), Chakraborti *et al.* (2001), Bersimis *et al.* (2007), Montgomery (2009), Qiu (2014), Tartakovsky *et al.* (2015), Woodall *et al.* (2017), Bersimis *et al.* (2018) and Chakrabortia and Graham (2019).

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One of popular used sequential tests is the following upper-sided CUSUM test which was proposed by Page (1954).

$$T_C(c) = \min\{n \ge 0 : \max_{1 \le k \le n} \sum_{i=n-k+1}^n Z_i \ge c\},$$
(1)

where c > 0 is a constant control limit, $Z_i = \log[p_{v_1}(X_i)/p_{v_0}(X_i)]$, $p_{v_0}(x)$ and $p_{v_1}(x)$ are prechange and post-change probability density functions respectively for a sequence of mutually independent observations $\{X_i, i \ge 1\}$, that is, there is a unknown change-point $\tau \ge 1$ such that $X_1, ..., X_{\tau-1}$ have the probability density function p_{v_0} , whereas, $X_{\tau}, X_{\tau+1}, ...$ have the probability density function p_{v_1} . By the renewal property of the CUSUM test T_C we have $sup_{k\ge 1}\mathbf{E}_k(T_C - k + 1|T_C \ge k) = \mathbf{E}_1(T_C)$ (see Siegmund 1985, P.25), where $\mathbf{E}_1(T_C)$ is the out-of-control average run length (ARL_1), \mathbf{P}_k and \mathbf{E}_k denote the probability and expectation respectively when the change from p_{v_0} to p_{v_1} occurs at the change-point $\tau = k$ for $k \ge 1$.

Though we know that the CUSUM test is optimal under Lorden's measure (see Moustakides 1986 and Ritov 1990), the out-of-control ARL₁ of the CUSUM test is not small, especially in detecting small mean shifts (see Table 1 in Section 4). In other words, the CUSUM test is insensitive in detecting small mean shifts. Then, how to increase the sensitivity of the CUSUM test ? Note that the control limit in the CUSUM test is a constant c which does not depend on the observation samples. Intuitively, if the control limit of the CUSUM test can become low as the samples mean of the observation sequence increases, then the alarm time of detecting the increasing mean shifts will be greatly shortened. Based on this idea, by selecting a decreasing function g(x) we may define the (upper-sided) CUSUM chart $T_C(cg)$ with the observation-adjusted control limits $cg(\hat{Z}_n)$ (abbreviated to the CUSUM-OAL chart) in the following

$$T_C(cg) = \min\{n \ge 0 : \max_{0 \le k \le n} \sum_{i=n-k+1}^n Z_i \ge cg(\hat{Z}_n)\},$$
(2)

where c > 0 is a constant and $\hat{Z}_n = \sum_{i=1}^n Z_i/n$. In other words, the control limits $cg(\hat{Z}_n)$ of the CUSUM-OAL test can be adjusted adaptively according to the observation information $\{\hat{Z}_n\}$. Note that the control limits $cg(\hat{Z}_n)$ may be negative. In the special case, the CUSUM-OAL chart $T_C(cg)$ becomes into the conventional CUSUM chart $T_C(c)$ in (1) when $g \equiv 1$. Similarly, we can define a down-sided CUSUM-OAL test. In this paper, we consider only the upper-sided CUSUM-OAL test since the properties of the down-sided CUSUM-OAL test can be obtained by the similar method.

The main purpose of the present paper is to show the good detection performance of the CUSUM-OAL test and to give the estimation of its the in-control and out-of-control ARLs.

The paper is organized as follows. In Section 2, we first present an optimal SLR sequential test, then define two sequences of the CUSUM-OAL tests and prove that one of the two sequences of CUSUM-OAL tests converges to the optimal test, another sequences of CUSUM-OAL tests converges to a combination of the optimal test and the CUSUM test. The estimation of the in-control and out-of-control ARLs of the CUSUM-OAL tests and their comparison are

given in Section 3. The detection performances of the three CUSUM-OAL tests and the conventional CUSUM test are illustrated in Section 4 by comparing their numerical out-of-control ARLs. Section 5 provides some concluding remarks. Proofs of the theorems are given in the Appendix.

2 AN OPTIMAL SLR TEST, TWO CUSUM-OAL TESTS AND THEIR LIMITING RELATIONSHIPS

Let \mathbf{P}_0 and \mathbf{E}_0 denote the probability and the expectation respectively with the probability density p_{v_0} when there is no change for all the time. It is known that $\mathbf{E}_0(Z_1) = \mathbf{E}_{v_0}(\log[p_{v_1}(X_1)/p_{v_0}(X_1)]) < 0$ for $p_{v_1}(x) \neq p_{v_0}(x)$. Without loss of generality, let $\mu_0 = \mathbf{E}_0(Z_1) < 0$.

It follows from Proposition 2.38 in Siegmund (1985) and (5.8)-(5.9) in Chow *et al*, P.108) that the following sequence test of sum of logarithmic likelihood ratio (SLR)

$$T_{SLR} = \min\{n \ge 1 : \prod_{k=1}^{n} \frac{p_{v_1}(X_k)}{p_{v_0}(X_k)} \ge B\} = \min\{n \ge 1 : \sum_{j=1}^{n} Z_j \ge c\}$$
(3)

$$= \min\{n \ge 1 : \sum_{j=1}^{n} (Z_j - \mu_0) \ge c + n|\mu_0|\}$$
(4)

for B > 1, is optimal in the following sense

$$\min_{T: \mathbf{P}_0(T < \infty) \ge \alpha} {\mathbf{E}_1(T)} = {\mathbf{E}_1(T_{SLR})}$$

for $\mathbf{P}_0(T_{SLR} < \infty) = \alpha$, where $c = \log B$ and $0 < \alpha < 1$.

In particular, if \mathbf{P}_0 is the standard normal distribution with mean shift $\mu > 0$ after changepoint, we have $Z_j - \mu_0 = \mu X_j$, where $\mu_0 = -\mu^2/2$. It follows from proposition 4 in Frisén (2003) that the SLR test T_{SLR} in (4) is also optimal (minimal ARL₁) with the same false alarm probability $\mathbf{P}_0(T < \tau)$.

It can be seen that the in-control average run length of T_{SLR} is infinite, that is, $ARL_0 = \mathbf{E}_0(T_{SLR}) = \infty$. However, the minimal ARL_1 with finite ARL_0 is a widely used optimality criterion in statistical quality control (see Montgomery, 2009) and detection of abrupt changes (see Basseville and Nikiforov, 1993). In order to get finite ARL_0 for T_{SLR} , we replace the constant control limit c of T_{SLR} in (3) or (4) with the dynamic control limit $n(\mu_0 - r)$ and obtain a modified SLR test $T_{SLR}(r)$ in the following

$$T_{SLR}(r) = \min\{n \ge 1 : \sum_{j=1}^{n} Z_j \ge n(\mu_0 - r)\} = \min\{n \ge 1 : \sum_{j=1}^{n} (Z_j - \mu_0) \ge -rn\}$$
(5)

for $r \geq 0$.

For comparison, the in-control \mathbf{ARL}_0 of all candidate sequential tests are constrained to be equal to the same desired level of type I error, the test with the lowest out-of-control \mathbf{ARL}_v has the highest power or the fastest monitoring (detection) speed.

In the following example 1, the numerical simulations of the out-of-control ARLs of the CUSUM-OAL tests $T_C(cg_{u,0})$ in detecting the mean shifts of observations with normal distribution will be compared with that of the SLR tests $T^*(r)$ and $T^*(0)$, and that of the CUSUM-SLR test $T_C(c) \wedge T^*(0) := \min\{T_C(c), T^*(0)\}$ in the following Table 1. These comparisons lead us to guess that there are some limiting relationships between $T_C(cg_{u,r})$ and $T^*(r)$, and $T_C(c\tilde{g}_u)$ and $T_C(c) \wedge T^*(0)$, respectively.

Example 1. Let X_1, X_2, \ldots be mutually independent following the normal distribution N(0, 1) if there is no change. After the change-point $\tau = 1$, the mean $\mathbf{E}_{\mu}(X_k)$ ($k \ge 1$) will change from $v_0 = 0$ to v = 0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 3. Here, we let $p_{v_0}(x) = e^{-x^2/2}/\sqrt{2\pi}$, $p_{v_1}(x) = e^{-(x-1)^2/2}/\sqrt{2\pi}$ and therefore, $Z_k = X_k - 1/2$ for $k \ge 1$, where $v_1 = 1$ is a given reference value which for the CUSUM test is the magnitude of a shift in the process mean to be detected quickly. We conducted the numerical simulation based on 1,000,000 repetitions.

The following Table 1 lists the simulation results of the ARLs of the tests $T_C(c)$, $T_C(c\tilde{g}_u)$ for $u = 1, 10, 10^2, 10^3, 10^4, T^*(0.0007), T_C(c) \wedge T^*(0)$ and $T^*(0)$ for detecting the mean shifts, where the mean shift 0.0 means that there is no change which corresponds to the in-control ARL₀ and all tests have the common ARL₀ ≈ 1000 except the test $T^*(0)$ which has ARL₀ = ∞ . The values in the parameters are the standard deviations of the tests.

Tests	u, r				Shifts				
	С	0.0	0.1	0.25	0.5	0.75	1.0	1.5	3.00
$T_C(c)$		1000.59	439.00	147.72	38.91	17.32	10.50	5.82	2.61
	c = 5.0742	(993.16)	(431.89)	(140.25)	(31.79)	(11.23)	(5.49)	(2.26)	(0.66)
	u = 1.0	1000.43	237.34	46.84	10.02	4.39	2.65	1.52	1.03
	c = 5.6125	(1510.55)	(391.64)	(79.68)	(15.37)	(5.72)	(2.90)	(1.10)	(0.14)
$T_C(c\widetilde{g_u})$	u = 10.0	999.17	18.16	4.57	2.19	1.57	1.31	1.11	1.01
	c = 7.7790	(4969.50)	(77.77)	(11.06)	(3.01)	(1.45)	(0.86)	(0.38)	(0.04)
	$u = 10^2$	1000.66	8.15	3.38	1.92	1.47	1.27	1.11	1.01
	c = 9.97	(13951.15)	(29.90)	(6.98)	(2.33)	(1.20)	(0.73)	(0.34)	(0.04)
	$u = 10^3$	1001.49	7.56	3.28	1.89	1.46	1.26	1.11	1.01
	c = 11.38	(25423.13)	(26.45)	(6.60)	(2.27)	(1.18)	(0.72)	(0.33)	(0.04)
	$u = 10^4$	1001.32	7.51	3.26	1.89	1.45	1.24	1.11	1.01
	c = 11.84	(30042.90)	(26.22)	(6.56)	(2.27)	(1.17)	(0.72)	(0.33)	(0.04)
$T^*(r)$	r = 0.0007	1001	7.44	3.25	1.88	1.45	1.25	1.09	1.01
		(43017)	(26.07)	(6.57)	(2.24)	(1.18)	(0.71)	(0.33)	(0.04)
$T_C(c) \wedge T^*(0)$	r = 0	1001.13	7.52	3.25	1.88	1.45	1.25	1.09	1.01
	c = 11.9271	(31090.71)	(26.88)	(6.57)	(2.24)	(1.17)	(0.72)	(0.33)	(0.04)
$T^{*}(0)$	r = 0	∞	7.47	3.25	1.88	1.45	1.25	1.09	1.01
		(∞)	(26.24)	(6.54)	(2.27)	(1.18)	(0.72)	(0.33)	(0.04)

Table 1. ARLs of $T_C(c)$, $T_C(c\tilde{g}_u)$, $T^*(r)$, $T_C(c) \wedge T^*(0)$ and $T^*(0)$ when $\tau = 1$.

From the last row in Table 1, it's a little surprising that though the ARL₀ of $T^*(0)$ is infinite, that is, $\mathbf{E}_0(T^*(0)) = \infty$, the detection speed of $T^*(0)$ is faster than that of the CUSUM chart

 T_C for all mean shifts, in particular, for detecting the small mean shift 0.1, the speed of $T^*(0)$ is only 7.47 which is very faster than the speed, 439, of the CUSUM test. Moreover, both control charts $T^*(0.0007)$ and $T_C(11.9271) \wedge T^*(0)$ not only have the nearly same detection performance as $T^*(0)$ but also can have the finite in-control ARL₀. Note particularly that when the number u in \tilde{g}_u is taken from 0 to 1, 10, 10², 10³, 10⁴, the detection speed of $T_C(c\tilde{g}_u)$ is getting faster and faster, approaching to that of $T_C(c) \wedge T^*(0)$. This inspires us to prove the following theoretic results.

Theorem 2. Let $\tau = 1$ and $\{X_k, k \ge 1\}$ be an i.i.d. observations sequence with $\mu_0 = \mathbf{E}_0(Z_1) < 0$. Then

$$\lim_{u \to \infty} T_C(cg_{u,r}) = T^*(r), \quad \lim_{u \to \infty} T_C(c\widetilde{g}_u) = T_C(c) \wedge T^*(0).$$
(6)

Theorem 2 shows that when the constant control limit c of the CUSUM test $T_C(c)$ is replaced with the observation-adjusted control limits $\{cg_{u,r}(\hat{Z}_n)\}$ and $\{c\tilde{g}_u(\hat{Z}_n)\}$ respectively, the corresponding two CUSUM-OAL tests $\{T_C(cg_{u,r})\}\$ and $\{T_C(c\tilde{g}_u)\}\$ will converge to the optimal SLR test $T^*(r)$ and the CUSUM-SLR test $T_C(c) \wedge T^*(0)$ as $u \to \infty$, respectively. In other words, the fastest alarm times that $\{T_C(cg_{u,r})\}\$ and $\{T_C(c\tilde{g}_u)\}\$ can be reached are $T^*(r)$ and $T_C(c) \wedge T^*(0)$, respectively.

Remark 2. Since $T_C(cg_{0,r}) = T_C(c\tilde{g}_0) = T_C(c)$ when u = 0, it follows that both $\{T_C(cg_{u,r}) : u \ge 0\}$ and $\{T_C(c\tilde{g}_u) : u \ge 0\}$ can be seen as two "long bridges" connecting $T_C(c)$ and $T^*(r)$, and $T_C(c)$ and $T_C(c) \wedge T^*(0)$, respectively.

3 ESTIMATION AND COMPARISON OF ARL OF THE CUSUM-OAL TEST

In this section we will give an estimation of the ARLs of the following CUSUM-OAL test that can be written as

$$T_C(cg) = \min\{n \ge 1 : \max_{1 \le k \le n} \sum_{i=n-k+1}^n Z_i \ge cg(\hat{Z}_n(ac))\},\tag{7}$$

where g(.) is a decreasing function, $\hat{Z}_n(ac) = \frac{1}{j} \sum_{i=n-j+1}^n Z_i, j = \min\{n, [ac]\}$ for $ac \ge 1, a > 0, c > 0$, and [x] denotes the smallest integer greater than or equal to x. Here $\hat{Z}_n(ac)$ is a sliding average of the statistics, $Z_i, n-j+1 \le i \le n$, which will become $\hat{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i$ when $a = \infty$.

Next we discuss on the post-change probability distribution in order to estimate the ARLs of $T_C(cg)$.

Usually we rarely know the post-change probability distribution P_v of the observation process before it is detected. But the possible change domain and its boundary (including the size and form of the boundary) about v may be determined by engineering knowledge, practical experience or statistical data. So we may assume that the region of parameter space V and a probability distribution Q on V are known. If we have no prior knowledge of the possible value of v after the change time τ , we may assume that v occurs equally on V, that is, the probability distribution Q is an equal probability distribution (or uniform distribution) on V. For example, let P_v be the normal distribution and $v = (\mu, \sigma)$, where μ and σ denote the mean and standard deviation respectively, we can take the set $V = \{(\mu, \sigma) : \mu_1 \leq \mu \leq \mu_2, 0 < \sigma_1 \leq \sigma \leq \sigma_2\}$ and Q is subject to the uniform distribution U(V) on V if v occurs equally on V, where the numbers μ_1, μ_2, σ_1 and σ_2 are known. It means that we know the domain of the possible post-change distributions, $P_v, v \in V$, i.e., the boundary ∂V of the parameter space Vis known.

Next we shall divide the parameter space V into three subsets V^+ , V^0 and V^- by the Kullback-Leibler information distance. Let

$$V^{-} = \{v : E_v(Z_1) < 0\}, \quad V^0 = \{v : E_v(Z_1) = 0\}, \quad V^+ = \{v : E_v(Z_1) > 0\}$$

where $E_v(Z_1) = I(P_v|P_{v_0}) - I(P_v|P_{v_1})$ and

$$I(P_v|P_{v_0}) = E_v(\log[\frac{P_v(X_1)}{P_{v_0}(X_1)}]), \quad I(P_v|P_{v_1}) = E_v(\log[\frac{P_v(X_1)}{P_{v_1}(X_1)}])$$

are two Kullblak-Leibler information distances between P_v , P_{v_0} and P_v , P_{v_1} . Since I(p|q) = 0if and only if p = q, where p and q are two probability measures, it follows that $E_{v_0}(Z_1) = -I(P_{v_0}|P_{v_1})$, and therefore, $v_0 \in V^-$ when $P_{v_1} \neq P_{v_0}$. When $v \in V^-$, i.e., $I(P_v|P_{v_0}) < I(P_v|P_{v_1})$, it means that P_v is closer to P_{v_0} than to P_{v_1} according to the Kullblak-Leibler information distance. There is a similar explanation for $v \in V^+$ or $\in V^0$.

Suppose the post-change distribution P_v and the function g(x) satisfy the following conditions:

(I) The probability P_v is not a point mass at $E_v(Z_1)$ and $P_v(Z_1 > 0) > 0$.

(II) The moment-generating function $h_v(\theta) = E_v(e^{\theta Z_1})$ satisfies $h_v(\theta) < \infty$ for some $\theta > 0$.

(III) The function g(x) is decreasing, its second order derivative function g''(x) is continuous and bounded, and there is a positive number x^* such that $g(x^*) = 0$.

Let
$$\tilde{Z}_1 = Z_1 + |g'(\mu)|(Z_1 - \mu)/a, \ \tilde{h}(\theta) = \mathbf{E}_v(e^{\theta Z_1})$$
 and

$$H_v(\theta) = \frac{au}{g(\mu)} \ln \tilde{h}_v(\theta) + \left(1 - \frac{au}{g(\mu)}\right) \ln h_v(\theta)$$

for $\theta \geq 0$, where $E_v(Z_1) = \mu < 0$, $a \leq g(\mu)/u$ and $u = H'_v(\theta_v^*)$, where $\theta_v^* > 0$ satisfies $H_v(\theta_v^*) = 0$. Note that $H_v(\theta)$ is a convex function and $H'_v(0) = \mu < 0$. It follows that there is a unique positive number $\theta_v^* > 0$ such that $H_v(\theta_v^*) = 0$.

Note that the following function

$$\Theta(x) = \theta(\frac{1}{x}) - xH_v(\theta(\frac{1}{x})) - 2\theta_v^*$$

satisfies that $\Theta(1/u) = \theta(u) - 2\theta_v^* = -\theta_v^*$, where $\theta(u) = \theta_v^*$,

$$\Theta'(x) = -\frac{1}{x^2}\theta'(\frac{1}{x}) - H_v(\theta(\frac{1}{x})) + \frac{1}{x}\theta'(\frac{1}{x})H_v'(\theta(\frac{1}{x})) = -H_v(\theta(\frac{1}{x}))$$

and therefore, $\Theta'(\theta(u)) = -H(\theta(u)) = -H(\theta_v^*) = 0$, $\Theta'(\theta(1/x)) > 0$ for x > 1/u and $\Theta'(\theta(1/x)) < 0$ for x > 1/u. Hence, there exists a positive number b defined in (??).

It can be seen, the main part of $\operatorname{ARL}_v(T_c(g))$ will be an exponential function, square function, and linear function of c when the process $\{Z_k : k \ge 0\}$ has no change or a "small change", a "medium change" and a "large change" from P_{v_0} to P_v , respectively. Here, the "small change" $(v \in V^-)$ means that P_v is closer to P_{v_0} than to P_{v_1} , i.e., $I(P_v|P_{v_0}) < I(P_v|P_{v_1})$, and the "large change" is just the opposite. The "medium change" $(v \in V^0)$ corresponds to $I(P_v|P_{v_0}) = I(P_v|P_{v_1})$.

In this paper, we will use another method to prove Theorem 3 since Wald's identity and the martingale method do not hold or can not work for showing the ARLs estimation of the test $T_c(g)$ when g is not constant.

Next we compare the detection performance of the CUSUM-OAL test $(\mathbf{ARL}_v(T_{c'}(g)))$ with that of the CUSUM test $(\mathbf{ARL}_v(T_C(c)))$ by using (??) in Theorem 4.1.

Let $\operatorname{ARL}_{v_0}(T_{c'}(g)) = \operatorname{ARL}_{v_0}(T_C(c))$ for large c' and c. We have $c = c'\theta_{v_0}^*g(\mu_0) + o(1)$. Hence

$$\operatorname{ARL}_v(T_C(c)) > \operatorname{ARL}_v(T_c(g))$$

for $s_v^* \theta_{v_0}^* > g(\mu) \theta_v^* / g(\mu_0)$ when $\mu_0 < \mu < 0$ and for $\theta_{v_0}^* > g(\mu) / g(\mu_0)$ when $\mu \ge 0$. This means that $\operatorname{ARL}_v(T_c(g))$ can be smaller than $\operatorname{ARL}_v(T_C(c))$ as long as $g(\mu) / g(\mu_0)$ is small for all $\mu > \mu_0$.

4 NUMERICAL SIMULATION AND A REAL EX-AMPLE ILLUSTRATION

4.1 Numerical Simulation of ARLs for $\tau \ge 1$

By the simulation results of ARLs in Table 1, we see that the detection performance of $T^*(r)$, $T_C(c) \wedge T^*(0)$, $T^*(0)$ and $T_C(c\tilde{g}_u)$ for large u is much better than that of the conventional CUSUM test T_C for $\tau = 1$.

The following Table 2 illustrates the simulation values of $\mathbf{E}_{\tau_i,v}$ and \mathbf{J}_{ACE} of nine tests in detecting two mean shifts v = 0.1 and v = 1 after six change-points, $\tau_i, 1 \leq i \leq 6$ with $\text{ARL}_0(T) = \mathbf{E}_0(T) \approx 500$.

Tests	v	$\mathbf{E}_{0,v} \ au_0=0$	$\mathbf{E}_{1,v}$ $\tau_1=1$	$\mathbf{E}_{10,v}$ $ au_2=10$	$\mathbf{E}_{50,v}\\\tau_3=50$	$\underset{\tau_{4}=100}{\mathbf{E}_{100,v}}$	$\mathbf{E}_{150,v} \\ \tau_5 = 150$	$\underset{\tau_{6}=200}{\mathbf{E}_{200,v}}$	\mathbf{J}_{ACE}
		parameter	$\frac{\tau_1=1}{c=4.3867}$	12-10	73-00	14-100	13-100	78-200	
	v = 0.1	498.55	247.25	241.73	243.52	243.87	244.23	243.28	243.98
$T_C(c)$	v=0.1	(493.25)	(241.76)	(243.46)	(263.86)	(288.53)	(314.22)	(339.73)	210.00
IC(C)	v=1	498.55	9.15	8.48	8.46	8.46	8.46	8.46	8.58
	v-1	(493.25)	(5.00)	(5.13)	(5.87)	(6.77)	(7.63)	(8.49)	0.00
		parameter	c=6.5839	(0.10)	(0.01)	(0.11)	(1.00)	(0.45)	
	v=0.1	498.11	8.06	41.52	95.15	136.72	167.20	193.34	107.00
$T_C(c\widetilde{g}_{100})$	v=0.1	(6540.43)	(29.29)	(184.62)	(501.30)	(781.77)	(1013.44)	(1225.66)	107.00
$IC(Cg_{100})$	v=1	498.11	(29.29) 1.25	4.15	9.03	12.38	14.69	16.60	9.68
	v-1	(6540.43)	(0.73)		(34.72)	(56.19)	(72.86)		9.08
				(10.64)	(34.72)	(30.19)	(12.80)	(88.02)	
	0.1		r=0.00137	20.02	07.94	109.01	150.00	170 70	00 55
/ T 1*()	v=0.1	499.43	7.44	38.03	87.34	123.01	150.69	172.79	96.55
$T^*(r)$		(16588.89)	(26.07)	(168.31)	(473.17)	(734.40)	(966.69)	(1164.92)	10.00
	v=1	499.43	1.25	4.11	9.02	12.73	15.51	17.86	10.08
		(16588.89)	(0.71)	(10.72)	(36.12)	(61.00)	(82.62)	(102.32)	
_ ()			c=10.4889						
$T_C(c)$	v=0.1	499.58	7.50	38.30	88.61	125.16	152.58	179.39	98.59
\wedge		(10932.09)	(26.43)	(170.16)	(476.22)	(746.80)	(966.95)	(1202.64)	
$T^{*}(0)$	v=1	499.58	1.25	4.11	9.04	12.10	13.88	15.10	9.25
		(10932.09)	(0.72)	(10.70)	(35.90)	(56.80)	(71.61)	(83.38)	
	v=0.1	∞	7.50	38.58	88.64	125.00	152.92	177.42	98.34
TT * (0)		(∞)	(26.54)	(170.58)	(475.69)	(743.32)	(972.55)	(1191.50)	
$T^*(0)$	v=1	∞	1.25	4.12	9.09	12.77	15.60	18.02	10.14
		(∞)	(0.72)	(10.75)	(36.28)	(60.67)	(82.23)	(101.86)	
		parameter	m = 50	c = 5.8093	p=1		· ·		
	v = 0.1	500.87	164.18	251.68	266.00	266.51	266.07	266.44	246.81
		(686.03)	(239.82)	(348.33)	(377.68)	(399.06)	(418.45)	(438.60)	
	v=1	500.87	2.68	7.34	9.41	9.34	9.35	9.36	7.91
		(686.03)	(2.94)	(7.25)	(9.45)	(10.04)	(10.70)	(11.36)	
		parameter	m=30	c=6.7701	p=1	× /	· /	. /	
	v = 0.1	500.64	174.61	259.52	264.36	264.87	264.67	264.55	248.76
T(1)		(666.64)	(244.09)	(345.62)	(366.62)	(388.06)	(407.65)	(428.29)	
$T_C(ch_m)$	v=1	500.64	2.80	8.07	9.36	9.37	9.36	9.36	8.05
		(666.64)	(3.18)	(7.68)	(8.82)	(9.52)	(10.18)	(10.84)	
		parameter	m=10	c=26.3031	p=1	(((
	v = 0.1	-	218.56	288.10	287.93	287.44	287.59	287.68	276.22
		(613.17)	(276.99)	(356.17)	(375.19)	(398.04)	(422.13)	(447.11)	
	v=1	500.12	3.77	9.35	9.31	9.30	9.30	9.30	8.39
	· - 1	(613.17)	(4.83)	(7.44)	(8.02)	(8.73)	(9.47)	(10.23)	0.00
	v=0.1	500.38	6.89	32.94	$\frac{(8.02)}{58.76}$	46.86	7.57	88.88	40.32
	v-0.1	(15373.79)	(19.02)	(116.99)	(248.01)		(130.62)		40.04
T_M^*		· /			· /	(223.84)		(2208.53)	6 97
111	v=1	500.38	1.25	4.11	9.07	12.79	1.25	9.12	6.27
		(15373.79)	(0.72)	(10.72)	(36.27)	(60.88)	(6.60)	(175.62)	

Table 2. Simulation of $\mathbf{E}_{\tau_i,v}$ and \mathbf{J}_{ACE} for detecting two mean shifts v = 0.1, v = 1.

Tablenotes: The parameters for T_M^* are k1=1, k2=150, $r_1 = 5.2 * 10^{-5}$, $r_2 = 1.1 * 10^{-5}$, and the expectation and standard deviation in both cases are 1717.06 with 13459.80 and 3918.33 with 16893.25, respectively.

4.2 A Real Example

5 CONCLUSION

The contributions of this paper can be summarized to the following three aspects.

(1) We present the optimal test $T^*(r)$ under generalized ARL₀ with finite $\mathbf{E}_0(T^*(r))$.

(2) To enhance the sensitivity of the CUSUM test for detecting the distribution change we propose a CUSUM-type test with a real-time observation control limit (CUSUM-OAL). Numerical simulations show that the out-control ARLs of the CUSUM-OAL tests are significantly smaller than the out-control ARLs of the CUSUM test. But the CUSUM-OAL tests have bigger standard deviations than the CUSUM test in the in-control state. Moreover, we obtain the estimations of the ARLs of the CUSUM-OAL tests. Both theoretical estimations and numerical simulations show that the CUSUM-OAL tests perform much better than the CUSUM test when $\tau = 1$.

Appendix : Proofs of Theorems

Proof of Theorem 1. Let $v \in V^-$. We first prove that

$$e^{cg(\mu)\theta_v^*(1+o(1))}/bc \le E_v(T_c(g)) \le cu^{-1}g(\mu)e^{cg(\mu)\theta_v^*(1+o(1))}$$
(A. 1)

for a large c.

Next we first prove the upward inequality of (36). Let $m_1 = \lceil cu^{-1}g(\mu) \rceil$, $m_k = \lceil km_1 \rceil$ for $k \ge 0$ and $m = \lceil tm_1 \exp\{cg(\mu)\theta_v^*(1+o(1))\}\rceil$ for t > 0, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Without loss of generality, the number $\lceil x \rceil$ will be replaced by x in the following when x is large. It follows that

$$P_{v}(T_{c}(g) > m) = P_{v}(\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac)), \quad 1 \le k \le n, 1 \le n \le m)$$

$$\leq P_{v}(\sum_{i=m_{j}-m_{1}+1}^{m_{j}} Z_{i} < cg(\hat{Z}_{m_{j}}(ac)), \quad 1 \le j \le m/m_{1})$$

$$= [P_{v}(\sum_{i=1}^{m_{1}} Z_{i} < cg(\hat{Z}_{m_{1}}(ac))]^{m/m_{1}}$$
(A. 2)

for a large c, where $\hat{Z}_{m_j}(ac) = (ac)^{-1} \sum_{i=m_j-ac+1}^{m_j} Z_i$ and the last quality holds since the events

$$\{\sum_{i=m_j-m_1+1}^{m_j} Z_i < cg(\hat{Z}_{m_j}(ac))\},\$$

 $1 \leq j \leq m/m_1$, are mutually independent and have an identity distribution. Since $\hat{Z}_{m_1}(ac) - \mu \rightarrow 0(a.s.)$ and $ac(\hat{Z}_{m_1}(ac) - \mu)^2 \Rightarrow \chi^2 (\chi^2$ -distribution) as $c \to \infty$, it follows that

$$g(\hat{Z}_{m_1}(ac)) = g(\mu) + g'(\mu)(\hat{Z}_{m_1}(ac) - \mu) + O(1/c)$$

and

$$P_v(\sum_{i=1}^{m_1} Z_i < cg(\hat{Z}_{m_1}(ac)) = P_v(\sum_{i=m_1-ac+1}^{m_1} \tilde{Z}_i + \sum_{i=1}^{m_1-ac} Z_i < c(g(\mu) + O(1/c)))$$

for a large c, where $\tilde{Z}_i = Z_i + a^{-1} |g'(\mu)| (Z_i - \mu)$. Let

$$\tilde{h}_v(\theta) = E_v(e^{\theta Z_1}), \ h_v(\theta) = E_v(e^{\theta Z_1}).$$

and

$$H_v(\theta) = \frac{au}{g(\mu)} \ln \tilde{h}_v(\theta) + (1 - \frac{au}{g(\mu)}) \ln h_v(\theta).$$

Note that $H_v(\theta)$ is a convex function and $H'_v(0) = \mu < 0$. This means that there is a unique positive number $\theta_v^* > 0$ such that $H_v(\theta_v^*) = 0$. Let $u = H'_v(\theta_v^*)$. It follows from (A.9) that

$$P_{v}\left(\sum_{i=m_{1}-ac+1}^{m_{1}}\tilde{Z}_{i}+\sum_{i=1}^{m_{1}-ac}Z_{i}\geq c(g(\mu)+O(1/c))\right)$$

= $P_{v}\left(\sum_{i=m_{1}-ac+1}^{m_{1}}\tilde{Z}_{i}+\sum_{i=1}^{m_{1}-ac}Z_{i}\geq m_{1}u(1+O(1/c))\right)$
 $\geq \exp\{-m_{1}(\theta u'-H_{v}(\theta)+\frac{1}{m_{1}}\log(F_{\theta}^{m_{1}}(m_{1}u')-F_{\theta}^{m_{1}}(m_{1}u))+o(1))\}$
= $\exp\{-cg(\mu)(\theta\frac{u'}{u}-\frac{1}{u}H_{v}(\theta)+o(1))\}$

for a large c. Taking $\theta\searrow \theta_v^*$ and $u'\searrow u,$ we have

$$P_v(\sum_{i=m_1-ac+1}^{m_1} \tilde{Z}_i + \sum_{i=1}^{m_1-ac} Z_i < c(g(\mu) + O(1/c))) \le 1 - \exp\{-cg(\mu)\theta_v^*(1+o(1))\}$$

for a large c. Thus, by (A.11) we have

$$P_v(T_c(g) > m) \le [1 - \exp\{-cg(\mu)\theta_v^*(1 + o(1))\}]^{m/m_1} \to e^{-t}.$$
 (A. 3)

as $c \to \infty$. By the properties of exponential distribution, we have

$$E_v(T_c(g))) \le cu^{-1}g(\mu)e^{cg(\mu)\theta_v^*(1+o(1))}$$

for a large c.

To prove the downward inequality of (A.10), let

$$U_{m} = \{\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac)), 1 \le k \le ac-1, bc \le n \le m\}$$

$$V_{m} = \{\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac)), ac \le k \le bc-1, bc \le n \le m\}$$

$$W_{m} = \{\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac)), bc \le k \le n, bc \le n \le m\}$$

$$S_{bc} = \{\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac)), 1 \le k \le n, 1 \le n \le bc-1\},$$

where b is defined in (??) and without loss of generality, we assume that b > a. Obviously, $\{T_c(g) > m\} = U_m V_m W_m S_{bc}$.

Let $k = xcg(\mu)$. By Chebyshev's inequality, we have

$$P_{v}(\sum_{i=n-k+1}^{n} Z_{i} < cg(\hat{Z}_{n}(ac))) = P_{v}\left(\sum_{i=n-k+1}^{n} \tilde{Z}_{i} + \sum_{i=n-ac+1}^{n-k} \tilde{\tilde{Z}}_{i} < cg(\mu)(1+o(1))\right)$$

$$\geq 1 - \exp\{-cg(\mu)(\theta - x\tilde{H}_{v}(\theta) + o(1))\}$$

for $1 \le k \le ac - 1$, $bc \le n \le m$, where $\tilde{\tilde{Z}}_i = -g'(\mu)(Z_i - \mu)/a$ and

$$\tilde{H}_v(\theta) = \ln \tilde{h}_v(\theta) + \left(\frac{ac}{k} - 1\right) \ln \hat{h}_v(\theta), \quad \hat{h}_v(\theta) = E_v(e^{\theta \tilde{Z}_i}).$$

Since $\tilde{H}_v(\theta)$ and $H_v(\theta)$ are two convex functions and

$$\begin{split} \tilde{H}'_v(0) - H'_v(0) &= 0, \\ \tilde{H}''_v(0) - H''_v(0) &= \sigma^2 [(1 + \frac{g'(\mu)}{a})^2 + (\frac{ac}{k} - 1) - \frac{au}{g(\mu)}(1 - \frac{g'(\mu)}{a})^2 + \frac{au}{g(\mu)} - 1] > 0, \end{split}$$

it follows that $\tilde{\theta}_v^* \ge \theta_v^*$, where $\tilde{\theta}_v^*$ and θ_v^* satisfy $\tilde{H}_v(\tilde{\theta}_v^*) = H_v(\theta_v^*) = 0$. Hence

$$P_{v}\left(\sum_{i=n-k+1}^{n} \tilde{Z}_{i} + \sum_{i=n-ac+1}^{n-k} \tilde{\tilde{Z}}_{i} < cg(\mu)(1+o(1))\right) \\ \ge 1 - \exp\{-cg(\mu)\theta_{v}^{*}(1+o(1))\}$$
(A. 4)

for $1 \le k \le ac - 1$, $bc \le n \le m$. Similarly, we can get

$$P_{v}\left(\sum_{i=n-ac+1}^{n} \tilde{Z}_{i} + \sum_{i=n-k+1}^{n-ac} Z_{i} < cg(\mu)(1+o(1))\right)$$

$$\geq 1 - \exp\{-cg(\mu)\theta_{v}^{*}(1+o(1))\}$$
(A. 5)

for $ac \leq k \leq bc - 1$, $bc \leq n \leq m$, and

$$P_{v}\left(\sum_{i=n-ac+1}^{n} \tilde{Z}_{i} + \sum_{i=n-k+1}^{n-ac} Z_{i} < cg(\mu)(1+o(1))\right) \\ \geq 1 - \exp\{-2cg(\mu)\theta_{v}^{*}(1+o(1))\}$$
(A. 6)

for $bc \leq k \leq n$, $bc \leq n \leq m$.

Let $m = tcg(\mu)\theta_v^*/bc$ for t > 0. By (A.13), (A.14), (A.15) and Theorem 5.1 in Esary, Proschan and Walkup (1967) we have

$$P_{v}(U_{m}V_{m}) \geq \prod_{n=bc}^{m} \prod_{k=1}^{bc-1} P_{v}(\sum_{i=n-k+1}^{n} \tilde{Z}_{i} - \frac{g'(\mu)}{a} \sum_{i=n-ac+1}^{n-k} (Z_{i} - \mu) < cg(\mu)(1 + o(1)))$$

$$\geq [1 - \exp\{-cg(\mu)\theta_{v}^{*}(1 + o(1))\}]^{bcm} \rightarrow e^{-t}$$

and

$$P_{v}(W_{m}) \geq \prod_{n=bc}^{m} \prod_{k=bc}^{n} P_{v}(\sum_{i=n-ac+1}^{n} \tilde{Z}_{i} + \sum_{i=n-k+1}^{n-ac} Z_{i} < cg(\mu)(1+o(1)))$$

$$\geq [1 - \exp\{-2cg(\mu)\theta_{v}^{*}(1+o(1))\}]^{(m-bc)^{2}} \to 1$$

as $c \to +\infty$.

Finally,

$$P_{v}(S_{bc}) \geq P_{v}(\sum_{i=n-k+1}^{n} Z_{i} < cg_{0}, \ 1 \le k \le n, \ 1 \le n \le bc-1)$$

$$\geq \prod_{n=1}^{bc-1} \prod_{k=1}^{n} (1 - \exp\{-cg_{0}\theta + k \ln h_{v}(\theta)\})$$

$$\geq [1 - \exp\{-cg_{0}\theta_{0}\}]^{(bc)^{2}} \to 1$$

as $c \to +\infty$, where $\theta_0 > 0$ satisfies $h_v(\theta_0) = 1$.

Thus

$$P_v(T_c(g) > m) = P_v(U_m V_m W_m S_{bc}) \searrow e^{-t}.$$

as $c \to \infty$. This implies that

$$E_v(T_c(g)) \ge e^{cg(\mu)\theta_v^*(1+o(1))}/bc$$

for a large c. This completes the proof of (A.10).

Let $v \in V^0$.

Let $m_1 = (cg(0))^2/\sigma^2$. It follows that

$$E_{v}(T_{c}(g)) = \sum_{n=0}^{\infty} P_{v}(T_{c}(g) > n)$$

$$\leq m_{1} + \sum_{n=m_{1}}^{2m_{1}} P_{v}(T_{c}(g) > n) + \dots + \sum_{n=km_{1}}^{(k+1)m_{1}} P_{v}(T_{c}(g) > n) + \dots$$

$$\leq m_{1}[1 + \sum_{k=1}^{\infty} P_{v}(T_{c}(g) > km_{1})].$$

Note that

$$P_{v}(T_{c}(g) > km_{1}) \leq P_{v}\left(\sum_{i=(j-1)m_{1}+1}^{jm_{1}} Z_{i} + \sum_{i=jm_{1}-ac+1}^{jm_{1}} Z_{i}' < cg(0)(1+o(1)), \ 1 \leq j \leq k\right)$$

$$= \left[P_{v}\left(\sum_{i=1}^{m_{1}} Z_{i} + \sum_{i=m_{1}-ac+1}^{m_{1}} Z_{i}' < cg(0)(1+o(1))\right)\right]^{k}$$

$$= \left[P_{v}\left(\sum_{i=1}^{m_{1}-ac} Z_{i} + \sum_{i=m_{1}-ac+1}^{m_{1}} (1+A)Z_{i} < cg(0)(1+o(1))\right)\right]^{k}$$

$$= \left[P_{v}\left(\frac{\sum_{i=1}^{m_{1}-ac} Z_{i}}{cg(0)} + \frac{\sum_{i=m_{1}-ac+1}^{m_{1}} (1+A)Z_{i}}{cg(0)} < (1+o(1))\right)\right]^{k}$$

for a large c, where A = |g'(0)|/a, and

$$\frac{\sum_{i=1}^{m_1 - ac} Z_i}{cg(0)} \Rightarrow X \sim N(0, 1), \quad \frac{\sum_{i=m_1 - ac+1}^{m_1} (1+A) Z_i}{cg(0)} \to 0$$

as $c \to \infty$. Thus

$$E_v(T_c(g)) \le m_1[1 + \sum_{k=1}^{\infty} [\Phi(1+o(1))]^k] = \frac{(cg(0))^2}{\sigma^2} \frac{(1+o(1))}{1-\Phi(1)},$$

where $\Phi(.)$ is the standard normal distribution.

Let $m_2 = (cg(0))^2 / (8\sigma^2 \ln c)$. Note that

$$\prod_{n=1}^{ac} \prod_{k=1}^{n} P_v \Big(\sum_{i=n-k+1}^{n} Z_i + c |g'(0)| n^{-1} \sum_{i=1}^{n} Z_i < cg(0)(1+o(1)) \Big) \\
\geq \left[P_v \Big(\sum_{i=1}^{ac} Z_i + \sum_{i=1}^{ac} Z'_i < cg(0)(1+o(1)) \Big) \right]^{(ac)^2} = (1+o(1))(1 - \Phi(\frac{\sqrt{cg(0)}}{\sqrt{a}(1+A)}))^{(ac)^2} \to 1$$

as $c \to \infty$, since

$$P_v(\frac{\sum_{i=1}^{ac} Z_i + \sum_{i=1}^{ac} Z'_i}{\sigma(1+A)\sqrt{ac}}) \Rightarrow X \sim N(0,1)$$

as $c \to \infty$. It follows that

$$E_{v}(T_{c}(g)) \geq \sum_{n=0}^{m_{2}} P_{v}(T_{c}(g) > n) \geq m_{2}P_{v}(T_{c}(g) > m_{2})$$

$$\geq m_{2}(1+o(1)) \prod_{n=ac+1}^{m_{1}} \prod_{k=1}^{n} P_{v} \Big(\sum_{i=n-k+1}^{n} Z_{i} + \sum_{i=n-ac+1}^{n} Z_{i}' < cg(0)(1+o(1)) \Big)$$

$$\geq m_{2}(1+o(1)) [P_{v} \Big(\sum_{i=1}^{m_{2}} Z_{i} + \sum_{i=m_{2}-ac+1}^{m_{2}} Z_{i}' < cg(0)(1+o(1)) \Big)]^{m_{2}^{2}}$$

$$= m_{2}(1+o(1)) [P_{v} \Big(\frac{\sum_{i=1}^{m_{2}} Z_{i} + \sum_{i=m_{2}-ac+1}^{m_{2}} Z_{i}'}{\sqrt{m_{2}\sigma}} < \sqrt{8 \ln c}(1+o(1)) \Big)]^{m_{2}^{2}}$$

$$= m_{2}(1+o(1)) [\Phi(\sqrt{8 \ln c})]^{m_{2}^{2}} = m_{2}(1+o(1)) [1 - \frac{1}{c^{4}\sqrt{8 \ln c}}]^{m_{2}^{2}} \rightarrow m_{2}(1+o(1))$$

as $c \to \infty$, where the third inequality comes from Theorem 5.1 in Esary, Proschan and Walkup (1967). Thus, we have

$$\frac{(cg(0))^2}{8\sigma^2 \ln c} (1+o(1)) \le E_v(T_c(g)) \le \frac{(cg(0))^2}{\sigma^2(1-\Phi(1))} (1+o(1)).$$

Let $v \in V^+$ and let

$$T_0 = \min\{n : \sum_{i=1}^n Z_i + \frac{ac}{(ac) \wedge n} \sum_{i=n-(ac) \wedge n+1}^n Z'_i \ge c\}.$$

The uniform integrability of $\{T_c(g)/c\}$ for $c \ge 1$, follows from the well-known uniform integrability of $\{T_0/c\}$ (see Gut (1988)).

By the Strong Large Number Theorem we have

$$\mu = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} Z_i}{n}$$

=
$$\lim_{n \to \infty} \max_{1 \le j \le n} \frac{1}{n} [\sum_{i=j}^{n} Z_i + g'(\mu) a^{-1} \sum_{i=n-ac+1}^{n} (Z_i - \mu)]$$

Note that $T_c(g) \to \infty$ as $c \to \infty$,

$$\max_{1 \le j \le T_c(g)} \left[\sum_{i=j}^{T_c(g)} Z_i + g'(\mu) a^{-1} \sum_{i=T_c(g)-ac+1}^{T_c(g)} (Z_i - \mu) \right] \ge cg(\mu)(1 + o(1)),$$

and

$$\max_{1 \le j \le T_c(g) - 1} \left[\sum_{i=j}^{T_c(g) - 1} Z_i + g'(\mu) a^{-1} \sum_{i=T_c(g) - ac}^{T_c(g) - 1} (Z_i - \mu)\right] \le cg(\mu)(1 + o(1)).$$

It follows that

$$\begin{split} \mu &\longleftarrow \max_{1 \le j \le T_c(g)} \frac{1}{T_c(g)} [\sum_{i=j}^{T_c(g)} Z_i + g'(\mu) a^{-1} \sum_{i=T_c(g)-ac+1}^{T_c(g)} (Z_i - \mu)] \\ \ge \quad \frac{cg(\mu)(1 + o(1))}{T_c(g)} \\ \ge \quad \max_{1 \le j \le T_k - 1} \frac{1}{T_c(g) - 1} [\sum_{i=j}^{T_c(g)-1} Z_i + g'(\mu) a^{-1} \sum_{i=T_c(g)-ac}^{T_c(g)-1} (Z_i - \mu)] \longrightarrow \mu \end{split}$$

as $c \to \infty$. By the uniform integrability of $\{T_c(g)/c\}$ and using Theorem A.1.1 in Gut's book (1988), we have

$$E_v(T_c(g)) = (1 + o(1))\frac{cg(\mu)}{\mu}$$

for a large c. This completes the proof of Theorem 2.

Proof of Theorem 4. Since g(x) < 0 for $x > a^*$, $a^* \le \mu^*$ and $\mu^* \ge 0$, it follows that

$$P_v\left(m\hat{Z}_m < cg(\hat{Z}_m), \ \hat{Z}_m > a^*\right) \le P_v(\hat{Z}_m < \mu^*)$$

and

$$\begin{aligned} P_v(T_c(g) > m) &= P_v \Big(\sum_{i=n-k+1}^n Z_i < cg(\hat{Z}_n), \ 1 \le k \le n, \ 1 \le n \le m \Big) \le P_v \Big(m \hat{Z}_m < cg(\hat{Z}_m) \Big) \\ &= P_v \Big(m \hat{Z}_m < cg(\hat{Z}_m), \ \hat{Z}_m \le a^* \Big) + P_v \Big(m \hat{Z}_m < cg(\hat{Z}_m), \ \hat{Z}_m > a^* \Big) \\ &\le 2P_v(\hat{Z}_m < \mu^*). \end{aligned}$$

Furthermore,

$$P_{v}(\hat{Z}_{m} < \mu^{*}) = P_{v}(\sum_{i}^{m} -Z_{i} > -m\mu^{*}) = P_{v}(\sum_{i}^{m} (\mu - Z_{i}) > m(\mu - \mu^{*}))$$
$$= P_{v}(e^{\theta \sum_{i}^{m} (\mu - Z_{i})} > e^{\theta m(\mu - \mu^{*})}) \le e^{-m[\theta(\mu - \mu^{*}) - \ln M(\theta)]},$$

where $M(\theta) = E_v(e^{\theta(\mu-Z_1)})$ and the last inequality follows from Chebychev's inequality. Note that $h(\theta) = \theta(\mu - \mu^*) - \ln M(\theta)$ attains its maximum value $h(\theta^*) = \theta^*(\mu - \mu^*) - \ln M(\theta^*) > 0$ at $\theta = \theta^* > 0$, where $h'(\theta^*) = 0$. So,

$$E_v(T_c(g)) = 1 + \sum_{m=1}^{\infty} P_v(T_c(g) > m) \le 1 + 2\sum_{m=1}^{\infty} e^{-m[\theta^*(\mu - \mu^*) - \ln M(\theta^*)]} = \frac{e^{\theta^*(\mu - \mu^*) - \ln M(\theta^*)} + 1}{e^{\theta^*(\mu - \mu^*) - \ln M(\theta^*)} - 1}.$$

Let k > 1. It follows that

$$E_{vk}(T_c(g) - k + 1)^+ = \sum_{m=1}^{\infty} P_{vk}(T_c(g) > m + k - 1, T_c(g) > k - 1)$$

$$\leq (a_0 + 1)(k - 1)P_0(T_c(g) > k - 1) + \sum_{m \ge (a_0 + 1)(k - 1)}^{\infty} P_{vk}(T_c(g) > m + k - 1).$$

Similarly, we have

$$\begin{aligned} &P_{vk}(T_c(g) > m+k-1) \\ &= P_{vk}\Big(\sum_{i=n-k+1}^n Z_i < cg(\hat{Z}_n), \ 1 \le k \le n, \ 1 \le n \le m+k-1\Big) \le 2P_{vk}(\hat{Z}_{m+k-1} < \mu^*) \\ &= 2P_{vk}\Big(\sum_{i=k-1}^{m+k-1} (\mu - Z_i) + \sum_{i=1}^{k-1} (\mu_0 - Z_i) > m(\mu - \mu^*) + (k-1)(\mu_0 - \mu^*)\Big) \\ &\le 2\exp\{-m\Big(\theta^*(\mu - \mu^*) - \ln M(\theta^*) + \frac{k-1}{m}[\mu_0 - \mu^* - \ln M_0(\theta^*)]\Big)\} \le e^{-mb} \end{aligned}$$

for $m \ge (a_0 + 1)(k - 1)$, since

$$\theta^*(\mu - \mu^*) - \ln M(\theta^*) + \frac{k-1}{m} [\mu_0 - \mu^* - \ln M_0(\theta^*)] \ge b$$

for $m \ge (a_0 + 1)(k - 1)$. Thus,

$$E_{vk}(T_c(g) - k + 1)^+ \leq (a_0 + 1)(k - 1)P_0(T_c(g) \geq k) + 2\sum_{\substack{m \geq (a_0 + 1)(k - 1)\\ \leq (a_0 + 1)(k - 1)P_0(T_c(g) \geq k) + \frac{2e^{-(a_0 + 1)(k - 1)b}}{1 - e^{-b}}}.$$

References

- [1] Basseville, M. and Nikiforov, I. (1993) Detection of Abrupt Changes : Theory and Applications. Prentice-Hall, Englewood Cliffs.
- [2] Bersimis, S., Sgora, A. and Psarakis, S. (2018) The application of multivariate statistical process monitoring in non-industrial processes. *Qual. Reliab. Engng. Int.*, 15, 526-549.
- [3] Bersimis, S., Psarakis, S. and Panaretos, J. (2007) Multivariate statistical process control charts: An Overview. *Qual. Reliab. Engng. Int.*, **23**, 517-543.
- [4] Celano, G. and Castagliola, P. (2018). An EWMA sign control chart with varying control limits for finite horizon processes. *Qual. Reliab. Engng. Int.*, 34, 1717-C1731
- [5] Chakrabortia, S. and Graham, M. A. (2019). Nonparametric (distribution-free) control charts: An updated overview and some results. *Qual. Engineering*, DOI: 10.1080/08982112.2018.1549330
- [6] Chatterjee, S. and Qiu, P. (2009). Distribution-free cumulative sum control charts using bootstrap-based control limits. Ann. Appl. Statistics. 3, 349-369.
- [7] Durrett, R. (2000). Probability Theory and Examples. Fourth Edition, Cambridge University Press.
- [8] Frisén, M. (2003) Statistical Surveillance, Optimality and Methods. Int. Statist. Rev., 71, 403-434.
- [9] Huang, W. P., Shu, L. J., Woodall, W. H. and Tsui, K. L. (2016). CUSUM procedures with probability control limits for monitoring processes with variable sample sizes. *IIE Trans.* 48, 759-771.
- [10] Lai, T. L. (2001) Sequential analysis: some classical problems and new challenges. Statist Sinica, 11, 303-408.
- [11] Lai, T. L. (1995). Sequential change-point detection in quality control and dynamical systems. J. R. Statist Soc. B. 57, 613-658.
- [12] Lai, T. L. (1998). Information Bounds and Quick Detection of Parameter Changes in Stochastic Systems. *IEEE Trans. Inf. Theory*, 44, 2917-2929.
- [13] Lee, P. H., Huang, Y. H., Kuo, T. I. and Wang, C. C. (2013). The effect of the individual chart with variable control limits on the river pollution monitoring. *Qual Quant*, 47, 1803-1812.
- [14] Margavio, T. M., Conerly, M. D., Woodall, W. H. and Drake, L. G. (1995). Alarm rates for quality control charts. *Stat. Probab. Lett.*, 24, 219-C224

- [15] Montgomery, D. C. (2009) Introduction to Statistical Quality Control. 6th ed. New York: John Wiley & Sons.
- [16] Page, E. S. (1954). Continuous inspection schemes, *Biometrika* 41, 100-115
- [17] Poor, H.V. and Hadjiliadis, O. (2009). Quickest Detection, Cambridge University Press, Camridge, New York.
- [18] Qiu, P. (2014) Introduction to Statistical Process Control. Boca Raton, FL: Chapman and Hall/CRC.
- [19] Shen, X., Zou, C. L., Jiang, W. and Tsung, F. G. (2013) Monitoring Poisson count data with probability control limits when sample sizes are time varying. Naval Research Logistics, 60, 625-C636
- [20] Shewhart, W. A. (1931) Economic Control of Quality of Manufactured Product. New York: Van Nostrand.
- [21] Siegmund, D. (1985). Sequential Analysis : Tests and Confidence Intervals. Springer, New York.
- [22] Sogandi, F., Aminnayeri, M., Mohammadpour, A. and Amiri, A. (2019) Risk-adjusted Bernoulli chart in multi-stagehealthcare processesbased onstate-space modelwith alatent riskvariable and dynamic probability control limits. *Comp. Industr. Engin.* 130, 699-713
- [23] Steiner, S. H. (1999). EWMA control charts with time-varying control limits and fast initial response. J. Quality Technology, 31, 75-86.
- [24] Stoumbos, Z. G., Reynolds, M. R., Ryan, T. P., and Woodall, W. H. (2000) The state of statistical process control as we proceed into the 21st century. J. Amer. Statist. Assoc., 95, 992-998.
- [25] Verdier, G., Hilgert, N. and Vila, J. P. (2008). Adaptive threshold computation for CUSUM-type procedures in change detection and isolation problems. *Computational Statistics. & Data Analysis*, 52, 4161-4171.
- [26] Wald, A. and Wolfowitz, J. (1948). Optimum character of the squential propability ratio test. Ann. Math. Sattist., 19, 326-339.
- [27] Woodall, W. H., Zhao, M. J., Paynabar, K., Sparks, R. and Wilson, J. D. (2017) An overview and perspective on social network monitoring. *IIE Trans.*, 49, 354-365.
- [28] Yang, W. W., Zou, C. L. and Wang, Z. J. (2017). Nonparametric profile monitoring using dynamic probability control limits. *Qual. Reliab. Engng. Int.*, 33, 1131-C1142
- [29] Zhang, X. and Woodal, W. H. (2015). Dynamic probability control limits for risk-adjusted Bernoulli CUSUM chart. Statist. Med., 34, 3336-C3348

[30] Zhang, X. and Woodal, W. H. (2017). Reduction of the effect of estimation error on incontrol performance for risk-adjusted Bernoulli CUSUM chart with dynamic probability control limits. *Qual. Reliab. Engng. Int.*, 33, 381-C386